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On the counting function for the Niven numbers

by

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1. Introduction. A positive integer n is said to be a *Niven number* (or a Harshad number) if it is divisible by the sum of its decimal digits.

In 1984, Kennedy and Cooper [7] established that the set of Niven numbers is of zero density. In 1985, the same authors [1] showed that, given any $t > 0$, we have $N(x) \geq \log^t x$ provided x is sufficiently large, where $N(x)$ stands for the number of Niven numbers not exceeding x, and in 1988, they [2] obtained an asymptotic formula for the number of Niven numbers $\leq x$ whose sum of digits equals k. In 1991, Vardi [9] proved that, for any given $\varepsilon > 0$,

$$N(x) \ll \frac{x}{(\log x)^{1/2-\varepsilon}}$$

and that there exists a positive constant α such that

$$N(x) > \alpha \frac{x}{(\log x)^{11/2}}$$

for infinitely many integers x, namely for all sufficiently large x of the form $x = 10^{10k+n+2}$, k and n being positive integers satisfying $10^n = 45k + 10$.

Recently, De Koninck and Doyon [3] established that, given any fixed $\varepsilon > 0$,

$$x^{1-\varepsilon} \ll N(x) \ll \frac{x \log\log x}{\log x},$$

and conjectured, using a heuristic argument, that, as $x \to \infty$,

$$N(x) = (\eta + o(1)) \frac{x}{\log x} \quad \text{with} \quad \eta = \frac{14}{27} \log 10. \tag{1}$$

More generally, given an integer $q \geq 2$, we shall say that a positive integer is a *q-Niven number* if it is divisible by the sum of its digits in base q.

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In this paper, we prove that (1) holds and moreover that, given any base $q \geq 2$, a similar result holds for $N_q(x)$, the number of q-Niven numbers not exceeding x. Hence, our main goal will be to prove the following result.

THEOREM 1. *As* $x \to \infty$,

$$N_q(x) = (\eta_q + o(1))\frac{x}{\log x} \quad \textit{with} \quad \eta_q = \frac{2\log q}{(q-1)^2}\sum_{j=1}^{q-1}(j, q-1). \tag{2}$$

Theorem 1 will follow from our results on the local distribution of $\alpha(n)$, the sum of the digits of n, when n runs over an arithmetic progression with growing modulus k. Similar techniques for the study of the sum of digits function residue classes have been used by other authors, namely Delange [4] and Gel'fond [6].

2. Notations and preliminary observations. Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$ and $\mathbb{C}$ stand for the set of positive integers, non-negative integers, real numbers and complex numbers, respectively.

Throughout this paper, let $q \geq 2$ be a fixed integer. The *q-ary expansion* of a non-negative integer n is defined as the unique sequence $\epsilon_0(n), \epsilon_1(n), \ldots$ for which

$$n = \sum_{j=0}^{\infty}\epsilon_j(n)q^j, \qquad \epsilon_j(n) \in \{0, 1, \ldots, q-1\}. \tag{3}$$

Let $\alpha(n) = \alpha_q(n)$ be the sum of the digits of n in base q, that is,

$$\alpha(n) = \epsilon_0(n) + \epsilon_1(n) + \ldots$$

Given $x \in \mathbb{R}$, $N \in \mathbb{N}$ and $z, w \in \mathbb{C}$, we set

$$S(x|z, w) := \sum_{0 \leq n < x} z^{\alpha(n)}w^n \quad \text{and} \quad S_N(z, w) := S(q^N|z, w). \tag{4}$$

It is clear that

$$S_N(z, w) = \prod_{l=0}^{N-1}\Big(\sum_{j=0}^{q-1} z^j\, w^{jq^l}\Big). \tag{5}$$

Let also

$$U(x|z, k, l) := \sum_{\substack{0 \leq n < x \\ n \equiv l \,(\mathrm{mod}\, k)}} z^{\alpha(n)} \quad \text{and} \quad U_N(z, k, l) := U(q^N|z, k, l). \tag{6}$$

