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Adelic equidistribution, characterization of equidistribution, and a general equidistribution theorem in non-archimedean dynamics

by

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1. Introduction. Let K be an algebraically closed field of any characteristic and complete with respect to a non-trivial and possibly non-archimedean absolute value $|\cdot|$, and let $f \in K(z)$ be a rational function of degree $d > 1$ on the projective line $\mathbb{P}^1 = \mathbb{P}^1(K)$ over K. The Berkovich projective line $\mathsf{P}^1 = \mathsf{P}^1(K)$ over K provides a compactification of the classical $\mathbb{P}^1$, containing $\mathbb{P}^1$ as a dense subset. Under the assumption that K is algebraically closed, K is archimedean if and only if $K \cong \mathbb{C}$, and then $\mathsf{P}^1(\mathbb{C}) \cong \mathbb{P}^1(\mathbb{C})$. The action of f on $\mathbb{P}^1$ canonically extends to a continuous, open, surjective and fiber-discrete endomorphism on P^1, preserving P^1 and $\mathsf{P}^1 \setminus \mathbb{P}^1$. The exceptional set of (the extended) f is

$$E(f) := \Big\{a \in \mathbb{P}^1 : \# \bigcup_{n\in\mathbb{N}} f^{-n}(a) < \infty\Big\},$$

which agrees with the set of all superattracting periodic points $a \in \mathbb{P}^1$ of f such that $\deg_{f^j(a)} f = d$ for any $j \in \mathbb{N}$. The *Berkovich* Julia set of f is

$$\mathsf{J}(f) := \Big\{\mathcal{S} \in \mathsf{P}^1 : \bigcap_{U:\,\text{open in}\,\mathsf{P}^1,\,\mathcal{S}\in U} \ \bigcup_{n\in\mathbb{N}} f^n(U) = \mathsf{P}^1 \setminus E(f)\Big\}$$

(cf. [9, Definition 2.8]). Let $\delta_{\mathcal{S}}$ be the Dirac measure on P^1 at a point $\mathcal{S} \in \mathsf{P}^1$. For each rational function $a \in K(z)$, which we will call a *possibly moving target*, on $\mathbb{P}^1$ and each $n \in \mathbb{N}$, let us consider the probability Radon measure

$$\nu_n^a = \nu_{f^n}^a := \frac{1}{d^n + \deg a} \sum_{w\in\mathbb{P}^1:\, f^n(w)=a(w)} \delta_w \tag{1.1}$$

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on P^1. Here the sum takes into account the (algebraic) multiplicity of each root of the equation $f^n(\cdot) = a(\cdot)$ in $\mathbb{P}^1$. In Section 2, among other generalities, we recall a variational characterization of the equilibrium (or canonical) measure μ_f of f on P^1 as a unique solution of a Gauss variational problem.

Our principal result determines the conditions on f and a under which the equidistribution property

$$\lim_{n\to\infty} \nu_n^a = \mu_f \quad \text{weakly on } \mathsf{P}^1 \tag{1.2}$$

holds. Let us denote the normalized chordal distance on $\mathbb{P}^1$ by $[z, w]$.

Theorem 1. *Let K be an algebraically closed field of any characteristic and complete with respect to a non-trivial absolute value. Let $f \in K(z)$ be a rational function on $\mathbb{P}^1$ of degree $d > 1$, and let $a \in K(z)$ be a rational function on $\mathbb{P}^1$. Then for every sequence $(n_j) \subset \mathbb{N}$ tending to ∞, the following three conditions are equivalent:*

(i) *The equidistribution property*

$$\lim_{j\to\infty} \nu_{n_j}^a = \mu_f \quad \text{on } \mathsf{P}^1 \tag{1.3}$$

holds. Equivalently, for each weak limit ν of a subsequence of $(\nu_{n_j}^a)$,

$$\nu = \mu_f; \tag{1.3$'$}$$

(ii) *each weak limit ν of a subsequence of $(\nu_{n_j}^a)$ satisfies*

$$\operatorname{supp} \nu \subset \mathsf{J}(f); \tag{1.4}$$

(iii) *under the additional assumption that K is non-archimedean, on $\mathsf{P}^1 \setminus \mathbb{P}^1$ we have*

$$\lim_{j\to\infty} \frac{1}{d^{n_j}} \log\,[f^{n_j}, a]_{\mathrm{can}}(\cdot) = 0. \tag{1.5}$$

Under these three conditions, we have

$$\lim_{j\to\infty} \frac{1}{d^{n_j}} \int_{\mathsf{P}^1} \log\,[f^{n_j}, a]_{\mathrm{can}}(\cdot)\, d\mu_f = 0. \tag{1.6}$$

Moreover, if a is constant, then (1.6) *holds without assuming* (1.3), (1.4) *or* (1.5).

Here, the proximity function $\mathcal{S} \mapsto [f^n, a]_{\mathrm{can}}(\mathcal{S})$ of f^n $(n \in \mathbb{N})$ and a on P^1 is the unique continuous extension of $z \mapsto [f^n(z), a(z)]$ on $\mathbb{P}^1$ to P^1. For its construction, see Proposition 2.9.

In Section 3, we prove Theorem 1 based on the above variational characterization of μ_f. Theorem 1 is partly motivated by the following dynamical Diophantine approximation result. For a number field k with a non-trivial absolute value (or place) v, set $K = \mathbb{C}_v$ with the extended v (e.g., $K = \mathbb{C}_p$ for $k = \mathbb{Q}$ with p-adic norm v) and assume that $f \in k(z)$, i.e., f has its coefficients in k. Then the dynamical Diophantine approximation theorem due to Silverman [19, Theorem E] and Szpiro–Tucker [21, Proposition 5.3 (in the preprint version, Proposition 4.3)] asserts that for every constant $a \in \mathbb{P}^1(\overline{k}) \setminus E(f)$ and every $z \in \mathbb{P}^1(\overline{k})$ which is wandering under f, i.e., $\#\{f^n(z) : n \in \mathbb{N}\} = \infty$, we have

$$\lim_{n\to\infty} \frac{1}{d^n} \log\,[f^n(z), a]_v = 0. \tag{1.7}$$

Here $\overline{k}$ denotes the algebraic closure of k, and the notation $[z, w]_v$ emphasizes the dependence of $[z, w]$ on v. Theorem 1 gives a partial generalization of (1.7) to general K for possibly non-constant a.

In Section 4, based on a variational argument and (1.7), we give a purely local proof of the following adelic equidistribution theorem for possibly moving targets, which is a special case of Favre and Rivera-Letelier's [9, Théorèmes A et B] (Theorems 1.1 and 1.2 below) for non-archimedean K of characteristic 0.

Theorem A. *Let k be a number field with a non-trivial absolute value v, and let $f \in k(z)$ be a rational function on $\mathbb{P}^1(\mathbb{C}_v)$ of degree $d > 1$ whose coefficients are in k. Then for every rational function $a \in \overline{k}(z)$ on $\mathbb{P}^1(\mathbb{C}_v)$ which is not identically equal to a value in $E(f)$ and whose coefficients are in $\overline{k}$, $\lim_{n\to\infty} \nu_n^a = \mu_{f,v}$ weakly on $\mathsf{P}^1(\mathbb{C}_v)$. Here the notation $\mu_{f,v}$ emphasizes the dependence of μ_f on v.*

For another application (quantitative equidistribution for non-exceptional algebraic constants) of the dynamical Diophantine approximation (1.7) to adelic dynamics, see [16].

For general K, the equidistribution theorem for *constant* $a \in \mathbb{P}^1 \setminus E(f)$ is due to Brolin [6], Lyubich [12], Freire, Lopes and Mañé [10] for archimedean K, and to Favre and Rivera-Letelier [9, Théorème A] for non-archimedean K.

Theorem 1.1. *Let K be an algebraically closed field of any characteristic and complete with respect to a non-trivial absolute value. Let $f \in K(z)$ be a rational function on $\mathbb{P}^1$ of degree $d > 1$, and $a \in K(z)$ be a constant function. Then $\lim_{n\to\infty} \nu_n^a = \mu_f$ weakly on P^1 if and only if*

$$a \in \mathbb{P}^1(K) \setminus E(f). \tag{1.8}$$

In Section 4, we give a proof of Theorem 1.1, the fundamental equivalence between (1.2) and (1.8) for constant a, based on a variational argument and on the classification of cyclic Berkovich Fatou components of f (see Theorem 2.17).

For general K of characteristic 0, the equidistribution theorem for *moving* targets is due to Lyubich [12, Theorem 3] (see also Tortrat [23, §IV]) for archimedean K, and to Favre and Rivera-Letelier [9, Théorème B] for non-archimedean K of characteristic 0.

THEOREM 1.2. *Let K be an algebraically closed field of characteristic* 0 *and complete with respect to a non-trivial absolute value. Let $f \in K(z)$ be a rational function on $\mathbb{P}^1$ of degree $d > 1$. Then for every non-constant rational function $a \in K(z)$ on $\mathbb{P}^1$, $\lim_{n\to\infty} \nu_n^a = \mu_f$ weakly on P^1.*

In Section 4, we also describe how a variational argument together with the dynamical uniformization on the quasiperiodicity domain $\mathcal{E}_f$ (see Theorem 4.5) yields Theorem 1.2. This is foundational in our study of the problem of density of the classical repelling periodic points in the classical Julia set in non-archimedean dynamics [15]. Our proof of Theorem 1.2 complements the original one given in [9, §3.4] (see also Remark 2.10).

In Section 5, we discuss the case where f and a are polynomials, and compute a concrete example.

We conclude this section with an open problem.

PROBLEM. Let K be an algebraically closed field of positive characteristic and complete with respect to a non-trivial absolute value. Let $f \in K(z)$ be a rational function on $\mathbb{P}^1$ of degree $d > 1$. Determine concretely all rational functions $a \in K(z)$ on $\mathbb{P}^1$ which are *exceptional for f* in that the equidistribution (1.2) does not hold.

We hope condition (1.5) will be helpful for studying this problem.

2. Background. For the foundations of potential theory on P^1, see [1, §5 and §8], [8, §7], [11, §1–§4], [24, Chapter III]. For a potential-theoretic study of dynamics on P^1, see [1, §10], [9, §3], [11, §5], [4, Chapitre VIII]. See also [2, 17] including non-archimedean dynamics.

Let K be an algebraically closed field of any characteristic and complete with respect to a non-trivial absolute value $|\cdot|$. Under the assumption that K is algebraically closed, $|K| := \{|z| : z \in K\}$ is dense in $\mathbb{R}_{\geq 0}$. We will say that K is *non-archimedean* if the strong triangle inequality $|z-w| \leq \max\{|z|, |w|\}$ holds for all $z, w \in K$. This in particular implies that the equality $|z-w| = \max\{|z|, |w|\}$ holds if $|z| \neq |w|$. When K is non-archimedean, for every $a, b \in K$ and every $r \geq 0$, $\{z \in K : |z-a| \leq r\} = \{z \in K : |z-b| \leq r\}$ if $|b-a| \leq r$, and the diameters of these sets with respect to $|\cdot|$ equal r. If K is not non-archimedean, then K is said to be *archimedean*. Under the assumption that K is algebraically closed, K is archimedean if and only if $K \cong \mathbb{C}$ as valued fields.

