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An example in Beurling's theory of generalised primes

by

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Introduction. Beurling prime systems are generalisations of the prime numbers in which we start with a sequence, say p_n, called the *generalised primes* (or *g-primes* for short), satisfying

$$1 < p_1 \le p_2 \le \cdots \quad \text{and} \quad p_n \to \infty.$$

From these numbers we form all possible products, called the *generalised integers* (or *g-integers*). We distinguish between two such products (made from different g-primes) even if they are numerically the same. Let $\pi(x)$ and $N(x)$ denote, respectively, the counting functions of the g-primes and of the g-integers up to and including x. Much of the interest and research is in deducing information about one of these functions from the other. Many authors have studied these systems (see for example [1], [5], [6], [10], [14] to name just a few) and found such connections, starting with Beurling's broad generalisation of the Prime Number Theorem [3].

The system we have described above is a *discrete* system, where π and N are step functions with integer jumps. More generally, one can consider the case where these functions are just increasing but not necessarily step functions, maybe even continuous (see §1 for the definition). In §3, we study a particular such system in which the generalised Chebyshev function is given by

$$\psi_0(x) = [x] - 1.$$

In this case, the 'g-primes' are highly regular and very close to being 'discrete', and we wish to investigate how regular the corresponding g-integer counting function $N(x)$ is.

Letting $\zeta_0(s) = \hat{N}(s)$ denote the Mellin transform of N, we find that

$$-\frac{\zeta_0'(s)}{\zeta_0(s)} = \hat{\psi}_0(s) = \zeta(s) - 1,$$

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where $\zeta(s)$ is the Riemann zeta function. Integrating one finds

$$\zeta_0(s) = \exp\left\{\sum_{n=2}^{\infty} \frac{1}{n^s \log n}\right\} \qquad (\operatorname{Re} s > 1).$$

This function has an analytic continuation to $\mathbb{C} \setminus \{1\}$ with a simple pole at 1, and has *no* zeros anywhere.

The problem is to study the distribution of the associated g-integer counting function, $N_0(x)$. The fact that $\psi_0(x)$ is sufficiently close to x implies that $N_0(x) \sim \tau x$ for some $\tau > 0$. We shall look more closely at how good this approximation is by producing O and Ω results for the error $N_0(x) - \tau x$. The problem is of interest in that it gives an explicit example where we can be quite precise about the asymptotics of *both* ψ and N, which is extremely rare for discrete systems.

In order to prove these results, we need to establish some connections between the growth of a function and its Mellin transform. This we do in §2.

Notation. Throughout this article we make use of the standard O, $\ll$, and Ω notation. For f, g defined on a neighbourhood of ∞ we write $f = O(g)$ or $f \ll g$ to mean $|f(x)| \le Cg(x)$ for some constant C and all x sufficiently large. We write $f = \Omega(g)$ if there exists $C > 0$ and $x_n \to \infty$ such that $|f(x_n)| > Cg(x_n)$ for all n.

1. G-prime systems. *Discrete* g-prime systems are those as described in the Introduction. Here $\pi(x)$ and $N(x)$ are counting functions—increasing step functions with integer jumps. The g-primes and integers are related in the following way. First let ψ denote the *generalised Chebyshev function*

$$\psi(x) = \sum_{p_n^k \le x} \log p_n$$

where the p_n are the g-primes and $k \in \mathbb{N}$. Now let $N_L(x) = \sum_{n_i \le x} \log n_i$, where the n_i are the g-integers. Then it can be readily verified that

$$N_L(x) = \int_0^x \psi(x/t)\, dN(t). \tag{1.1}$$

This identity is used to generalise these systems to 'continuous' ([1]) systems.

Let S denote the set of functions $f : \mathbb{R} \to \mathbb{R}$ which are zero on $(-\infty, 1)$, right-continuous, and of local bounded variation. Let S^+ denote the subset of S consisting of increasing functions. Also, for $\alpha \in \mathbb{R}$, let $S_\alpha = \{f \in S : f(1) = \alpha\}$, while $S_\alpha^+ = S^+ \cap S_\alpha$.