Observe that, using the standard notation $e(y) := e^{2\pi i y}$, we have

$$U(x|z, k, l) = \frac{1}{k}\sum_{s=0}^{k-1} e(-ls/k)S(x|z, e(s/k)). \tag{7}$$

Furthermore, if we set

$$A(x|k,l,t) := \#\{n < x : n \equiv l \pmod{k} \text{ and } \alpha(n) = t\}, \tag{8}$$

then

$$A(x|k,l,t) = \int_0^1 U(x|e(\xi),k,l)e(-t\xi)\,d\xi. \tag{9}$$

A function $g : \mathbb{N}_0 \to \mathbb{C}$ is said to be *q-multiplicative* if $g(0) = 1$ and

$$g(n) = \prod_{j=0}^{\infty} g(\epsilon_j(n)q^j) \qquad (n = 1, 2, \ldots).$$

Now for a q-multiplicative function g, set $M(x) = M_q(x) = \sum_{0\le n<x} g(n)$. Given a positive integer x, write

$$x = b_1q^{N_1} + b_2q^{N_2} + \ldots + b_sq^{N_s}, \tag{10}$$

where $N_1 > \ldots > N_s,\ b_j \in \{1, \ldots, q-1\}$. Set

$$\begin{aligned}
x_0 &= x,\\
x_1 &= b_2q^{N_2} + \ldots + b_sq^{N_s},\\
x_2 &= b_3q^{N_3} + \ldots + b_sq^{N_s},\\
&\ \vdots\\
x_{s-1} &= b_sq^{N_s},\\
x_s &= 0
\end{aligned}$$

and

$$\xi_j = \sum_{c=0}^{b_j-1} g(cq^{N_j}) \qquad (j = 1, \ldots, s).$$

Using these notations, it is easy to observe that

$$M(x) = \xi_1M(q^{N_1}) + g(b_1q^{N_1})M(x_1), \tag{11}$$

and by iteration,

$$\begin{aligned}
M(x) = \xi_1M(q^{N_1}) &+ g(b_1q^{N_1})\xi_2M(q^{N_2}) + g(b_1q^{N_1})g(b_2q^{N_2})\xi_3M(q^{N_3})\\
&+ \ldots + g(b_1q^{N_1})\ldots g(b_{s-1}q^{N_{s-1}})\xi_sg(b_sq^{N_s}).
\end{aligned} \tag{12}$$

Note that $S(x|z,w)$ is such a function.

3. Preliminary lemmas. For $y \in \mathbb{R}$, let $\|y\|$ be the distance of y to the closest integer. Let $\xi \in [0,1)$ be fixed.

LEMMA 1. *Let $R \in \mathbb{N}$. Given two coprime positive integers $s < k$ with $(k,q) = 1$ and $k \nmid q-1$, assume that*

$$\left\|\xi + \frac{s}{k} q^u\right\| < \frac{1}{8q} \quad \textit{for } u = h, h+1, \dots, h+R. \tag{13}$$

Then $q^R \le k/4$.

Proof. From (13), it follows that

$$\left\|\frac{s}{k} q^u(q-1)\right\| \le \left\|\left(\xi + \frac{s}{k} q^{u+1}\right) - \left(\xi + \frac{s}{k} q^u\right)\right\| < \frac{1}{4q} \qquad (u = h, h+1, \dots, h+R-1). \tag{14}$$

Since $k \nmid q-1$, the left hand side of (14) is non-zero and therefore it is $\ge 1/k$.

Now from (14), we have

$$\left\|\frac{s}{k} q^{u+1}(q-1)\right\| = q\left\|\frac{s}{k} q^u(q-1)\right\| \qquad (u = h, h+1, \dots, h+R-2), \tag{15}$$

and therefore

$$\left\|\frac{s}{k} q^{R-1+h}(q-1)\right\| = q^{R-1}\left\|\frac{s}{k} q^h(q-1)\right\| < \frac{1}{4q}. \tag{16}$$

Hence combining this with our observation that the left hand side of (14) must be $\ge 1/k$, we conclude that

$$\frac{1}{k} \le \left\|\frac{s}{k} q^h(q-1)\right\| < \frac{1}{4q^R},$$

that is $q^R \le k/4$, as claimed.

Lemma 2. *Let* $A(x|k,l,t)$ *be as in* (8) *and* $S(x|z,w)$ *as in* (4). *Then*

$$\left|A(x|k,l,t) - \frac{1}{k}A(x|1,0,t)\right| \le \max_{1\le s\le k-1} \max_{|z|=1} |S(x|z, e(s/k))|.$$

Proof. This follows immediately from (9) and (7).

