On subset sums of a fixed set

by

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1. Introduction. Let $A$ be a strictly increasing sequence of positive integers. Let $P(A) = \{ \sum \varepsilon_i a_i : a_i \in A, \varepsilon_i = 0 \text{ or } 1 \}$ be the set of all the subset sums of $A$. $A$ is said to be subcomplete if $P(A)$ contains an infinite arithmetic progression. P. Erdős conjectured that if $a_{n+1}/a_n \to 1$, then $A$ is subcomplete. But J. W. S. Cassels [1] proved that for every $\varepsilon > 0$ there exists a sequence $A$ for which $a_{n+1} - a_n = o(a_n^{1/2+\varepsilon})$ and $A$ is not subcomplete. Let $A(n) = A \cap [1, n]$. The cardinality of a finite set $S$ is denoted by $|S|$. In 1962 Erdős [2] proved that if $|A(n)| > cn^{(\sqrt{5}-1)/2}$ ($c > 0$), then $A$ is subcomplete. In 1966 J. Folkman [3] improved this to $|A(n)| > n^{1/2+\varepsilon}$ ($\varepsilon > 0$), and recently, N. Hegyvári [5] showed $|A(n)| > 300\sqrt{n}\log n$ for $n > n_0$. A similar result was also proved by Łuczak and Schoen [6] independently. In this note we improve Hegyvári’s result:

Theorem 1. There exists an absolute constant $C > 1$ such that if $A = \{a_1 < a_2 < \ldots \}$ is an infinite sequence of positive integers with $|A(n)| > C\sqrt{n}$ for $n > C^2$, then $A$ is subcomplete.

Corollary. There exists an absolute constant $c > 0$ such that if $A = \{a_1 < a_2 < \ldots \}$ is an infinite sequence of positive integers with $a_n < cn^2$ for $n > c^{-1}$, then $A$ is subcomplete.

This is best possible (cf. [5]). The Corollary gives a partial answer to a question in [3].

2. Preliminaries

Lemma 1. Let $B = \{b_1 < b_2 < \ldots \}$ be an infinite sequence of positive integers, and let $n'_0, d$ be positive integers with $n'_0 > 4$. Assume that $b_i \geq$
i + 4d for all i and \(|B(n)| \geq 2\sqrt{dn}\) for \(n \geq n'_0\). Then there exists an infinite sequence \(\{v_1 < v_2 < \ldots\}\) in \(P(B)\) for which \(d \mid v_n\) and \(v_{n+1} - v_n < dn'_0\).

**Proof.** By the Erdős–Ginzburg–Ziv theorem (cf. [7]), for each \(i \geq 1\), there exists \(d\) integers \(b_{i1}, \ldots, b_{id}\) in \(\{b_{2(i-1)d+1}, \ldots, b_{id}\}\) such that \(u_i = b_{i1} + \ldots + b_{id}\) is divisible by \(d\). Then \(u_1 < u_2 < \ldots\) Let \(U = \{u_1, u_2, \ldots\}\) and \(P(U) = \{v_1 < v_2 < \ldots\}\). Then \(d \mid v_n\) for all \(n\). Now we show that \(v_{n+1} - v_n < dn'_0\).

Since \(|B(n)| \geq 2\sqrt{dn}\) for \(n \geq n'_0\), it follows that if \(b_n \geq n'_0\), then \(b_n \leq n^2/(4d)\). Hence \(b_{2d} < n'_0\). Let \(n_1\) be the largest integer with \(b_{2n_1d} < n'_0\). Then \(n_1 \geq 1\). For \(n > 1\), let \(j_1\) be the least integer such that \(\sum_{1 \leq i \leq j_1} u_i > v_n\); \(j_2\) be the least integer such that \(\sum_{1 \leq i \leq j_2} u_i > v_n - u_{j_1}\); and so on. Thus we have defined \(j_1 > \ldots > j_t = 1\) such that

\[
\sum_{1 \leq i \leq j_k} u_i > v_n - u_{j_1} - \ldots - u_{j_{k-1}}, \quad k = 2, 3, \ldots, t.
\]

