Primes with preassigned digits

by

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1. Introduction and statement of the results. Let \( g \geq 2 \) be an integer. For \( k \in \mathbb{N} \) every integer \( n \in [g^{k-1}, g^k) \) can be uniquely written as

\[
n = \sum_{\nu=0}^{k-1} z_\nu g^\nu \quad (z_\nu \in \{0, \ldots, g-1\}, z_{k-1} > 0),
\]

briefly \( n = z_{k-1} \ldots z_0 \). Sierpiński [6] showed, that for any given \( b, b' \in \{0, \ldots, g-1\}, (b, g) = 1 \), and \( b' > 0 \), there are infinitely many primes \( p \) which have \( b \) as the last digit \( z_0 \) and \( b' \) as the first. This is an immediate consequence of Dirichlet’s theorem on primes in arithmetic progressions.

The question can be generalized as follows. For \( a \in \mathbb{N} \) and \( k \geq a \) let

\[
0 \leq l_1 < \cdots < l_a \leq k - 1, \quad \vec{l} = (l_1, \ldots, l_a), \quad \vec{b} = (b_1, \ldots, b_a) \in \{0, \ldots, g-1\}^a \quad \text{with} \quad (b_1, g) = 1 \quad \text{if} \quad l_1 = 0,
\]

\[
b_a > 0 \quad \text{if} \quad l_a = k - 1.
\]

Such vectors \( \vec{l} \) and \( \vec{b} = (b_1, \ldots, b_a) \) will be called admissible. Write

\[
\pi_{k,a,\vec{l},\vec{b}} = \# \{g^{k-1} \leq p < g^k : p = z_{k-1} \cdots z_0, z_{l_j} = b_j \ (j = 1, \ldots, a) \},
\]

\[
f_1(l) = \begin{cases} 
1/\varphi(g) & \text{if } l = 0, \\
1/g & \text{if } 1 \leq l < k - 1, \\
1/(g - 1) & \text{if } l = k - 1,
\end{cases}
\]

\[
f(\vec{l}) = \prod_{j=1}^{a} f_1(l_j).
\]

It seems reasonable to expect, for any fixed \( a \) and admissible \( \vec{l} \) and \( \vec{b} \),

\[
(C) \quad \pi_{k,a,\vec{l},\vec{b}} \sim f(\vec{l})(\pi(g^k) - \pi(g^{k-1})) \quad (k \to \infty).
\]

It will be shown that (C) is true for \( a = 1 \) and \( a = 2 \).

**Theorem 1.** For \( a \in \{1, 2\} \) and admissible \( \vec{l} \) and \( \vec{b} \) of length \( a \) we have

\[
\pi_{k,a,\vec{l},\vec{b}} = f(\vec{l})(1 - g^{-1}) g^k / \ln g^k + O\left( g^k / k^2 \right).
\]
The O-constants here and in what follows may depend on $g$, but not on $\vec{l}$.

There is some hope that (C) can be proved for $a = 3$ or even for bigger values. If one assumes the Riemann hypothesis for the characters mod $g^d$ ($d \in \mathbb{N}_0$) then much more can be shown.

**Theorem 2.** Assume the Riemann hypothesis for the $L$-functions with characters mod $g^d$ ($d \in \mathbb{N}_0$). Then, for any $\varepsilon \in (0, 1)$, $k \geq k_0(\varepsilon)$, and all admissible vectors $\vec{l}, \vec{b}$ of length $a$, $1 \leq a \leq (1 - \varepsilon)k^{1/2}$, we have

$$\pi_{k,a,\vec{l},\vec{b}} = f(\vec{l}) \left(1 - \frac{1}{g}\right) \frac{g^k}{\ln g^k} + O(g^{k-a}k^{-2}).$$

It is not clear whether any form of the density hypothesis for the $L$-functions mod $g^d$ will imply a result of similar strength.