Now for $1 \le s < k$, set

$$s_h = \max_{0\le j\le q-1} \left\|j\xi + q^h \frac{s}{k}\right\|.$$

Lemma 3. *There exists a constant* $c = c(q)$ *such that*

$$\left|\frac{1}{q}\sum_{j=0}^{q-1} e(\xi j) e\left(\frac{s}{k} q^h j\right)\right| \le q^{-cs_h}.$$

Proof. This follows immediately from the definition of s_h.

4. Local distribution of $\alpha(n)$ as n runs through a congruence class $l \pmod k$

4.1. We first consider the case $(k, q(q-1)) = 1$.

THEOREM 2. *Assume that* $(k, q(q-1)) = 1$. *Then, for each integer* $l \in [0, k-1]$ *and* $t \in \mathbb{N}$, *we have*

$$\left| A(x|k,l,t) - \frac{1}{k} A(x|1,0,t) \right| \le x e^{-c_1 \frac{\log x}{\log 2k}}, \tag{17}$$

where $c_1 = c_1(c, q)$ *is a suitable positive constant independent of* k, l *and* t.

Proof. Let x be written as in (10). Then, from (12), we have

$$|S(x|z, e(s/k))| \le q \sum_{j=1}^{s} |S_{N_j}(z, e(s/k))|.$$

To estimate each expression $|S_{N_j}(z, e(s/k))|$, we use Lemmas 1–3.

For $k = 2, 3, 4$, we set $R = 0$, while for each $k \ge 5$, we set

$$R = \left[\frac{\log(kq/4)}{\log q} \right].$$

From Lemma 1, we know that

$$\max_{h \le u \le h+R} s_u \ge \frac{1}{8q}.$$

Therefore

$$|S_{N_j}(z, e(s/k))| \le q^{N_j} \cdot q^{-\frac{c}{8q}\left[\frac{N_j}{R+1}\right]},$$

which completes the proof of Theorem 2.

REMARK. It is interesting to observe that the following assertion is also true:

If $(k, q(q-1)) = 1$, then

$$\max_{|z|=1} \left| \sum_{\substack{n<x \\ n \equiv l \,(\mathrm{mod}\, k)}} z^{\alpha(n)} - \frac{1}{k} \sum_{n<x} z^{\alpha(n)} \right| \le x e^{-c_1 \frac{\log x}{\log 2k}}.$$

4.2. We now consider the case $(k, q) > 1$. Actually we shall reduce this case to the one of Section 4.1. Indeed, let $k = k_1 k_2$, where k_1 is the largest divisor of k coprime to q and $k_2 = k/k_1$. Further let h be the smallest positive integer such that $k_2 \mid q^h$. Then the congruence class $l \pmod{k}$ can be written as the union of some congruence classes $\mathrm{mod}\, k_1 q^h$, namely

$$\{n : n \equiv l \pmod{k}\} = \bigcup_{j=1}^{q^h/k_2} \{n : n \equiv l^{(j)} \pmod{k_1 q^h}\}. \tag{18}$$

First define $l_1^{(j)}$ and $l_2^{(j)}$ implicitly by

$$l^{(j)} = l_1^{(j)} + q^h l_2^{(j)}, \qquad 0 \le l_1^{(j)} < q^h,$$

and then write a positive integer $n \equiv l^{(j)} \pmod{k_1 q^h}$ as

$$n = l_1^{(j)} + q^h m \equiv l_1^{(j)} + q^h l_2^{(j)} \pmod{k_1 q^h},$$

which is equivalent to

$$m \equiv l_2^{(j)} \pmod{k_1}. \tag{19}$$

Using this setup, we obtain the following result.

LEMMA 4. *We have*

$$\sum_{\substack{n<x \\ n\equiv l \,(\mathrm{mod}\, k)}} z^{\alpha(n)} = \sum_{j=1}^{q^h/k_2} z^{\alpha(l_1^{(j)})} \sum_{\substack{m<x/q^h \\ m\equiv l_2^{(j)} (\mathrm{mod}\, k_1)}} z^{\alpha(m)} \tag{20}$$

and

$$A(x|k,l,t) = \sum_{j=1}^{q^h/k_2} A\Big(\frac{x}{q^h}\Big| k_1, l_2^{(j)}, t - \alpha(l_1^{(j)})\Big). \tag{21}$$

4.3. We now consider the case $k = k_1k_2$, where $(k,q) = 1$, $(k_1, q-1) = 1$ and all the prime factors of k_2 are divisors of $q-1$.

