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Isogeny orbits in a family of abelian varieties

by

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1. Introduction. In this paper we are interested in a curve of an abelian scheme that contains infinitely many isogeny orbits of a finitely generated group of a simple abelian variety. We prove that it is special. Zilber–Pink have conjectured, roughly speaking, that a subvariety containing many special points must be special. This generalizes many well-known problems including conjectures of Mordell–Lang, Manin–Mumford and André–Oort. Special points considered in this paper are isogeny orbits which are closely related to the generalized Hecke orbits considered by Zilber–Pink. Therefore our result fits into the context of their conjecture.

Let S be a smooth irreducible algebraic curve over $\overline{\mathbb{Q}}$, and $\pi : A \to S$ be an abelian scheme. An *abelian scheme* $A \to S$ refers to a smooth proper group scheme with geometrically connected fibers. Then A can be regarded as a smooth family of abelian varieties over S. Take an abelian variety A' defined over $\overline{\mathbb{Q}}$ and a finitely generated group $\Gamma \subset A'(\overline{\mathbb{Q}})$. We call a point $q \in A_t(\overline{\mathbb{Q}})$, where $t \in S(\overline{\mathbb{Q}})$, *special* if there exist an isogeny $\phi : A' \to A_t$ and $\gamma \in \Gamma$ with $\phi(\gamma) = q$. In this paper we prove

Theorem 1.1. *Assume that A' is simple and A is nonisotrivial. If an irreducible Zariski closed algebraic curve X of A, over $\overline{\mathbb{Q}}$, dominates S and contains infinitely many special points, then there exists a positive integer n such that $[n]X(\overline{\mathbb{Q}}) = 0$.*

Although not used in our proof, under our assumptions a theorem of Orr [O] implies that X is indeed finite over a special curve. His result together with ours gives a more precise description of X. Moreover, a recent work of Gao [G] touches this kind of problem by a different approach. In particular, if Γ is rank 1 then our result is covered by his Theorem 1.6.

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We cannot remove the condition that A' is simple, because the general version of Bertrand's Lemma 3.1 does not readily provide a sufficient lower bound for the canonical heights of all elements of the finitely generated group. The proof of our result is based on the comparison of heights, for which we need Bertrand's lemma to quantify that the height of a nontorsion special point gets bigger very fast as the degree of isogeny gets bigger. This is not the case if A' is not simple, where certain abelian subvarieties of A' must be taken into account.

In the next section we prove a partially stronger result in the case of a family of elliptic curves, and indicate two major obstructions preventing that argument from working in the case of families of abelian varieties. In Section 3 we use the polyhedral reduction theory to give a new proof of the result of Bertrand, which is crucial for this paper. In Section 4 we present the proof of the main theorem. The basic strategy of our paper is to compare different heights in number theory, including geometric Faltings height, Néron–Tate height and Weil height.

Throughout this paper $h_F(A)$ refers to the geometric Faltings height of an abelian variety A over $\overline{\mathbb{Q}}$, $h_{X,D} : X(\overline{\mathbb{Q}}) \to \mathbb{R}$ refers to a Weil height function of a variety over $\overline{\mathbb{Q}}$ with respect to the divisor D, and $\hat{h}_{A,D}$: $A(\overline{\mathbb{Q}}) \to \mathbb{R}$ refers to the canonical height function of an abelian variety over $\overline{\mathbb{Q}}$ with respect to a symmetric divisor D. Two linearly equivalent divisors, respectively isomorphic line bundles, are connected by $\sim$, respectively $\cong$. The group of linear equivalence classes of divisors or line bundles of X is $\mathrm{Pic}(X)$. For a complete nonsingular curve X over $\overline{\mathbb{Q}}$, we have a canonical surjective homomorphism $\deg : \mathrm{Pic}(X) \to \mathbb{Z}$. Let $\mathcal{L}$ be an invertible sheaf of an abelian variety A, $\chi(\mathcal{L})$ be its Euler characteristic, and $\lambda_{\mathcal{L}}$ be the morphism from $A(\overline{\mathbb{Q}})$ to $\mathrm{Pic}(A)$. The subgroup $\mathrm{Pic}^0(A)$ of $\mathrm{Pic}(A)$ consists of invertible sheaves $\mathcal{L}$ for which $\lambda_{\mathcal{L}} \equiv 0$. The Néron–Severi group of A is denoted by $\mathrm{NS}(A)$. Points of the dual abelian or Picard variety $A^\vee$ of A parametrize the elements of $\mathrm{Pic}^0(A)$. A homomorphism of abelian varieties $\phi : A \to B$ gives rise to a dual $\phi^\vee : B^\vee \to A^\vee$. Write $\mathrm{End}^0(A) = \mathrm{End}(A) \otimes \mathbb{Q}$, on which we have another deg function. The set of isomorphism classes of pairs (A, λ) with A an abelian variety of dimension g and λ a polarization of A of degree $d = \chi(\lambda)$ is parametrized by the Siegel modular variety $M_{g,d}$; in particular $M_{1,1} = \mathbb{A}^1$. If Γ is a finitely generated abelian group then we let Γ_t, respectively Γ_{nt}, be the torsion subgroup, respectively one of its complements.