2. **Proof of Theorem 1**

2.1. The proof will be given in the more complicated case $a = 2$. Let $0 \leq l < l' \leq k - 1$, $\vec{l} = (l, l')$, $b, b' \in \{0, \ldots, g - 1\}$, $(b, g) = 1$ if $l = 0$, $b' > 0$ if $l' = k - 1$, $\vec{b} = (b, b')$. Instead of $\pi_k$ we will study

$$\psi_k := \psi_{k,2,\vec{l},\vec{b}} = \sum_{g^{k-1} \leq n < g^k, \quad z_l = b, z_{l'} = b'} \Lambda(n),$$

and show

$$(2.1.1) \quad \psi_k = f(\vec{l})g^k \left(1 - \frac{1}{g}\right) + O(g^kk^{-1}).$$

Let $k$ be sufficiently large and write $x = g^k$.

2.2. **First case:** $l > \frac{1}{g}k$. Every $n$ to be counted in $\psi_k$ can be written as

$$(2.2.1) \quad n = n_2g^{l+1} + bg^l + n_1, \quad \text{where} \quad 0 \leq n_1 < g^l, \quad g^{k-l-2} \leq n_2 < g^{k-l-1},$$

and $n_2$ has the digit $b'$ at the place with index $l' - l - 1$. In 2.2, $n_2$ will run through these numbers. This gives

$$\psi_k = \sum_{n_2} (\psi((gn_2 + b + 1)g^l) - \psi((gn_2 + b)g^l)) + O(xk^{-1}),$$

where

$$\psi(y) = \sum_{n \leq y} \Lambda(n).$$

The error term, which of course could be estimated much better, results from the possible contribution of the end points of the intervals.
Let $\rho = \rho(\zeta) = \beta + i\gamma$ denote non-trivial zeros of $\zeta(s)$ (and similarly let $\rho(\chi)$ be zeros of $L(s, \chi)$). Then, for $2 \leq T \leq x$,

$$\psi_k = g^1 \#\{n_2\} - \sum_{\rho(\zeta), |\gamma| \leq T} g^{-1} g^{1\rho} \sum_{n_2} ((gn_2 + b + 1)^\rho - (gn_2 + b)^\rho)$$

$$+ O\left(\frac{x}{T} k^2 \#\{n_2\}\right) + O(xk^{-1}).$$

The main term is equal to that in (2.1.1). The contribution of the error terms is $\ll xk^{-1}$ if

$$T = xg^{-l}k^3.$$  

For $|\rho| \leq xg^{-l}k^{-1} =: T_1$ the difference in the $n_2$-sum can be expanded to

$$\rho(gn_2 + b)^{\rho - 1} + \frac{\rho(\rho - 1)}{2}(gn_2 + b)^{\rho - 2} + \cdots.$$  

We will treat the first term. The higher terms can be treated in the same manner. Because of the choice of $T_1$ they lead to smaller bounds. For $T_1 < |\rho| \leq T$ there is no cancellation. It is therefore sufficient to study the following expression:

$$\Sigma_\rho := \sum_{\rho(\zeta), |\gamma| \leq T_1} g^{1\rho} \left| \sum_{n_2} (gn_2 + b)^{\rho - 1} \right|$$

$$+ \frac{g^l}{x} \sum_{\rho(\zeta), T_1 < |\gamma| \leq T} g^{l_\beta} \left| \sum_{n_2} (gn_2 + b)^\rho \right|.$$  

For these sums and similar ones in the other cases we will apply zero density bounds and a mean value theorem for Dirichlet polynomials.

