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Primes with preassigned digits

by

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1. Introduction and statement of the results. Let $g \geq 2$ be an integer. For $k \in \mathbb{N}$ every integer $n \in [g^{k-1}, g^k)$ can be uniquely written as

$$n = \sum_{\nu=0}^{k-1} z_\nu g^\nu \quad (z_\nu \in \{0, \ldots, g-1\},\, z_{k-1} > 0),$$

briefly $n = z_{k-1} \ldots z_0$. Sierpiński [6] showed, that for any given $b, b' \in \{0, \ldots, g-1\}$, $(b, g) = 1$, and $b' > 0$, there are infinitely many primes p which have b as the last digit z_0 and b' as the first. This is an immediate consequence of Dirichlet's theorem on primes in arithmetic progressions.

The question can be generalized as follows. For $a \in \mathbb{N}$ and $k \geq a$ let $0 \leq l_1 < \cdots < l_a \leq k-1$, $\vec{l} = (l_1, \ldots, l_a)$, and

$$\begin{aligned} &b_1, \ldots, b_a \in \{0, \ldots, g-1\} \quad \text{with } (b_1, g) = 1 \text{ if } l_1 = 0,\\ &b_a > 0 \quad \text{if } l_a = k-1. \end{aligned} \tag{1.1}$$

Such vectors $\vec{l}$ and $\vec{b} = (b_1, \ldots, b_a)$ will be called *admissible*. Write

$$\pi_{k,a,\vec{l},\vec{b}} = \#\{g^{k-1} \leq p < g^k : p = z_{k-1} \ldots z_0,\, z_{l_j} = b_j \ (j = 1, \ldots, a)\},$$

$$f_1(l) = \begin{cases} 1/\varphi(g) & \text{if } l = 0,\\ 1/g & \text{if } 1 \leq l < k-1,\\ 1/(g-1) & \text{if } l = k-1, \end{cases} \qquad f(\vec{l}) = \prod_{j=1}^{a} f_1(l_j).$$

It seems reasonable to expect, for any fixed a and admissible $\vec{l}$ and $\vec{b}$,

$$\pi_{k,a,\vec{l},\vec{b}} \sim f(\vec{l})(\pi(g^k) - \pi(g^{k-1})) \quad (k \to \infty). \tag{C}$$

It will be shown that (C) is true for $a = 1$ and $a = 2$.

THEOREM 1. *For $a \in \{1, 2\}$ and admissible $\vec{l}$ and $\vec{b}$ of length a we have*

$$\pi_{k,a,\vec{l},\vec{b}} = f(\vec{l})(1 - g^{-1}) \frac{g^k}{\ln g^k} + O\left(\frac{g^k}{k^2}\right).$$

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The O-constants here and in what follows may depend on g, but not on $\vec{l}$.

There is some hope that (C) can be proved for $a = 3$ or even for bigger values. If one assumes the Riemann hypothesis for the characters mod g^d $(d \in \mathbb{N}_0)$ then much more can be shown.

THEOREM 2. *Assume the Riemann hypothesis for the L-fuctions with characters mod* g^d $(d \in \mathbb{N}_0)$. *Then, for any* $\varepsilon \in (0,1)$, $k \geq k_0(\varepsilon)$, *and all admissible vectors* $\vec{l}, \vec{b}$ *of length* a, $1 \leq a \leq (1-\varepsilon)k^{1/2}$, *we have*

$$\pi_{k,a,\vec{l},\vec{b}} = f(\vec{l})\left(1 - \frac{1}{g}\right)\frac{g^k}{\ln g^k} + \mathrm{O}(g^{k-a}k^{-2}).$$

It is not clear whether any form of the density hypothesis for the L-functions mod g^d will imply a result of similar strength.