2. Isogeny orbits in a family of elliptic curves. In this section we let S be a smooth irreducible algebraic curve over $\overline{\mathbb{Q}}$, and $\pi : A \to S$ an abelian scheme of relative dimension one. Then A can be regarded as a smooth family of elliptic curves over S. Take an elliptic curve A' defined over $\overline{\mathbb{Q}}$ and a $p \in A'(\overline{\mathbb{Q}})$. We call $q \in A_t(\overline{\mathbb{Q}})$, where $t \in S(\overline{\mathbb{Q}})$, *special* if there exists either an isogeny $\phi : A' \to A_t$ with $\phi(p) = q$, or an isogeny $\phi : A_t \to A'$ with $\phi(q) = p$. We prove

Proposition 2.1. *Assume A is nonisotrivial. If an irreducible Zariski closed algebraic curve X of A, over $\overline{\mathbb{Q}}$, contains infinitely many special points, then either X is some special fiber A_t that is isogenous to A', or there exists a positive integer n such that $[n]X(\overline{\mathbb{Q}}) = 0$.*

Proof. Firstly we assume that X is not any fiber A_t, otherwise there is nothing to prove. Secondly we notice that it suffices to prove the result under the assumption that X is the image of a section $s : S \to A$ of $\pi : A \to S$. Indeed, in the general case let X' be a smooth resolution of X; then $A \times_S X' \to X'$ is also a smooth family of elliptic curves over X', and we write $f : X' \to A$ to be the natural morphism.

$$\begin{tikzcd}
X' \arrow[dr, dotted, "s"] \arrow[drr, bend left, "\mathrm{id}"] \arrow[ddr, bend right, "f"'] & & \\
& A \times_S X' \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & X' \arrow[d, "\pi\circ f"'] \\
& A \arrow[r, "\pi"] & S
\end{tikzcd}$$

The above commutative diagram provides a section $s : X' \to A \times_S X'$ of $A \times_S X'/X'$. Moreover, it is easy to check that $s(X') \subset A \times_S X'$ contains infinitely many special points if and only if $X \subset A$ does, and that $[n]X = 0$ if and only if $([n] \circ s)X = 0$. Therefore the general case reduces to the special case that X comes from a section.

If p is a torsion point and A' has complex multiplication, then our assertion is a special case of a result of André [A] (see also [Pi]). If p is a torsion point and A' has no complex multiplication, then by a lemma of Habegger [H, Lemma 5.8] there are only finitely many elliptic curves isogenous to A' with bounded height (the proof of Habegger's statement relies heavily on the work of Szpiro and Ullmo). This makes André's argument valid line by line after replacing Poonen's lemma [Po] by Habegger's lemma. From here on we assume that p is not torsion.

Given an elliptic curve E over $\overline{\mathbb{Q}}$ we write $\hat{h}_E : E(\overline{\mathbb{Q}}) \to \mathbb{R}_{\geq 0}$ for the canonical height function with respect to the divisor given by the zero section (0) of $E/\overline{\mathbb{Q}}$. For any other symmetric divisor D of E, the associated canonical height function satisfies

$$\hat{h}_{E,D} = \deg D \cdot \hat{h}_E,$$

as $D \sim \deg D(0)$ for any symmetric divisor D of E. Before proceeding we notice that if $\phi : E' \to E$ is an isogeny of elliptic curves over $\overline{\mathbb{Q}}$ with $\phi(a) = b$, where a, b are closed points, then

$$\hat{h}_E(b) = \hat{h}_{E',\phi^*(0)}(a) = \deg(\phi^*(0))\hat{h}_{E'}(a) = \deg \phi \cdot \hat{h}_{E'}(a).$$

Let D be the divisor given by the zero section of the abelian scheme A/S. Then the canonical height function on $A_t(\overline{\mathbb{Q}})$ with respect to $\deg D_t$ is simply $\hat{h}_{A_t}$, for any $t \in S(\overline{\mathbb{Q}})$. In view of the nonisotriviality of A, the modular map $j : S(\overline{\mathbb{Q}}) \to \mathbb{A}^1(\overline{\mathbb{Q}})$ is nonconstant. Without ambiguity we write $h : \mathbb{A}^1(\overline{\mathbb{Q}}) \to \mathbb{R}$ for the standard Weil height function on $\mathbb{A}^1$ and $h : s \in S(\overline{\mathbb{Q}}) \to h(j(s)) \in \mathbb{R}$ for the Weil height function on S with respect to $j^*((0))$.

