ACTA ARITHMETICA
161.2 (2013)

Multiplicative relations on binary recurrences

by

FLORIAN LUCA (Mexico, DF) and VOLKER ZIEGLER (Linz)

1. Introduction. A *linear recurrence* is a sequence $\{f_n\}_{n\geq 0}$ such that for some $k \geq 1$ we have

$$f_{n+k} = r_1 f_{n+k-1} + \cdots + r_k f_n$$

for all $n \geq 0$, where $r_1, \ldots, r_k$ are given complex numbers with $r_k \neq 0$. When $r_1, \ldots, r_k$ are integers and $f_0, \ldots, f_{k-1}$ are also integers, f_n is an integer for all $n \geq 0$. It is known that if we write

$$F(X) = X^k - r_1 X^{k-1} - \cdots - r_k = \prod_{i=1}^{t} (X - \alpha_i)^{\sigma_i}, \tag{1.1}$$

where $\alpha_1, \ldots, \alpha_t$ are distinct complex numbers, and $\sigma_1, \ldots, \sigma_t$ are positive integers whose sum is k, then there exist polynomials $g_1(X), \ldots, g_t(X)$ whose coefficients are in $\mathbb{Q}(\alpha_1, \ldots, \alpha_t)$ such that $g_i(X)$ is of degree at most $\sigma_i - 1$ for $i = 1, \ldots, t$, and such that furthermore the formula

$$f_n = \sum_{i=1}^{t} g_i(n) \alpha_i^n \tag{1.2}$$

holds for all $n \geq 0$. We may certainly assume that $g_i(X)$ is not the zero polynomial for any $i = 1, \ldots, t$. A sequence $\{f_n\}_{n\geq 0}$ for which α_i/α_j is not a root of unity for any $1 \leq i < j \leq t$ is called *nondegenerate*. From now on, all sequences $\{f_n\}_{n\geq 0}$ that will appear are nondegenerate. Arithmetic and Diophantine properties of nondegenerate linear recurrences $\{f_n\}_{n\geq 0}$ have been intensively studied. In particular, there is a rich literature on Diophantine equations involving terms of nondegenerate linear recurrences, and we shall sample a few such results below.

For instance, the equation $f_n = 0$ has been studied by several mathematicians. A famous theorem due to Skolem, Mahler and Lech [25, 14, 9]

2010 *Mathematics Subject Classification*: 11B37, 11D57, 11D75, 11J25.
Key words and phrases: binary recurrences, Diophantine equations, Diophantine inequalities.

DOI: 10.4064/aa161-2-4

asserts that the above equation has only finitely many solutions n. The number of such solutions is called the *zero-multiplicity* of the sequence $\{f_n\}_{n\geq 0}$. More generally, equations of the type $Af_n = Bf_m$ and $Af_n+Bf_m+Cf_k = 0$ with fixed nonzero coefficients A, B, C were investigated by Laurent [8] and Schlickewei and Schmidt [21]. Furthermore, given two nondegenerate linear recurrences $\{f_n\}_{n\geq 0}$ and $\{g_n\}_{n\geq 0}$, equations of the type $Af_n = Bg_m$ were studied by several authors, such as Schlickewei and Schmidt in [22].

Effective results were established for the equations $f_n = 0$, or $Af_n = Bg_m$ in case both $\{f_n\}_{n\geq 0}$ and $\{g_n\}_{n\geq 0}$ have dominant roots, which means that each has a root of multiplicity 1 whose absolute value is strictly larger than the absolute values of all other roots. See, for example, Mignotte [16, 17], Lewis and Turk [10] or Shorey [23] and many others for effective results on Diophantine equations with linear recurrences of low order.

For a full account on linear equations in terms of recurrence sequences we refer the reader to the book of Everest et al. [6].

Not only linear equations in terms of nondegenerate linear recurrences were studied, but also multiplicative equations in terms of such recurrences were considered (see, for example, [20]). Recently, Bérczes and Ziegler [1] looked at geometric progressions on Lucas sequences; that is, they studied the equation $u_n u_m u_k^{-2} = 1$, where $\{u_n\}_{n\geq 0}$ is a Lucas sequence, which is a linear recurrence of order two with $u_0 = 0$ and $u_1 = 1$.

In the present paper, we generalize the results of Bérczes and Ziegler [1] by considering general binary recurrences as well as more general multiplicative relations between terms of such sequences.