Write as usual, for $q \geq 1$, $\chi \mod q$, $1/2 \leq \sigma \leq 1$, $T \geq 2$,

$$N(\sigma, T, \chi) = \#\{\rho = \beta + i\gamma : L(\rho, \chi) = 0, \beta \geq \sigma, |\gamma| \leq T\}.$$  

Then we have

$$\sum_{\chi \mod q} N(\sigma, T, \chi) \ll \varepsilon \begin{cases} (qT)^{3(1-\sigma)/2-\sigma} \ln^9 (qT) & \text{if } 1/2 \leq \sigma \leq 3/4, \\
(qT)^{3(1-\sigma)/(3\sigma-1)+\varepsilon} & \text{if } 3/4 \leq \sigma \leq 4/5, \\
(qT)^{(2+\varepsilon)(1-\sigma)} & \text{if } 4/5 \leq \sigma \leq 1, \end{cases}$$  

(see Montgomery [4, Theorem 12.1], Huxley [1], Jutila [3]).

Let $a_n \in \mathbb{C}$ for $1 \leq n \leq N$, and set $D(s) = \sum_{n \leq N} a_n n^{-s}$. Let $A$ be a set of complex numbers $s = \sigma + i\tau$ where $\sigma \geq \sigma_0, |\tau - \tau'| \geq \delta$ for $s \neq s' \in A$,
and $T_0 - \delta/2 \leq \tau \leq T_0 + T + \delta/2$. Then

$$(2.2.7) \sum_{s \in A} |D(s)|^2 \ll \ln \ln N \cdot (\delta^{-1} + \ln N) \cdot (T + N) \sum_{n \leq N} |a_n|^2 n^{-2\sigma_0}$$

(Montgomery [4, Theorem 7.5]).

Put

$$\sigma_0 = \sigma_0(\zeta) = 1 - k^{-3/4} =: 1 - \delta_0.$$ 

The Vinogradov–Korobov zero free region (see Ivić [2, Theorem 6.1]) ensures

$$\zeta(s) \neq 0 \quad \text{for} \quad |\text{Im} \ s| \leq x, \ Re \ s \geq \sigma_0.$$ 

We have

$$\Sigma_{\varrho} \ll k \max_{1/2 \leq \sigma \leq \sigma_0} g^{\frac{1}{\sigma}} \left\{ (N(\sigma, T))^1/2 \left( \sum_{\varrho, \gamma | T_1} \left| \sum_{n_2} (g n_2 + b)^{\varrho-1} \right|^2 \right)^{1/2} \right.$$ 

$$+ \frac{g^l k}{x} (N(\sigma, T))^1/2 \left( \sum_{\varrho, \gamma | T_1} \left| \sum_{n_2} (g n_2 + b)^{\varrho} \right|^2 \right)^{1/2} \right\}.$$ 

We split up the set of zeros into $\ll k$ classes $A$ which fulfill the conditions of (2.2.7) with $\delta = 1$. Hence, by (2.2.7) and (2.2.6)

$$\Sigma_{\varrho} \ll \epsilon k^3 \max_{1/2 \leq \sigma \leq \sigma_0} g^{\frac{1}{\sigma}} T^{(6/5+\epsilon)(1-\sigma)} \left( \frac{x}{g^l} k^3 \right)^{1/2}$$

$$\times \left\{ \left( \frac{x}{g^l} \right)^{(2\sigma-1)/2} + \frac{g^l k}{x} \left( \frac{x}{g^l} k^3 \right)^{(2\sigma+1)/2} \right\}$$

$$\ll k^{10} \max_{1/2 \leq \sigma \leq \sigma_0} x^\sigma (x g^{-1})^{(6/5+\epsilon)(1-\sigma)}$$

$$\ll k^{10} x^{1/2+3/5+\epsilon} g^{-3l/5} + x^{\sigma_0+(6/5+\epsilon)\delta_0} g^{-6l\delta_0/5}.$$ 

This is $\ll xk^{-1}$ for $g^l > x^{1/5}$ if $\epsilon > 0$ is chosen sufficiently small. Hence Theorem 1 is true in this case.