There are two types of special points: the back orbit of p and the forward orbit of p. We first assume that X contains infinitely many back orbits q_i $(i = 1, 2, \ldots)$ of p. If $q_i \in A_{t_i}$, where $t_i \in S(\overline{\mathbb{Q}})$, then $q_i = s(t_i)$. Let $\phi_i : A_{t_i} \to A'$ be the isogeny that satisfies $\phi_i(q_i) = p$. Then

$$\hat{h}_{A_{t_i}}(q_i) = \frac{\hat{h}_{A'}(p)}{\deg \phi_i}. \tag{2.1}$$

The lemma of Habegger [H, Lemma 5.8] shows that there are only finitely many elliptic curves over $\overline{\mathbb{Q}}$ within the isogeny class of E' with bounded Weil height. Using the nonisotriviality of A, given any elliptic curve E_1, we see that there are only finitely many i such that E_{t_i} is isomorphic to E_1 over $\overline{\mathbb{Q}}$. These two facts clearly lead to

$$\lim_{i\to\infty} h(t_i) = \infty. \tag{2.2}$$

By Theorem B of [S1] there is a constant C such that

$$\lim_{h(t)\to\infty} \frac{\hat{h}_{A_t}(s(t))}{h(t)} = C. \tag{2.3}$$

Because of (2.2) we can apply (2.3) to t_i and obtain

$$\lim_{i\to\infty} \frac{\hat{h}_{A_t}(q_i)}{h(t_i)} = C.$$

Using (2.1) we have

$$\lim_{i\to\infty} \frac{\hat{h}_{A'}(p)}{\deg \phi_i \cdot h(t_i)} = C.$$

As p is not torsion, we have $h_{A'}(p) > 0$, and therefore the above identity gives $C = 0$. Recall that in Silverman's specialization theorem [S1] the constant C being 0 means that the canonical height of X regarded as a point in the abelian variety A_η over the generic point is zero. By the nonisotriviality of A and the fact that A is of relative dimension one, the $\overline{\mathbb{Q}}(S)/\overline{\mathbb{Q}}$ trace of A is trivial. This implies that when X is regarded as a point of the abelian variety A_η over the generic fiber, X must be a torsion point. There exists

$n \in \mathbb{Z}_{>0}$ such that $[n]X(\overline{\mathbb{Q}}) = 0$, contrary to our assumption that p is not torsion.

Now we assume that X contains infinitely many forward orbits $\{q_i\}_{i=1}^{\infty}$ of p. Let $\phi_i : A' \to A_{t_i}$ be the isogeny that satisfies $\phi_i(p) = q_i$. Then

$$\hat{h}_{A_{t_i}}(q_i) = \deg \phi_i \cdot \hat{h}_{A'}(p). \tag{2.4}$$

The inequality of Faltings [F] implies that

$$h_F(A_{t_i}) \le h_F(A') + \log(\deg \phi_i)/2. \tag{2.5}$$

We claim that $\lim_{i\to\infty} \deg \phi_i = \infty$. Otherwise there are infinitely many i such that A_{t_i} are isomorphic to each other over $\overline{\mathbb{Q}}$, contradicting the fact that A is nonisotrivial. We also claim that

$$\lim_{i\to\infty} \frac{h_F(A_{t_i})}{h(t_i)}$$

is a positive constant. This follows from the property (2.2), the fact that A is nonisotrivial and Proposition 2.1 of [S2]. These claims combined with (2.4), (2.5) and $\hat{h}_{A'}(p) > 0$ lead to

$$\lim_{i\to\infty} \frac{\hat{h}_{A_{t_i}}(s(t_i) = q_i)}{h(t_i)} = \infty.$$

Since (2.2) is still valid, this contradicts Silverman's specialization theorem (2.3). ∎

The above argument does not work in the context of families of abelian varieties for the following reasons. Firstly the relation between canonical heights of points on isogenious abelian varieties is not as simple as in (2.1). Secondly the lemma of Habegger is not known in the higher-dimensional case. More precisely, we do not know whether within an isogeny class of abelian varieties there are only finitely many ones with bounded height. Without this result we have no validity of (2.2) in general, which is essential if we want to directly apply Silverman's specialization theorem.

3. Polyhedral reduction theory and canonical heights. In this section we give a proof of Lemma 3.1 below, based on the polyhedral reduction theory [AMRY]. Actually this lemma is not new. G. Rémond pointed out to us that it is equivalent to the main theorem of Bertrand [B] in the case of simple abelian varieties, linked by the theorem of Mordell–Weil. We still present the proof here, as our approach is rather distinct from Bertrand's original one.

Throughout this section, A is an abelian variety over $\overline{\mathbb{Q}}$. Any symmetric line bundle $\mathcal{L}$ of A defines a canonical height function $\hat{h}_{A,\mathcal{L}}$ that is quadratic on $A(\overline{\mathbb{Q}})$. We remark that $\hat{h}_{A,\mathcal{L}}$ depends only on the class of $\mathcal{L}$ in $\mathrm{NS}(A)$.