For the rest of the paper, we are concerned with binary recurrences, namely sequences $\{u_n\}_{n\geq 0}$ which satisfy the recurrence

$$u_{n+2} = r_1 u_{n+1} + r_2 u_n \tag{1.3}$$

for all $n \geq 0$, where $r_1, r_2 \neq 0$, and u_0, u_1 are fixed integers not both zero. We assume that $r_1^2 + 4r_2 \neq 0$, and therefore in formula (1.1) we have $t = 2$. Thus, formula (1.2) becomes

$$u_n = c\alpha^n + d\beta^n, \quad \text{where} \quad (c, d) = \left(\frac{u_1 - \beta u_0}{\alpha - \beta}, \frac{\alpha u_0 - u_1}{\alpha - \beta}\right), \tag{1.4}$$

and where α and β are the roots of the characteristic polynomial

$$F(x) = X^2 - r_1 X - r_2.$$

We shall assume that $|\alpha| \geq |\beta| > 0$ and $cd \neq 0$.

In order to formulate our main theorem, we need to define some parameters associated to our sequence $\{u_n\}_{n\geq 0}$. Let $\mathbb{K} := \mathbb{Q}(\alpha)$, and put $D := [\mathbb{K} : \mathbb{Q}] \in \{1, 2\}$. Let p denote a prime and $\mathbb{P}$ be the set of all prime numbers. Further, put

$$S := \{p \in \mathbb{P} : p \,|\, r_2(r_1^2 + 4r_2)(u_1^2 - r_1u_1u_0 - r_2u_0^2)\}.$$

Observe that $S \neq \emptyset$ because $r_2(r_1^2 + 4r_2) \neq \pm 1$ and

$$u_1^2 - r_1u_1u_0 - r_2u_0^2 = (u_1 - \alpha u_0)(u_1 - \beta u_0) = -cd(\alpha - \beta)^2$$

is a nonzero integer. Further, S is finite. The set S has the following interpretation. For a prime number p and a nonzero rational number x let $\nu_p(x)$ be the exponent of p in the prime factorization of x. Put $|x|_p = p^{-\nu_p(x)}$ if $x \neq 0$ and $|0|_p = 0$ for the standard p-adic valuation on $\mathbb{Q}$. Extend $|x|_p$ multiplicatively in some way to all algebraic numbers x. Then $|x|_p = 1$ for all $x \in \{\alpha, \beta, c, d\}$ provided that $p \notin S$. In case $\mathbb{K} = \mathbb{Q}$, the set S can also be thought of as containing all the primes dividing either the numerator or the denominator of one of α, β, c, d. Note that the denominator of c and d divides $|\alpha - \beta| = \sqrt{r_1^2 + 4r_2}$. In the case when $\mathbb{K}$ is quadratic, S can be thought of as containing the rational primes p sitting above prime ideals $\mathfrak{p}$ of $\mathcal{O}_{\mathbb{K}}$ which appear with nonzero (positive or negative) exponents in the factorization of one of the four principal ideals in $\mathbb{K}$ generated by α, β, c or d, respectively.

Note that the ideal $(\alpha - \beta)$ contains all prime ideals with negative exponent in the prime ideal factorization of (c) and (d). Further, note that $|\mathrm{N}_{\mathbb{K}/\mathbb{Q}}(\alpha - \beta)| = |r_1^2 + 4r_2|$. For each $p \in S$, we put

$$o_p := \nu_p(r_2) + \nu_p(r_1^2 + 4r_2) + \nu_p(u_1^2 - r_1u_1u_0 - r_2u_0^2).$$

The quantity o_p has the following meaning. It is an upper bound for the p-adic valuation of the product of α, β and the denominators and numerators of c and d respectively, provided $\mathbb{K} = \mathbb{Q}$. In the case when $\mathbb{K}$ is a quadratic number field, o_p is an upper bound for the sum of the $\mathfrak{p}$-exponents of the prime ideal factorizations of the principal ideals generated by α, β and the denominators and numerators of c and d respectively, where $\mathfrak{p}$ is a prime of $\mathbb{K}$ below some $p \in S$.