2. Proof of Theorem 1

2.1. The proof will be given in the more complicated case $a = 2$. Let $0 \leq l < l' \leq k-1$, $\vec{l} = (l, l')$, $b, b' \in \{0, \dots, g-1\}$, $(b, g) = 1$ if $l = 0$, $b' > 0$ if $l' = k-1$, $\vec{b} = (b, b')$. Instead of π_k we will study

$$\psi_k := \psi_{k,2,\vec{l},\vec{b}} = \sum_{\substack{g^{k-1} \leq n < g^k \\ z_l = b,\, z_{l'} = b'}} \Lambda(n),$$

and show

$$\psi_k = f(\vec{l})g^k\left(1 - \frac{1}{g}\right) + \mathrm{O}(g^k k^{-1}). \tag{2.1.1}$$

Let k be sufficiently large and write $x = g^k$.

2.2. *First case:* $l > \frac{1}{5}k$. Every n to be counted in ψ_k can be written as

$$\begin{aligned} &n = n_2 g^{l+1} + bg^l + n_1, \quad \text{where} \\ &0 \leq n_1 < g^l, \quad g^{k-l-2} \leq n_2 < g^{k-l-1}, \end{aligned} \tag{2.2.1}$$

and n_2 has the digit b' at the place with index $l' - l - 1$. In 2.2, n_2 will run through these numbers. This gives

$$\psi_k = \sum_{n_2} (\psi((gn_2 + b + 1)g^l) - \psi((gn_2 + b)g^l)) + \mathrm{O}(xk^{-1}),$$

where

$$\psi(y) = \sum_{n \leq y} \Lambda(n).$$

The error term, which of course could be estimated much better, results from the possible contribution of the end points of the intervals.

Let $\varrho = \varrho(\zeta) = \beta + i\gamma$ denote non-trivial zeros of $\zeta(s)$ (and similarly let $\varrho(\chi)$ be zeros of $L(s,\chi)$). Then, for $2 \le T \le x$,

$$\psi_k = g^l \,\#\{n_2\} - \sum_{\varrho(\zeta),\,|\gamma|\le T} \varrho^{-1} g^{l\varrho} \sum_{n_2} ((gn_2+b+1)^{\varrho} - (gn_2+b)^{\varrho}) + \mathrm{O}\Big(\frac{x}{T} k^2 \,\#\{n_2\}\Big) + \mathrm{O}(xk^{-1}). \tag{2.2.2}$$

The main term is equal to that in (2.1.1). The contribution of the error terms is $\ll xk^{-1}$ if

$$T = xg^{-l}k^3. \tag{2.2.3}$$

For $|\varrho| \le xg^{-l}k^{-1} =: T_1$ the difference in the n_2-sum can be expanded to

$$\varrho(gn_2+b)^{\varrho-1} + \frac{\varrho(\varrho-1)}{2}(gn_2+b)^{\varrho-2} + \cdots .$$

We will treat the first term. The higher terms can be treated in the same manner. Because of the choice of T_1 they lead to smaller bounds. For $T_1 < |\varrho| \le T$ there is no cancellation. It is therefore sufficient to study the following expression:

$$\varSigma_{\varrho} := \sum_{\varrho(\zeta),\,|\gamma|\le T_1} g^{l\beta} \Big| \sum_{n_2} (gn_2+b)^{\varrho-1} \Big| + \frac{g^l k}{x} \sum_{\varrho(\zeta),\,T_1<|\gamma|\le T} g^{l\beta} \Big| \sum_{n_2} (gn_2+b)^{\varrho} \Big|. \tag{2.2.4}$$

For these sums and similar ones in the other cases we will apply zero density bounds and a mean value theorem for Dirichlet polynomials.

Write as usual, for $q \ge 1$, $\chi \bmod q$, $1/2 \le \sigma \le 1$, $T \ge 2$,

$$N(\sigma,T,\chi) = \#\{\varrho = \beta + i\gamma : L(\varrho,\chi) = 0,\ \beta \ge \sigma, |\gamma| \le T\}.$$

Then we have

$$\sum_{\chi \bmod q} N(\sigma,T,\chi) \ll_{\varepsilon} \begin{cases} (qT)^{3(1-\sigma)/(2-\sigma)} \ln^9(qT) & \text{if } 1/2 \le \sigma \le 3/4, \\ (qT)^{3(1-\sigma)/(3\sigma-1)+\varepsilon} & \text{if } 3/4 \le \sigma \le 4/5, \\ (qT)^{(2+\varepsilon)(1-\sigma)} & \text{if } 4/5 \le \sigma \le 1, \end{cases} \tag{2.2.5}$$

$$\ll_{\varepsilon} (qT)^{(12/5+\varepsilon)(1-\sigma)} \tag{2.2.6}$$

(see Montgomery [4, Theorem 12.1], Huxley [1], Jutila [3]).