Indeed, if a symmetric line bundle $\mathcal{L}$ maps to zero in the short exact sequence

$$0 \to \mathrm{Pic}^0(A) \to \mathrm{Pic}(A) \to \mathrm{NS}(A) \to 0,$$

then $\mathcal{L} \in \mathrm{Pic}^0(A)$. This leads to $[-1]^*\mathcal{L} = \mathcal{L}^{-1}$. Together with $[-1]^*\mathcal{L} = \mathcal{L}$ we have $\mathcal{L}^2 = 0$, and therefore $\hat{h}_{A,\mathcal{L}} \equiv 0$. Hence $\hat{h}_{A,\mathcal{L}}$ is also a well-defined function when $\mathcal{L}$ is an element in $\mathrm{NS}(A)$.

When A is a simple abelian variety, a recent result of Kawaguchi and Silverman [KS] tells us that for any nonzero nef symmetric $\mathcal{L} \in \mathrm{Pic}(A) \otimes \mathbb{R}$ the canonical height $\hat{h}_{A,\mathcal{L}}(x)$ is zero if and only if $x \in A(\overline{\mathbb{Q}})_t$.

The endomorphism algebra $\mathrm{End}^0(A)$ is semisimple and contains $\mathrm{End}(A)$ as a lattice. The unit group $(\mathrm{End}^0(A) \otimes \mathbb{R})^\times$ is reductive, and $\mathrm{Aut}(A)$ is an arithmetic group. The function deg extends to a homogeneous function of degree $2g$ on $\mathrm{End}^0(A) \otimes \mathbb{R}$.

For $x \in A(\overline{\mathbb{Q}})$ let $T_x : A \to A$ be the translation induced by x on A. For any line bundle $\mathcal{L}$ the theorem of square leads to a group homomorphism

$$\lambda_{\mathcal{L}} : A(\overline{\mathbb{Q}}) \to A^\vee(\overline{\mathbb{Q}})$$

which takes x to $T_x^*\mathcal{L} \otimes \mathcal{L}^{-1}$. It is an isogeny if and only if $\mathcal{L}$ is ample.

From here on we fix an ample line bundle $\mathcal{N}$, which defines a Rosati involution $\dagger$ of $\phi \in \mathrm{End}^0(A)$ by $\phi^\dagger = \lambda_{\mathcal{N}}^{-1} \circ \phi^\vee \circ \lambda_{\mathcal{N}}$. The map

$$\mathrm{NS}(A)_{\mathbb{Q}} \to \mathrm{End}^0(A)$$

defined by $\mathcal{L} \mapsto \lambda_{\mathcal{N}}^{-1} \circ \lambda_{\mathcal{L}}$ identifies $\mathrm{NS}(A)_{\mathbb{Q}}$ with the subset of $\mathrm{End}^0(A)$ of elements fixed by $\dagger$. Given $\phi \in \mathrm{Aut}(A)$ and $\mathcal{L} \in \mathrm{Pic}(A)$, it is straightforward to check that $\lambda_{\phi^*(\mathcal{L})} = \phi^\vee \circ \lambda_{\mathcal{L}} \circ \phi$. This extends to an action of $\mathrm{End}^0(A)$ on $\mathrm{NS}(A)_{\mathbb{Q}} \subset \mathrm{End}^0(A)$ given by $\alpha^\phi = \phi^\dagger \circ \alpha \circ \phi$. The bilinear form

$$\langle \phi, \psi \rangle \mapsto \mathrm{Tr}(\phi \circ \psi^\dagger)$$

on $\mathrm{End}^0(A) \times \mathrm{End}^0(A)$ is positive definite.

As a finite-dimensional algebra over $\mathbb{R}$ with a positive involution, $\mathrm{End}^0(A) \otimes \mathbb{R}$ is isomorphic to $\prod_i M_{r_i}(\mathbb{R}) \times \prod_j M_{s_j}(\mathbb{C}) \times \prod_k M_{t_k}(\mathbb{H})$ where the involution is given by conjugations. Under this identification the real vector space $\mathrm{N}^1(A) = \mathrm{NS}(A) \otimes \mathbb{R}$ and the ample cone $\mathrm{Amp}(A)$ are isomorphic to $\prod_i H_{r_i}(\mathbb{R}) \times \prod_j H_{s_j}(\mathbb{C}) \times \prod_k H_{t_k}(\mathbb{H})$ and $\prod_i P_{r_i}(\mathbb{R}) \times \prod_j P_{s_j}(\mathbb{C}) \times \prod_k P_{t_k}(\mathbb{H})$ respectively, where H_r is the space of symmetric, or respectively Hermitian symmetric, matrices and P_r consists of the positive ones. Let $G(\mathrm{Amp}(A))$ be the automorphism group of the cone $\mathrm{Amp}(A)$, and $G(\mathrm{Amp}(A))^0$ its identity component. The homomorphism $(\mathrm{End}^0(A) \otimes \mathbb{R})^\times \to G(\mathrm{Amp}(A))^0$ is surjective. Let $\mathrm{Amp}_+(A)$ be the convex hull of the rational points of $\overline{\mathrm{Amp}(A)}$, which could be larger than $\mathrm{Amp}(A)$. According to Ash's main result of polyhedral reduction theory, there exists a (topologically closed) rational polyhedral cone $F \subset \mathrm{Amp}_+(A)$ such that $\mathrm{Aut}(A) \cdot F = \mathrm{Amp}_+(A)$. For more details we refer to [AMRY] and [Pr].