Moreover, let

$$P := \max\{p : p \in S\} \quad \text{and} \quad O := \max\{o_p : p \in S\}.$$

For an algebraic number γ with minimal polynomial

$$f(X) := a_0 \prod_{i=1}^{d} (X - \gamma^{(i)}) \in \mathbb{Z}[X],$$

where as usual we write $\gamma^{(1)}, \ldots, \gamma^{(d)}$ for all the conjugates of γ, let

$$h(\gamma) := \frac{1}{d}\Big(\log a_0 + \sum_{i=1}^{t} \log \max\{|\gamma^{(i)}|, 1\}\Big)$$

be its regular Weil height. We define

$$B_1 := \max\{2h(\alpha/\beta), 2\big|\log|\alpha|\big|, 2\big|\log|\beta|\big|, \pi, \log P\},$$
$$B_2 := \max\{2h(c/d), h(c/d)+h(\alpha/\beta), \big|\log|c|\big|, \big|\log|d|\big|, \pi, \log P\}.$$

Our main result is the following.

THEOREM 1.1. *Let $\{u_n\}_{n\geq 0}$ be the nondegenerate binary sequence satisfying the recurrence* (1.3) *and whose general term is given by* (1.4). *Assume further that α/β and c/d are multiplicatively independent and $|\alpha|\geq|\beta|>0$. Moreover, let $r=\gcd(r_1^2,r_2)$. Then the Diophantine equation*

$$u_{n_1}^{x_1}\cdots u_{n_L}^{x_L}=1 \tag{1.5}$$

in nonnegative integer unknowns

$$n_1,\dots,n_L, \quad n_i\neq n_j \quad \text{for } 1\leq i<j\leq L, \quad \text{and} \quad \max_{1\leq i\leq L}|x_i|\leq K,$$

where K is a given parameter, has only finitely many solutions $(n_1,\dots,n_L)$.

Moreover, all solutions can be determined effectively. Let us put

$$X := \max\{n_1,\dots,n_L\}. \tag{1.6}$$

In the case of $|\alpha|=|\beta|$, we have

$$X \leq \max\left\{C, \left(\frac{2KL\log 2 + B_2 + 4KL|S|O\log P}{KL(B_1+B_2)}\right)^2, \frac{P^{10}}{1.4\log P}, 10^{16}, 2O+1\right\}, \tag{1.7}$$

where

$$C = \left(\frac{3KL(B_1+B_2)+4.94\cdot 10^5 B_1B_2 + \frac{0.084KL|S|B_1B_2(P^2+P)}{(\log P)^3}}{\log|\alpha|-\frac{1}{2}\log r}\right)^2. \tag{1.8}$$

In the case of $|\alpha|>|\beta|$, we have the bound

$$X \leq \max\left\{C, \left(\frac{(2KL+1)\log 2 + 4KL|S|O\log P}{KL(B_1+B_2)}\right)^2, \frac{P^{10}}{1.4\log P}, 10^8, 2O+1, \frac{\log 2+\log|d/c|}{\log|\alpha|-\log|\beta|}\right\} \tag{1.9}$$

with

$$C = \left(KL\frac{3(B_1+B_2)+209|S|B_1B_2(P^2+P)/(\log P)^3}{\log|\alpha|-\frac{1}{2}\log r}\right)^2. \tag{1.10}$$

Let us note that always $|\alpha|^2>|r|$. In the case of $|\alpha|>|\beta|$ this is obvious since $|\alpha|^2>|\alpha\beta|=|r_2|\geq|r|$. Therefore, we are left with the case $|\alpha|=|\beta|$. In this case α and β are complex conjugates since otherwise we would have $\alpha/\beta=\pm 1$, which is excluded. Hence, $r_2<0$ and $r_1^2<4|r_2|$. Assuming

$|\alpha^2| = |r|$ yields $|\alpha|^2 = |r_2| = \gcd(r_1^2, r_2)$, i.e., $r_2 \mid r_1^2$. Therefore, $r_2 = -r_1^2$, $r_2 = -r_1^2/2$, or $r_2 = -r_1^2/3$. In all three cases, α/β is a root of unity, which contradicts our assumptions.

We demonstrate the strength of our method by solving the following Diophantine equation.

THEOREM 1.2. *Let $u_n = 2^n + 3$ for all $n \geq 0$. Then the Diophantine equation*

$$u_{n_1} u_{n_2}^2 u_{n_3}^3 u_{n_4}^{-4} u_{n_5}^{-5} = 1 \tag{1.11}$$

has exactly two solutions, namely

$$(n_1, n_2, n_3, n_4, n_5) = (1, 2, 5, 1, 2), (2, 1, 5, 2, 1).$$

In the next section, we prove several results on arithmetic properties of binary recurrences as well as their growth rate. In Section 3, we use these results to prove Theorem 1.1, and in Section 4 we prove Theorem 1.2. The last section is devoted to comments and open problems.