([1]) Strictly speaking, they are not necessarily continuous.

For $f, g \in S$, the *convolution* (or *Mellin–Stieltjes convolution*) is defined by

$$(f * g)(x) = \int_0^x f(x/t)\, dg(t).$$

We note that S is closed under $*$ and that $*$ is commutative and associative. The identity (with respect to $*$) is $i(x) = 1$ for $x \geq 1$ and zero otherwise. Thus (1.1) is just $N_L = \psi * N$.

For $g \in S_0$ the *exponential* (with respect to $*$) is defined by

$$\exp_* g = \sum_{n=0}^{\infty} \frac{g^{*n}}{n!},$$

where $g^{*n} = g * g^{*(n-1)}$ and $g^{*0} = i$, the series converging absolutely. It is known that, given $f \in S_1$, there exists $g \in S_0$ such that $f = \exp_* g$. Also $f = \exp_* g$ if and only if $f * g_L = f_L$, where $f_L \in S$ is the function defined for $x \geq 1$ by

$$f_L(x) = \int_1^x \log t\, df(t).$$

(See for example [2, pp. 50–70] and [4].)

Returning to discrete g-prime systems define, as usual,

$$\Pi(x) = \int_{p_1-}^{x} \frac{1}{\log t}\, d\psi(t) = \sum_{p_n^k \leq x} \frac{1}{k} = \sum_{k=1}^{\infty} \frac{1}{k} \pi(x^{1/k}),$$

so that $\psi = \Pi_L$. We see that $N_L = \Pi_L * N$; that is, $N = \exp_* \Pi$. This leads to the following:

Definition. (i) An *outer g-prime system* is a pair of functions Π, N with $\Pi \in S_0^+$ and $N \in S_1^+$ such that $N = \exp_* \Pi$.

Of course, if $\Pi \in S_0^+$ then $\exp_* \Pi \in S_1^+$, so (Π, N) is an outer g-prime system (with $N = \exp_* \Pi$). The above definition is somewhat more general than the usual 'generalised primes', since we have not yet mentioned the equivalent of the prime counting function $\pi(x)$.

(ii) A *g-prime system* is an outer g-prime system for which there exists $\pi \in S_0^+$ such that

$$\Pi(x) = \sum_{k=1}^{\infty} \frac{1}{k} \pi(x^{1/k}). \tag{1.2}$$

(iii) For an outer g-prime system (Π, N), define the *Beurling zeta function* to be the Mellin transform of N:

$$\hat{N}(s) = \int_0^{\infty} x^{-s}\, dN(x) = e^{\hat{\Pi}(s)}.$$

(The equality of the two expressions follows from the fact that $\widehat{\exp_* f} = \exp \hat{f}$.)

REMARKS 1. (a) With Π given in terms of π by (1.2) we have, by Möbius inversion,

$$\pi(x) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \Pi(x^{1/k}).$$

In fact this sum always converges for $\Pi \in S^+$ (since $\Pi(x^{1/k})$ decreases with k and $\sum_{k=1}^{\infty} \mu(k)/k$ converges), so π may always be defined for an outer g-prime system. But in general π need not be increasing. Thus an outer g-prime system is more general than a g-prime system.

(b) For an outer g-prime system (Π, N), let $\psi = \Pi_L$ denote the *generalised Chebyshev function*. Then $\psi \in S_0^+$ and $\psi * N = N_L$. Taking Mellin transforms gives $\hat{\psi}(s) = -\hat{N}'(s)/\hat{N}(s)$.

Some relevant results. Let (Π, N) be an outer g-prime system with $\psi = \Pi_L$.

(a) Beurling's Prime Number Theorem [2]: If $N(x) = cx + O(x(\log x)^{-\gamma})$ for some $c > 0$ and $\gamma > 3/2$, then $\psi(x) \sim x$.

(b) Conversely, it follows from results by Diamond [5] that if $\psi(x) = x + O(x(\log x)^{-\delta})$ for some $\delta > 0$, then $N(x) \sim cx$ for some $c > 0$.