Define \(u_{j_0} = 0\). Since \(u_{j_0} < v_n < u_{j_0} + u_{j_1} + \ldots + u_{j_t}\), there exists an integer \(l\) with \(0 \leq l < t\) such that

\[
(1) \quad u_{j_0} + u_{j_1} + \ldots + u_{j_l} \leq v_n < u_{j_0} + u_{j_1} + \ldots + u_{j_{l+1}}.
\]

Suppose that \(j_{l+1} > n_1\). Then \(b_{2j_{l+1}d} \geq n'_0\) and

\[
v_n - u_{j_0} - \ldots - u_{j_l} < u_{j_{l+1}} \leq db_{2j_{l+1}d} \leq d \left(\frac{(2j_{l+1}d)^2}{4d}\right) = d \sum_{1 \leq i \leq j_{l+1}-1} (2(i-1)d + 4d + 1)
\]

\[
\leq d \sum_{1 \leq i \leq j_{l+1}-1} b_{2(i-1)d+1} \leq \sum_{1 \leq i \leq j_{l+1}-1} u_i,
\]

contrary to the definition of \(j_{l+1}\). Hence \(j_{l+1} \leq n_1\). Thus, by (1) and \(j_{l+1} \leq n_1\),

\[
v_{n+1} - v_n \leq u_{j_{l+1}} \leq db_{2j_{l+1}d} < dn'_0.
\]

This completes the proof of Lemma 1.

**Lemma 2.** Let \(0 < a_1 < \ldots < a_k \leq n\) be an increasing sequence of integers. Assume that \(n > 2500\) and \(k > 200\sqrt{n}\log n\). Then there exist integers \(d, y, z\) such that \(1 \leq d \leq 50\sqrt{n}\log n\), \(z - y > \frac{3}{4}n\log n\) and

\[
\{td : y \leq t \leq z\} \subset P(\{a_1, \ldots, a_k\}).
\]

**Proof.** By Theorem 4 in [8] (see also [4]) there exist integers \(d, y, z\) with

\[
1 \leq d \leq 10^4nk^{-1}, \quad z \geq 7^{-1} \cdot 10^{-4}k^2, \quad y \leq 7 \cdot 10^4znk^{-2}
\]

and

\[
\{td : y \leq t \leq z\} \subset P(\{a_1, \ldots, a_k\}).
\]
For these $d, y, z$ we have
\[
1 \leq d \leq 50 \sqrt{\frac{n}{\log n}}, \quad z - y \geq 7^{-1} \cdot 10^{-4} k^2 (1 - 7 \cdot 10^4 nk^{-2}) \geq \frac{3}{7} n \log n.
\]
This completes the proof of Lemma 2.

**Lemma 3.** Let $A(D_i, H_i) = \{a_i + tD_i : 0 \leq t \leq H_i\}$ ($i = 0, 1, \ldots, l$) be arithmetic progressions of integers. Assume that
\[
H_i \geq D_0 + D_{i+1}, \quad i = 0, 1, \ldots
\]
Then there exists an arithmetic progression $A(d_l, h_l) = \{a + td_l : 0 \leq t \leq h_l\} \subset A(D_0, H_0) + \ldots + A(D_l, H_l)$ with $d_l = (D_0, \ldots, D_l)$ and $h_l \geq H_l - D_0$.

This follows from the proof of Lemma 4 in [5].

**3. Proof of Theorem 1.** In this section we prove the following theorem which implies Theorem 1.

**Theorem 2.** Let $A = \{a_1, a_2, \ldots\}$ be an infinite sequence of positive integers, $n_0$ be an integer with $n_0 \geq e^{60}$ and $w = 50 \sqrt{n_0 \log n_0}$. Assume that $|A(n)| \geq 202 \sqrt{12wn}$ for $n_0 \leq n \leq e^{12wn_0}$ and $|A(n)| \geq 808 \sqrt{wn}$ for $n > e^{12wn_0}$. Then $A$ is subcomplete.