2. Auxillary results on binary recurrences. Since the proof of Theorem 1.1 depends on various arithmetic properties of binary recurrences, we collect some lemmas concerning this topic. We start with the following lemma (cf. [24, Lemma A.10]).

LEMMA 2.1. *Let $r = \gcd(r_1^2, r_2)$ and put $\alpha' = \alpha^2/r$ and $\beta' = \beta^2/r$. Then α' and β' are algebraic integers and the principal ideals (α') and (β') are coprime in the ring of integers of $\mathbb{K}$.*

In order to keep track of the p-adic valuation of u_n for $p \in S$, and in view of Lemma 2.1, we introduce the following new sequence $\{w_n\}_{n\geq 0}$. As in Lemma 2.1, let $r = \gcd(r_1^2, r_2)$, $\alpha' = \alpha^2/r$, $\beta' = \beta^2/r$. Then we write

$$u_n = w_n r^{\lfloor n/2 \rfloor},$$

where

$$w_n = \begin{cases} c\alpha'^{n/2} + d\beta'^{n/2} & \text{if } n \text{ is even,} \\ c\alpha\alpha'^{(n-1)/2} + d\beta\beta'^{(n-1)/2} & \text{if } n \text{ is odd.} \end{cases}$$

With this notation, we start by computing the p-adic valuation of the binary recurrence $\{w_n\}_{n\geq 0}$.

LEMMA 2.2. *Let $p \in S$. Then either*

$$\nu_p(w_n) \leq 768 \frac{p^2+p}{(\log p)^4} \log A_1 \log A_2 (\log n + \log\log p + 0.4)^2 + 2o_p, \tag{2.1}$$

or

$$n \leq \max\left\{ \frac{p^{10}}{1.4 \log p}, e^{10}, 2o_p + 1 \right\}, \tag{2.2}$$

where

$$\begin{aligned} \log A_1 &:= \max\{2h(\alpha/\beta), \log p\}, \\ \log A_2 &:= \max\{h(c/d) + h(\alpha/\beta), \log p\}. \end{aligned} \tag{2.3}$$

In order to prove Lemma 2.2, one may use Yu's theorem on linear forms in p-adic logarithms [26] obtaining a result of the form

$$\nu_p(u_n) \leq K(p, \alpha, \beta, c, d) \log n,$$

where the constant $K := K(p, \alpha, \beta, c, d)$ is rather large. To obtain a smaller constant K but at a cost of working with a factor of $(\log n)^2$ instead of $\log n$, we use the following result due to Bugeaud and Laurent [4]. In what follows, for a nonzero algebraic number γ in an algebraic number field $\mathbb{K}$ and a prime ideal $\mathfrak{p}$ of $\mathbb{K}$, we write $\nu_{\mathfrak{p}}(\gamma)$ for the exponent of $\mathfrak{p}$ in the factorization in prime ideals of the fractional principal ideal generated by γ in $\mathbb{K}$.

THEOREM 2.3 (Bugeaud and Laurent [4]). *Let α_1 and α_2 be two multiplicatively independent algebraic numbers such that $\nu_{\mathfrak{p}}(\alpha_i) = 0$. Let D_p denote the quotient $[\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}]$ and the residue class degree f of the extension $\mathbb{Q}_p(\alpha_1, \alpha_2)/\mathbb{Q}_p$. Further, let A_1, A_2 be real numbers > 1 with $\log A_i \geq \max\{h(\alpha_i), (\log p)/D_p\}$ for $i = 1, 2$. Put*

$$b' := \frac{b_1}{D_p \log A_2} + \frac{b_2}{D_p \log A_1}. \tag{2.4}$$

Then

$$\nu_{\mathfrak{p}}(\alpha_1^{b_1} \alpha_2^{b_2} + 1) \leq \frac{24p(p^f - 1)D_p^4}{(p-1)(\log p)^4} B^2 \log A_1 \log A_2, \tag{2.5}$$

with

$$B := \max\left\{\log b' + \log\log p + 0.4, 10, \frac{10 \log p}{D_p}\right\}. \tag{2.6}$$

In [13], Lemma 2.2 was proved, but without computing the constants explicitly. To our knowledge, the constants in [13] have not been computed yet.

Proof of Lemma 2.2. Let $\mathfrak{p}$ be a prime lying above $p \in S$. First, we note that, by Lemma 2.1, we may assume that either $\mathfrak{p} \nmid \alpha'$, or $\mathfrak{p} \nmid \beta'$.