2.3. Second case: $l \leq \frac{1}{5} k$, $l' > \frac{4}{5} k$. The numbers $n$ to be counted here can be written as

$$n = n_2 g^{l'+1} + b' g^{l'} + \sum_{\nu = l+1}^{l'-1} z_\nu g^\nu + b g^l + n_1$$

(2.3.1)

$$1 \leq n_1 \leq g^l - 1, \quad (n_1, g) = 1, \quad g^{k-l'-2} \leq n_2 < g^{k-l'-1} - g^{l'+1}.$$ 

In this part $n_1$ and $n_2$ will denote these numbers. We get
(2.3.2) \[ \psi_k = \sum_{n_1, n_2} (\psi(n_2g^{l'+1} + (b' + 1)g^l, g^{l+1}, bg^l + n_1) - \psi(n_2g^{l'+1} + b'g^l, g^{l+1}, bg^l + n_1)) + O(x/k) \]

\[ = \sum_{n_1, n_2} \frac{g^{l'}}{\varphi(g^{l+1})} - \frac{1}{\varphi(g^{l+1})} \sum_{\chi \mod g^{l+1}} \sum_n \overline{\chi}(n_1 + bg^l) \]

\[ \times \sum_{n_2} \left( \sum_{\varrho(\chi); |\gamma| \leq T} \frac{g^{l'}\varrho}{\varrho} ((n_2g + b' + 1)^e - (n_2g + b')^e) + O\left(\frac{x}{T}k^2\right) \right) \]

\[ + O(xk^{-1}) \quad (2 \leq T \leq x). \]

Again the main term gives the expected value. By the Pólya–Vinogradov inequality we have

(2.3.3) \[ \sum_{n_1} \chi(n_1 + bg^l) \ll \begin{cases} g^l/k & \text{if } \chi = \chi_0, \\ g^l/2k & \text{if } \chi \neq \chi_0. \end{cases} \]

Therefore the contribution of the error term \(O(xT^{-1}k^2)\) in (2.3.2)—which does not depend on \(n_1\)—is \(\ll \#\{n_2\}g^{l/2}kxT^{-1}k^2\). This is \(\ll xk^{-1}\) for

(2.3.4) \[ T = xg^{l/2-l'}k^4. \]

Note that \(T < x\) for \(k\) sufficiently large.

Let \(\Sigma_\varrho\) be the \(\varrho\)-sum in (2.3.2). For \(\varrho = \beta + i\gamma\) with \(|\gamma| \leq xg^{-l'}k^{-1} =: T_1\) we have

\[ \varrho^{-1}((n_2g + b' + 1)^e - (n_2g + b')^e) \ll (xg^{-l'})^{\beta-1}. \]

Put

\[ \sigma_0(l) = \begin{cases} 1 - k^{-3/4} & \text{if } g^l \leq k^{30}, \\ 1 & \text{otherwise} \end{cases} \]

(in particular, \(\sigma_0(\zeta) = \sigma_0(0)\)). Then \(\sigma_0(l)\) describes a zero free region of \(L(s, \chi), \chi \mod g^l\) (see Prachar [5, §6, Satz 6.2]). Hence

\[ \Sigma_\varrho \ll k^3 \sum_{1/2 \leq \sigma \leq \sigma_0(\zeta)} (x^\sigma N(\sigma, T_1) + xg^{-l'} \max_{T_1 < U \leq T} U^{-1} x^\sigma N(\sigma, U)) \]

\[ + k^4g^{-l/2} \sum_{1/2 \leq \sigma \leq \sigma_0(l)} (x^\sigma \sum_{\chi \mod g^{l+1}} N(\sigma, T_1, \chi_1)) \]

\[ + xg^{-l'} \max_{T_1 < U \leq T} U^{-1} x^\sigma \sum_{\chi \mod g^{l+1}} N(\sigma, U, \chi_1) \].

It is easy to see that (2.2.6), combined with the zero free regions, is sufficient to show that the last quantity is \(\ll xk^{-1}\) in the case \(g^l, xg^{-l'} \leq x^{1/5}\).