Let $a_n \in \mathbb{C}$ for $1 \le n \le N$, and set $D(s) = \sum_{n\le N} a_n n^{-s}$. Let $\mathcal{A}$ be a set of complex numbers $s = \sigma + i\tau$ where $\sigma \ge \sigma_0$, $|\tau - \tau'| \ge \delta$ for $s \ne s' \in \mathcal{A}$, and $T_0 - \delta/2 \le \tau \le T_0 + T + \delta/2$. Then

$$\sum_{s\in\mathcal{A}} |D(s)|^2 \ll \ln\ln N \cdot (\delta^{-1} + \ln N)\cdot (T+N) \sum_{n\le N} |a_n|^2 n^{-2\sigma_0} \tag{2.2.7}$$

(Montgomery [4, Theorem 7.5]).

Put

$$\sigma_0 = \sigma_0(\zeta) = 1 - k^{-3/4} =: 1 - \delta_0.$$

The Vinogradov–Korobov zero free region (see Ivić [2, Theorem 6.1]) ensures

$$\zeta(s) \ne 0 \quad \text{for } |\operatorname{Im} s| \le x, \operatorname{Re} s \ge \sigma_0.$$

We have

$$\begin{aligned}\Sigma_\varrho \ll k \max_{1/2\le\sigma\le\sigma_0} g^{l\sigma}\Big\{ (N(\sigma,T_1))^{1/2}\Big(\sum_{\substack{\varrho,\,|\gamma|\le T_1\\ \sigma\le\beta\le\sigma+k^{-1}}} \Big|\sum_{n_2} (gn_2+b)^{\varrho-1}\Big|^2\Big)^{1/2} \\ + \frac{g^l k}{x}(N(\sigma,T))^{1/2}\Big(\sum_{\substack{\varrho,\,|\gamma|\le T_1\\ \sigma\le\beta\le\sigma+k^{-1}}} \Big|\sum_{n_2} (gn_2+b)^{\varrho}\Big|^2\Big)^{1/2}\Big\}.\end{aligned}$$

We split up the set of zeros into $\ll k$ classes $\mathcal{A}$ which fulfill the conditions of (2.2.7) with $\delta = 1$. Hence, by (2.2.7) and (2.2.6)

$$\begin{aligned}\Sigma_\varrho &\ll_\varepsilon k^3 \max_{1/2\le\sigma\le\sigma_0} g^{l\sigma} T^{(6/5+\varepsilon)(1-\sigma)} \Big(\frac{x}{g^l}k^3\Big)^{1/2} \\ &\quad\times \Big\{ \Big(\frac{x}{g^l}\Big)^{(2\sigma-1)/2} + \frac{g^l k}{x}\Big(\frac{x}{g^l}k^3\Big)^{(2\sigma+1)/2}\Big\} \\ &\ll k^{10} \max_{1/2\le\sigma\le\sigma_0} x^\sigma (xg^{-l})^{(6/5+\varepsilon)(1-\sigma)} \\ &\ll k^{10}(x^{1/2+3/5+\varepsilon} g^{-3l/5} + x^{\sigma_0 + (6/5+\varepsilon)\delta_0} g^{-6l\delta_0/5}).\end{aligned}$$

This is $\ll xk^{-1}$ for $g^l > x^{1/5}$ if $\varepsilon > 0$ is chosen sufficiently small. Hence Theorem 1 is true in this case.