(c) If the stronger relation $\psi(x) = x + O(x^\alpha)$ holds for some $\alpha < 1$, then $N(x) = cx + O(x \exp\{-d\sqrt{\log x \log\log x}\})$ for some $c, d > 0$ (see [8]).

2. Growth relations between a function and its Mellin transform. Let $f \in S_1^+$ and suppose that

$$F(s) = \hat{f}(s) = \int_0^{\infty} x^{-s}\, df(x) \tag{2.1}$$

converges absolutely for $\sigma = \operatorname{Re} s > 1$, and that there exists an analytic continuation to $H_\alpha \backslash \{1\}$ for some $\alpha < 1$ except for a simple pole at $s = 1$ with residue ρ. (Here H_α is the half-plane of complex numbers whose real part is greater than α.) We prove the following connections between 'polynomial growth' of F on curves close to $\sigma = 1$ and the size of the 'error' in $f(x) - \rho x$. We note here that we have assumed there is a simple pole at 1. The result can easily be adjusted to include any finite number of poles on the 1-line, including none.

The first result, which argues from the size of F to the size of f, uses standard methods from complex analysis.

(2) This is usually formulated for g-prime systems, but actually proved for outer g-prime systems. No use of $\pi(x)$ being increasing is made, only that of $\Pi(x)$.

THEOREM 2.1. *With the above conventions, suppose that* $F(\sigma \pm it) \ll t^a$ *for some* $a < 1$ $(t \geq 1)$, *for* $\sigma \geq 1 - \delta(\log t)$ *where* $\delta : [0, \infty) \to (0, 1 - \alpha)$ *is continuously differentiable, decreasing, and* $\delta'(x) \ll 1$. *Then*

$$f(x) = \rho x + O(xe^{-\frac{b}{2}h^{-1}(\frac{\log x}{b})})$$

where $b = 1 - a$ *and* $h(x) = x/\delta(x)$.

Proof. We make use of the inverse Mellin transform, not for $f(x)$ but its integral, thus:

$$f_1(x) = \int_0^x f = \frac{1}{2\pi i} \int_{(c)} \frac{F(s)}{s(s+1)} x^{s+1}\, ds \qquad (x > 0)$$

where the path is the vertical line $\operatorname{Re} s = c$ and $c > 1$. (3) Note that the integral converges absolutely as $F(s)$ is bounded on any such line. Now push the contour to the left as far as the curve C made up of $s = 1 - \delta(\log |t|) + it$ for $|t| \geq 1$, and the vertical section $s = 1 - \delta(0) + it$ for $|t| < 1$. We pick up the residue at the simple pole at 1 to obtain

$$f_1(x) = \frac{\rho}{2} x^2 + \frac{1}{2\pi i} \int_C \frac{F(s)}{s(s+1)} x^{s+1}\, ds.$$

It follows that (using $\delta'(x) \ll 1$)

$$\left| f_1(x) - \frac{\rho}{2} x^2 \right| \ll \int_1^\infty \frac{|F(1 - \delta(\log t) + it)|}{t^2} x^{2-\delta(\log t)}\, dt + x^{2-\delta(0)}$$

$$\ll x^2 \int_1^\infty \frac{x^{-\delta(\log t)}}{t^{2-a}}\, dt + x^{2-\delta(0)} = x^2 \int_0^\infty e^{-bu - \delta(u) \log x}\, du + x^{2-\delta(0)}$$

where $b = 1 - a$. Split up the integral into the ranges $[0, A]$ and $[A, \infty)$. On $[0, A]$, the integral is bounded by $e^{-\delta(A) \log x} \int_0^A e^{-bu}\, du \ll e^{-\delta(A) \log x}$. On $[A, \infty)$ it is bounded by $\int_A^\infty e^{-bu}\, du \ll e^{-bA}$. The optimal choice is to take A so that $\delta(A) \log x = bA$, that is,

$$h(A) = \frac{A}{\delta(A)} = \frac{1}{b} \log x, \quad \text{or} \quad A = h^{-1}\left(\frac{1}{b} \log x\right).$$

This gives

$$f_1(x) = \frac{\rho}{2} x^2 + O(x^2 e^{-bh^{-1}(\frac{\log x}{b})}).$$

The usual trick of passing from f_1 to f using the fact that f is increasing now gives the result. ∎

Note that in the above theorem, it clearly suffices to have $\delta(\cdot)$ *eventually* decreasing.