In the case of $\mathfrak{p} \mid \alpha'$, but $\mathfrak{p} \nmid \beta'$, we immediately see that if n is even we have $\nu_{\mathfrak{p}}(d\beta'^{n/2}) < \nu_{\mathfrak{p}}(c\alpha'^{n/2})$ if $n > 2o_p$; hence, $\nu_{\mathfrak{p}}(w_n) = \nu_{\mathfrak{p}}(d\beta'^{n/2}) \leq o_p$. In the case of n odd we have $\nu_{\mathfrak{p}}(d\beta\beta'^{(n-1)/2}) < \nu_{\mathfrak{p}}(c\alpha\alpha'^{(n-1)/2})$ provided $n > 2o_p + 1$ and again $\nu_{\mathfrak{p}}(w_n) \leq o_p$. The case when $\mathfrak{p} \nmid \alpha$ and $\mathfrak{p} \mid \beta$ is similar.

Let us now consider the case $\mathfrak{p} \nmid \alpha'$ and $\mathfrak{p} \nmid \beta'$. If neither $\nu_{\mathfrak{p}}(c) = \nu_{\mathfrak{p}}(d)$ nor $\nu_{\mathfrak{p}}(c\alpha) = \nu_{\mathfrak{p}}(d\beta)$, we have $\nu_p(w_n) \leq 2o_p$. Therefore, we are left with either $\nu_{\mathfrak{p}}(c) = \nu_{\mathfrak{p}}(d)$ and n even, or $\nu_{\mathfrak{p}}(c\alpha) = \nu_{\mathfrak{p}}(d\beta)$ and n odd. In the first case we consider the expression

$$\frac{c}{d}\left(\frac{\alpha'}{\beta'}\right)^{n/2}+1,$$

and apply Theorem 2.3 to it. Obviously, $D_p \leq 2$ and $b' \leq n$. Assuming that

$$n \geq \max\left\{\frac{p^{10}}{1.48\log p}, e^{10}\right\},$$

we get

$$\max\{\log b' + \log\log p + 0.4, 10, 10\log p\} \leq \log n + \log\log p + 0.4.$$

Hence, by Theorem 2.3, we have

$$\nu_{\mathfrak{p}}\left(\frac{c}{d}\left(\frac{\alpha'}{\beta'}\right)^{n/2}+1\right) \leq 384\frac{p^2+p}{(\log p)^4}\log A_1 \log A_2(\log n + \log\log p + 0.4)^2,$$

with A_1 and A_2 given by (2.3), and therefore

$$\nu_p(w_n) \leq 768\frac{p^2+p}{(\log p)^4}\log A_1 \log A_2(\log n + \log\log p + 0.4)^2 + 2o_p,$$

which proves the lemma in this case.

The proof for the case $\nu_{\mathfrak{p}}(c\alpha) = \nu_{\mathfrak{p}}(d\beta)$ and n odd is similar and is therefore omitted. Note that

$$h(\alpha'/\beta') = h(\alpha^2/\beta^2) = 2h(\alpha/\beta)$$

and

$$\max\{h(c/d), h(\alpha c/\beta d)\} \leq h(c/d) + h(\alpha/\beta). \blacksquare$$

For nonnegative integers m and n, let $D_{m,n}$ denote the greatest common divisor of w_m and w_n which is free of primes $p \in S$. The following lemma already appears in several papers of the first author (see e.g. [11, 12, 13]) in some similar form. However, for completeness we include a proof.

Lemma 2.4. *Let $m \neq n$ with $\max\{m,n\} \geq 3$ and assume $|\alpha| \geq |\beta|$. We then have*

$$D_{m,n} < 4\exp\Big(\big(4\log|\alpha| + 2\max\{|\log|c||, |\log|d||\}\big)\sqrt{\max\{m,n\}}\Big).$$

Proof. First, we observe that $u_m \equiv w_m \equiv u_n \equiv w_n \equiv 0 \pmod{D_{m,n}}$, hence

$$u_m = c\alpha^m + d\beta^m \equiv 0 \pmod{D_{m,n}},$$
$$u_n = c\alpha^n + d\beta^n \equiv 0 \pmod{D_{m,n}}.$$

Since by definition $D_{m,n}$ and $c\beta$ are coprime, we get

$$\left(\frac{\alpha}{\beta}\right)^m \equiv -\frac{d}{c} \pmod{D_{n,m}}, \tag{2.7}$$

$$\left(\frac{\alpha}{\beta}\right)^n \equiv -\frac{d}{c} \pmod{D_{n,m}}. \tag{2.8}$$