2.3. *Second case:* $l \le \frac{1}{5}k$, $l' > \frac{4}{5}k$. The numbers n to be counted here can be written as

$$\begin{aligned} &n = n_2 g^{l'+1} + b' g^{l'} + \sum_{\nu=l+1}^{l'-1} z_\nu g^\nu + bg^l + n_1 \quad \text{where} \\ &1 \le n_1 \le g^l - 1, \quad (n_1, g) = 1, \quad g^{k-l'-2} \le n_2 < g^{k-l'-1} - g^{l'+1}.\end{aligned} \tag{2.3.1}$$

In this part n_1 and n_2 will denote these numbers. We get

$$(2.3.2)\qquad \psi_k = \sum_{n_1,n_2} \big(\psi(n_2 g^{l'+1} + (b'+1)g^{l'}, g^{l+1}, bg^l + n_1) - \psi(n_2 g^{l'+1} + b'g^{l'}, g^{l+1}, bg^l + n_1)\big) + \mathrm{O}(x/k)$$

$$= \sum_{n_1,n_2} \frac{g^{l'}}{\varphi(g^{l+1})} - \frac{1}{\varphi(g^{l+1})} \sum_{\chi \bmod g^{l+1}} \sum_{n_1} \overline{\chi}(n_1 + bg^l) \times \sum_{n_2} \bigg(\sum_{\varrho(\chi), |\gamma| \le T} \frac{g^{l'\varrho}}{\varrho}\, ((n_2 g + b' + 1)^\varrho - (n_2 g + b')^\varrho) + \mathrm{O}\Big(\frac{x}{T} k^2\Big)\bigg) + \mathrm{O}(xk^{-1}) \quad (2 \le T \le x).$$

Again the main term gives the expected value. By the Pólya–Vinogradov inequality we have

$$(2.3.3)\qquad \sum_{n_1} \chi(n_1 + bg^l) \ll \begin{cases} g^l & \text{if } \chi = \chi_0, \\ g^{l/2}k & \text{if } \chi \ne \chi_0. \end{cases}$$

Therefore the contribution of the error term $\mathrm{O}(xT^{-1}k^2)$ in (2.3.2)—which does not depend on n_1—is $\ll \#\{n_2\} g^{l/2} k x T^{-1} k^2$. This is $\ll xk^{-1}$ for

$$(2.3.4)\qquad T = x g^{l/2 - l'} k^4.$$

Note that $T < x$ for k sufficiently large.

Let Σ_ϱ be the ϱ-sum in (2.3.2). For $\varrho = \beta + i\gamma$ with $|\gamma| \le x g^{-l'} k^{-1} =: T_1$ we have

$$\varrho^{-1}((n_2 g + b' + 1)^\varrho - (n_2 g + b')^\varrho) \ll (x g^{-l'})^{\beta - 1}.$$

Put

$$\sigma_0(l) = \begin{cases} 1 - k^{-3/4} & \text{if } g^l \le k^{30}, \\ 1 & \text{otherwise} \end{cases}$$

(in particular, $\sigma_0(\zeta) = \sigma_0(0)$). Then $\sigma_0(l)$ describes a zero free region of $L(s,\chi)$, $\chi \bmod g^l$ (see Prachar [5, §6, Satz 6.2]). Hence

$$\Sigma_\varrho \ll k^3 \max_{1/2 \le \sigma \le \sigma_0(\zeta)} \big(x^\sigma N(\sigma, T_1) + x g^{-l'} \max_{T_1 < U \le T} U^{-1} x^\sigma N(\sigma, U)\big) + k^4 g^{-l/2} \max_{1/2 \le \sigma \le \sigma_0(l)} \Big(x^\sigma \sum_{\chi \bmod g^{l+1}} N(\sigma, T_1, \chi_1) + x g^{-l'} \max_{T_1 < U \le T} U^{-1} x^\sigma \sum_{\chi \bmod g^{l+1}} N(\sigma, U, \chi)\Big).$$

It is easy to see that (2.2.6), combined with the zero free regions, is sufficient to show that the last quantity is $\ll xk^{-1}$ in the case $g^l, xg^{-l'} \le x^{1/5}$.

