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On subset sums of a fixed set

by

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1. Introduction. Let A be a strictly increasing sequence of positive integers. Let $P(A) = \{\sum \varepsilon_i a_i : a_i \in A,\ \varepsilon_i = 0 \text{ or } 1\}$ be the set of all the subset sums of A. A is said to be *subcomplete* if $P(A)$ contains an infinite arithmetic progression. P. Erdős conjectured that if $a_{n+1}/a_n \to 1$, then A is subcomplete. But J. W. S. Cassels [1] proved that for every $\varepsilon > 0$ there exists a sequence A for which $a_{n+1} - a_n = o(a_n^{1/2+\varepsilon})$ and A is not subcomplete. Let $A(n) = A \cap [1, n]$. The cardinality of a finite set S is denoted by $|S|$. In 1962 Erdős [2] proved that if $|A(n)| > cn^{(\sqrt{5}-1)/2}$ $(c > 0)$, then A is subcomplete. In 1966 J. Folkman [3] improved this to $|A(n)| > n^{1/2+\varepsilon}$ $(\varepsilon > 0)$, and recently, N. Hegyvári [5] showed $|A(n)| > 300\sqrt{n \log n}$ for $n > n_0$. A similar result was also proved by Łuczak and Schoen [6] independently. In this note we improve Hegyvári's result:

THEOREM 1. *There exists an absolute constant $C > 1$ such that if $A = \{a_1 < a_2 < \ldots\}$ is an infinite sequence of positive integers with $|A(n)| > C\sqrt{n}$ for $n > C^2$, then A is subcomplete.*

COROLLARY. *There exists an absolute constant $c > 0$ such that if $A = \{a_1 < a_2 < \ldots\}$ is an infinite sequence of positive integers with $a_n < cn^2$ for $n > c^{-1}$, then A is subcomplete.*

This is best possible (cf. [5]). The Corollary gives a partial answer to a question in [3].

2. Preliminaries

LEMMA 1. *Let $B = \{b_1 < b_2 < \ldots\}$ be an infinite sequence of positive integers, and let n_0', d be positive integers with $n_0' > 4$. Assume that $b_i \geq$*

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$i+4d$ for all i and $|B(n)|\ge 2\sqrt{dn}$ for $n\ge n_0'$. Then there exists an infinite sequence $\{v_1<v_2<\ldots\}$ in $P(B)$ for which $d\,|\,v_n$ and $v_{n+1}-v_n<dn_0'$.

Proof. By the Erdős–Ginzburg–Ziv theorem (cf. [7]), for each $i\ge 1$, there exists d integers $b_{i_1},\ldots,b_{i_d}$ in $\{b_{2(i-1)d+1},\ldots,b_{2id}\}$ such that $u_i=b_{i_1}+\ldots+b_{i_d}$ is divisible by d. Then $u_1<u_2<\ldots$ Let $U=\{u_1,u_2,\ldots\}$ and $P(U)=\{v_1<v_2<\ldots\}$. Then $d\,|\,v_n$ for all n. Now we show that $v_{n+1}-v_n<dn_0'$.

Since $|B(n)|\ge 2\sqrt{dn}$ for $n\ge n_0'$, it follows that if $b_n\ge n_0'$, then $b_n\le n^2/(4d)$. Hence $b_{2d}<n_0'$. Let n_1 be the largest integer with $b_{2n_1d}<n_0'$. Then $n_1\ge 1$. For $n\ge 1$, let j_1 be the least integer such that $\sum_{1\le i\le j_1}u_i>v_n$; j_2 be the least integer such that $\sum_{1\le i\le j_2}u_i>v_n-u_{j_1}$; and so on. Thus we have defined $j_1>\ldots>j_t=1$ such that

$$\sum_{1\le i\le j_k}u_i>v_n-u_{j_1}-\ldots-u_{j_{k-1}},\qquad k=2,3,\ldots,t.$$

Define $u_{j_0}=0$. Since $u_{j_0}<v_n<u_{j_0}+u_{j_1}+\ldots+u_{j_t}$, there exists an integer l with $0\le l<t$ such that

$$u_{j_0}+u_{j_1}+\ldots+u_{j_l}\le v_n<u_{j_0}+u_{j_1}+\ldots+u_{j_{l+1}}. \tag{1}$$