Let $\|\cdot\|$ be the maximum norm on K^2 if K is non-archimedean, and the Euclidean norm on $\mathbb{C}^2$ if K is archimedean ($\cong \mathbb{C}$). Put $p \wedge q := p_0 q_1 - p_1 q_0$ for

$p=(p_0,p_1), q=(q_0,q_1)\in K^2$; let π be the canonical projection $K^2\setminus\{0\}\to\mathbb{P}^1 = \mathbb{P}^1(K)$, and put $\infty := \pi(0,1)$. The normalized *chordal distance* on $\mathbb{P}^1$ is

$$[z,w] := \frac{|p\wedge q|}{\|p\|\cdot\|q\|}\in[0,1],$$

where $p\in\pi^{-1}(z), q\in\pi^{-1}(w)$. We usually identify K with $\mathbb{P}^1\setminus\{\infty\}$ by the injection $z\mapsto\pi(1,z)$ on K.

For non-archimedean K, the Berkovich projective line $\mathsf{P}^1=\mathsf{P}^1(K)$ is defined as an analytic space in the sense of Berkovich; see Berkovich's original monograph [3], as well as [1, §1, §2] for P^1. For archimedean K, we have $\mathsf{P}^1=\mathbb{P}^1$.

Fact 2.1 (Berkovich's classification of points in P^1). Suppose that K is non-archimedean. A subset $\mathcal{B}=\{z\in K : |z-a|\le r\}$ in K for some $a\in K$ and some $r:=\operatorname{diam}(\mathcal{B})\ge 0$ is called a (K-closed) *disk*. Any two intersecting disks $\mathcal{B},\mathcal{B}'$ satisfy either $\mathcal{B}\subset\mathcal{B}'$ or $\mathcal{B}\supset\mathcal{B}'$.

A point $\mathcal{S}$ in the Berkovich projective line P^1 is either ∞ or a cofinal class (or tail) of non-increasing and nested sequences $(\mathcal{B}_j)$ of disks. Here, two non-increasing and nested sequences $(\mathcal{B}_j),(\mathcal{B}'_k)$ of disks are *cofinally equivalent* either if (i) $\bigcap_j\mathcal{B}_j=\bigcap_k\mathcal{B}'_k\neq\emptyset$ or if (ii) $\bigcap_j\mathcal{B}_j=\bigcap_k\mathcal{B}'_k=\emptyset$, for any $j\in\mathbb{N}$, $\mathcal{B}_j$ contains $\mathcal{B}'_N$ for some $N\in\mathbb{N}$, and for any $k\in\mathbb{N}$, $\mathcal{B}'_k$ contains $\mathcal{B}_{N'}$ for some $N'\in\mathbb{N}$. The cofinal class of a non-increasing and nested sequence of disks $(\mathcal{B}_j)$ is identified with the disk $\mathcal{B}=\bigcap_{j\in\mathbb{N}}\mathcal{B}_j$ if it is non-empty. The projective line $\mathbb{P}^1$ is regarded as the set of all disks $\mathcal{B}$ with $\operatorname{diam}(\mathcal{B})=0$ and the point ∞ (cf. [1, §1], [2, §6.1], [9, §2]).

Let Ω_{can} be the Fubini–Study area element on $\mathbb{P}^1\cong\mathsf{P}^1$ normalized as $\Omega_{\mathrm{can}}(\mathbb{P}^1)=1$ for archimedean $K\cong\mathbb{C}$, and the Dirac measure $\delta_{\mathcal{S}_{\mathrm{can}}}$ on P^1 at the Gauss (or canonical) point $\mathcal{S}_{\mathrm{can}}\in\mathsf{P}^1$ determined by the disk $\{z\in K : |z|\le 1\}$ for non-archimedean K.

Definition 2.2 (the generalized Hsia kernel). Suppose that K is non-archimedean. For the cofinal class $\mathcal{S}$ of a non-increasing and nested sequence $(\mathcal{B}_j)$ of disks, set $\operatorname{diam}(\mathcal{S}):=\lim_{j\to\infty}\operatorname{diam}(\mathcal{B}_j)$. Then the function $\operatorname{diam}(\cdot)$ is continuous on $\mathsf{P}^1\setminus\{\infty\}$.

For the cofinal classes $\mathcal{S},\mathcal{S}'$ of non-increasing and nested sequences of disks $(\mathcal{B}_j),(\mathcal{B}'_k)$, respectively, let $\mathcal{S}\wedge\mathcal{S}'\in\mathsf{P}^1$ be the smallest cofinal class of a non-increasing and nested sequence $(\mathcal{B}''_\ell)$ of disks such that for every $\ell\in\mathbb{N}$, $\mathcal{B}''_\ell$ contains $\mathcal{B}_N\cup\mathcal{B}'_{N'}$ for some $N,N'\in\mathbb{N}$. Here the cofinal class of $(\mathcal{B}''_\ell)$ is said to be smaller than that of $(\mathcal{B}'''_m)$ if for every $m\in\mathbb{N}$, $\mathcal{B}'''_m$ contains $\mathcal{B}''_{N''}$ for some $N''\in\mathbb{N}$.

For each $w\in\mathbb{P}^1\setminus\{\infty\}$, the function $|\cdot-w|:=\operatorname{diam}(\cdot\wedge w)$ on $\mathsf{P}^1\setminus\{\infty\}$ is a unique continuous extension of $|\cdot-w|$ on $\mathbb{P}^1\setminus\{\infty\}$. We denote $|\cdot-0|$ by $|\cdot|$ in the case $w=0$.

The *generalized Hsia kernel* $[\mathcal{S},\mathcal{S}']_{\mathrm{can}}$ on P^1 with respect to the Gauss point $\mathcal{S}_{\mathrm{can}}$ is defined as

$$[\mathcal{S},\mathcal{S}']_{\mathrm{can}} := \frac{\mathrm{diam}(\mathcal{S}\wedge\mathcal{S}')}{\max\{1,|\mathcal{S}|\}\max\{1,|\mathcal{S}'|\}} \in [0,1]$$

for $\mathcal{S},\mathcal{S}' \in \mathsf{P}^1\setminus\{\infty\}$, $[\mathcal{S},\infty]_{\mathrm{can}} := 1/\max\{1,|\mathcal{S}|\}$ for $\mathcal{S}\in\mathsf{P}^1\setminus\{\infty\}$, and $[\infty,\infty]_{\mathrm{can}} := [\infty,\infty] = 0$ (see [1, §4], [9, §2.4]).

By convention, for archimedean K, $[z,w]_{\mathrm{can}}$ is defined to be $[z,w]$.

Fact 2.3. The extension $[\mathcal{S},\mathcal{S}']_{\mathrm{can}}$ is upper semicontinuous on $\mathsf{P}^1\times\mathsf{P}^1$, continuous nowhere in the diagonal of $(\mathsf{P}^1\setminus\mathbb{P}^1)\times(\mathsf{P}^1\setminus\mathbb{P}^1)$ (indeed, $[\mathcal{S},\mathcal{S}]_{\mathrm{can}}$ is continuous nowhere on $\mathsf{P}^1\setminus\mathbb{P}^1$), but continuous elsewhere on $\mathsf{P}^1\times\mathsf{P}^1$. On the other hand, $[\mathcal{S},\mathcal{S}']_{\mathrm{can}}$ is separately continuous in each variable, and vanishes if and only if $\mathcal{S}=\mathcal{S}'\in\mathbb{P}^1$ (see [1, Proposition 4.10]).

We normalize the Laplacian Δ on P^1 so that for every $\mathcal{S}\in\mathsf{P}^1$,

$$\Delta\log[\cdot,\mathcal{S}]_{\mathrm{can}} = \delta_{\mathcal{S}} - \Omega_{\mathrm{can}} \tag{2.1}$$

on P^1 (for the construction of Δ on P^1 for non-archimedean K, see [1, §5], [8, §7.7], [22, §3]; in [1] the opposite sign convention on Δ is adopted).

Since we are interested in dynamics of rational functions, we introduce only Berkovich (open or closed) connected affinoids in P^1.

Fact 2.4. Suppose that K is non-archimedean. A Berkovich closed disk D is either $\{\mathcal{S}\in\mathsf{P}^1\setminus\{\infty\} : |\mathcal{S}-w|\le r\}$ or $\{\mathcal{S}\in\mathsf{P}^1\setminus\{\infty\} : |\mathcal{S}-w|\ge r\}\cup\{\infty\}$ for some $w\in\mathbb{P}^1\setminus\{\infty\}$ and some $r\ge 0$, and is said to be *strict* (or *rational*) if $r\in|K|$. Similarly, a Berkovich open disk is either $\{\mathcal{S}\in\mathsf{P}^1\setminus\{\infty\} : |\mathcal{S}-w|<r\}$ or $\{\mathcal{S}\in\mathsf{P}^1\setminus\{\infty\} : |\mathcal{S}-w|>r\}\cup\{\infty\}$ for some $w\in\mathbb{P}^1\setminus\{\infty\}$ and some $r\ge 0$, and is said to be *strict* (or *rational*) if $r\in|K|$.

A *Berkovich open* (resp. *closed*) *connected affinoid* U in P^1 is the intersection of finitely many Berkovich open (resp. closed) disks and P^1, and is said to be *strict* if in addition all the Berkovich open (resp. closed) disks determining U are strict (or rational).

A Berkovich open connected affinoid is also called either a *simple domain* or an *open fundamental domain* in P^1. The set of all strict Berkovich open connected affinoids generates the topology of P^1 (cf. [1, § 2.6], [2, § 6], [9, § 2.1]). For non-archimedean K, the relative topology of $\mathbb{P}^1$ in P^1 agrees with the metric topology on $\mathbb{P}^1$ induced by the chordal distance on $\mathbb{P}^1$. Both $\mathbb{P}^1$ and $\mathsf{P}^1\setminus\mathbb{P}^1$ are dense in P^1.

From rigid analysis, we take the following.

Definition 2.5. For non-archimedean K, a *closed* (resp. *open*) *connected affinoid* in $\mathbb{P}^1$ is the intersection of $\mathbb{P}^1$ and a Berkovich closed (resp. open) connected affinoid U in P^1, and is said to be *strict* if U is strict. A (K-valued) holomorphic function T on a strict closed connected affinoid

V in $\mathbb{P}^1$ is defined by a uniform limit on V (with respect to $[\cdot,\cdot]$) of a sequence of rational functions on $\mathbb{P}^1$ with no pole in V. By definition, a holomorphic function T on an open subset D in $\mathbb{P}^1$ is a function on D which restricts to a holomorphic function on any strict closed connected affinoid V in D.

Fact 2.6. For non-archimedean K, the modulus $|T|$ of a holomorphic function T on a strict closed connected affinoid V in $\mathbb{P}^1$ attains both its maximum and minimum values on V (the maximum modulus principle, cf. [5, §6.2.1, §7.3.4]). If in addition T is non-constant, then it has at most finitely many zeros in V (this follows from the Weierstrass preparation theorem, cf. [2, Theorem 3.5]).

Let $\phi \in K(z)$ be a rational function on $\mathbb{P}^1$. For non-archimedean K, the analytic structure on P^1 induces the extended action of ϕ on P^1. For non-constant ϕ, the extended action of ϕ on P^1 is continuous, open, surjective, and fiber-discrete, and preserves $\mathbb{P}^1$ and $\mathsf{P}^1 \setminus \mathbb{P}^1$ (see [1, Corollaries 9.9, 9.10], [9, §2.2]).