**Proof.** Let $n_i = n_{i-1}^2$ ($i = 1, 2, \ldots$). Let $l$ be the integer with $n_{l-1} < e^{6w} \leq n_l$. Then $n_l = n_{l-1}^2 < e^{12w}$. Let $n_{l-1} = 0$. Hence
\[
|A(n_i) \setminus A(n_{i-1})| \geq 202 \sqrt{n_i \log n_i - n_{i-1}} \geq 202 \sqrt{n_i \log n_i - n_i} > 200 \sqrt{n_i \log n_i + 1}, \quad i = 0, 1, \ldots, l.
\]
Take $B_i = A(n_i) \setminus A(n_{i-1})$ for $i = 0, 1, \ldots, l - 1$ and $B_l \subset A(n_l) \setminus A(n_{l-1})$ with $|B_i| = [200 \sqrt{n_l \log n_l}] + 1$. By Lemma 2, for each $i$, there exists an arithmetic progression
\[
A_i = \{a_i + D_i k : 0 \leq k \leq H_i\} \subset P(B_i)
\]
with $D_i \leq 50 \sqrt{n_i \log n_i}$ and $H_i \geq \frac{3}{7} n_i \log n_i$. Since
\[
H_i \geq \frac{3}{7} n_i \log n_i > 50 \sqrt{\frac{n_{i+1}}{\log n_{i+1}}} + w \geq D_{i+1} + D_0,
\]
by Lemma 3 there exists an arithmetic progression
\[
A(d, h) = \{a + dk : 0 \leq k \leq h\} \subset A_0 + \ldots + A_l
\]
with $d \mid D_0$ and $h \geq H_l - D_0$.

For $n_l \leq n \leq e^{12w}$ we have
\[
|A(n) \setminus (B_l \cup A(n_{l-1}))| \geq 202 \sqrt{n \log n} - 200 \sqrt{n_l \log n_l} - 1 - n_{l-1}
\]
\[
\geq 2 \sqrt{n \log n} - 1 - \sqrt{n_l} \geq 2 \sqrt{6 \sqrt{wn} - 1} - \sqrt{n_l}
\]
\[
\geq 2 \sqrt{wn} \geq 2 \sqrt{D_0 n} \geq 2 \sqrt{dn}.
\]
For \( n > e^{12w} \) we have
\[
|A(n) \setminus (B_l \cup A(n_{l-1}))| \geq 808\sqrt{wn} - 200\sqrt{n_l \log n_l} - 1 - \sqrt{n_l} \\
\geq 808\sqrt{wn} - 200\sqrt{12 \sqrt{wn} - 1} - \sqrt{n_l} \geq 2\sqrt{wn}.
\]

Let \( A \setminus (B_l \cup A(n_{l-1})) = \{b_1 < b_2 < \ldots \} \). Then
\[
b_1 \geq n_{l-1} = \sqrt{n_l} \geq e^{3w} \geq 4w + 1 \geq 4D + 1 \geq 4d + 1.
\]
Hence \( b_i \geq b_1 + i - 1 \geq i + 4d \). By Lemma 1 there exists an infinite sequence
\[
\{v_1 < v_2 < \ldots \} \subseteq P(A \setminus (B_l \cup A(n_{l-1})))
\]
with \( d \mid v_i \) and \( v_{i+1} - v_i < n_l d \). Since \( D_0(n_l + 1) < 2wn_l < \frac{3}{7}n_l \log n_l \), we have
\[
n_l d \leq n_l D_0 < \frac{3}{7}n_l \log n_l - D_0 \leq H_l - D_0 \leq h.
\]
Hence
\[
A(d, h) + \{v_1, v_2, \ldots \} = \{a + v_1 + dk : k = 0, 1, \ldots \} \\
\subseteq P(B_l \cup A(n_{l-1})) + P(A \setminus (B_l \cup A(n_{l-1}))) \\
\subseteq P(A).
\]
This completes the proof of Theorem 2.

**Proof of the Corollary.** Let \( C \) be as in Theorem 1 and \( c = \frac{1}{4}C^{-2} \). If \( ci^2 \leq n \), then \( a_i \leq n \). Hence \( |A(n)| \geq \lfloor \sqrt{n/c} \rfloor \geq \lfloor 2C \sqrt{n} \rfloor > C \sqrt{n} \). Then the Corollary follows from Theorem 1.

**4. Remark.** I believe (but have no proof) that for every \( K > 0 \) there exists an \( n_0 \) and a sequence \( A \) which is not subcomplete such that \( |A(n)| \geq K \sqrt{n} \) for \( n \geq n_0 \).

**References**


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