LEMMA 5. *We have*

$$U(x|z,k,l) = \frac{1}{k}\sum_{\tau=1}^{k_2} e(-l\tau/k_2) S(x|z, e(\tau/k_2)) + O(xe^{-c_1\frac{\log x}{\log 2k}}) \tag{22}$$

and

$$U(x|z,k,l) = \frac{1}{k_1} U(x|z,k_2,l) + O(xe^{-c_1\frac{\log x}{\log 2k}}). \tag{23}$$

Proof. It is clear that (23) follows from (22) and (7). Therefore we only need to prove (22). Recall the representation of $U(x|z,k,l)$ given by (7). For each $1 \le s < k$, write $s/k = s^*/k^*$, where $(s^*, k^*) = 1$. If k^* has a prime factor which does not divide k_2, then arguing as in the proof of Theorem 2, we obtain

$$|S(x|z, e(s/k))| \le xe^{-c_1\frac{\log x}{\log 2k}}.$$

Therefore, it remains only to consider those s which are multiples of k_1, in which case we simply write $s = \tau k_1$, where $\tau = 0, 1, \ldots, k_2 - 1$, and the proof is complete.

COROLLARY. *If $k = k_1k_2$ with $(k,q) = 1$, $(k_1, q-1) = 1$ and all the prime factors of k_2 are divisors of $q-1$, then*

$$A(x|k,l,t) = \frac{1}{k_1} A(x|k_2,l,t) + O(xe^{-c_1\frac{\log x}{\log 2k}}). \tag{24}$$

4.4. Assume now that the prime divisors of k divide $q-1$. For each positive integer m, let $\kappa(m) = (m, q-1)$ and set $K = k/\kappa(k)$. Then, repeating the argument used above and again using Lemmas 1–3, we can conclude that

$$A(x|k,l,t) = \frac{1}{K}\, A(x|\kappa(k),l,t) + O(xe^{-c_1 \frac{\log x}{\log 2k}}).$$

4.5. Assume finally that $k \mid q-1$. Since in this case, we have $q^\nu \equiv 1 \pmod{k}$ for each $\nu \in \mathbb{N}_0$, it follows that $n \equiv l \pmod{k}$ implies that $\alpha(n) \equiv l \pmod{k}$. Consequently,

$$A(x|k,l,t) = \begin{cases} \#\{n < x : \alpha(n) = t\} & \text{if } t \equiv l \pmod{k}, \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

We now have the proper setup to build the proof of Theorem 1.

5. The proof of Theorem 1. Given x, define N_x as the unique integer satisfying $q^{N_x} \le x < q^{N_x+1}$, so that $N_x = \left[\frac{\log x}{\log q}\right]$.

Further define

$$B(x|t) := \#\{n < x : \alpha(n) = t \text{ with } t \mid n\},$$
$$a(x|t) := A(x|1,0,t) = \#\{n < x : \alpha(n) = t\}.$$

Using Theorem 6, Chapter VII, of V. V. Petrov [8] on local distribution of sums of identically distributed random variables, and by an easy computation we obtain the following.

Lemma 6. *Let*

$$m = \frac{q-1}{2} \quad \text{and} \quad \sigma^2 = \frac{1}{q}\sum_{j=1}^{q-1} j^2 - m^2 = \frac{q^2-1}{12}.$$

Then

$$a(x|t) = \frac{x}{\sqrt{N_x}}\,\varphi\left(\frac{t - mN_x}{\sigma\sqrt{N_x}}\right) + O\left(\frac{x(\log N_x)^{3/2}}{N_x}\right) \tag{26}$$

uniformly in t, where $\varphi(y) = (1/\sqrt{2\pi})e^{-y^2/2}$ is the density function of the Gaussian law.

Remark. For a similar result in a more general setup, see Drmota and Gajdosik [5].