Fact 2.7. Suppose that K is non-archimedean and ϕ is non-constant. Then ϕ maps a Berkovich disk (resp. Berkovich connected affinoid) onto either P^1 or a Berkovich disk (resp. Berkovich connected affinoid), preserving their openness, closedness, and strictness. Each component U of $\phi^{-1}(V)$ for any Berkovich connected affinoid V is a Berkovich connected affinoid, and the restriction $\phi : U \to V$ is proper and surjective ([1, Corollary 9.11 and Lemma 9.12], [2, Proposition 6.13], [17, Proposition 2.6]). The local (algebraic) degree $\deg_{z_0} \phi \in \mathbb{N}$ of ϕ at each $z_0 \in \mathbb{P}^1$ also uniquely extends to the function $\deg_{\mathcal{S}} \phi \in \mathbb{N}$ for all $\mathcal{S} \in \mathsf{P}^1$ so that for any Berkovich open connected affinoid V and every component U of $\phi^{-1}(V)$, the function

$$V \ni \mathcal{S}_0 \mapsto \sum_{\mathcal{S} \in \phi^{-1}(\mathcal{S}_0) \cap U} \deg_{\mathcal{S}} \phi \in \mathbb{N}$$

is constant ([1, §2, §9] and [9, §2.1, Proposition-Définition 2.1]. See also [2, §6.3], [11, §4]). We denote this constant by $\deg(\phi : U \to V)$.

If $\deg \phi > 0$, then the extended $\phi : \mathsf{P}^1 \to \mathsf{P}^1$ and the local degree $\deg_{\mathcal{S}} \phi$ of ϕ at each $\mathcal{S} \in \mathsf{P}^1$ induce a push-forward ϕ_* and pullback ϕ^* on the space of continuous functions on P^1, on the space of δ-subharmonic functions on P^1 (functions on P^1 which can locally be written as the difference of two subharmonic functions), and on the space of Radon measures on P^1 (see [1, §9.4, §9.5], [9, §2.2]). When $\deg \phi = 0$, for a Radon measure μ on P^1, we set $\phi^* \mu := 0$ by convention. It is fundamental that for each non-constant ϕ, the Laplacian Δ behaves functorially under ϕ^* in that for any δ-subharmonic function h on P^1,

$$\Delta \phi^* h = \phi^* \Delta h$$

on P^1 (for non-archimedean K, see [1, §9.5], [9, §2.4]).

Definition 2.8. A *lift* $F_\phi = ((F_\phi)_0, (F_\phi)_1) : K^2 \to K^2$ of ϕ is a homogeneous polynomial endomorphism of K^2 such that

$$\pi \circ F_\phi = \phi \circ \pi$$

and that $F_\phi^{-1}(0) = \{0\}$ if $\deg \phi > 0$. Such an F_ϕ is unique up to scaling by an element of $K^* = K \setminus \{0\}$, and $\deg F_\phi = \deg \phi$. The function

$$\log \|F_\phi\| - (\deg \phi) \log \| \cdot \|$$

on $K^2 \setminus \{0\}$ descends to one on $\mathbb{P}^1$, which in turn extends continuously to a function $T_{F_\phi} : \mathsf{P}^1 \to \mathbb{R}$ satisfying

$$\Delta T_{F_\phi} = \phi^* \Omega_{\mathrm{can}} - (\deg \phi) \Omega_{\mathrm{can}} \tag{2.2}$$

on P^1; indeed, for each $w \in \mathbb{P}^1 \setminus \{\infty\}$, since $|\cdot - w| = [\cdot, w]_{\mathrm{can}} [\cdot, \infty]_{\mathrm{can}}^{-1} [w, \infty]^{-1}$ on P^1, we have $\Delta \log |\cdot - w| = \delta_w - \delta_\infty$ on P^1. The homogeneous polynomial $(F_\phi)_0(p_0, p_1) \in K[p_0, p_1]$ factors into $\deg \phi$ homogeneous linear factors in $K[p_0, p_1]$. Hence the function $\log |(F_\phi)_0(p_0, p_1)| - (\deg \phi) \log |p_0|$ on $K^2 \setminus \{0\}$ descends to one on $\mathbb{P}^1$, which in turn extends to a δ-subharmonic function S_{F_ϕ} on P^1 satisfying $\Delta S_{F_\phi} = \phi^* \delta_\infty - (\deg \phi) \delta_\infty$ on P^1. This yields (2.2) since $T_{F_\phi} = S_{F_\phi} - \log [\phi(\cdot), \infty]_{\mathrm{can}} + (\deg \phi) \log [\cdot, \infty]_{\mathrm{can}}$ on P^1.

Let $\phi_i \in K(z)$, $i \in \{1, 2\}$, be rational functions on $\mathbb{P}^1$ of degree d_i. We call the following extension $[\phi_1, \phi_2]_{\mathrm{can}}$ to P^1 of the function $z \mapsto [\phi_1(z), \phi_2(z)]$ on $\mathbb{P}^1$ the *proximity function of* ϕ_1 *and* ϕ_2 *on* P^1.

Proposition 2.9. *For each $n \in \mathbb{N}$, the function $[\phi_1(\cdot), \phi_2(\cdot)]$ on $\mathbb{P}^1$ extends continuously to a function $[\phi_1, \phi_2]_{\mathrm{can}}(\cdot)$ on P^1 which takes its values in $[0, 1]$ and, if $\phi_1 \not\equiv \phi_2$ and $\max\{d_1, d_2\} > 0$, satisfies*

$$\Delta \log [\phi_1, \phi_2]_{\mathrm{can}}(\cdot) = \sum_{w \in \mathbb{P}^1 :\, \phi_1(w) = \phi_2(w)} \delta_w - \phi_1^* \Omega_{\mathrm{can}} - \phi_2^* \Omega_{\mathrm{can}}. \tag{2.3}$$

Here the sum $\sum_{w \in \mathbb{P}^1 :\, \phi_1(w) = \phi_2(w)} \delta_w$ takes into account the multiplicity of each root of $\phi_1 = \phi_2$ in $\mathbb{P}^1$.

Proof. Let F_1 and F_2 be lifts of ϕ_1 and ϕ_2, respectively. Then there are points $q_j = q_j^{F_1, F_2} \in K^2 \setminus \{0\}$ $(j = 1, \ldots, d_1 + d_2)$ such that

$$F_1(p) \wedge F_2(p) = \prod_{j=1}^{d_1 + d_2} (p \wedge q_j)$$

on K^2. Here, $\pi(q_j)$ is a root of $\phi_1 = \phi_2$ in $\mathbb{P}^1$ for each $j \in \{1, \ldots, d_1 + d_2\}$. On $\mathbb{P}^1$,

$$\log [\phi_1(\cdot), \phi_2(\cdot)] = \sum_{j=1}^{d_1 + d_2} (\log [\cdot, \pi(q_j)] + \log \|q_j\|) - T_{F_1} | \mathbb{P}^1 - T_{F_2} | \mathbb{P}^1, \tag{2.4}$$

where $T_{F_i} = \log \|F_i\| - d_i \log \|\cdot\|$ (extended continuously to P^1), $i \in \{1,2\}$, is the function introduced in Definition 2.8. The right hand side of (2.4) extends $[\phi_1(\cdot), \phi_2(\cdot)]$ on $\mathbb{P}^1$ to $[\phi_1, \phi_2]_{\mathrm{can}}(\cdot)$ on P^1 continuously so that

$$\log [\phi_1, \phi_2]_{\mathrm{can}}(\cdot) = \sum_{j=1}^{d_1+d_2} (\log [\cdot, \pi(q_j)]_{\mathrm{can}} + \log \|q_j\|) - T_{F_1} - T_{F_2}$$

on P^1 (see Fact 2.3), and satisfies (2.3) in view of (2.1) and (2.2). The density of $\mathbb{P}^1$ in P^1 implies that $[\phi_1, \phi_2]_{\mathrm{can}}(\cdot) \in [0,1]$ on P^1. ∎

REMARK 2.10 (discontinuity of $[\phi_1(\cdot), \phi_2(\cdot)]_{\mathrm{can}}$). If $\phi_2 \equiv a \in \mathbb{P}^1$ on $\mathbb{P}^1$, then $[\phi_1(\cdot), a]_{\mathrm{can}}$ coincides with $[\phi_1, a]_{\mathrm{can}}(\cdot)$ since they are continuous on P^1 and identical on the dense subset $\mathbb{P}^1$ in P^1. We point out that if K is non-archimedean and both ϕ_1 and ϕ_2 are non-constant, then $[\phi_1(\cdot), \phi_2(\cdot)]_{\mathrm{can}}$, which is the evaluation of $[\mathcal{S}_1, \mathcal{S}_2]_{\mathrm{can}}$ at $\mathcal{S}_1 = \phi_1(\cdot)$ and $\mathcal{S}_2 = \phi_2(\cdot)$ in P^1, is not always continuous on P^1, so is not always identical with $[\phi_1, \phi_2]_{\mathrm{can}}(\cdot)$. This discrepancy seems to have been overlooked in the proof of Theorem 1.2 in [9, §3.4].

An example is $\phi_1 = \phi_2 = \mathrm{Id}_{\mathbb{P}^1}$; see Fact 2.3. More generally, let ϕ_1 and ϕ_2 be non-constant polynomials such that $\phi_1(0) = \phi_2(0) = 0$ and $\phi_1'(0) = \phi_2'(0) \neq 0$. Fix $r > 0$ small enough that on $\{z \in K : |z| < 2r\}$,

$$[\phi_1(z), \phi_2(z)]_{\mathrm{can}} = [\phi_1(z), \phi_2(z)] = |\phi_1(z) - \phi_2(z)| \le \tfrac{1}{2}|\phi_1'(0)|r,$$

and that for the point $\mathcal{S}_r \in \mathsf{P}^1 \setminus \mathbb{P}^1$ determined by the disk $\{z \in K : |z| \le r\}$,

$$[\phi_1(\mathcal{S}_r), \phi_2(\mathcal{S}_r)]_{\mathrm{can}} = \mathrm{diam}(\phi_1(\mathcal{S}_r) \wedge \phi_2(\mathcal{S}_r)) = |\phi_1(\mathcal{S}_r)| = |\phi_1'(0)|r > 0.$$

Since any open neighborhood of $\mathcal{S}_r$ in P^1 intersects $\{z \in K : |z| < 2r\}$, we have $\liminf_{\mathcal{S} \to \mathcal{S}_r} [\phi_1(\mathcal{S}), \phi_2(\mathcal{S})]_{\mathrm{can}} \le |\phi_1'(0)|r/2 < [\phi_1(\mathcal{S}_r), \phi_2(\mathcal{S}_r)]_{\mathrm{can}}$. Hence the function $[\phi_1(\cdot), \phi_2(\cdot)]_{\mathrm{can}}$ on P^1 is not continuous at $\mathcal{S}_r$.

Let $f \in K(z)$ be a rational function on $\mathbb{P}^1$ of degree $d > 1$, and let F be a lift of f.

DEFINITION 2.11. The *dynamical Green function* of F on P^1 is

$$g_F := \sum_{n=0}^{\infty} \frac{1}{d^n} (f^n)^* \left(\frac{1}{d} T_F \right) = \lim_{n \to \infty} \frac{1}{d^n} T_{F^n} \in \mathbb{R}, \tag{2.5}$$

which converges uniformly on P^1 ([1, §10.1], [9, §3.1]).

The function g_F is continuous on P^1. For every $n \in \mathbb{N}$, we have $g_{F^n} = g_F$. For an arbitrary lift of f, given by cF for some $c \in K^*$, we have $g_{cF} = g_F + (\log |c|)/(d-1)$.