(3) Here $\int_{(c)}$ means $\int_{c-i\infty}^{c+i\infty}$.

Now we prove a kind of converse result, in that bounds for the 'error' $f(x)-\rho x$ imply polynomial growth for F in some region close to the line $\operatorname{Re} s=1$ (and to the right of it).

THEOREM 2.2. *Suppose that $F=\hat{f}$ has an analytic continuation to $H_\alpha\setminus\{1\}$ for some $\alpha<1$ except for a simple pole at 1 with residue ρ, and $F(\sigma\pm it)\ll e^t$ as $t\to\infty$ for $\sigma>\alpha$. Further assume that*

$$f(x)=\rho x+O(xe^{-k(x)})$$

where k is strictly increasing and positive and $k(e^x)/x$ is decreasing. Then there exist $a,c>0$ such that

$$F(\sigma\pm it)\ll t^c$$

as $t\to\infty$, in the region where

$$\sigma\geq 1-\frac{k(e^{at})}{at}.$$

Proof. We shall actually show that $F(\sigma\pm it)\ll t^c$ in the region where $t^{-b}\ll 1-\sigma\leq k(e^{at})/(at)$ for some $a,b>0$. (Note that $k(e^{at})/(at)\gg 1/t$ in any case.) But $F(\sigma+it)\ll t^{b+1}$ for $\sigma-1\gg t^{-b}$ (4), hence a straightforward Phragmén–Lindelöf argument, together with the bound $F(\sigma+it)\ll e^t$, shows that the polynomial bound holds throughout the region where $|\sigma-1|\ll t^{-b}$. So for the rest of the proof, we may take $1-\sigma\gg t^{-b}$ whenever necessary.

We start from the formula

$$\int_0^\infty x^{-s}e^{-\lambda x}\,df(x) = \frac{1}{2\pi i}\int_{(c)}\Gamma(w)F(s+w)\lambda^{-w}\,dw \qquad (\lambda>0,\ c>\max\{0,1-\sigma\}), \tag{2.2}$$

which can be proved by inserting the Mellin transform (2.1) on the right, swapping the order of integration, and using the formula

$$e^{-y}=\frac{1}{2\pi i}\int_{(c)}\Gamma(w)y^{-w}\,dw.$$

This is justified as $|\Gamma(c+iy)F(s+c+iy)|$ is exponentially small for large $|y|$, on the assumption that $F(\sigma+it)\ll e^t$.

Take $\sigma\in(\alpha,1)$. Integrate the LHS of (2.2) by parts, while on the RHS push the contour to the left as far as $\operatorname{Re} w=-\beta$ past the simple poles at 0 and $1-s$, picking up the residues $F(s)$ and $\rho\Gamma(1-s)\lambda^{s-1}$ respectively, where $0<\beta<\sigma-\alpha$. This is justified since, along $[-\beta+iy,c+iy]$, we have

(4) Here $|F(s)|=|s\int_1^\infty \frac{f(x)}{x^{s+1}}\,dx|\ll t\int_1^\infty x^{-\sigma}\,dx\ll\frac{t}{\sigma-1}\ll t^{b+1}$ for $\sigma-1\gg t^{-b}$.

$|\Gamma(w)F(s+w)\lambda^{-w}| \ll |y|^A \lambda^{-B} e^{-\pi|y|/2} e^{|t|+|y|} \to 0$ as $|y| \to \infty$. Rearranging gives

$$F(s) = \int_0^\infty \frac{f(x)(s+\lambda x)e^{-\lambda x}}{x^{s+1}}\,dx \\ - \rho\Gamma(1-s)\lambda^{s-1} - \frac{1}{2\pi i}\int_{(-\beta)} \Gamma(w)F(s+w)\lambda^{-w}\,dw.$$

Now insert $f(x) = \rho x + E(x)$. The part involving ρx cancels with the $\rho\Gamma(1-s)\lambda^{s-1}$ term to give

$$F(s) = \int_0^\infty \frac{E(x)e^{-\lambda x}(s+\lambda x)}{x^{s+1}}\,dx - \frac{1}{2\pi i}\int_{(-\beta)} \Gamma(w)F(s+w)\lambda^{-w}\,dw \tag{2.3}$$

valid for $0 < \beta < \sigma - \alpha$.