Now, x being fixed, we define the interval I as follows:

$$I = \left[\frac{q-1}{2}\,N_x - \frac{N_x}{\log^2 N_x}, \frac{q-1}{2}\,N_x + \frac{N_x}{\log^2 N_x}\right].$$

A simple probabilistic argument shows that

$$\#\{n < x : \alpha(n) \notin I\} \ll \frac{x}{\log x \log\log x}. \tag{27}$$

Therefore, it is clear that

$$N_q(x) = \sum_{t\in I} B(x|t) + O\left(\frac{x}{\log x \log\log x}\right). \tag{28}$$

Let us factorise each $t \in I$ as $t = t_1t_2t_3$, where $(t_1, q(q-1)) = 1$, the prime factors of t_2 divide q, and the prime factors of t_3 divide $q-1$.

Fixing $t \in I$, let h be the smallest positive integer such that $t_2 \mid q^h$. Note that

$$q^h < N_x^{c_3} \quad \text{for a suitable positive constant } c_3 = c_3(q). \tag{29}$$

To see this, first observe that t_2 must have a divisor to the h-th power, and therefore $N_x > t_2 \geq 2^h$, which means that $h < \log N_x/\log 2$. Hence $q^h < q^{\log N_x/\log 2} < N_x^{c_3}$, which proves (29).

Using (21), we obtain

$$A(x|t,0,t) = \sum_{j=1}^{q^h/t_2} A\left(\frac{x}{q^h}\middle| t_1t_3, l_2^{(j)}, t-\alpha(l_1^{(j)})\right), \tag{30}$$

where

$$l^{(j)} := (t_1t_3)t_2 j = l_1^{(j)} + q^h l_2^{(j)} \quad (0 \leq l_1^{(j)} < q^h). \tag{31}$$

Using (24), we have

$$\begin{aligned} A\left(\frac{x}{q^h}\middle| t_1t_3, l_2^{(j)}, t-\alpha(l_1^{(j)})\right) \\ = \frac{1}{t_1} A\left(\frac{x}{q^h}\middle| t_3, l_2^{(j)}, t-\alpha(l_1^{(j)})\right) + O\left(\frac{x}{q^h} e^{-\frac{c_1}{2}\cdot\frac{\log x}{\log 2t}}\right). \end{aligned} \tag{32}$$

Since $\kappa(t_3)$ divides t and $l^{(j)}$, $\alpha(l_1^{(j)}) \equiv l_1^{(j)} \pmod{\kappa(k_3)}$, $l^{(j)} = l_1^{(j)} + q^h l_2^{(j)}$ and $q^h \equiv 1 \pmod{\kappa(t_3)}$, it follows that

$$t \equiv \alpha(l_1^{(j)}) \equiv l_2^{(j)} \pmod{\kappa(t_3)}.$$

Therefore the main term on the right hand side of (32) is, because of (25),

$$\frac{1}{t_1}\cdot\frac{\kappa(t_3)}{t_3}\, a\left(\frac{x}{q^h}\middle| t-\alpha(l_1^{(j)})\right).$$

Consequently, using (30), we obtain

$$A(x|t,0,t) = \frac{\kappa(t_3)}{t_1t_3}\sum_{j=1}^{q^h/t_2} a\left(\frac{x}{q^h}\middle| t-\alpha(l_1^{(j)})\right) + O(xe^{-\frac{c_1}{2}\cdot\frac{\log x}{\log 2t}}). \tag{33}$$

Using Lemma 6, and after observing that

$$(34)\qquad \begin{aligned} &l_1^{(j)} < q^h < N_x^{c_3},\\ &\alpha(l_1^{(j)}) = O(\log l_1^{(j)}) = O(\log N_x),\\ &|\varphi(\xi_1) - \varphi(\xi_2)| \ll |\xi_1 - \xi_2|, \end{aligned}$$

we find that, for each $t \in I$,

$$(35)\qquad a\left(\frac{x}{q^h}\middle| t - \alpha(l_1^{(j)})\right) = a\left(\frac{x}{q^h}\middle| t\right) + O\left(\frac{x}{q^h} \cdot \frac{(\log N_x)^{3/2}}{N_x}\right).$$