Using the theorem of Ash we give a new proof of

LEMMA 3.1 (Bertrand). *Let A be a simple abelian variety defined over $\overline{\mathbb{Q}}$, and let $\Gamma \subset A(\overline{\mathbb{Q}})$ be a finitely generated group. Then there exists a constant $C > 0$ depending on Γ and A such that for any symmetric ample divisor $\mathcal{M} \in \mathrm{Pic}(A)$ and nontorsion $x \in \Gamma$,*

$$\hat{h}_{A,\mathcal{M}}(x) \geq C(\chi(\mathcal{M}))^{1/g}.$$

Proof. It is well-known that $\mathrm{End}(A)$ is of finite rank, hence $\mathrm{End}(A)(\Gamma)$ is also a finitely generated group. Therefore it suffices to prove the lemma under the assumption that Γ is invariant under $\mathrm{Aut}(A)$.

By the polyhedral reduction theory [AMRY], there is a rational polyhedral fundamental domain $F \subset \mathrm{Amp}_+(A)$ under the action of $\mathrm{Aut}(A)$. The rationality of F guarantees that there is a basis

$$\{v_1, \ldots, v_t\} \subset \mathrm{NS}(A) \cap \overline{\mathrm{Amp}(A)}$$

such that if $w \in F \cap \mathrm{NS}(A)$, then there are nonnegative real numbers r_i with $w = \sum_{i=1}^t r_i v_i$.

We have $\Gamma = \Gamma_t + \Gamma_{nt}$. Because v_i are nef, a result of Kawaguchi–Silverman [KS] tells us that $\hat{h}_{A,v_i}$ is a positive bilinear function on the finitely generated abelian Γ_{nt}, therefore there exists a positive constant c_1 such that

$$\hat{h}_{A,v_i}(\gamma) \geq c_1 \tag{3.1}$$

for all $1 \leq i \leq t$ and all nonzero $\gamma \in \Gamma_{nt}$. Furthermore for any $\mathcal{M} \in \mathrm{Amp}(A)$ and $x = x_1 + x_2 \in \Gamma$ we have $\hat{h}_{A,\mathcal{M}}(x) = \hat{h}_{A,\mathcal{M}}(x_2)$. Consequently, the inequality (3.1) is valid for all $1 \leq i \leq t$ and nontorsion $\gamma \in \Gamma$.

Take a symmetric ample divisor with image $\mathcal{M} \subset F$. Then there are nonnegative real numbers r_i such that $\mathcal{M} = \sum_{i=1}^t r_i v_i$. In particular for any nontorsion $\gamma \in \Gamma$ we have

$$\hat{h}_{A,\mathcal{M}}(\gamma) \geq c_1 \max_{i=1}^{t}\{r_i\}. \tag{3.2}$$

The degree function $\deg : \mathrm{End}^0(A) \otimes \mathbb{R} \to \mathbb{R}$ is homogeneous of degree $2g$. Let $\deg \mathcal{M}$ be the degree of the image of $\mathcal{M}$ in $\mathrm{End}^0(A) \otimes \mathbb{R}$. Then

$$\deg \mathcal{M} = (\deg \lambda_{\mathcal{M}})/\deg \lambda_{\mathcal{N}} = c_2 \chi(\mathcal{M})^2 \tag{3.3}$$

where $c_2 = 1/\deg \lambda_{\mathcal{N}}$. The homogeneity of $\deg : \mathrm{End}^0(A) \otimes \mathbb{R} \to \mathbb{R}$ implies that there is a positive constant c_3 which depends only on v_i but not on $\mathcal{M}$ such that

$$\deg \mathcal{M} \leq c_3 \sum_{i=1}^{t} r_i^{2g}. \tag{3.4}$$

Now (3.2)–(3.4) obviously yield a constant $C > 0$ such that

$$\hat{h}_{A,\mathcal{M}}(\gamma) \geq C(\chi(\mathcal{M}))^{1/g}$$

for all $\mathcal{M} \subset F \cap \mathrm{Amp}(A)$ and nontorsion $\gamma \in \Gamma$.

For general $\mathcal{M} \in \mathrm{Amp}(A)$, there exists $\sigma \in \mathrm{Aut}(A)$ with $\sigma^*(\mathcal{M}) \in F$. We have assumed that $\sigma^{-1}(x) \in \Gamma$, and it is clear that $\chi(\sigma^*(\mathcal{M})) = \chi(\mathcal{M})$. Therefore, for any nontorsion $\gamma \in \Gamma$,

$$\hat{h}_{A,\mathcal{M}}(\gamma) = \hat{h}_{A,\sigma^*(\mathcal{M})}(\sigma^{-1}(\gamma)) \geq C(\chi(\mathcal{M}))^{1/g},$$

which proves what has been claimed in the lemma. ■

Notice that the proof is completely geometric and makes no use of the number field at all. Indeed, by this same proof we can even remove the condition that Γ is defined over $\overline{\mathbb{Q}}$, with the usual height replaced by Moriwaki's arithmetic function height.