Next, we claim the following:

Claim 2.5. *Let m, n be nonnegative integers not both zero and suppose that $X \geq \max\{3, m, n\}$ is an integer. Then there exist integers $(u, v) \neq (0, 0)$ such that $\max\{|u|, |v|\} \leq \sqrt{X}$ and $0 \leq mu + nv \leq 2\sqrt{X}$.*

Let (u, v) be the pair of integers satisfying the above claim. Raising congruence (2.7) to the uth power and congruence (2.8) to the vth power and multiplying the resulting expressions, we obtain

$$\left(\frac{\alpha}{\beta}\right)^{mu+nv} - (-1)^{u+v}\left(\frac{d}{c}\right)^{u+v} \equiv 0 \pmod{D_{m,n}}.$$

The expression on the left is not zero. Indeed, if it were, we would get $(\alpha/\beta)^{2(um+vn)} = (d/c)^{2(u+v)}$, and since α/β and c/d are multiplicatively independent, we would get $u + v = 0$ and $mu + nv = 0$; so $(m - n)u = 0$. Since $m \neq n$, we get $u = 0$ and then $v = 0$, which is impossible since $(u, v) \neq (0, 0)$. Put

$$c_1 := (\alpha - \beta)c \quad \text{and} \quad d_1 := (\alpha - \beta)d.$$

The above relation implies that $\alpha^{mu+nv}c_1^{u+v} - (-1)^{u+v}\beta^{mu+nv}d_1^{u+v}$ is a nonzero algebraic integer which is a multiple of $D_{m,n}$. Thus,

$$\alpha^{mu+nv}c_1^{u+v} - (-1)^{u+v}\beta^{mu+nv}d_1^{u+v} = D_{m,n}\gamma$$

for some algebraic integer γ. Taking norms in $\mathbb{K}$, we get

$$|N_{\mathbb{K}/\mathbb{Q}}(\alpha^{mu+nv}c_1^{u+v} \pm \beta^{mu+nv}d_1^{u+v})| = D_{m,n}^D |N_{\mathbb{K}/\mathbb{Q}}(\gamma)| \geq D_{m,n}^D.$$

Since either $D = 1$, or $D = 2$ in which case the conjugates of α, β, c, d are β, α, d, c, respectively, we easily see that

$$D_{m,n} \leq 2\exp\big((2\log|\alpha| + 2\max\{\log|c_1|, \log|d_1|\})\sqrt{X}\big). \tag{2.9}$$

Since

$$\max\{\log|c_1|, \log|d_1|\} = \log|\alpha - \beta| + \max\{\log|c|, \log|d|\},$$

and $|\alpha - \beta| \leq 2|\alpha|$, we get the desired conclusion. ∎

It remains to prove the claim.

Proof of Claim 2.5. Let $\tilde{u}$ and $\tilde{v}$ be integers with $0 \leq \tilde{u}, \tilde{v} \leq \lfloor\sqrt{X}\rfloor$. Then $\tilde{u}m + \tilde{v}n$ belongs to $[0, 2X^{3/2}]$. Since there are $(\lfloor\sqrt{X}\rfloor + 1)^2 > X$ pairs $(\tilde{u}, \tilde{v})$, by Dirichlet's box principle there exist two such pairs $(\tilde{u}_1, \tilde{v}_1) \neq (\tilde{u}_2, \tilde{v}_2)$ with

$$|(\tilde{u}_1 m + \tilde{v}_1 n) - (\tilde{u}_2 m + \tilde{v}_2 n)| \leq \frac{2X^{3/2}}{X} = 2\sqrt{X}.$$

Putting $(u, v) = (\tilde{u}_1 - \tilde{u}_2, \tilde{v}_1 - \tilde{v}_2)$, we get the desired conclusion. ∎

Although the next lemma has already been proved by Kiss [7], and improved by Phong [18], we reprove it and state bounds suitable for our purposes.

LEMMA 2.6. *Assume* $|\alpha| = |\beta|$. *Then*

$$|u_n| \geq |c|\, |\alpha|^n n^{-\tilde{c}}$$

with

$$\tilde{c} = 1.34 \cdot 10^{12} \max\{2h(\beta/\alpha), \pi\} \max\{2h(d/c), \pi\}$$

provided $n \geq 10^6$. *If* $|\alpha| > |\beta|$, *then*

$$|u_n| \geq \frac{|c|}{2} |\alpha|^n$$

provided

$$n > \frac{\log 2 + \log |d/c|}{\log |\alpha/\beta|}.$$

The following theorem due to Matveev [15] is useful.