Suppose that $j_{l+1}>n_1$. Then $b_{2j_{l+1}d}\ge n_0'$ and

$$\begin{aligned}v_n-u_{j_0}-\ldots-u_{j_l}<u_{j_{l+1}}&\le db_{2j_{l+1}d}\le d\,\frac{(2j_{l+1}d)^2}{4d}\\&\le d\sum_{1\le i\le j_{l+1}-1}(2(i-1)d+4d+1)\\&\le d\sum_{1\le i\le j_{l+1}-1}b_{2(i-1)d+1}\le\sum_{1\le i\le j_{l+1}-1}u_i,\end{aligned}$$

contrary to the definition of j_{l+1}. Hence $j_{l+1}\le n_1$. Thus, by (1) and $j_{l+1}\le n_1$,

$$v_{n+1}-v_n\le u_{j_{l+1}}\le db_{2j_{l+1}d}<dn_0'.$$

This completes the proof of Lemma 1.

LEMMA 2. *Let $0<a_1<\ldots<a_k\le n$ be an increasing sequence of integers. Assume that $n>2500$ and $k>200\sqrt{n\log n}$. Then there exist integers d,y,z such that $1\le d\le 50\sqrt{n/\log n}$, $z-y>\frac{3}{7}n\log n$ and*

$$\{td:y\le t\le z\}\subset P(\{a_1,\ldots,a_k\}).$$

Proof. By Theorem 4 in [8] (see also [4]) there exist integers d, y, z with

$$1\le d\le 10^4nk^{-1},\qquad z\ge 7^{-1}\cdot 10^{-4}k^2,\qquad y\le 7\cdot 10^4znk^{-2}$$

and

$$\{td:y\le t\le z\}\subset P(\{a_1,\ldots,a_k\}).$$

For these d, y, z we have

$$1 \le d \le 50\sqrt{\frac{n}{\log n}}, \qquad z - y \ge 7^{-1} \cdot 10^{-4} k^2 (1 - 7 \cdot 10^4 n k^{-2}) \ge \frac{3}{7}\, n \log n.$$

This completes the proof of Lemma 2.

LEMMA 3. *Let* $A(D_i, H_i) = \{a_i + tD_i : 0 \le t \le H_i\}$ $(i = 0, 1, \ldots, l)$ *be arithmetic progressions of integers. Assume that*

$$H_i \ge D_0 + D_{i+1}, \quad i = 0, 1, \ldots$$

Then there exists an arithmetic progression $A(d_l, h_l) = \{a + td_l : 0 \le t \le h_l\} \subset A(D_0, H_0) + \ldots + A(D_l, H_l)$ *with* $d_l = (D_0, \ldots, D_l)$ *and* $h_l \ge H_l - D_0$.

This follows from the proof of Lemma 4 in [5].

3. Proof of Theorem 1. In this section we prove the following theorem which implies Theorem 1.

THEOREM 2. *Let* $A = \{a_1, a_2, \ldots\}$ *be an infinite sequence of positive integers*, n_0 *be an integer with* $n_0 \ge e^{60}$ *and* $w = 50\sqrt{n_0/\log n_0}$. *Assume that* $|A(n)| \ge 202\sqrt{12wn}$ *for* $n_0 \le n \le e^{12wn_0}$ *and* $|A(n)| \ge 808\sqrt{wn}$ *for* $n > e^{12wn_0}$. *Then* A *is subcomplete.*

Proof. Let $n_i = n_{i-1}^2$ $(i = 1, 2, \ldots)$. Let l be the integer with $n_{l-1} < e^{6w} \le n_l$. Then $n_l = n_{l-1}^2 < e^{12w}$. Let $n_{-1} = 0$. Hence

$$\begin{aligned} |A(n_i) \setminus A(n_{i-1})| &\ge 202\sqrt{n_i \log n_i} - n_{i-1} \ge 202\sqrt{n_i \log n_i} - \sqrt{n_i} \\ &> 200\sqrt{n_i \log n_i} + 1, \qquad i = 0, 1, \ldots, l. \end{aligned}$$

Take $B_i = A(n_i) \setminus A(n_{i-1})$ for $i = 0, 1, \ldots, l-1$ and $B_l \subset A(n_l) \setminus A(n_{l-1})$ with $|B_l| = [200\sqrt{n_l \log n_l}] + 1$. By Lemma 2, for each i, there exists an arithmetic progression

$$A_i = \{a_i + D_i k : 0 \le k \le H_i\} \subset P(B_i)$$

with $D_i \le 50\sqrt{n_i/\log n_i}$ and $H_i \ge \frac{3}{7} n_i \log n_i$. Since

$$H_i \ge \frac{3}{7}\, n_i \log n_i > 50\sqrt{\frac{n_{i+1}}{\log n_{i+1}}} + w \ge D_{i+1} + D_0,$$

by Lemma 3 there exists an arithmetic progression

$$A(d, h) = \{a + dk : 0 \le k \le h\} \subset A_0 + \ldots + A_l$$

with $d \mid D_0$ and $h \ge H_l - D_0$.