DEFINITION 2.12. The probability Radon measure

$$\mu_f := \Delta g_F + \Omega_{\mathrm{can}} = \lim_{n \to \infty} \frac{1}{d^n} (f^n)^* \Omega_{\mathrm{can}} \tag{2.6}$$

on P^1 is called the *equilibrium measure* of f on P^1. Here the last limit is a weak one on P^1.

FACT 2.13. By the continuity of g_F, the measure μ_f has no atoms in $\mathbb{P}^1$. Moreover, μ_f is both balanced and invariant under f in the sense that

$$f^*\mu_f = (\deg f)\mu_f \quad \text{and} \quad f_*\mu_f = \mu_f, \tag{2.7}$$

respectively (see [1, §10], [7, §2], [9, §3.1] for non-archimedean K).

We define the *F-kernel* on P^1 to be

$$\Phi_F(\mathcal{S}, \mathcal{S}') := \log [\mathcal{S}, \mathcal{S}']_{\mathrm{can}} - g_F(\mathcal{S}) - g_F(\mathcal{S}')$$

for $\mathcal{S}, \mathcal{S}' \in \mathsf{P}^1$. The function Φ_F is upper semicontinuous on $\mathsf{P}^1 \times \mathsf{P}^1$, and for each $\mathcal{S} \in \mathsf{P}^1 \setminus \mathbb{P}^1$, $\Phi_F(\mathcal{S}, \cdot)$ is continuous on P^1 (see Fact 2.3). We have

$$\sup_{(\mathcal{S},\mathcal{S}') \in \mathsf{P}^1 \times \mathsf{P}^1} |\Phi_F(\mathcal{S}, \mathcal{S}') - \log [\mathcal{S}, \mathcal{S}']_{\mathrm{can}}| \le 2 \sup_{\mathsf{P}^1} g_F < \infty,$$

and from (2.1) and (2.6), $\Delta \Phi_F(\cdot, \mathcal{S}) = \delta_{\mathcal{S}} - \mu_f$ for each $\mathcal{S} \in \mathsf{P}^1$. For a Radon measure μ on P^1, the *F-potential* on P^1 and the *F-energy* of μ are

$$U_{F,\mu}(\cdot) := \int_{\mathsf{P}^1} \Phi_F(\cdot, \mathcal{S}')\, d\mu(\mathcal{S}'), \qquad I_F(\mu) := \int_{\mathsf{P}^1} U_{F,\mu}\, d\mu,$$

respectively (see also [1, §8.10], [9, §2.4]). The function $U_{F,\mu}$ is upper semicontinuous on P^1 and has the following continuity property: for every $z_0 \in \mathbb{P}^1 \setminus \{\infty\}$ and every $r \ge 0$, if $\mathcal{S}_r(z_0)$ is the point in P^1 corresponding to the disk $\mathcal{B}_r(z_0) := \{z \in K : |z - z_0| \le r\}$, we have

$$\lim_{r \to 0} U_{F,\mu}(\mathcal{S}_r(z_0)) = U_{F,\mu}(z_0) \tag{2.8}$$

(see [1, Proposition 6.12]). By Fubini's theorem,

$$\Delta U_{F,\mu} = \mu - \mu(\mathsf{P}^1)\mu_f.$$

A probability Radon measure μ on P^1 is called an *F-equilibrium mass distribution on* P^1 if the F-energy $I_F(\mu)$ of this μ equals

$$V_F := \sup\{I_F(\nu) : \nu \text{ is a probability Radon measure on } \mathsf{P}^1\},$$

which is $> -\infty$ since $I_F(\Omega_{\mathrm{can}}) > -\infty$.

We recall Baker and Rumely's characterization of μ_f as the unique solution of a Gauss variational problem; see [1, Theorem 8.67 and Proposition 8.70] for non-archimedean K. For a discussion of the Gauss variational problem, see e.g. [18].

LEMMA 2.14. *There is a unique F-equilibrium mass distribution on P^1, which coincides with the equilibrium measure μ_f of f. Indeed, on P^1,*

$$U_{F,\mu_f} \equiv V_F. \tag{2.9}$$

The functions Φ_F, $U_{F,\mu}$ and g_F depend on the lift F of f. We will now introduce more canonical functions Φ_f, U_μ, and g_f, which do not depend on the choice of the lift F. The *f-kernel* on P^1 (the negative of the Arakelov Green function for f in [1, §10.2]) is

$$\Phi_f := \Phi_F - V_F.$$

It is independent of the choice of F. For each Radon measure μ on P^1, we define the *f-potential*

$$U_\mu := \int_{\mathsf{P}^1} \Phi_f(\cdot, \mathcal{S}')\, d\mu(\mathcal{S}')$$

on P^1. We still have $\Delta U_\mu = \mu - \mu(\mathsf{P}^1)\mu_f$. From Lemma 2.14, we obtain

LEMMA 2.15. *For each Radon measure μ on P^1, we have $U_\mu \geq 0$ on $\operatorname{supp}\mu$ if and only if $\mu = \mu_f$. Moreover, $U_{\mu_f} \equiv 0$ on P^1.*

The *dynamical Green function* g_f of f (a canonical version of g_F) is defined as

$$g_f(\mathcal{S}) := g_F(\mathcal{S}) + \tfrac{1}{2}V_F = \tfrac{1}{2}(\log[\mathcal{S}, \mathcal{S}]_{\mathrm{can}} - \Phi_f(\mathcal{S}, \mathcal{S})),$$

which is independent of the choice of F and still satisfies

$$\Delta g_f = \mu_f - \Omega_{\mathrm{can}}. \tag{2.10}$$

For every $(\mathcal{S}, \mathcal{S}') \in \mathsf{P}^1 \times \mathsf{P}^1$,

$$\Phi_f(\mathcal{S}, \mathcal{S}') = \log[\mathcal{S}, \mathcal{S}']_{\mathrm{can}} - g_f(\mathcal{S}) - g_f(\mathcal{S}').$$

Our definition (2.6) of μ_f agrees with Favre and Rivera-Letelier's [9, Proposition-Définition 3.2]:

LEMMA 2.16. *For every $\mathcal{S} \in \mathsf{P}^1 \setminus \mathbb{P}^1$, weakly on P^1,*

$$\lim_{k\to\infty} \frac{(f^n)^*\delta_{\mathcal{S}}}{d^n} = \mu_f.$$

Proof. For every $\mathcal{S} \in \mathsf{P}^1$ and every $n \in \mathbb{N}$, from the balanced property $f^*\mu_f = d \cdot \mu_f$,

$$\Delta\Phi_f(f^n(\cdot), \mathcal{S}) = (f^n)^*(\delta_{\mathcal{S}} - \mu_f) = (f^n)^*\delta_{\mathcal{S}} - d^n\mu_f$$

on P^1. Suppose that $\mathcal{S} \in \mathsf{P}^1 \setminus \mathbb{P}^1$. Then since $[\mathcal{S}, \mathcal{S}]_{\mathrm{can}} > 0$,

$$\sup_{\mathcal{S}' \in \mathsf{P}^1} |\Phi_f(f^n(\mathcal{S}'), \mathcal{S})| \leq |\log[\mathcal{S}, \mathcal{S}]_{\mathrm{can}}| + 2\sup_{\mathsf{P}^1} g_f < \infty,$$

so $\lim_{n\to\infty} \Phi_f(f^n(\cdot), \mathcal{S})/d^n = 0$ uniformly on P^1. By the continuity of Δ on uniformly convergent sequences of δ-subharmonic functions (for non-archimedean K, see [1, Corollary 5.39], [9, Proposition 2.17]), as $n \to \infty$,

$$\frac{(f^n)^*\delta_{\mathcal{S}}}{d^n} - \mu_f = \Delta \frac{1}{d^n}\Phi_f(f^n(\cdot), \mathcal{S}) \to 0$$

weakly on P^1. ∎

The *Berkovich Fatou set* $\mathsf{F}(f)$ of f is by definition $\mathsf{P}^1 \setminus \mathsf{J}(f)$, which is open in P^1. A *Berkovich Fatou component* W of f is a component of $\mathsf{F}(f)$. Given such a W, $f(W)$ is also a Berkovich Fatou component of f, and so is each component of $f^{-1}(W)$. We call W a *cyclic* Berkovich Fatou component of f if $f^p(W) = W$ for some $p \in \mathbb{N}$.

For archimedean K, the classification of cyclic Fatou components (immediate (super)attractive basins of attracting cycles, immediate attractive basins of parabolic cycles, Siegel disks, and Herman rings) of f is essentially due to Fatou (cf. [14, Theorem 5.2]). The following is its non-archimedean counterpart due to Rivera-Letelier; see [9, Proposition 2.16] and its *esquisse de démonstration*, and also [2, Remark 7.10].

THEOREM 2.17. *Suppose that K is non-archimedean. Then for each cyclic Berkovich Fatou component W of f, either W contains an attracting periodic point of f in $W \cap \mathbb{P}^1$ (attracting case), or $\deg(f^p : W \to W) = 1$ for some $p \in \mathbb{N}$ satisfying $f^p(W) = W$. Moreover, only one case occurs. In the former case, W is called an immediate (super)attractive basin of f, and in the latter case, W is called a singular domain of f.*

All of $E(f)$, $\mathsf{J}(f)$, $\mathsf{F}(f)$, and $\operatorname{supp}\mu_f$ are *completely invariant* under f. Here, a subset E in P^1 is said to be *completely invariant* under f if $f(E) \subset E$ and $f^{-1}(E) \subset E$. The following equality is fundamental.

THEOREM 2.18. $\mathsf{J}(f) = \operatorname{supp}\mu_f$. *Moreover, for each $a \in E(f)$, no weak limit point of (ν_n^a) on P^1 equals μ_f.*

Proof. Since μ_f has no atoms in $\mathbb{P}^1$ and $E(f)$ is a countable subset in $\mathbb{P}^1$, $\operatorname{supp}\mu_f \not\subset E(f)$. Then $\mathsf{J}(f) \subset \overline{\bigcup_{n\in\mathbb{N}} f^{-n}((\operatorname{supp}\mu_f) \setminus E(f))}$, which is contained in $\operatorname{supp}\mu_f$. Hence $\mathsf{J}(f) \subset \operatorname{supp}\mu_f$.

For archimedean K, Ω_{can} is the normalized Fubini–Study metric on $\mathsf{P}^1 = \mathbb{P}^1$. By Marty's theorem [13, Théorème 5], which is an infinitesimal version of Montel's theorem, $\mathsf{F}(f)$ coincides with the maximal open subset in $\mathbb{P}^1$ where the family of chordal derivatives of f^n, $n \in \mathbb{N}$,

$$\left\{\mathbb{P}^1 \ni z \mapsto \sqrt{\frac{(f^n)^*\Omega_{\mathrm{can}}}{\Omega_{\mathrm{can}}}(z)} = \lim_{w\to z} \frac{[f^n(z), f^n(w)]}{[z,w]} \in [0,\infty) : n \in \mathbb{N}\right\}$$

is locally uniformly bounded. Hence by the definition (2.6) of μ_f, we have $\mathsf{F}(f) \subset \mathsf{P}^1 \setminus \operatorname{supp}\mu_f$, i.e., $\operatorname{supp}\mu_f \subset \mathsf{J}(f)$.