We estimate the two terms on the RHS of (2.3). Then we choose λ appropriately. Without loss of generality take $t \ge 1$. The second term is bounded in modulus by

$$\frac{\lambda^\beta}{2\pi}\int_{-\infty}^{\infty} |\Gamma(-\beta+iy)|\,|F(\sigma-\beta+i(y+t))|\,dy \\ \ll \lambda^\beta \int_{-\infty}^{\infty} \frac{e^{-\pi|y|/2}}{(|y|+1)^{\beta+1/2}} e^{|y+t|}\,dy \ll \lambda^\beta e^t.$$

To make this $\ll t^c$, we shall take $\lambda = e^{-t/\beta}$. For the first integral of (2.3) consider first the range $[0,1]$. This is

$$\ll t\int_0^1 \frac{|E(x)|}{x^{\sigma+1}}\,dx = \rho t\int_0^1 x^{-\sigma}\,dx = \frac{\rho t}{1-\sigma} \ll t^{b+1},$$

on the assumption that $1-\sigma \gg t^{-b}$. Thus, with $\lambda = e^{-t/\beta}$ and $\beta \in (0, \sigma-\alpha)$,

$$F(s) \ll \int_1^\infty \frac{e^{-k(x)-\lambda x}(t+\lambda x)}{x^\sigma}\,dx + t^{b+1}.$$

Split the integral into the ranges $[1, 1/\lambda]$ and $[1/\lambda, \infty)$. For $[1, 1/\lambda]$ it is

$$\ll t\int_1^{1/\lambda} \frac{e^{-k(x)}}{x^\sigma}\,dx = t\int_0^{t/\beta} \exp\{(1-\sigma)y - k(e^y)\}\,dy.$$

Now take $1-\sigma \le \frac{k(e^{t/\beta})}{t/\beta}$. Since $k(e^x)/x$ is decreasing, we must have $(1-\sigma)y \le k(e^y)$ for $0 \le y \le t/\beta$. Thus the above integral is at most $t/\beta \ll t$.

There remains the range $[1/\lambda, \infty)$. We have

$$\int_{1/\lambda}^{\infty} \frac{e^{-k(x)-\lambda x}(t+\lambda x)}{x^\sigma}\,dx$$
$$\ll \lambda t e^{-k(1/\lambda)} \int_{1/\lambda}^{\infty} x^{1-\sigma} e^{-\lambda x}\,dx = \lambda^{\sigma-1} t e^{-k(1/\lambda)} \int_{1}^{\infty} y^{1-\sigma} e^{-y}\,dy$$
$$\le \Gamma(2-\sigma) t \exp\{(1-\sigma)t/\beta - k(e^{t/\beta})\} \le \Gamma(2-\sigma) t \ll t$$

in the range where $1-\sigma \le \frac{k(e^{t/\beta})}{t/\beta}$. ∎

Examples. We illustrate the above results with some examples.

(i) Take $\delta(x) = x^{-\alpha}$ where $\alpha \ge 0$. Then $h(x) = x^{1+\alpha}$, and Theorem 2.1 says (for suitable $b, c > 0$) that

$$F(\sigma+it) \ll t^c \text{ for } 1-\sigma \le 1/(\log t)^\alpha \Rightarrow f(x)-\rho x \ll x e^{-b(\log x)^{1/(1+\alpha)}}. \tag{2.4}$$

(ii) Take $k(x) = (\log x)^\beta$ where $0 < \beta \le 1$. Thus $k(e^x)/x = x^{-(1-\beta)}$ decreases, and Theorem 2.2 says (essentially) that

$$f(x) - \rho x \ll x e^{-(\log x)^\beta} \Rightarrow F(\sigma+it) \ll t^c \text{ for } 1-\sigma \ll 1/t^{1-\beta}. \tag{2.5}$$

Note the discrepancy between the regions. In (2.4) the region of polynomial growth of F needed to force $f(x) - \rho x \ll x e^{-(\log x)^\beta}$ is much larger than that gained in (2.5)—except in the case where $\beta = 1$ ($\alpha = 0$). This undoubtedly has to do with the apriori assumption in Theorem 2.2 about the growth of F. If we assumed more, we can expect a bigger region where F is of polynomial growth.