For $n_l \le n \le e^{12w}$ we have

$$\begin{aligned} |A(n) \setminus (B_l \cup A(n_{l-1}))| &\ge 202\sqrt{n \log n} - 200\sqrt{n_l \log n_l} - 1 - n_{l-1} \\ &\ge 2\sqrt{n \log n_l} - 1 - \sqrt{n_l} \ge 2\sqrt{6}\,\sqrt{wn} - 1 - \sqrt{n_l} \\ &\ge 2\sqrt{wn} \ge 2\sqrt{D_0 n} \ge 2\sqrt{dn}. \end{aligned}$$

For $n > e^{12w}$ we have

$$\begin{aligned}|A(n) \setminus (B_l \cup A(n_{l-1}))| &\geq 808\sqrt{wn} - 200\sqrt{n_l \log n_l} - 1 - \sqrt{n_l} \\ &\geq 808\sqrt{wn} - 200\sqrt{12}\,\sqrt{wn} - 1 - \sqrt{n_l} \geq 2\sqrt{wn}.\end{aligned}$$

Let $A \setminus (B_l \cup A(n_{l-1})) = \{b_1 < b_2 < \ldots\}$. Then

$$b_1 \geq n_{l-1} = \sqrt{n_l} \geq e^{3w} \geq 4w + 1 \geq 4D + 1 \geq 4d + 1.$$

Hence $b_i \geq b_1 + i - 1 \geq i + 4d$. By Lemma 1 there exists an infinite sequence

$$\{v_1 < v_2 < \ldots\} \subseteq P(A \setminus (B_l \cup A(n_{l-1})))$$

with $d \mid v_i$ and $v_{i+1} - v_i < n_l d$. Since $D_0(n_l + 1) < 2wn_l < \frac{3}{7} n_l \log n_l$, we have

$$n_l d \leq n_l D_0 < \frac{3}{7} n_l \log n_l - D_0 \leq H_l - D_0 \leq h.$$

Hence

$$\begin{aligned}A(d,h) + \{v_1, v_2, \ldots\} &= \{a + v_1 + dk : k = 0, 1, \ldots\} \\ &\subseteq P(B_l \cup A(n_{l-1})) + P(A \setminus (B_l \cup A(n_{l-1}))) \\ &\subseteq P(A).\end{aligned}$$

This completes the proof of Theorem 2.

Proof of the Corollary. Let C be as in Theorem 1 and $c = \frac{1}{4}C^{-2}$. If $ci^2 \leq n$, then $a_i \leq n$. Hence $|A(n)| \geq [\sqrt{n/c}] \geq [2C\sqrt{n}] > C\sqrt{n}$. Then the Corollary follows from Theorem 1.

4. Remark. I believe (but have no proof) that for every $K > 0$ there exists an n_0 and a sequence A which is not subcomplete such that $|A(n)| \geq K\sqrt{n}$ for $n \geq n_0$.

References

[1] J. W. S. Cassels, *On the representation of integers as sums of distinct summands taken from a fixed set*, Acta Sci. Math. (Szeged) 21 (1960), 111–124.

[2] P. Erdős, *On the representation of large integers as sums of distinct summands taken from a fixed set*, Acta Arith. 7 (1962), 345–354.

[3] J. Folkman, *On the representation of integers as sums of distinct terms from a fixed sequence*, Canad. J. Math. 18 (1966), 643–655.

[4] G. Freiman, *New analytical results in subset-sum problem*, Discrete Math. 114 (1993), 205–218.

[5] N. Hegyvári, *On the representation of integers as sums of distinct terms from a fixed set*, Acta Arith. 92 (2000), 99–104.

[6] T. Łuczak and T. Schoen, *On the maximal density of sum-free sets*, ibid. 95 (2000), 225–229.

[7] M. B. Nathanson, *Additive Number Theory*, Grad. Texts in Math. 165, Springer, New York, 1996.
[8] A. Sárközy, *Finite addition theorems II*, J. Number Theory 48 (1994), 197–218.

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