Suppose that K is non-archimedean. If $\mathsf{J}(f) \subset \mathbb{P}^1$, then $\mathsf{F}(f)$ is itself the unique Berkovich Fatou component of f, which is completely invariant under f. Since $\deg(f : F(f) \to F(f)) = \deg f > 1$, by Theorem 2.17, $\mathsf{F}(f)$ is the immediate attractive basin of an attracting fixed point $a \in \mathbb{P}^1$. Since $\mathcal{S}_{\mathrm{can}} \in \mathsf{P}^1 \setminus \mathbb{P}^1 \subset \mathsf{F}(f) \setminus \{a\}$, we have $\bigcap_{N\in\mathbb{N}} \overline{\bigcup_{n\geq N} f^{-n}(\mathcal{S}_{\mathrm{can}})} \subset \partial\mathsf{F}(f) = \mathsf{J}(f)$. Moreover, since $\Omega_{\mathrm{can}} = \delta_{\mathcal{S}_{\mathrm{can}}}$ in this case, by the definition (2.6) of μ_f,

$\operatorname{supp}\mu_f \subset \bigcap_{N\in\mathbb{N}}\overline{\bigcup_{n\ge N} f^{-n}(\mathcal{S}_{\mathrm{can}})}$. Hence $\operatorname{supp}\mu_f \subset \mathsf{J}(f)$. Finally, if $\mathsf{J}(f)\not\subset\mathbb{P}^1$, then by Lemma 2.16, we have $\operatorname{supp}\mu_f \subset \overline{\bigcup_{n\in\mathbb{N}} f^{-n}(\mathsf{J}(f)\setminus\mathbb{P}^1)}$, which is contained in $\mathsf{J}(f)$.

Hence we have $\operatorname{supp}\mu_f\subset\mathsf{J}(f)$ in both archimedean and non-archimedean cases, and the proof of the former assertion is complete.

Recall that for any $a\in E(f)$, the backward orbit $\bigcup_{n\in\mathbb{N}} f^{-n}(a)$ is finite and contained in $\mathsf{F}(f)$. Hence any weak limit point $\nu=\lim_{j\to\infty}\nu^a_{n_j}$ has its support in $\mathsf{F}(f)$, so $\nu\ne\mu_f$ by the former assertion. ∎

Finally, for a rational function $f\in K(z)$ on $\mathbb{P}^1$ of degree $d>1$ and a rational function $a\in K(z)$ on $\mathbb{P}^1$, we introduce the (logarithmic) proximity function $\log[f^n,a]_{\mathrm{can}}(\cdot)$ of $f^n(\cdot)$ and $a(\cdot)$ weighted by g_f:

$$\Phi(f^n,a)_f(\cdot):=\log[f^n,a]_{\mathrm{can}}(\cdot)-g_f\circ f^n-g_f\circ a.$$

The function $\Phi(f^n,a)_f(\cdot)$ extends the function $z\mapsto\Phi_f(f^n(z),a(z))$ on $\mathbb{P}^1$ continuously to P^1 and plays a crucial role in the rest of the paper. It agrees with $\Phi_f(f^n(\cdot),a)$ when a is constant. For each $n\in\mathbb{N}$, we have

$$\sup_{\mathsf{P}^1}|\Phi(f^n,a)_f(\cdot)-\log[f^n,a]_{\mathrm{can}}(\cdot)|\le\sup_{\mathsf{P}^1}|2g_f|<\infty. \tag{2.11}$$

Lemma 2.19 (cf. [20, (1.4)]). *For every $n\in\mathbb{N}$,*

$$\frac{1}{d^n}\Phi(f^n,a)_f(\cdot)=U_{(1+(\deg a)/d^n)\nu^a_n}-\frac{1}{d^n}U_{a^*\mu_f}+\frac{1}{d^n}\int_{\mathsf{P}^1}\Phi(f^n,a)_f(\cdot)\,d\mu_f \tag{2.12}$$

on P^1. Similarly, the function $U_{a^*\mu_f}=a^*g_f+U_{a^*\Omega_{\mathrm{can}}}-\int_{\mathsf{P}^1}(a^*g_f)\,d\mu_f$ is continuous (*hence bounded*) *on P^1.*

Proof. For each $n\in\mathbb{N}$, from (2.3) and (2.10),

$$\Delta\Phi(f^n,a)_f(\cdot)=(d^n+\deg a)\nu^a_n-(f^n)^*\mu_f-a^*\mu_f,$$

and using the balanced property $f^*\mu_f=d\cdot\mu_f$, we have

$$\Delta\Phi(f^n,a)_f(\cdot)=\Delta(U_{(d^n+\deg a)\nu^a_n}-U_{a^*\mu_f}).$$

Hence the function

$$\frac{1}{d^n}\Phi(f^n,a)_f(\cdot)-\left(U_{(1+(\deg a)/d^n)\nu^a_n}(\cdot)-\frac{1}{d^n}U_{a^*\mu_f}(\cdot)\right)$$

is constant on P^1 (for non-archimedean K, this holds on $\mathsf{P}^1\setminus\mathbb{P}^1$ by a basic property of Δ (see [1, Lemma 5.24], [9, §2.4]) and indeed on P^1 by continuity (2.8)). We determine the constant by integrating this against $d\mu_f$ on P^1: by Fubini's theorem and the fact that $U_{\mu_f}\equiv0$, the integrals of the second and third terms in $d\mu_f$ vanish. Hence (2.12) holds.

Similarly, from $\Delta U_{a^*\mu_f} = a^*\mu_f - (\deg a)\mu_f = \Delta(a^*g_f + U_{a^*\Omega_{\mathrm{can}}})$, the function $U_{a^*\mu_f} - (a^*g_f + U_{a^*\Omega_{\mathrm{can}}})$ is constant on P^1. The constant is determined by integrating this function against $d\mu_f$ on P^1. ∎

3. Proof of Theorem 1. Let K be an algebraically closed field of any characteristic and complete with respect to a non-trivial absolute value. Let $f \in K(z)$ be a rational function on $\mathbb{P}^1 = \mathbb{P}^1(K)$ of degree $d > 1$, and $a \in K(z)$ a rational function on $\mathbb{P}^1$. Let (n_j) be a sequence in $\mathbb{N}$ tending to ∞, and ν be any weak limit of a subsequence of $(\nu^a_{n_j})$ on $\mathsf{P}^1 = \mathsf{P}^1(K)$. This is a probability Radon measure on P^1, and the equidistribution property (1.3) is equivalent to

$$\nu = \mu_f. \tag{1.3$'$}$$

Taking a subsequence of (n_j) if necessary, we can assume that $\nu = \lim_{j\to\infty} \nu^a_{n_j}$ weakly on P^1 and that the limit

$$\lim_{j\to\infty} \frac{1}{d^{n_j}} \int_{\mathsf{P}^1} \Phi(f^{n_j}, a)_f \, d\mu_f \tag{3.1}$$

exists in $[-\infty, 0]$.

Lemma 3.1. *On* P^1,

$$\begin{aligned} \limsup_{j\to\infty} \frac{1}{d^{n_j}} \log\,[f^{n_j}, a]_{\mathrm{can}}(\cdot) &= \limsup_{j\to\infty} \frac{1}{d^{n_j}} \Phi(f^{n_j}, a)_f(\cdot) \\ &\le U_\nu + \lim_{j\to\infty} \frac{1}{d^{n_j}} \int_{\mathsf{P}^1} \Phi(f^{n_j}, a)_f \, d\mu_f \le \min\{U_\nu, 0\}. \end{aligned} \tag{3.2}$$

Moreover, on $\mathsf{P}^1 \setminus \mathbb{P}^1$,

$$\lim_{j\to\infty} \frac{1}{d^{n_j}} \log\,[f^{n_j}, a]_{\mathrm{can}}(\cdot) = U_\nu + \lim_{j\to\infty} \frac{1}{d^{n_j}} \int_{\mathsf{P}^1} \Phi(f^{n_j}, a)_f \, d\mu_f. \tag{3.3}$$

Proof. By a cut-off argument, on P^1,

$$\limsup_{j\to\infty} U_{\nu^a_{n_j}} \le U_\nu; \tag{3.4}$$

indeed, for every $N \in \mathbb{N}$, $U_{\nu^a_{n_j}} \le \int_{\mathsf{P}^1} \max\{-N, \Phi_f(\cdot, \mathcal{S}')\}\, d\nu^a_{n_j}(\mathcal{S}')$ on P^1, and since for every $\mathcal{S} \in \mathsf{P}^1$ the function $\mathcal{S}' \mapsto \max\{-N, \Phi_f(\mathcal{S}, \mathcal{S}')\}$ is continuous on P^1, we have

$$\limsup_{j\to\infty} U_{\nu^a_{n_j}} \le \int_{\mathsf{P}^1} \max\{-N, \Phi_f(\cdot, \mathcal{S}')\}\, d\nu(\mathcal{S}')$$

on P^1. Taking $N \to \infty$, we obtain (3.4) by the monotone convergence theorem.

On the other hand, for every $\mathcal{S} \in \mathsf{P}^1 \setminus \mathbb{P}^1$, the function $\mathcal{S}' \mapsto \Phi_f(\mathcal{S}, \mathcal{S}')$ is continuous on P^1, so we have $\lim_{j\to\infty} U_{\nu^a_{n_j}} = U_\nu$ on $\mathsf{P}^1 \setminus \mathbb{P}^1$.

By the comparison (2.11) and $[f^n, a]_{\mathrm{can}} \le 1$,

$$\limsup_{j\to\infty} \frac{1}{d^{n_j}} \Phi(f^{n_j}, a)_f(\cdot) = \limsup_{j\to\infty} \frac{1}{d^{n_j}} \log\,[f^{n_j}, a]_{\mathrm{can}}(\cdot) \le 0$$

on P^1. Now taking $\limsup_{j\to\infty}$ of ((2.12) for $n = n_j$), we have (3.2) on P^1, and also (3.3) on $\mathsf{P}^1 \setminus \mathbb{P}^1$. ∎

If a is constant, then by convention, we identify a with its value in $\mathbb{P}^1$.

Lemma 3.2. *If a is constant, then $\int_{\mathsf{P}^1} \Phi_f(f^n(\cdot), a)\, d\mu_f = 0$ for every $n \in \mathbb{N}$, and $U_\nu \ge 0$ on $\mathsf{J}(f)$.*

Proof. Let $a \in \mathbb{P}^1$. Then for every $n \in \mathbb{N}$, by the invariance $f_*\mu_f = \mu_f$ and the fact that $U_{\mu_f} \equiv 0$ on P^1, we have

$$\int_{\mathsf{P}^1} \Phi_f(f^n(\cdot), a)\, d\mu_f = U_{(f^n)_*\mu_f}(a) = U_{\mu_f}(a) = 0.$$

Hence by Fatou's lemma and (3.2), this implies that

$$\begin{aligned} 0 &= \lim_{j\to\infty} \frac{1}{d^{n_j}} \int_{\mathsf{P}^1} \Phi_f(f^{n_j}(\cdot), a)\, d\mu_f \\ &\le \int_{\mathsf{P}^1} \limsup_{j\to\infty} \frac{1}{d^{n_j}} \Phi_f(f^{n_j}(\cdot), a)\, d\mu_f \le \int_{\{U_\nu<0\}\cap \mathsf{J}(f)} U_\nu\, d\mu_f. \end{aligned}$$

Since $\mathsf{J}(f) \subset \operatorname{supp} \mu_f$ (by Theorem 2.18), $\{U_\nu < 0\} \cap \mathsf{J}(f) = \emptyset$. ∎

We show the following counterpart of Lemma 3.2 for non-constant a.