2.4. **Third case**: \(g^l' \leq x^{4/5}, g^l \leq x^{1/5}\). Assume, for simplicity, \(0 < l < l'\). The other cases can be treated in the same manner with minor modifications.
Here we write the numbers $n$ to be counted as

$$n = \sum_{\nu=\nu'+1}^{k-1} z_{\nu} g^{\nu} + b' g^{\nu'} + n_2 g^{l+1} + b g^l + n_1$$

where

$$1 \leq n_1 < g^l, \quad (n_1, g) = 1, \quad 0 \leq n_2 < g^{\nu'-(l+1)}.$$

Therefore

$$\psi_{k,2,l,b} = \sum_{n_1, n_2} \left( \psi(x, g^{\nu'+1}, b' g^{\nu'} + n_2 g^{l+1} + b g^l + n_1) - \psi(x, g^{\nu'-1}, b' g^{\nu'} + n_2 g^{l+1} + b g^l + n_1) \right) + O(xk^{-1})$$

$$= x \left( 1 - \frac{1}{g} \right) \frac{1}{\varphi(g)} g^{-\nu'} \# \{ n_1 \} \cdot \# \{ n_2 \}$$

$$- \frac{1}{\varphi(g) g^\nu} \sum_{\chi \mod g^{l'+1}} \sum_{n_1, n_2} \chi(b' g^{\nu'} + n_2 g^{l+1} + b g^l + n_1)$$

$$\times \left( \sum_{g(x), |\gamma| \leq T} \frac{1}{q} x^e (1 - g^{-e}) + O \left( \frac{x}{T} k^2 \right) \right) + O(xk^{-1}).$$

The contribution of the error terms is, by the orthogonality relation for characters, $\ll g^{l'} g^{3l'/2} (x/T) k^2 + xk^{-1}$. This is $\ll xk^{-1}$ if one chooses

$$T = g^{l'/2} k^3.$$

For $\chi \mod g^{l'+1}$, $\chi \neq \chi_0$, we consider the sum

$$\Sigma_\chi := \sum_{n_1, n_2} \chi(b' g^{\nu'} + n_2 g^{l+1} + b g^l + n_1).$$

$\Sigma_\chi$ results from summation over $\ll g^{\nu'-l}$ intervals of length $\ll g^l$. By Pólya–Vinogradov the sum over every such interval is $\ll g^{l'/2k}$, hence

$$\Sigma_\chi \ll g^{3l'/2-l} k.$$

On the other hand,

$$\Sigma_\chi = \sum_{n_1} \sum_{n \leq g^{l'}} \chi(n + b' g^{\nu'})$$

$$= \frac{1}{\varphi(g) g^l} \sum_{n_1} \sum_{\chi_1 \mod g^{l'+1}} \sum_{n \leq g^{\nu'}} \chi(n + b' g^{\nu'}) \chi_1(n_1 + b g^l).$$

The sum over $n_1$ is $\ll g^{l'/2k}$ if $\chi_1 \neq \chi_0$, but $\sum_n \chi(\cdot) \ll g^{l'/2k}$. Therefore

$$\Sigma_\chi \ll g^{l'/2k} g^{-l} (g^l + g^{l'/2k}) \ll g^{(l+l')/2} k^2.$$
Combined with (2.4.4) this implies

\[(2.4.5) \quad \Sigma_{\chi} \ll k^2 \min(g^{3l/2-l}, g^{(l+\nu)/2}) \ll k^2 g^{5l/6}.\]

The contribution of the \(g\)-sum to (2.4.1) is

\[(2.4.6) \quad \ll \sum_{g(\zeta), |\gamma| \leq T} x^{\beta} + g^{l^\nu} \sum_{\chi \equiv g^{l^0+1}, \chi \neq \chi_0} |\Sigma_{\chi}| \sum_{g(\chi), |\gamma| \leq T} x^{\beta}.\]

For \(g^{l^\nu} \leq k^{30}\) we use, similarly to the second case, \(\sigma_0(l') = 1 - k^{-3/4}\). Now (2.2.6) and the last inequality in (2.4.5) show that (2.4.6) is \(\ll xk^{-1}\).