THEOREM 2.7 (Matveev [15]). *Denote by* $\alpha_1, \ldots, \alpha_n$ *algebraic numbers, neither* 0 *nor* 1, *by* $\log \alpha_1, \ldots, \log \alpha_n$ *some fixed determination of their logarithms, by* D *the degree over* $\mathbb{Q}$ *of the number field* $\mathbb{K} := \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$, *and by* $b_1, \ldots, b_n$ *rational integers. Furthermore, let* $\kappa = 1$ *if* $\mathbb{K}$ *is real and* $\kappa = 2$ *otherwise. Choose*

$$A_i \geq \max\{Dh(\alpha_i), |\log \alpha_i|\} \qquad (1 \leq i \leq n)$$

and

$$B := \max\{1, \max\{|b_j| A_j / A_n : 1 \leq j \leq n\}\}.$$

Assume that $b_n \neq 0$ *and* $\log \alpha_1, \ldots, \log \alpha_n$ *are linearly independent over* $\mathbb{Z}$. *Then*

$$\log |b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n| \geq -C(n) C_0 W_0 D^2 \Omega$$

with

$$\begin{aligned} \Omega &= A_1 \cdots A_n, \\ C(n) = C(n, \kappa) &= \frac{16}{n!\kappa} e^n (2n + 1 + 2\kappa)(n + 2)(4(n + 1))^{n+1} \left(\frac{1}{2} en\right)^{\kappa}, \\ C_0 = \log(e^{4.4n+7} n^{5.5} D^2 \log(eD)), &\qquad W_0 = \log(1.5eBD \log(eD)). \end{aligned}$$

Proof of Lemma 2.6. First, let us assume that $|\alpha| > |\beta|$. In this case, we have

$$|u_n| \geq |c|\,|\alpha|^n \left(1 - \frac{|d|}{|c|}\left(\frac{|\beta|}{|\alpha|}\right)^n\right) > \frac{|c|}{2}|\alpha|^n$$

provided

$$n > \frac{\log 2 + \log|d/c|}{\log|\alpha/\beta|}.$$

Therefore, we are left with the case $|\alpha| = |\beta|$. Recall that if $|\alpha| = |\beta|$, then α and β are complex conjugates. Hence, so are c and d, which implies $|c| = |d|$. Therefore, we have

$$|u_n| \geq |c|\,|\alpha|^n |1 + (d/c)\theta^n|,$$

where $\theta = \beta/\alpha$ is of modulus 1 and not a root of unity. Let us write

$$\Lambda := \log(d/c) + n\log\theta + ki\pi.$$

We use Matveev's Theorem 2.7 and choose the parameters

$$A_1 := \max\{2h(\beta/\alpha), \pi\}, \qquad A_2 := \max\{2h(d/c), \pi\}, \qquad A_3 := \pi.$$

Therefore $B \leq n + 1 \leq 2n$, and we obtain

$$\log|\Lambda| > -1.14 \cdot 10^{12}(\log n + 2.32)\max\{2h(\beta/\alpha), \pi\}\max\{2h(d/c), \pi\}.$$

Let us assume for the moment that $|\Lambda| < 1/3$. Then

$$|1 + (c/d)\theta^n| = |1 \pm e^{\Lambda}| > \left|1 - 1 + |\Lambda| - \frac{|\Lambda|^2}{2}\right| > \frac{5|\Lambda|}{6}.$$

Hence,

$$|1 + (d/c)\theta^n| > n^{-1.34\cdot 10^{12}\max\{2h(\beta/\alpha),\pi\}\max\{2h(d/c),\pi\}}.$$

Note that $\log n + 2.32 < 1.168\log n$ and $5/6 > n^{-1}$ if $n \geq 10^6$.