Lemma 3.3. *If a is non-constant, then $U_\nu \ge 0$ on $\mathsf{J}(f)$.*

Proof. Assume that $\{U_\nu < 0\} \cap \mathsf{J}(f) \ne \emptyset$. Then since $\{U_f < 0\}$ is open,

$$\bigcup_{n\in\mathbb{N}} f^n(\{U_\nu < 0\} \cap \mathbb{P}^1) = \Big(\bigcup_{n\in\mathbb{N}} f^n(\{U_\nu < 0\})\Big) \cap \mathbb{P}^1 \supset \mathbb{P}^1 \setminus E(f).$$

If there exists $z_1 \in E(f)$, then $\bigcup_{n\in\mathbb{N}} f^n(\{U_\nu < 0\} \cap \mathbb{P}^1)$ intersects the immediate attractive basin of z_1, so by (3.2), $a \equiv z_1$. This contradicts that a is non-constant, and so we have $E(f) = \emptyset$.

Let z_0 be a fixed point of f in $\mathbb{P}^1 = \mathbb{P}^1 \setminus E(f)$. Then by the assumption $\{U_\nu < 0\} \cap \mathsf{J}(f) \ne \emptyset$ and the definition of $\mathsf{J}(f)$, $\mathsf{J}(f) \cap \{U_\nu < 0\} \subset \big(\bigcap_{\ell\in\mathbb{N}} \overline{\bigcup_{j\ge\ell} f^{-n}(z_0)}\big) \cap \{U_\nu < 0\}$. Hence if $\#(\bigcup_{n\in\mathbb{N}} f^{-n}(z_0) \cap \{U_\nu < 0\}) < \infty$, then $\mathsf{J}(f) \cap \{U_\nu < 0\}$ is a non-empty and finite subset in $\mathbb{P}^1$. Since $\mathsf{J}(f) \subset \operatorname{supp} \mu_f$ (by Theorem 2.18), this contradicts that μ_f has no atoms in $\mathbb{P}^1$.

Hence there is an $N \in \mathbb{N}$ such that $f^{-N}(z_0) \cap \{U_\nu < 0\} \not\subset a^{-1}(z_0)$ since $\#(\bigcup_{n\in\mathbb{N}} f^{-n}(z_0) \cap \{U_\nu < 0\}) = \infty$ and $\#a^{-1}(z_0) < \infty$. Let $z_{-N} \in (f^{-N}(z_0) \cap \{U_\nu < 0\}) \setminus a^{-1}(z_0)$. Then

$$\limsup_{j\to\infty} \frac{1}{d^{n_j}} \log\,[f^{n_j}(z_{-N}), a(z_{-N})] = \limsup_{j\to\infty} \frac{1}{d^{n_j}} \log\,[z_0, a(z_{-N})] = 0,$$

which contradicts (3.2) at z_{-N} since $U_\nu(z_{-N}) < 0$.

Hence $\{U_\nu < 0\} \cap \mathsf{J}(f) = \emptyset$, and the proof is complete. ∎

Lemma 3.4. *If* $(1.3')$ *holds, then*

$$\lim_{j\to\infty} \frac{1}{d^{n_j}} \int_{\mathsf{P}^1} \Phi(f^{n_j}, a)_f(\cdot)\, d\mu_f = 0. \tag{3.5}$$

Indeed, (3.5) *holds for every* $a \in \mathbb{P}^1$ *without assuming* $(1.3')$.

Proof. If a is constant, then this follows from the former assertion in Lemma 3.2 without assuming $(1.3')$.

Suppose that a is non-constant. If $(1.3')$ holds but (3.5) does not hold, then by (3.2) and $U_\nu = U_{\mu_f} \equiv 0$,

$$\limsup_{j\to\infty} \frac{1}{d^{n_j}} \log\,[f^{n_j}, a]_{\mathrm{can}}(\cdot) \le U_\nu + \lim_{j\to\infty} \frac{1}{d^{n_j}} \int_{\mathsf{P}^1} \Phi(f^{n_j}, a)_f\, d\mu_f < 0 \tag{3.6}$$

on P^1. If there exists $z_1 \in E(f)$, then (3.6) holds on the immediate attractive basin of z_1, so $a \equiv z_1$. This is a contradiction, and we have $E(f) = \emptyset$.

Let $z_0 \in \mathbb{P}^1 = \mathbb{P}^1 \setminus E(f)$ be a fixed point of f. Then $\infty > \#a^{-1}(z_0) < \#\bigcup_{n\in\mathbb{N}} f^{-n}(z_0) = \infty$, so there is an $N \in \mathbb{N}$ such that $f^{-N}(z_0) \not\subset a^{-1}(z_0)$. Let $z_{-N} \in f^{-N}(z_0) \setminus a^{-1}(z_0)$. Then

$$\limsup_{j\to\infty} \frac{1}{d^{n_j}} \log\,[f^{n_j}(z_{-N}), a(z_{-N})] = \limsup_{j\to\infty} \frac{1}{d^{n_j}} \log\,[z_0, a(z_{-N})] = 0,$$

which contradicts (3.6) at z_{-N}. ∎

We can now complete the proof of Theorem 1. If (1.4) holds, then by the latter assertion in Lemma 3.2, Lemma 3.3, and Lemma 2.15, the condition $(1.3')$ holds. The reverse implication follows by Theorem 2.18.

Suppose now that K is non-archimedean. If $(1.3')$ holds, then $U_\nu = U_{\mu_f} \equiv 0$ on P^1, and by (3.3) and Lemma 3.4, we have

$$\lim_{j\to\infty} \frac{1}{d^{n_j}} \log\,[f^{n_j}, a]_{\mathrm{can}}(\cdot) = U_\nu + \lim_{j\to\infty} \frac{1}{d^{n_j}} \int_{\mathsf{P}^1} \Phi(f^{n_j}, a)_f\, d\mu_f = 0,$$

i.e., (1.5), on $\mathsf{P}^1 \setminus \mathbb{P}^1$. Conversely, if (1.5) holds on $\mathsf{P}^1 \setminus \mathbb{P}^1$, then by (3.2), we have $\{U_\nu < 0\} \setminus \mathbb{P}^1 = \emptyset$, so $\{U_\nu < 0\} = \emptyset$. Hence by Lemma 2.15, $(1.3')$ holds. If one (so ultimately all) of (1.3), (1.4) and (1.5) holds, then by Lemma 3.4

and (2.11), the final (1.6) holds; indeed, (1.6) holds for every $a \in \mathbb{P}^1$ without assuming (1.3), (1.4) or (1.5).

This completes the proof of Theorem 1.

4. Proof of Theorems A, 1.1 and 1.2. We give some addenda to our argument in Section 3. Let K be an algebraically closed field of arbitrary characteristic, and complete with respect to a non-trivial absolute value. Let $f \in K(z)$ be a rational function on $\mathbb{P}^1$ of degree $d > 1$, and $a \in K(z)$ a rational function on $\mathbb{P}^1$. If a is constant, we identify a with its value in $\mathbb{P}^1$. Let $\nu = \lim_{j\to\infty} \nu^a_{n_j}$ be the weak limit of a subsequence $(\nu^a_{n_j})$ of (ν^a_n) on $\mathsf{P}^1 = \mathsf{P}^1(K)$. Taking a subsequence of (n_j) if necessary, we can assume that the limit (3.1) exists in $[-\infty, 0]$.

We first give a purely local proof of Theorem A based on (1.7) and Lemma 2.15.

Proof of Theorem A. Under the assumption in Theorem A, we set $K = \mathbb{C}_v$. The set of all points in $\mathbb{P}^1(\overline{k})$ which are wandering under f and, if in addition a is non-constant, do not belong to $a^{-1}(E(f))$, is dense in P^1. Since U_ν is upper semicontinuous, the inequality (3.2), combined with the dynamical Diophantine approximation result (1.7), implies that $U_\nu \geq 0$ on P^1. Hence by Lemma 2.15, (1.3′) holds. ∎

Next, we prove Theorem 1.1.

Proof of Theorem 1.1. We will show that $(\operatorname{supp}\nu) \cap \{U_\nu < 0\} = \emptyset$. This means that, by Lemma 2.15, (1.3′) will hold.

Suppose first that $a \in \mathsf{J}(f) \cap \mathbb{P}^1$. Then as $f^{-1}(\mathsf{J}(f)) = \mathsf{J}(f)$, we have $\operatorname{supp}\nu \subset \mathsf{J}(f)$. Hence by Lemma 3.2, $(\operatorname{supp}\nu) \cap \{U_\nu < 0\} = \emptyset$.

Suppose that $a \in (\mathsf{F}(f) \cap \mathbb{P}^1) \setminus E(f)$. By the upper semicontinuity of U_ν, $\{U_\nu < 0\}$ is open. From (3.2),

$$\limsup_{j\to\infty} \frac{1}{d^{n_j}} \log\,[f^{n_j}(\cdot), a]_{\mathrm{can}} \leq U_\nu(\cdot) < 0 \tag{4.1}$$

on $\{U_\nu < 0\}$. This implies that $\lim_{j\to\infty} f^{n_j} = a$ on $\{U_\nu < 0\} \cap \mathbb{P}^1$, so $\{U_\nu < 0\} \cap \mathbb{P}^1 \subset \mathsf{F}(f)$, and that the Berkovich Fatou component W of f containing a is cyclic under f, i.e., $f^p(W) = W$ for some $p \in \mathbb{N}$. Then from the classification of cyclic (Berkovich) Fatou components (see Theorem 2.17 for non-archimedean K), it follows that either a is the unique attracting fixed point of f^p in W (*attracting case*), or $\deg(f^p : W \to W) = 1$ (*singular case*). In the attracting case, by (4.1), a is the superattracting fixed point of f^p in W satisfying $\deg_a f^p = d^p$. This contradicts the assumption $a \in \mathbb{P}^1 \setminus E(f)$.

Hence the singular case occurs. Let U be a component of $\{U_\nu < 0\}$ and put $N := \min\{n \in \mathbb{N} \cup \{0\} : f^n(U) \subset W\}$. Then for every $n > N$, there is at most one root of $f^{n-N}(\cdot)=a$ in W, which is simple if it exists. Hence

$$0\le \nu(U)\le \limsup_{j\to\infty}\frac{1\cdot d^N}{d^{n_j}}=0.$$

This implies that $(\operatorname{supp}\nu)\cap\{U_\nu<0\}=\emptyset$. ∎

REMARK 4.1. For a purely potential-theoretical proof of Theorem 1.1 for non-archimedean K, see [11, §5].

An application of Theorem 1.1 is the following.

LEMMA 4.2. *The Berkovich Julia set* $\mathsf{J}(f)$ *of* f *coincides with*

$$\Big\{\mathcal{S}\in\mathsf{P}^1:\bigcap_{(n_j)\subset\mathbb{N}:\text{ infinite}}\ \bigcap_{U:\text{ open in }\mathsf{P}^1,\,\mathcal{S}\in U}\ \bigcup_{j\in\mathbb{N}} f^{n_j}(U)=\mathsf{P}^1\setminus E(f)\Big\},\tag{4.2}$$

which is a priori contained in $\mathsf{J}(f)$.

Proof. By Theorem 2.18, $\mathsf{J}(f)\subset\operatorname{supp}\mu_f$. By Theorems 1.1 and 2.16, $\operatorname{supp}\mu_f$ is contained in (4.2). Clearly, (4.2) is contained in $\mathsf{J}(f)$. ∎

Suppose now that a is non-constant.

Proof of Theorem 1.2 for archimedean $K\cong\mathbb{C}$. We will show that $U_\nu\ge 0$ on $\operatorname{supp}\nu$. Then by Lemma 2.15, (1.3′) will hold.