This approach might be generalized to give a proof of the full theorem of Bertrand [B] on abelian varieties, and we shall come back to this point in the near future.

4. Proof of the main theorem. It is unknown to us whether in an isogeny class of abelian varieties there are only finitely many ones with bounded height. Therefore we cannot directly use Silverman's specialization theorem as before. Instead we shall use some arguments of [S1] to prove our main theorem.

Before going to the proof we point out that if the curve X in Theorem 1.1 is not finite over S, then Faltings' theorem on the Mordell–Lang conjecture easily implies that X is a translate of an algebraic subgroup of a special fiber that is isogenous to A'. The simplicity of A' forces A/S to be an elliptic scheme, and then X goes to a special elliptic fiber of A/S.

Proof of Theorem 1.1. Firstly we may assume that Γ is invariant under $\mathrm{Aut}(A')$. Secondly by the same trick used in the proof of Proposition 2.1 we assume that X is the image of a section $s : S \to A$ of $\pi : A \to S$. We write $\epsilon : S \to A$ for the zero section. Throughout the proof we shall compare different kinds of heights that are constructed under some number field as field of definition. It will be essential for us to check carefully that all these heights and corresponding inequalities are indeed independent of the number field chosen.

In the decomposition $\Gamma = \Gamma_t + \Gamma_{nt}$, Γ_t is finite, and so there exists a positive integer n such that $[n]\Gamma_t = 0$. If there are infinitely many $t \in S(\overline{\mathbb{Q}})$ such that there exists $\gamma_t \in \Gamma_t$ and an isogeny $\phi_t : A' \to A_t$ with $\phi_t(x_t) = s(t)$, then we also have $[n]s(t) = 0$. This implies that $[n]X(\overline{\mathbb{Q}})$ intersects the zero section infinitely many times. This leads to $[n]X(\overline{\mathbb{Q}}) = 0$.

Now we assume that there are infinitely many distinct $t_i \in S(\overline{\mathbb{Q}})$ $(i \in \mathbb{N})$ such that there exist $\gamma_i \in \Gamma \setminus \Gamma_t$ and isogenies $\phi_i : A' \to A_{t_i}$ with $\phi_i(\gamma_i) = s(t_i)$.

By a theorem of Grothendieck [R2, Theorem XI.1.4], A is projective over S (we thank the referee for this reference). Let $\mathcal{L}_1$ be a relatively ample bundle on A/S, and $\iota : A \to A$ the map that sends x to $-x$. As everything is defined over $\overline{\mathbb{Q}}$, there exists $\mathcal{R} \in A^\vee$ such that $\mathcal{R}^2 = \iota^*\mathcal{L}_1 \otimes \mathcal{L}_1^{-1}$. Replacing $\mathcal{L}_1$ by a power of $\mathcal{L}_1 \otimes \mathcal{R}$ we assume that $\mathcal{L}_1$ is a symmetric very ample invertible sheaf on A. If we replace S by a suitable étale open set, there will exist a symmetric theta structure. Multiplying the theta structure with a power of two, we assume that all its elementary divisors are divisible by 4. Let δ be the type of $\mathcal{L}_1$. Then by the theory of theta functions [I], [Mu] there exists a canonical closed immersion

$$A \hookrightarrow \mathbb{P}(V_\delta) \times S$$

where V_δ is the vector space defined in [Mu]. Let $\overline{A}$ be the closure of A in $\mathbb{P}(V_\delta) \times \overline{S}$ and use i to denote the immersion

$$\overline{A} \hookrightarrow \mathbb{P}(V_\delta) \times \overline{S}.$$

Take $\mathcal{L}'$ to be a very ample line bundle of $\overline{S}$, and let π_i be the projection of $\mathbb{P}(V_\delta) \times \overline{S}$ to its ith factor. We write $\mathcal{L} = i^*(\pi_1^*(\mathcal{O}_{\mathbb{P}}(1)) \otimes \pi_2^*(\mathcal{L}'))$, which is very ample on $\overline{A}$. Moreover, $\mathcal{L}_s$ at any $s \in S$ is isomorphic to the symmetric sheaf $(\mathcal{L}_1)_s$. Later we shall apply Silverman's arguments in [S1] and Lemma 3.1 to $\mathcal{L}$ and $\mathcal{L}_s$.

The Euler characteristic $\chi(\mathcal{L}_t)$ is a constant function of $t \in S(\overline{\mathbb{Q}})$, and we denote it by d. Because A is nonisotrivial, the modular map $j : S \to M_{g,d}$ is not constant. We claim that

$$\lim_{i \to \infty} \deg \phi_i = \infty.$$

Indeed, otherwise there are infinitely many $t \in S(\overline{\mathbb{Q}})$ such that A_t are all isomorphic to each other over $\overline{\mathbb{Q}}$. According to a geometric finiteness theorem [Mi, Theorem 18.1], given any abelian variety A^0 and $d \in \mathbb{N}$ there are only finitely many isomorphism classes of polarized abelian varieties (A^0, λ) with λ of degree d. These two facts together force the modular map $j : S \to M_{g,d}$ to be constant. This contradicts the nonisotriviality of A and proves the claim.