3. An example of generalised primes. As mentioned in the introduction, we wish to study the generalised prime system for which the generalised Chebyshev function is given by

$$\psi_0(x) = [x] - 1 \quad \text{for } x \ge 1,$$

and zero otherwise. Here

$$\Pi_0(x) = \sum_{1<n\le x} \frac{1}{\log n},$$

so that indeed we have an outer g-prime system, according to Definition (i). Further, note that, by setting

$$\pi_0(x) = \sum_{\substack{1<n\le x\\ n \text{ not a perfect power}}} \frac{1}{\log n}$$

(where a *perfect power* is a number of the form p^q with $p, q \in \mathbb{N}$ and $p, q > 1$), we have

$$\sum_{k=1}^{\infty} \frac{1}{k} \pi_0(x^{1/k}) = \sum_{\substack{k \geq 1,\, n > 1 \\ n^k \leq x \\ n \text{ not a perfect power}}} \frac{1}{k \log n} = \sum_{1 < m \leq x} \frac{1}{\log m} = \Pi_0(x).$$

So this is indeed a g-prime system. The Beurling zeta function ζ_0 is, for $\operatorname{Re} s > 1$, given by

$$\zeta_0(s) = e^{\hat{\Pi}_0(s)} = \exp\left\{ \sum_{n=2}^{\infty} \frac{1}{n^s \log n} \right\}.$$

Thus also

$$-\frac{\zeta_0'(s)}{\zeta_0(s)} = \zeta(s) - 1.$$

This shows that the LHS above has an analytic continuation to the whole plane except for a simple pole at 1. Standard complex analysis then implies that ζ_0 is holomorphic in $\mathbb{C} \setminus \{1\}$ with a simple pole at $s = 1$ and has *no* zeros.

Let $N_0(x)$ denote the associated g-integer counting function. Thus $\zeta_0(s) = \hat{N}_0(s)$. By a simple application of the Wiener–Ikehara Theorem (see for example [1]) it follows that

$$N_0(x) \sim \tau x \quad \text{as } x \to \infty,$$

for some $\tau > 0$. Indeed τ is the residue of $\zeta_0(s)$ at $s = 1$. The question is now: how small is the difference $N_0(x) - \tau x$?

***O*-results for $N_0(x) - \tau x$.** We can get an immediate improvement by using the fact that $\psi_0(x) = x + O(1)$ and result (c) at the end of Section 1, namely

$$N_0(x) = \tau x + O(x e^{-c\sqrt{\log x \log\log x}})$$

for some $c > 0$. We show below that the error can be strengthened further by using knowledge of the Riemann zeta function.

THEOREM 3.1. *With N_0 and τ as above, we have*

$$N_0(x) = \tau x + O(x e^{-c(\log x)^{3/5} (\log\log x)^{2/5}})$$

for some $c > 0$. Furthermore, on the Riemann Hypothesis,

$$N_0(x) = \tau x + O(x e^{-ck(x)}), \quad \textit{where} \quad k(x) = \frac{\log x \log\log\log x}{\log\log x},$$

for any $c < 1/4$.