By the upper semicontinuity of U_ν, $\{U_\nu<0\}$ is open. Let U be a component of $\{U_\nu<0\}$. By Lemma 3.3, $U\subset\mathsf{F}(f)$. From (3.2), we have $\lim_{j\to\infty}f^{n_j}=a$ on U. Since a is non-constant, this implies that there are an $N\in\mathbb{N}$ and a cyclic Fatou component Y of f such that Y is a Siegel disk or a Herman ring of f, and that for every $j\ge N$, $f^{n_j}(U)\subset Y$. Then $a(U)\subset Y$. For some $k_0\in\mathbb{N}$, we have $f^{k_0}(Y)=Y$, and for every $j\ge N$, we have $k_0|(n_j-n_N)$.

Let $h:Y\to\mathbb{C}$ be a holomorphic injection (a linearization map) such that for some $\alpha\in\mathbb{R}\setminus\mathbb{Q}$, setting $\lambda=e^{2i\pi\alpha}$, we have $h\circ f^{k_0}=\lambda\cdot h$ on Y. Taking a subsequence of (n_j) if necessary, $\lambda_0:=\lim_{j\to\infty}\lambda^{(n_j-n_N)/k_0}\in\mathbb{C}$ exists and

$$h\circ a=\lim_{j\to\infty}h\circ f^{n_j}=\lambda_0\cdot(h\circ f^{n_N})$$

on U. Moreover, for every $j\in\mathbb{N}$ large enough, $\lambda^{(n_j-n_N)/k_0}\ne\lambda_0$ and

$$h\circ f^{n_j}-h\circ a=(\lambda^{(n_j-n_N)/k_0}-\lambda_0)\cdot(h\circ f^{n_N})$$

on U. Since h has at most one zero in Y, which is simple if it exists, we have

$$0\le\nu(U)\le\limsup_{j\to\infty}\frac{1\cdot d^{n_N}}{d^{n_j}+\deg a}=0.$$

This implies that $\{U_\nu<0\}\cap(\operatorname{supp}\nu)=\emptyset$. ∎

Suppose now that K is non-archimedean. In the following definition, $\mathcal{E}_f$ is a *Berkovich version* of Rivera-Letelier's quasiperiodicity domain of f.

DEFINITION 4.3. Let $\mathcal{E}_f$ be the set of points in P^1 having a neighborhood U such that for some $(n_j) \subset \mathbb{N}$ tending to ∞,

$$\lim_{j\to\infty} \sup_{U\cap\mathbb{P}^1} [f^{n_j}, \mathrm{Id}_{\mathbb{P}^1}] = 0. \tag{4.3}$$

LEMMA 4.4. *$\mathcal{E}_f$ is open, $f(\mathcal{E}_f) \subset \mathcal{E}_f$, and $\mathcal{E}_f$ is covered by singular domains of f. In particular, $\mathcal{E}_f \cap \mathbb{P}^1 \neq \mathbb{P}^1$.*

Proof. From the definition, $\mathcal{E}_f$ is open in P^1. For every open subset U in P^1, $[f^{n_j}, \mathrm{Id}] \circ f = [f^{n_j+1}, f] \leq L[f^{n_j}, \mathrm{Id}]$ on $U \cap \mathbb{P}^1$, where $L > 0$ is a Lipschitz constant of $f|\mathbb{P}^1$ with respect to the chordal distance. Hence if (4.3) holds on U, then $\lim_{j\to\infty} \sup_{f(U)\cap\mathbb{P}^1} [f^{n_j}, \mathrm{Id}_{\mathbb{P}^1}] = 0$, so $f(\mathcal{E}_f) \subset \mathcal{E}_f$.

By Lemma 4.2 and (4.3), $\mathcal{E}_f \cap \mathbb{P}^1 \subset \mathsf{F}(f)$. Moreover, by (4.3), $\mathcal{E}_f$ is indeed covered by some cyclic Berkovich Fatou components W of f, and by Theorem 2.17 and (4.3), each W is a singular domain.

Since $\mathcal{E}_f$ is covered by singular domains of f, f has no critical points in $\mathcal{E}_f \cap \mathbb{P}^1$, so from $\deg f > 1$, we have $\mathcal{E}_f \cap \mathbb{P}^1 \neq \mathbb{P}^1$. ∎

For non-archimedean K of characteristic 0, a non-archimedean counterpart of the uniformization of a Siegel disk or a Herman ring of f is given by Rivera-Letelier's iterative logarithm of f on $\mathcal{E}_f$.

THEOREM 4.5 ([17, §3.2, §4.2]; see also [9, Théorème 2.15]). *Suppose that K has characteristic 0 and residual characteristic p. Let $f \in K(z)$ be a rational function on $\mathbb{P}^1$ of degree > 1 and suppose that $\mathcal{E}_f \neq \emptyset$, which implies $p > 0$ by* [9, Lemme 2.14]. *Then for every component Y of $\mathcal{E}_f$ not containing ∞, there are a $k_0 \in \mathbb{N}$, a continuous action $T : \mathbb{Z}_p \times (Y \cap K) \ni (\omega, y) \mapsto T^\omega(y) \in Y \cap K$ and a non-constant K-valued holomorphic function T_* on $Y \cap K$ such that for every $m \in \mathbb{Z}$, $(f^{k_0})^m = T^m$ on $Y \cap K$, that for each $\omega \in \mathbb{Z}_p$, T^ω is a biholomorphism on $Y \cap K$ and that for every $\omega_0 \in \mathbb{Z}_p$,*

$$\lim_{\mathbb{Z}_p \ni \omega \to \omega_0} \frac{T^\omega - T^{\omega_0}}{\omega - \omega_0} = T_* \circ T^{\omega_0} \tag{4.4}$$

locally uniformly in $Y \cap K$.

We also need the following.

LEMMA 4.6. *For every compact subset C in $\{U_\nu < 0\}$,*

$$\lim_{j\to\infty} \sup_C [f^{n_j}, a]_{\mathrm{can}}(\cdot) = 0.$$

Proof. By a lemma of Hartogs (cf. [9, Proposition 2.18], [1, Proposition 8.57]) and (3.4), for every compact subset C in P^1,

$$\limsup_{j\to\infty} \sup_C U_{\nu^a_{n_j}} \leq \sup_C U_\nu. \tag{4.5}$$

By Lemma 2.19,

$$\sup_C \frac{1}{d^{n_j}} \Phi(f^{n_j}, a)_f(\cdot)$$
$$= \sup_C U_{(1+(\deg a)/d^{n_j})\nu^a_{n_j}} + \frac{1}{d^{n_j}} \sup_C |U_{a^*\mu_f}| + \frac{1}{d^{n_j}} \int_{\mathsf{P}^1} \Phi(f^{n_j}, a)_f \, d\mu_f.$$

Let us take $\limsup_{j\to\infty}$ of both sides. Then by (2.11), the estimate (4.5), and the boundedness of $U_{a^*\mu_f}$, we have

$$\limsup_{j\to\infty} \frac{1}{d^{n_j}} \log \sup_C [f^{n_j}, a]_{\mathrm{can}}(\cdot) \le \sup_C U_\nu.$$

If $C \subset \{U_\nu < 0\}$, then by the upper semicontinuity of U_ν, $\sup_C U_\nu < 0$. This completes the proof. ∎

Suppose now that K is non-archimedean and of characteristic zero. By Lemma 4.4, we can assume $\infty \notin \mathcal{E}_f$ without loss of generality.

Proof of Theorem 1.2 for non-archimedean K of characteristic zero. We will show that $U_\nu \ge 0$ on $\operatorname{supp} \nu$. Then by Lemma 2.15, (1.3′) will hold.

By the upper semicontinuity of U_ν, $\{U_\nu < 0\}$ is open. Let U be a component of $\{U_\nu < 0\}$. For every compact subset C in $\{U_\nu < 0\}$, $\sup_C U_\nu < 0$.

LEMMA 4.7. $a(U) \subset \mathcal{E}_f$.

Proof. Fix $z_0 \in U \cap \mathbb{P}^1$. By Lemma 4.6, there is a Berkovich open disk D relatively compact in U and containing z_0 such that

$$\lim_{j\to\infty} \sup_{\mathsf{D}} [f^{n_j}, a]_{\mathrm{can}}(\cdot) = 0, \tag{4.6}$$

and without loss of generality, we can assume that D is so small that $a(D)$ is a Berkovich open disc (see Fact 2.7). Fix a Berkovich open disk D' relatively compact in $a(\mathsf{D})$ and containing $a(z_0)$. Then by (4.6), for every $j \in \mathbb{N}$ large enough, $f^{n_j}(\mathsf{D})$ is a Berkovich open disk intersecting $a(\mathsf{D})$, and moreover containing D'. Hence, since $[f^{n_{j+1}-n_j}, \mathrm{Id}] \circ f^{n_j} = [f^{n_{j+1}}, f^{n_j}] \le [f^{n_{j+1}}, a](\cdot) + [f^{n_j}, a](\cdot)$ on $\mathbb{P}^1$, we have

$$\sup_{\mathsf{D}'\cap\mathbb{P}^1} [f^{n_{j+1}-n_j}, \mathrm{Id}] \le \sup_{\mathsf{D}\cap\mathbb{P}^1} [f^{n_{j+1}}, a](\cdot) + \sup_{\mathsf{D}\cap\mathbb{P}^1} [f^{n_j}, a](\cdot),$$

so by (4.6), $\limsup_{j\to\infty} \sup_{\mathsf{D}'\cap\mathbb{P}^1} [f^{n_{j+1}-n_j}, \mathrm{Id}] = 0$. This implies $a(U) \subset \mathcal{E}_f$. ∎

Let Y be the component of $\mathcal{E}_f$ containing $a(U)$. Let $p > 0$, $k_0 \in \mathbb{N}$, T, T_* be as in Theorem 4.5 associated to this Y.

For any Berkovich closed connected affinoid V in U, by Lemma 4.6, $\lim_{j\to\infty} \sup_V [f^{n_j}, a]_{\mathrm{can}}(\cdot) = 0$. Then there exists an $N \in \mathbb{N}$ such that for every $j \ge N$, the Berkovich closed connected affinoid $f^{n_j}(V)$ is contained in Y, and $k_0 \mid (n_j - n_N)$.

For every $j \geq N$, $f^{n_j} = T^{(n_j - n_N)/k_0} \circ f^{n_N}$ on $V \cap \mathbb{P}^1$. Taking a subsequence of (n_j) if necessary, the limit

$$\lim_{j\to\infty} \frac{n_j - n_N}{k_0} =: \omega_0$$

exists in $\mathbb{Z}_p$, and $a = \lim_{j\to\infty} f^{n_j} = \lim_{j\to\infty} T^{(n_j-n_N)/k_0} \circ f^{n_N} = T^{\omega_0} \circ f^{n_N}$ on $V \cap \mathbb{P}^1$. For every $j \geq N$,

$$f^{n_j} - a = (T^{(n_j-n_N)/k_0} - T^{\omega_0}) \circ f^{n_N} \tag{4.7}$$

on $V \cap \mathbb{P}^1$. Increasing N if necessary, we also have $(n_j - n_N)/k_0 \neq \omega_0$.

Let Z_* be the set of all zeros in the closed connected affinoid $f^{n_N}(V) \cap K$ of the non-constant holomorphic function $T_* \circ T^{\omega_0}$ on $Y \cap K$. Then $\# Z_* < \infty$ (see Fact 2.6). Hence $\# f^{-n_N}(Z_*) < \infty$, and we can assume that $f^{-n_N}(Z_*) \subset K$ without loss of generality.

Now we also assume that the Berkovich closed connected affinoid V is strict.