In the case \(k^{30} < g^{l^\nu} \leq x^{1/3}\) we argue in the same manner with \(\sigma_0(l') = 1\). For \(x^{1/3} < g^{l^\nu} \leq x^{19/45}\) one uses \(\Sigma_{\chi} \ll k^2 g^{(l+\nu)/2}\). In the case \(x^{19/45} < g^{l^\nu} \leq x^{4/5}\)

the \(\zeta\)-part in (2.4.6) can be bounded by (2.2.6) with \(\sigma_0(\zeta) = 1 - k^{-3/4}\). The \(\chi\)-part \(\Sigma_{g, \chi}\) requires a bit more care. (2.2.5) gives

\[
\Sigma_{g, \chi} \ll \varepsilon k^{11} g^{(l+\nu)/2} \max_{U \leq T^{1/2/\leq \sigma\leq 3/4}} U^{-1}(U g^{l'})^{3(1-\sigma)/(2-\sigma)} x^{\sigma} \\
+ \max_{3/4 \leq \sigma \leq 4/5} U^{-1}(U g^{l'})^{3(1-\sigma)/(3\sigma-1)-\varepsilon} x^{\sigma} \\
+ \max_{4/5 \leq \sigma \leq 1} U^{-1}(U g^{l'})^{(2+\varepsilon)(1-\sigma)} x^{\sigma},
\]

The contribution of the part with \(4/5 \leq \sigma \leq 1\) is \(\ll xk^{-1}\) for the whole interval. The exponent of \(U\) is \(\leq 0\) for \(1/2 \leq \sigma \leq 4/5\). Write \(g^{l'} = x^\xi, 19/45 \leq \xi \leq 4/5\). Then we have, using \(g^{l'} \leq x^{1/5}\),

\[(2.4.8) \quad \Sigma_{g, \chi} \ll k^{11} x^{1/10} \max_{1/2 \leq \sigma \leq 3/4} x^{\sigma+\xi(3(1-\sigma)/2-\sigma)-1/2} \\
+ \max_{3/4 \leq \sigma \leq 4/5} x^{\sigma+\xi(3(1-\sigma)/3\sigma-1)-1/2 } + xk^{-1}.
\]

The function \(G(\sigma) = \sigma + \xi(3(1-\sigma)/2-\sigma)-1/2\) is decreasing in \([3/4, 4/5]\) for \(\xi \in [19/45, 4/5]\). Hence the second term in (2.4.8) is

\(\ll k^{11} x^{17/20+\xi/10+\varepsilon} \ll xk^{-1}\).

The function \(H(\sigma) = \sigma + \xi(3(1-\sigma)/3\sigma-1)-1/2\) is increasing in \([1/2, 3/4]\) for \(\xi \leq 25/48\). Again \(\sigma = 3/4\) leads to a sufficient bound.

For \(25/48 < \xi \leq 4/5\), \(G(\sigma)\) has a maximum at

\(\sigma^* = 2 - \sqrt{3\xi} \in [1/2, 3/4]\).

We have \(1/10 + H(\sigma^*) = 21/10 - 2\sqrt{3\xi} + 5\xi/2\). In \([25/48, 4/5]\) this function is increasing and gives a value \(< 1\) at \(\xi = 4/5\).