Proof. The proof depends on bounds for Riemann's zeta function in the region to the left of, and close to, the 1-line. By a result of Richert [11],

$$\zeta(\sigma+it) \ll (1+t^{100(1-\sigma)^{3/2}})(\log t)^{2/3} \quad \text{for } 0 \le \sigma \le 2,\ t \ge 2, \tag{3.1}$$

while on the Riemann Hypothesis, this can be improved to (see [13, Chapter 14])

$$\log \zeta(\sigma+it) \ll \frac{(\log t)^{2-2\sigma}-1}{(1-\sigma)\log\log t} + \log\log\log t \quad \text{for } 1/2 < \sigma_0 \le \sigma < 1. \tag{3.2}$$

From the above, we have

$$\begin{aligned}\log|\zeta_0(\sigma+it)| = \operatorname{Re}\{\log\zeta_0(\sigma+it)\} &= -\operatorname{Re}\int_\sigma^2 \frac{\zeta_0'}{\zeta_0}(x+it)\,dx + O(1)\\ &\ll (1+t^{100(1-\sigma)^{3/2}})(\log t)^{2/3}.\end{aligned}$$

This is $o(\log t)$ if, say, $100(1-\sigma)^{3/2} \le \frac{\log\log t}{4\log t}$. Thus we can apply Theorem 2.1 to ζ_0 with

$$\delta(x) = \left(\frac{\log x}{400x}\right)^{2/3}$$

and any $a \in (0,1)$. With $h(x) = x/\delta(x)$, we find $h^{-1}(y) \sim cy^{3/5}(\log y)^{2/5}$ for some $c > 0$, and hence the first (unconditional) result follows from Theorem 2.1.

Now assume the truth of the Riemann Hypothesis, so that (3.2) holds. Using the fact that $(e^u-1)/u \le e^u$ for $u > 0$, this implies that for $\sigma < 1$,

$$\begin{aligned}\log|\zeta_0(\sigma+it)| &\ll \exp\{A(\log t)^{2(1-\sigma)} + A\log\log\log t\}\\ &= (\log\log t)^A \exp\{A(\log t)^{2(1-\sigma)}\}\end{aligned}$$

for some A. The RHS is $o(\log t)$ if we take

$$1-\sigma \le \frac{\log\log\log t - B}{2\log\log t}$$

for some B sufficiently large. For then $A(\log t)^{2(1-\sigma)} \le Ae^{-B}\log\log t$ and so, taking $B = \log 2A$,

$$\log|\zeta_0(\sigma+it)| \ll (\log\log t)^A\sqrt{\log t} = o(\log t).$$

We can apply Theorem 2.1 with

$$\delta(x) = \frac{\log\log x - B}{2\log x}$$

and any $a \in (0,1)$. As $h(x) \sim \frac{2x\log x}{\log\log x}$ and $h^{-1}(x) \sim \frac{x\log\log x}{2\log x}$, we have

$$N_0(x) - \tau x \ll xe^{-\frac{1-\varepsilon}{4}\frac{\log x\log\log\log x}{\log\log x}}$$

for any $\varepsilon > 0$. ∎

REMARK. In the above it is clear that for the second part, we do not need the full force of the Riemann Hypothesis, but only (3.2), and even then only for σ close to 1.

Ω-results for $N_0(x)-\tau x$. On the other hand, the difference $N_0(x)-\tau x$ cannot be too small. Indeed, if $N_0(x)-\tau x \ll x^\beta$ for some $\beta<1$, then the system is 'well-behaved', which implies that

$$-\frac{\zeta_0'(\sigma+it)}{\zeta_0(\sigma+it)} \ll (\log t)^A \qquad (\beta<\sigma<1)$$

for some A (see [8, Theorem 2.3]). But the LHS is $\zeta(\sigma+it)-1$, which is sometimes of much larger order than a power of $\log t$. Thus $N_0(x)-\tau x = \Omega(x^{1-\delta})$ for every $\delta>0$. A further improvement is possible using Theorem 2.2.