Lemma 4.8. $(\operatorname{supp} \nu) \cap ((\operatorname{int} V) \setminus f^{-n_N}(Z_*)) = \emptyset$.

Proof. For each $\epsilon > 0$ in $|K^*|$, set

$$V_\epsilon := V \setminus \bigcup_{w \in f^{-n_N}(Z_*)} \{\mathcal{S} \in \mathsf{P}^1 \setminus \{\infty\} : |\mathcal{S} - w| < \epsilon\},$$

which is a strict Berkovich closed connected affinoid. Then $f^{n_N}(V_\epsilon)$ is a strict Berkovich closed connected affinoid in Y. Hence by the maximum modulus principle, the minimum

$$\min\{|T_* \circ T^{\omega_0}(z)| : z \in f^{n_N}(V_\epsilon) \cap K\} > 0$$

exists (see Fact 2.7) and is positive by the choice of V_ϵ. Then from the uniform convergence (4.4) on $f^{n_N}(V_\epsilon) \cap K$, for every $j \in \mathbb{N}$ large enough,

$$|T^{(n_j-n_N)/k_0} - T^{\omega_0}| > 0$$

on $f^{n_N}(V_\epsilon) \cap K$, which together with (4.7) implies that there is no root of $f^{n_j} = a$ in $V_\epsilon \cap \mathbb{P}^1$. Hence $(\operatorname{supp} \nu) \cap \operatorname{int} V_\epsilon = \emptyset$, which implies that $(\operatorname{supp} \nu) \cap ((\operatorname{int} V) \setminus f^{-n_N}(Z_*)) = \emptyset$. ■

Lemma 4.9. $(\operatorname{supp} \nu) \cap ((\operatorname{int} V) \cap f^{-n_N}(Z_*)) = \emptyset$.

Proof. Let $z_0 \in (\operatorname{int} V) \cap f^{-n_N}(Z_*)$. If z_0 is a root of $f^{n_j} = a$, then by (4.7) and the uniform convergence (4.4) on V, the multiplicity of z_0 as a root of $f^{n_j} = a$ is bounded from above by

$$(\deg_{f^{n_N}(z_0)}(T_* \circ T^{\omega_0})) \cdot d^{n_N} - 1. \tag{4.8}$$

For any Berkovich open disk D in V containing z_0 and satisfying the condition $\overline{\mathsf{D}} \cap f^{-n_N}(Z_*) = \{z_0\}$, from the upper bound (4.8) and Lemma 4.8,

$$\begin{aligned}0 &\le \limsup_{j\to\infty} \nu^a_{n_j}(\mathsf{D}) \le \limsup_{j\to\infty} \nu^a_{n_j}(\{z_0\}) + \limsup_{j\to\infty} \nu^a_{n_j}(\mathsf{D} \setminus \{z_0\}) \\ &\le \limsup_{j\to\infty} \frac{(\deg_{f^{n_N}(z_0)}(T_* \circ T^{\omega_0})) \cdot d^{n_N}}{d^{n_j}} + \nu((\operatorname{int} V) \setminus f^{-n_N}(Z_*)) = 0.\end{aligned}$$

Hence $\nu(\mathsf{D}) = 0$ if D is small enough, so $z_0 \notin \operatorname{supp} \nu$. ∎

From Lemmas 4.8 and 4.9, $(\operatorname{int} V) \cap (\operatorname{supp} \nu) = \emptyset$. This implies that $U \cap (\operatorname{supp} \nu) = \emptyset$, so $\{U_\nu < 0\} \cap (\operatorname{supp} \nu) = \emptyset$. ∎

Now the proof of Theorem 1.2 is complete.

5. The case of polynomials. Let K be an algebraically closed field of any characteristic and complete with respect to a non-trivial absolute value.

For every polynomial $\phi \in K[z]$ on $\mathbb{P}^1$, the factorization of ϕ extends $|\phi|$ continuously to $\mathsf{P}^1 \setminus \{\infty\}$ using the extended $|\cdot - w|$ on $\mathsf{P}^1 \setminus \{\infty\}$ for each $w \in \mathbb{P}^1 \setminus \{\infty\}$. For polynomials $\phi_i \in K[z]$ $(i \in \{1, 2\})$, $\phi_1 - \phi_2$ is also a polynomial. Hence the continuous extension $\mathcal{S} \mapsto |\phi_1 - \phi_2|_{\mathrm{can}}(\mathcal{S})$ to $\mathsf{P}^1 \setminus \{\infty\}$ of the function $z \mapsto |\phi_1(z) - \phi_2(z)|$ on $\mathbb{P}^1 \setminus \{\infty\}$ exists so that on $\mathsf{P}^1 \setminus \{\infty\}$,

$$|\phi_1 - \phi_2|_{\mathrm{can}}(\cdot) = [\phi_1, \phi_2]_{\mathrm{can}}(\cdot) \max\{1, |\phi_1(\cdot)|\} \max\{1, |\phi_2(\cdot)|\}. \tag{5.1}$$

Let $f \in K[z]$ be a polynomial on $\mathbb{P}^1$ of degree $d > 1$. The *Berkovich filled-in Julia set* of f is

$$\mathsf{K}(f) := \Big\{\mathcal{S} \in \mathsf{P}^1 : \lim_{n\to\infty} f^n(\mathcal{S}) \neq \infty\Big\}.$$

Noting that $f(\infty) = \infty \in E(f)$, let $\mathsf{A}_\infty = \mathsf{A}_\infty(f)$ be the fixed immediate attractive basin of f containing ∞. Then $f^{-1}(\mathsf{A}_\infty) = \mathsf{A}_\infty$ since $\deg(f : \mathsf{A}_\infty \to \mathsf{A}_\infty) = \deg_\infty f = d$. Hence A_∞ is completely invariant under f, and $\mathsf{K}(f) = \mathsf{P}^1 \setminus \mathsf{A}_\infty$. Moreover, $\partial \mathsf{A}_\infty = \partial \mathsf{K}(f) = \mathsf{J}(f)$. Indeed, by Theorem 2.18, $\mathsf{J}(f) \subset \operatorname{supp} \mu_f$. Fix $\mathcal{S} \in \mathsf{A}_\infty \cap \mathbb{P}^1$. Then by Theorem 1.1, $\operatorname{supp} \mu_f \subset \bigcap_{N\in\mathbb{N}} \overline{\bigcup_{n\ge N} f^{-n}(\mathcal{S})}$, which is contained in $\partial \mathsf{A}_\infty \subset \mathsf{J}(f)$.

For each $R > 0$ in $|K^*|$, let $\mathsf{D}^*_R := \{\mathcal{S} \in \mathsf{P}^1 \setminus \{\infty\} : |\mathcal{S}| > R\}$ and $\mathsf{D}_R := \mathsf{D}^*_R \cup \{\infty\}$. If $R > 0$ is large enough, then since ∞ is a (super)attracting fixed point of f, we have $\inf_{z\in \mathsf{D}^*_R \cap \mathbb{P}^1} |f(z)| > R$. Hence by the continuity of $|f(\cdot)|$, $\inf_{\mathsf{D}^*_R} |f(\cdot)| > R$. This implies that $\mathsf{D}_R \Subset f^{-1}(\mathsf{D}_R)$. Since $\mathsf{A}_\infty = \bigcup_{n\in\mathbb{N}} f^{-n}(\mathsf{D}_R)$, for every Berkovich closed disk D in $\mathsf{A}_\infty \setminus \{\infty\}$ we have $\liminf_{n\to\infty} \inf_{\mathsf{D}} |f^n(\cdot)| > R$. Hence

$$\liminf_{n\to\infty} \inf_{\mathsf{D}} |f^n(\cdot)| = \infty. \tag{5.2}$$

Lemma 5.1. *Suppose that K is non-archimedean. For every polynomial $f \in K[z]$ on $\mathbb{P}^1$ of degree $d > 1$ and every polynomial $a \in K[z]$ on $\mathbb{P}^1$, the* *condition* (1.5) *holds on* $\mathsf{A}_\infty(f)\setminus\{\infty\}$, *and on* $\mathsf{K}(f)$,

$$\sup_{n\in\mathbb{N}}\left|\log{[f^n,a]_{\mathrm{can}}}(\cdot)-\log|f^n-a|_{\mathrm{can}}(\cdot)\right|<\infty.$$

In particular, (1.5) *holds on* $\mathsf{P}^1\setminus\mathbb{P}^1$ *if and only if the condition*

$$\lim_{j\to\infty}\frac{1}{d^{n_j}}\log|f^{n_j}-a|_{\mathrm{can}}(\cdot)=0 \tag{1.5$'$}$$

holds on $\mathsf{K}(f)\setminus\mathbb{P}^1$.

Proof. For every Berkovich closed disk D in $\mathsf{A}_\infty\setminus\{\infty\}$, fix an $R>0$ in $|K^*|$ so large that $R>\max\{1,\sup_{\mathsf{D}}|a(\cdot)|\}$. By (5.1) and (5.2), for every $n\in\mathbb{N}$ large enough, on $\mathsf{D}\cap\mathbb{P}^1$,

$$\log{[f^n(\cdot),a(\cdot)]}=\log|f^n(\cdot)|-\log|f^n(\cdot)|-\log\max\{1,|a(\cdot)|\}\geq-\log R.$$

Hence $\log{[f^n,a]_{\mathrm{can}}}(\cdot)\geq-\log R$ on D since both sides are continuous. This implies that (1.5) holds on $\mathsf{A}_\infty(f)\setminus\{\infty\}$.

Next, fix an $R>0$ in $|K^*|$ so large that $\mathsf{D}_R\subset\mathsf{A}_\infty$. Then $\bigcup_{n\in\mathbb{N}}f^n(\mathsf{K}(f))\subset\mathsf{P}^1\setminus\mathsf{D}_R$. Hence by (5.1),

$$\sup_{n\in\mathbb{N}}\left|\log{[f^n,a]_{\mathrm{can}}}(\cdot)-\log|f^n-a|_{\mathrm{can}}(\cdot)\right|\\ \leq\log\max\{1,R\}+\log\max\{1,|a(\cdot)|\}<\infty$$

on $\mathsf{K}(f)$. ∎

We conclude this section with an example. Suppose that K has characteristic $p>0$, and set $f(z)=z+z^p$ and $a=\mathrm{Id}$. Then $\mathsf{K}(f)=\{\mathcal{S}\in\mathsf{P}^1\setminus\{\infty\}:|\mathcal{S}|\leq1\}$. For each $j\in\mathbb{N}$, $f^{p^j}(z)=z+z^{p^{p^j}}$ and the equality

$$\log|f^{p^j}-\mathrm{Id}|_{\mathrm{can}}=p^{p^j}\log|\cdot|$$

holds on $\mathbb{P}^1\setminus\{\infty\}$. By the continuity of both sides, this extends to $\mathsf{P}^1\setminus\{\infty\}$. In particular, (1.5$'$) does not hold on $\mathsf{K}(f)\setminus\mathbb{P}^1$. Hence the equidistribution property (1.2) for $f(z)=z+z^p$ and $a=\mathrm{Id}$ does not hold.

Of course, this could be more directly seen since

$$\lim_{j\to\infty}\nu^a_{p^j}=\lim_{j\to\infty}\frac{1}{p^{p^j}+1}(p^{p^j}\delta_0+\delta_\infty)=\delta_0$$

weakly on P^1, but $\operatorname{supp}\mu_f=\mathsf{J}(f)=\partial\mathsf{K}(f)=\{\mathcal{S}_{\mathrm{can}}\}$.

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