This shows that (2.1.1) is true in all subcases of the third case. Theorem 1 is proved.
3. Proof of Theorem 2. Assume \( r \geq r_0(\varepsilon) \) and \( a > 2 \). We will treat the case in which there is a \( j \) with \( 1 < j, j + 1 < a \) such that \( l_j + 1 - l_j \) is maximal amongst the \( a + 1 \) numbers \( l_1, l_2 - l_1, \ldots, l_a - l_{a-1}, k - 1 - l_a \), i.e. \( l_j + 1 - l_j \geq k/(a+1) \). We write \( l = l_j, l' = l_{j+1}, b = b_j, b' = b_{j+1} \). The numbers \( n \) to be counted in \( \psi_{k,a,l,b} \) can be written as

\[
n = n_2 g^{l+1} + b' g^l + \sum_{\nu=l+1}^{l'-1} z_{\nu} g^\nu + bg^l + n_1,
\]

where \( 0 \leq n_1 < g^l \) with the digits \( b_1, \ldots, b_{j-1} \) at the corresponding places, and \( g^{k-l'-2} \leq n_2 < g^{k-l'-1} \) with the digits \( b_{j+2}, \ldots, b_a \). We have

\[
N_1 := \#\{n_1\} \approx g^{l-j} \quad \text{and} \quad N_2 := \#\{n_2\} \approx g^{k-l'-(a-j)}.
\]

The explicit formula yields

\[
(3.1) \quad \psi_{k,a,l,b} = \sum_{n_1,n_2} (\varphi(g^{l+1}))^{-1} g^{l'} - (\varphi(g^{l+1}))^{-1} \sum_{\chi \mod g^{l+1}} \left( \sum_{n_1} \chi(n_1 + bg^l) \right) \sum_{n_2} \sum_{e(\chi), |\gamma| \leq \varepsilon} \left( \frac{g^{l'} g}{\bar{\nu}} ((n_2 g + b' + 1)^e - (n_2 g + b')^e) + O(k^2) \right) + O(k).
\]

Because

\[
(3.2) \quad \sum_{\chi \mod g^{l+1}} \left| \sum_{n_1} \chi(n_1 + bg^l) \right| \ll g^l N_1^{1/2}
\]

the contribution of the error terms is

\[
(3.3) \quad \ll N_1^{1/2} N_2 k^2 \leq N_1 N_2 k^2 \ll \frac{x}{g^a} g^{-(l'-l) k^2}.
\]

Again, for \( |\gamma| \leq T_1 := x g^{-l' k^{-1}} \), the difference \( (n_2 g + b' + 1)^e - (n_2 g + b')^e \) can be simplified by Taylor’s formula. For \( T_1 \leq \eta \leq \eta' \leq 2\eta \leq x \) we will consider the sum

\[
\Sigma_\eta := (\varphi(g^{l+1}))^{-1} \sum_{\chi \mod g^{l+1}} \left( \sum_{n_1} \chi(n_1 + b_1 g^l) \right) \sum_{e(\chi), \eta < |\gamma| \leq \eta'} \sum_{n_2} \left( n_2 g + b' \right)^e \cdot
\]

The Riemann hypothesis, (3.2), and (2.2.7) imply

\[
\Sigma_\eta \ll g^{(l'-l)/2} N_1^{1/2} U^{-1/2} k^{1/2} \left( \sum_{\chi \mod g^{l+1}} \sum_{e(\chi), \eta < |\gamma| \leq \eta'} \left| \sum_{n_2} \left( n_2 g + b' \right)^e \right|^2 \right)^{1/2}
\]

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\[
\ll k^2 g^{(l'-l)/2} N_1^{1/2} U^{-1/2} (g^l U (xg^{-l'}) N_2)^{1/2} \\
\ll k^2 x g^{-(a+l'-l)/2} \quad (x = g^k).
\]

There are \(\ll k\) sums \(\Sigma_U\). Therefore, the \(\chi\)-term in (3.1) is
\[
\ll x g^{-a} k^{-1} \cdot g^{a/2-(l'-l)/2} k^A.
\]
This is \(\ll x g^{-a} k^{-1}\) for \(k \geq k_0(\varepsilon)\) and \(a \leq (1-\varepsilon) k^{1/2}\).

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