Concerning isogenies ϕ_i of abelian varieties, Faltings' inequality [F] gives

$$h_F(A_{t_i}) \le h_F(A') + \log(\deg \phi_i)/2. \tag{4.1}$$

Under the isogeny, the canonical heights satisfy $\hat{h}_{A_{t_i}, \mathcal{L}_{t_i}}(s(t_i)) = \hat{h}_{A', \phi_i^*(\mathcal{L}_{t_i})}(\gamma_i)$. The Euler characteristics satisfy $\chi(\phi_i^*(\mathcal{L}_{t_i})) = \deg \phi_i \cdot \chi(\mathcal{L}_{t_i}) = d \deg \phi_i$. We apply the lemma in the last section to the fixed abelian variety A' to find a positive constant c_1 depending only on A', $\mathcal{L}$ and Γ such that, for all $i \in \mathbb{N}$,

$$\hat{h}_{A_{t_i}, \mathcal{L}_{t_i}}(s(t_i)) \geq c_1(\deg \phi_i)^{1/g}. \tag{4.2}$$

As mentioned before, $\mathcal{L}$ gives a projective embedding of $\overline{A}$. By this condition a theorem of Silverman–Tate [S1, Theorem A] applies, and consequently there exist positive constants c_2 and c_3 depending only on A/S, $\mathcal{L}$ and s such that for all $i \in \mathbb{N}$,

$$|\hat{h}_{A_{t_i}, \mathcal{L}_{t_i}}(s(t_i)) - h_{\overline{A}, \mathcal{L}}(s(t_i))| < c_2 h_{\overline{S}, \epsilon^*(\mathcal{L})}(t_i) + c_3. \tag{4.3}$$

Notice that Silverman's theorem is over $\overline{\mathbb{Q}}$, and therefore the constants c_2 and c_3 can be chosen for all t_i. Because $h_{\overline{A}, \mathcal{L}}(s(t_i)) = h_{\overline{S}, s^*(\mathcal{L})}(t_i)$ and because both $\epsilon^*(\mathcal{L})$ and $s^*(\mathcal{L})$ are ample, there exist positive constants c_4 and c_5 such that

$$h_{\overline{A}, \mathcal{L}}(s(t_i)) \leq c_4 h_{\overline{S}, \epsilon^*(\mathcal{L})}(t_i) + c_5. \tag{4.4}$$

Because this inequality is of geometric nature, the constants c_4 and c_5 can be chosen independent of the points t_i ($i \in \mathbb{N}$) and of their fields of definition.

As indicated in [Z], Zarhin's trick works for families, and therefore $B = (A \times A^\vee)^4$ is an abelian scheme over S with principal polarization. Because a constant family of abelian varieties contains no nonconstant subfamily, B is also nonisotrivial. The modular map $J : S \to M_{g,1}$ attached to B with respect to this principal polarization is nonconstant. Let $\mathcal{N}$ be an ample line bundle of the Baily–Borel compactification of $M_{g,1}$. By another inequality (Lemma 4.1 below) of Faltings [F], there exist positive constants $\bar{c}_1$ and $\bar{c}_2$ such that for all $i \in \mathbb{N}$,

$$h_{\overline{S}, J^*(\mathcal{N})}(t_i) - h_F((A_{t_i} \times A_{t_i}^\vee)^4) \leq \bar{c}_1 + \bar{c}_2 \log(\max(1, h_{\overline{S}, J^*(\mathcal{N})}(t_i))). \tag{4.5}$$

By a result of Raynaud [R1, Corollaire 2.1.3], for all $i \in \mathbb{N}$,

$$h_F((A_{t_i} \times A_{t_i}^\vee)^4) = 8 h_F(A_{t_i}). \tag{4.6}$$

The modular map J extends to a map from $\overline{S}$ to the Baily–Borel compactification of $M_{g,1}$, and the zero section map ϵ extends to a map from $\overline{S}$ to $\overline{A}$, and both $J^*(\mathcal{N})$ and $\epsilon^*(\mathcal{L})$ are positive on $\overline{S}$. By the fact that there exist positive numbers $\bar{c}_3$ and $\bar{c}_5$ such that $\bar{c}_3 J^*(\mathcal{N}) - \epsilon^*(\mathcal{L})$ and $\bar{c}_5 \epsilon^*(\mathcal{L}) - J^*(\mathcal{N})$ are ample, there exist real constants $\bar{c}_4$ and $\bar{c}_6$ such that for all $i \in \mathbb{N}$,

$$h_{\overline{S}, \epsilon^*(\mathcal{L})}(t_i) < \bar{c}_3 h_{\overline{S}, J^*(\mathcal{N})}(t_i) + \bar{c}_4, \tag{4.7}$$

$$h_{\overline{S}, J^*(\mathcal{N})}(t_i) < \bar{c}_5 h_{\overline{S}, \epsilon^*(\mathcal{L})}(t_i) + \bar{c}_6. \tag{4.8}$$