Therefore, using (33),

$$(36)\qquad A(x|t,0,t) = \frac{q^h \kappa(t_3)}{t}\, a\left(\frac{x}{q^h}\middle| t\right) + O\left(\frac{x}{t} \cdot \frac{(\log N_x)^{3/2}}{N_x}\right).$$

Furthermore, by Lemma 6, we have

$$(37)\qquad \begin{aligned} &\left| q^h a\left(\frac{x}{q^h}\middle| t\right) - a(x|t)\right|\\ &\quad \ll \left| \frac{x}{\sqrt{N_x - h}}\, \varphi\left(\frac{t - m(N_x - h)}{\sigma\sqrt{N_x - h}}\right) - \frac{x}{\sqrt{N_x}}\, \varphi\left(\frac{t - mN_x}{\sigma\sqrt{N_x}}\right)\right|\\ &\qquad + O\left(\frac{x}{N_x} (\log N_x)^{3/2}\right). \end{aligned}$$

But the expression $|\ldots|$ on the right hand side of (37) is no larger than the error term, which implies that

$$(38)\qquad \left| q^h a\left(\frac{x}{q^h}\middle| t\right) - a(x|t)\right| \ll \frac{x}{N_x}\, (\log N_x)^{3/2}.$$

Hence, using (36) and (38), we obtain

$$(39)\qquad A(x|t,0,t) = \frac{\kappa(t_3)}{t}\, a(x|t) + O\left(\frac{x}{tN_x}\, (\log N_x)^{3/2}\right).$$

From (28) and (39), we then have, since $N_x = [\log x/\log q]$,

$$(40)\qquad \begin{aligned} N_q(x) &= \sum_{t\in I} \frac{\kappa(t_3)}{t}\, a(x|t) + O\left(\frac{x}{N_x \log^2 N_x}\, (\log N_x)^{3/2}\right)\\ &= \frac{2}{N_x(q-1)} \sum_{t\in I} \kappa(t_3) a(x|t) + O\left(\frac{x}{(\log x)(\log\log x)^{1/2}}\right)\\ &= \frac{2\log q}{\log x} \cdot \frac{1}{q-1} \sum_{t\in I} \kappa(t_3) a(x|t) + O\left(\frac{x}{(\log x)(\log\log x)^{1/2}}\right). \end{aligned}$$

Since $a(x|t) = (1 + o(1))a(x|t+1)$ uniformly for $t \in I$, $\kappa(t_3) = \kappa(t)$, and

$\kappa(t)$ is periodic mod $q-1$, it follows that

$$(41)\qquad \sum_{t\in I}\kappa(t_3)a(x|t) = \frac{1}{q-1}\,(1+o(1))\sum_{t\in I}\kappa(t)\sum_{j=0}^{q-2}a(x|t-j)$$
$$= (1+o(1))\sum_{r\in I}a(x|r)\cdot\frac{1}{q-1}\sum_{j=0}^{q-2}\kappa(r+j)+E(x),$$

where $E(x)\ll\sum' a(x|s)$, where this last sum runs over those s such that $|s-I_i|\le q-1$, the I_i's being the endpoints of I, that is, $I=[I_1,I_2]$. Since $\max a(x|t)\ll x/\sqrt{\log x}$ and since the number of s's counted in $\sum' a(x|s)$ is bounded by a multiple of q, it follows that

$$(42)\qquad E(x)\ll\frac{x}{\sqrt{\log x}}.$$

Moreover, observe that, because of (27),

$$(43)\qquad \sum_{r\in I}a(x|r) = x+O\left(\frac{x}{\log x\log\log x}\right).$$

Finally, observe that

$$(44)\qquad \frac{1}{q-1}\sum_{j=0}^{q-2}\kappa(r+j)=\frac{1}{q-1}\sum_{j=1}^{q-1}\kappa(j)$$

is a constant.

Therefore, it follows from (40)–(44) that

$$N_q(x)=(1+o(1))\,\frac{2x}{\log x}\cdot\frac{1}{(q-1)^2}\sum_{j=1}^{q-1}\kappa(j),$$

which implies (2). The proof of Theorem 1 is thus complete.

6. Final remark. A similar result can be established if one replaces $\alpha(n)$ by a q-additive function $f(n)$ taking integer values and satisfying $f(bq^j)=f(b)$ for all positive integers j.

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