2.4. *Third case*: $g^{l'} \le x^{4/5}$, $g^l \le x^{1/5}$. Assume, for simplicity, $0 < l < l'$. The other cases can be treated in the same manner with minor modifications.

Here we write the numbers n to be counted as

$$n = \sum_{\nu=l'+1}^{k-1} z_\nu g^\nu + b'g^{l'} + n_2 g^{l+1} + bg^l + n_1$$

where

$$1 \le n_1 < g^l, \qquad (n_1, g) = 1, \qquad 0 \le n_2 < g^{l'-(l+1)}.$$

Therefore

$$\begin{aligned}(2.4.1)\qquad \psi_{k,2,\vec{l},\vec{b}} &= \sum_{n_1,n_2} (\psi(x, g^{l'+1}, b'g^{l'} + n_2 g^{l+1} + bg^l + n_1) \\ &\quad - \psi(xg^{-1}, g^{l'+1}, b'g^{l'} + n_2 g^{l+1} + bg^l + n_1)) + \mathrm{O}(xk^{-1}) \\ &= x\left(1 - \frac{1}{g}\right) \frac{1}{\varphi(g)}\, g^{-l'}\, \#\{n_1\} \cdot \#\{n_2\} \\ &\quad - \frac{1}{\varphi(g) g^{l'}} \sum_{\chi \bmod g^{l'+1}} \sum_{n_1,n_2} \overline{\chi}(b'g^{l'} + n_2 g^{l+1} + bg^l + n_1) \\ &\quad \times \left(\sum_{\varrho(\chi), |\gamma| \le T} \frac{1}{\varrho}\, x^\varrho (1 - g^{-\varrho}) + \mathrm{O}\left(\frac{x}{T}\, k^2\right)\right) + \mathrm{O}(xk^{-1}).\end{aligned}$$

The contribution of the error terms is, by the orthogonality relation for characters, $\ll g^{-l'} g^{3l'/2} (x/T) k^2 + xk^{-1}$. This is $\ll xk^{-1}$ if one chooses

$$(2.4.2)\qquad T = g^{l'/2} k^3.$$

For $\chi \bmod g^{l'+1}$, $\chi \ne \chi_0$, we consider the sum

$$(2.4.3)\qquad \varSigma_\chi := \sum_{n_1,n_2} \chi(b'g^{l'} + n_2 g^{l+1} + bg^l + n_1).$$

$\varSigma_\chi$ results from summation over $\ll g^{l'-l}$ intervals of length $\ll g^l$. By Pólya–Vinogradov the sum over every such interval is $\ll g^{l'/2}k$, hence

$$(2.4.4)\qquad \varSigma_\chi \ll g^{3l'/2 - l} k.$$

On the other hand,

$$\begin{aligned}\varSigma_\chi &= \sum_{n_1} \sum_{\substack{n \le g^{l'} \\ n \equiv n_1 + bg^l\ (g^{l+1})}} \chi(n + b'g^{l'}) \\ &= \frac{1}{\varphi(g) g^l} \sum_{n_1} \sum_{\chi_1 \bmod g^{l+1}} \sum_{n \le g^{l'}} \chi(n + b'g^{l'}) \overline{\chi}_1(n_1 + bg^l).\end{aligned}$$

The sum over n_1 is $\ll g^{l/2}k$ if $\chi_1 \ne \chi_0$, but $\sum_n \chi(\ ) \ll g^{l'/2}k$. Therefore

$$\varSigma_\chi \ll g^{l'/2} k g^{-l} (g^l + g^l \cdot g^{l/2} k) \ll g^{(l+l')/2} k^2.$$

Combined with (2.4.4) this implies

$$\Sigma_\chi \ll k^2 \min(g^{3l'/2-l}, g^{(l+l')/2}) \ll k^2 g^{5l'/6}. \tag{2.4.5}$$

The contribution of the ϱ-sum to (2.4.1) is

$$\ll \sum_{\varrho(\zeta),\,|\gamma|\le T} \frac{x^\beta}{|\gamma|} + g^{-l'} \sum_{\chi \bmod g^{l'+1},\,\chi\ne\chi_0} |\Sigma_\chi| \sum_{\varrho(\chi),\,|\gamma|\le T} \frac{x^\beta}{|\gamma|}. \tag{2.4.6}$$

For $g^{l'} \le k^{30}$ we use, similarly to the second case, $\sigma_0(l') = 1 - k^{-3/4}$. Now (2.2.6) and the last inequality in (2.4.5) show that (2.4.6) is $\ll xk^{-1}$.