THEOREM 3.2. *With N_0 as above, we have*

$$N_0(x)-\tau x = \Omega(xe^{-cl(x)}) \quad \textit{where} \quad l(x)=\frac{\log x\log_4 x}{\log_3 x},$$

for every $c>2$. ([5])

Proof. If the result is false, then $N_0(x)-\tau x = o(xe^{-cl(x)})$. Now apply Theorem 2.2 with $k(x)=cl(x)$. We have $k(e^x)/x = \frac{c\log_3 x}{\log_2 x}$, which is decreasing. The conditions of the theorem are met, and so

$$\zeta_0(\sigma+it)\ll t^A \quad \text{whenever} \quad 1-\sigma\le \frac{c\log_3 at}{\log_2 at}\sim\frac{c\log_3 t}{\log_2 t} \tag{3.3}$$

for some $A,a>0$. We show this is incompatible with known Ω-results for $\zeta(\sigma+it)$. For this, we quote a special case of a result of Ivić [9, p. 241]: *there exist arbitrarily large $t\in[T,2T]$ such that with $\sigma=1-\frac{\mu\log_3 T}{\log_2 T}$,*

$$\log|\zeta(\sigma+it)|\gg(\log\log t)^{\mu-1}. \tag{3.4}$$

(On the Riemann Hypothesis, it is $\ll(\log\log t)^{2\mu}/\log_3 t$, so one cannot expect to do much better.) Now apply the Borel–Carathéodory Theorem (see [12]) to $\log\zeta_0$ and the circles with centre $2+it$ and radii $R=1+\frac{(1-\varepsilon)c\log_3 t}{\log_2 t}$ and $r=R-1/\log_2 t$ (here $\varepsilon>0$). On the larger circle, by (3.3) we have

$$\mathrm{Re}\{\log\zeta_0(\sigma+it)\}=\log|\zeta_0(\sigma+it)|\ll\log t,$$

hence on and inside the smaller circle we have

$$|\log\zeta_0(\sigma+it)|\ll\log t\log\log t.$$

([5]) Here $\log_k x$ denotes the kth iterated log, that is, $\log_0 x=x$, $\log_{k+1}x=\log\log_k x$.

By Cauchy's integral formula,

$$\zeta(s)-1=-\frac{\zeta_0'(s)}{\zeta_0(s)}=-\frac{1}{2\pi i}\int_C \frac{\log\zeta_0(s+z)}{z^2}\,dz$$

for any sufficiently small circular contour C around 0. Taking $s=1-\frac{\mu\log_3 T}{\log_2 T}+it$ with $\mu<c$ and $t\in[T,2T]$, and C the circle with radius $\delta=1/\log_2 t$, we have (for t sufficiently large)

$$\left|\zeta\left(1-\frac{\mu\log_3 T}{\log_2 T}+it\right)\right|\le 1+\frac{1}{\delta}\max_{|z|=\delta}|\log\zeta_0(\sigma+it+z)|\ll \log t\,(\log\log t)^2.$$

Because of (3.4), this is a contradiction for every $\mu>2$. ∎

Remarks. (i) By the methods in [7], a slight strengthening in large values of ζ near the 1-line is possible. In particular, in (3.4) the RHS may be improved to $(\log\log t)^{\mu}/\log_3 t$. This leads to a small improvement in Theorem 3.2, namely the result holds for every $c>1$.

(ii) We see that, especially on the Riemann Hypothesis, there is a little gap between the O- and Ω-results. It would be interesting to investigate further which result is closer to the truth.

The value of τ. Finally, a few words about the value of τ. An easy exercise shows that τ is given by

$$\sum_{1<n\le x}\frac{1}{n\log n}=\log\log x+\gamma+\log\tau+o(1),$$

where γ is Euler's constant. For, denoting the LHS sum by $S(x)$, elementary real analysis implies $S(x)=\log\log x+\lambda+o(1)$ as $x\to\infty$ for some λ. But for $\delta>0$, we have

$$\begin{aligned}\log\zeta_0(1+\delta)&=\delta\int_1^\infty\frac{S(x)}{x^{1+\delta}}\,dx=\delta\int_1^\infty\frac{\log\log x}{x^{1+\delta}}\,dx+\delta\int_1^\infty\frac{\lambda+o(1)}{x^{1+\delta}}\,dx\\&=\delta\int_0^\infty\frac{\log y}{e^{\delta y}}\,dy+\lambda+o(1)=\log\frac{1}{\delta}-\gamma+\lambda+o(1).\end{aligned}$$

The LHS is $\log(1/\delta)+\log\tau+o(1)$ as $\delta\to 0$. Hence the result follows.

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