With (4.7) applied to the first term of the left hand side of (4.5); (4.6) applied to the second term of the left hand side of (4.5); and (4.8) applied to the second term of the right hand side of (4.5), there exist positive constants

c_6, c_7 and c_8 such that for all $i \in \mathbb{N}$,

$$c_6 h_{\overline{S},\epsilon^*(\mathcal{L})}(t_i) - h_F(A_{t_i}) \le c_7 + c_8 \log(\max(1, h_{\overline{S},\epsilon^*(\mathcal{L})}(t_i))). \tag{4.9}$$

Notice that although this is not explicitly mentioned in [F], one can check carefully (or see the lemma below) that c_8 is independent of the number field K used there. Combining (4.2)–(4.4) shows that there exist positive constants c_9 and c_{10} such that for all $i \in \mathbb{N}$,

$$c_9(\deg \phi_i)^{1/g} \le h_{\overline{S},\epsilon^*(\mathcal{L})}(t_i) + c_{10}. \tag{4.10}$$

Combining (4.1) and (4.9) implies that there exist positive constants c_{11}, c_{12} and c_{13} such that for all $i \in \mathbb{N}$,

$$h_{\overline{S},\epsilon^*(\mathcal{L})}(t_i) \le c_{11} + c_{12} \log(\max(1, h_{\overline{S},\epsilon^*(\mathcal{L})}(t_i))) + c_{13} \log(\deg \phi_i). \tag{4.11}$$

It is clear that (4.10) contradicts (4.11) as $\deg \phi_i$ goes to infinity. ■

Lastly we sketch a calculation to make sure that the positive constants c_3, c_4 obtained in [F, p. 356] (we shall write c_7 and c_8 instead of c_3 and c_4 used there) are independent of the number fields. The referee has pointed out that more concrete estimates comparing Faltings heights and certain Weil heights have already appeared in the literature [Pa]. For our purpose a slight modification of Faltings' argument already suffices, and this remark applies also to some more general schemes than the Siegel modular ones.

LEMMA 4.1. *Let $X \subset \mathbb{P}^n_{\mathbb{Z}}$ be Zariski closed, $Y \subset X$ closed, $\| \;\|$ an Hermitian metric on $\mathcal{O}(1)|(X(\mathbb{C}) - Y(\mathbb{C}))$ with logarithmic singularities along Y, and $\| \;\|_1$ an Hermitian metric on $\mathcal{O}(1)|X(\mathbb{C})$. For $x \in X(\overline{\mathbb{Q}}) - Y(\overline{\mathbb{Q}})$ one defines $h(x)$ and $h_1(x)$ as in* [F]. *There exist positive constants c_7 and c_8 such that for all $x \in X(\overline{\mathbb{Q}}) - Y(\overline{\mathbb{Q}})$ we have*

$$|h(x) - h_1(x)| \le c_7 + c_8 \log(\max(1, h_1(x))).$$

Proof. After replacing $\mathcal{O}(1)$ by $\mathcal{O}(s)$ with some positive integer s, we may assume that Y is the intersection of X with a linear subspace, and the set of common zeros of $f_1, \ldots, f_r \in \Gamma(X/\mathbb{Z}, \mathcal{O}(1))$ is exactly Y (see [F]). By multiplying the metric we assume $\|f_i\|_1 \le 1$. A rational point $x \in X(K) - Y(K)$ corresponds to $\rho : \mathrm{Spec}(R) \to X$, where R is the integer ring of the number field K. We assume $f_1(x) \ne 0$. By definition we have

$$[K : \mathbb{Q}] h_1(x) \ge \sum_{\sigma} - \log \|f_1\|_1(\sigma(x))$$

and $[K : \mathbb{Q}]|h(x) - h_1(x)| = |\sum_\sigma \log(\|f_1\|/\|f_1\|_1)(\sigma(x))|$, where σ runs through all embeddings $K \hookrightarrow \mathbb{C}$. Because of logarithmic singularities of the metric there exist positive constants c_7', c_8' such that for every number field K and a K-rational point x,

$$[K : \mathbb{Q}]|h(x) - h_1(x)| \le c_7'[K : \mathbb{Q}] + c_8' \sum_{\sigma} \log\bigl(- \log \|f_1\|_1(\sigma(x))\bigr).$$

Furthermore,

$$\sum_{\sigma} \log\bigl(-\log \|f_1\|_1(\sigma(x))\bigr) \le \log\Bigl(\sum_{\sigma} -\log \|f_1\|_1(\sigma(x))/[K:\mathbb{Q}]\Bigr)^{[K:\mathbb{Q}]} \le [K:\mathbb{Q}] \log h_1(x).$$

The above inequalities prove the desired claim. ■

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