In the case $k^{30} < g^{l'} \le x^{1/3}$ we argue in the same manner with $\sigma_0(l') = 1$. For $x^{1/3} < g^{l'} \le x^{19/45}$ one uses $\Sigma_\chi \ll k^2 g^{(l+l')/2}$. In the case

$$x^{19/45} < g^{l'} \le x^{4/5} \tag{2.4.7}$$

the ζ-part in (2.4.6) can be bounded by (2.2.6) with $\sigma_0(\zeta) = 1 - k^{-3/4}$. The χ-part $\Sigma_{\varrho,\chi}$ requires a bit more care. (2.2.5) gives

$$\begin{aligned}\Sigma_{\varrho,\chi} \ll_\varepsilon k^{11} g^{(l-l')/2} \max_{U\le T}\Big(&\max_{1/2\le\sigma\le 3/4} U^{-1}(Ug^{l'})^{3(1-\sigma)/(2-\sigma)} x^\sigma \\ &+ \max_{3/4\le\sigma\le 4/5} U^{-1}(Ug^{l'})^{3(1-\sigma)/(3\sigma-1)-\varepsilon} x^\sigma \\ &+ \max_{4/5\le\sigma\le 1} U^{-1}(Ug^{l'})^{(2+\varepsilon)(1-\sigma)} x^\sigma\Big).\end{aligned}$$

The contribution of the part with $4/5 \le \sigma \le 1$ is $\ll xk^{-1}$ for the whole interval. The exponent of U is ≤ 0 for $1/2 \le \sigma \le 4/5$. Write $g^{l'} = x^\xi$, $19/45 \le \xi \le 4/5$. Then we have, using $g^l \le x^{1/5}$,

$$\begin{aligned}\Sigma_{\varrho,\chi} \ll k^{11} x^{1/10}\Big(&\max_{1/2\le\sigma\le 3/4} x^{\sigma+\xi(\frac{3(1-\sigma)}{2-\sigma}-\frac12)} \\ &+ \max_{3/4\le\sigma\le 4/5} x^{\sigma+\xi(\frac{3(1-\sigma)}{3\sigma-1}-\frac12)+\varepsilon}\Big) + xk^{-1}.\end{aligned} \tag{2.4.8}$$

The function $G(\sigma) = \sigma + \xi\big(\frac{3(1-\sigma)}{3\sigma-1} - \frac12\big)$ is decreasing in $[3/4, 4/5]$ for $\xi \in [19/45, 4/5]$. Hence the second term in (2.4.8) is

$$\ll k^{11} x^{17/20+\xi/10+\varepsilon} \ll xk^{-1}.$$

The function $H(\sigma) = \sigma + \xi\big(\frac{3(1-\sigma)}{2-\sigma} - \frac12\big)$ is increasing in $[1/2, 3/4]$ for $\xi \le 25/48$. Again $\sigma = 3/4$ leads to a sufficient bound.

For $25/48 < \xi \le 4/5$, $G(\sigma)$ has a maximum at

$$\sigma^* = 2 - \sqrt{3\xi} \in [1/2, 3/4].$$

We have $1/10 + H(\sigma^*) = 21/10 - 2\sqrt{3\xi} + 5\xi/2$. In $[25/48, 4/5]$ this function is increasing and gives a value < 1 at $\xi = 4/5$.

This shows that (2.1.1) is true in all subcases of the third case. Theorem 1 is proved.

3. Proof of Theorem 2. Assume $r \ge r_0(\varepsilon)$ and $a > 2$. We will treat the case in which there is a j with $1 < j, j+1 < a$ such that $l_{j+1} - l_j$ is maximal amongst the $a+1$ numbers $l_1, l_2 - l_1, \ldots, l_a - l_{a-1}, k-1-l_a$, i.e. $l_{j+1} - l_j \ge k/(a+1)$. We write $l = l_j,\ l' = l_{j+1},\ b = b_j,\ b' = b_{j+1}$. The numbers n to be counted in $\psi_{k,a,\vec{l},\vec{b}}$ can be written as

$$n = n_2 g^{l'+1} + b' g^{l'} + \sum_{\nu=l+1}^{l'-1} z_\nu g^\nu + b g^l + n_1,$$

where $0 \le n_1 < g^l$ with the digits $b_1, \ldots, b_{j-1}$ at the corresponding places, and $g^{k-l'-2} \le n_2 < g^{k-l'-1}$ with the digits $b_{j+2}, \ldots, b_a$. We have

$$N_1 := \#\{n_1\} \approx g^{l-j} \quad \text{and} \quad N_2 := \#\{n_2\} \approx g^{k-l'-(a-j)}.$$

The explicit formula yields

$$\begin{aligned}(3.1)\quad \psi_{k,\vec{a},\vec{l},\vec{b}} = {} & \sum_{n_1,n_2} (\varphi(g^{l+1}))^{-1} g^{l'} \\ & - (\varphi(g^{l+1}))^{-1} \sum_{\chi \bmod g^{l+1}} \Big(\sum_{n_1} \overline{\chi}(n_1 + b g^l) \Big) \sum_{n_2} \\ & \sum_{\varrho(\chi), |\gamma| \le x} \Big(\frac{g^{l'\varrho}}{\varrho} ((n_2 g + b' + 1)^\varrho - (n_2 g + b')^\varrho) + \mathrm{O}(k^2) \Big) \\ & + \mathrm{O}(k).\end{aligned}$$

Because

$$(3.2)\qquad \sum_{\chi \bmod g^{l+1}} \Big| \sum_{n_1} \chi(n_1 + b g^l) \Big| \ll g^l N_1^{1/2}$$

the contribution of the error terms is

$$(3.3)\qquad \ll N_1^{1/2} N_2 k^2 \le N_1 N_2 k^2 \ll \frac{x}{g^a}\, g^{-(l'-l)} k^2.$$

Again, for $|\gamma| \le T_1 := x g^{-l'} k^{-1}$, the difference $(n_2 g + b' + 1)^\varrho - (n_2 g + b')^\varrho$ can be simplified by Taylor's formula. For $T_1 \le U < U' \le 2U \le x$ we will consider the sum

$$\Sigma_U := (\varphi(g^{l+1}))^{-1} \sum_{\chi \bmod g^{l+1}} \Big(\sum_{n_1} \overline{\chi}(n_1 + b_1 g^l) \Big) \sum_{\substack{\varrho(\chi) \\ U < |\gamma| \le U'}} \frac{g^{l'\varrho}}{\varrho} \sum_{n_2} (n_2 g + b')^\varrho.$$

The Riemann hypothesis, (3.2), and (2.2.7) imply

$$\Sigma_U \ll g^{(l'-l)/2} N_1^{1/2} U^{-1/2} k^{1/2} \Big(\sum_{\chi \bmod g^{l+1}} \sum_{\substack{\varrho(\chi) \\ U < |\gamma| \le U'}} \Big| \sum_{n_2} (n_2 g + b')^\varrho \Big|^2 \Big)^{1/2}$$

$$\ll k^2 g^{(l'-l)/2} N_1^{1/2} U^{-1/2} (g^l U(xg^{-l'}) N_2)^{1/2}$$
$$\ll k^2 x g^{-(a+l'-l)/2} \quad (x = g^k).$$

There are $\ll k$ sums Σ_U. Therefore, the χ-term in (3.1) is

$$\ll xg^{-a}k^{-1} \cdot g^{a/2-(l'-l)/2}k^4.$$

This is $\ll xg^{-a}k^{-1}$ for $k \geq k_0(\varepsilon)$ and $a \leq (1-\varepsilon)k^{1/2}$.

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