In the case $|\Lambda| \geq 1/3$, we have

$$|1 + (c/d)\theta^n| = |1 \pm e^{\Lambda}| > 1 - e^{-1/3} > 0.27 > n^{-1}$$

provided $n \geq 10^6$. ∎

3. The general case. This section is devoted to the proof of Theorem 1.1. We will make the following assumptions. If $|\alpha| = |\beta|$, then

$$X := \max\{n_1, \ldots, n_L\} > \max\left\{\frac{P^{10}}{1.4\log P}, 10^6, 2O + 1\right\}.$$

If $|\alpha| > |\beta|$, then

$$X := \max\{n_1, \ldots, n_L\} > \max\left\{\frac{P^{10}}{1.4\log P}, 10^6, 2O + 1, \frac{\log 2 + \log|d/c|}{\log|\alpha| - \log|\beta|}\right\}.$$

Considering equation (1.5) in terms of w_n we obtain

$$w_{n_1}^{x_1}\cdots w_{n_L}^{x_L} r^z = 1 \quad \text{with} \quad z=\sum_{i=1}^{L} x_i\lfloor n_i/2\rfloor. \tag{3.1}$$

In a first step, we want to estimate r^z. Since $r=\gcd(r_1^2,r_2)$ is composed of primes from S, we obtain, by Lemma 2.2 and (3.1),

$$\begin{aligned} r^{|z|} &\le \prod_{i=1}^{L}\prod_{\substack{p\in S\\ p\,|\,r}} 4p^{\nu_p(w_{n_i}^{|x_i|})}\\ &\le \exp\biggl(KL|S|\biggl(768\frac{P^2+P}{(\log P)^3}B_1B_2(\log n+\log\log p+0.4)^2+2O\log P\biggr)\biggr). \end{aligned}$$

Since $w_{n_i}^{|x_i|}$ divides $r^{|z|}\prod_{1\le j\le L,\, i\ne j} w_{n_j}^{|x_j|}$ we have

$$w_{n_i}\le r^{|z|}\prod_{\substack{1\le j\le L\\ i\ne j}}\gcd(w_{n_i},w_{n_j})^K \le r^{|z|}\prod_{\substack{1\le j\le L\\ i\ne j}}\Bigl(D_{n_i,n_j}\prod_{p\in S}p^{\nu_p(w_{n_i})}\Bigr)^K$$

for any $i=1,\dots,L$. Combining this with the upper bound for $r^{|z|}$ and with Lemmas 2.2 and 2.4, we get

$$\begin{aligned} w_{n_i} \le{}& 4^{KL}\exp(2KL(B_1+B_2)\sqrt{X})\\ &\times\exp\biggl(KL|S|\biggl(1536\biggl(\frac{P^2+P}{(\log P)^3}B_1B_2(\log X+\log\log P+0.4)^2\biggr)\\ &+4O\log P\biggr)\biggr). \end{aligned} \tag{3.2}$$

Let us consider the case $|\alpha|=|\beta|$ first. Comparing the bound (3.2) with Lemma 2.6, we get an upper bound for X. In particular, note that $w_n\ge u_n r^{-n/2}$, which yields

$$\begin{aligned} &\log|c|+X\bigl(\log|\alpha|-\tfrac12\log r\bigr)-\log X(1.34\cdot10^{12}B_1B_2)<2KL\log 2\\ &\quad+2KL(B_1+B_2)\sqrt{X}+KL|S|\biggl(6144B_1B_2\frac{P^2+P}{(\log P)^3}(\log X)^2+4O\log P\biggr), \end{aligned}$$

since we are assuming that $X>\frac{P^{10}}{1.4\log P}$, and that yields

$$\log X>\log\log P+0.4.$$

Let us assume X is large, in particular $X>10^{16}$, and let us additionally assume that

$$\sqrt{X} > \frac{2KL\log 2 + B_2 + 4KL|S|O\log P}{KL(B_1+B_2)}$$
$$> \frac{2KL\log 2 + \big|\log|c|\big| + 4KL|S|O\log P}{KL(B_1+B_2)}.$$

Then we have

$$X\big(\log|\alpha| - \tfrac{1}{2}\log r\big) - \log X\cdot(1.34\cdot 10^{12}B_1B_2)$$
$$< 3KL(B_1+B_2)\sqrt{X} + 6144KL|S|B_1B_2\frac{P^2+P}{(\log P)^3}(\log X)^2.$$

Since the function

$$X \mapsto \frac{(1.34\cdot 10^{12}B_1B_2)\log X + 6144KL|S|B_1B_2\frac{P^2+P}{(\log P)^3}(\log X)^2}{\sqrt{X}}$$

is decreasing for $X > 10^{16}$, we obtain

$$X\big(\log|\alpha| - \tfrac{1}{2}\log r\big)$$
$$< \sqrt{X}\left(3KL(B_1+B_2) + 4.94\cdot 10^5 B_1B_2 + 0.084KL|S|B_1B_2\frac{P^2+P}{(\log P)^3}\right);$$

that is,

$$X < \left(\frac{3KL(B_1+B_2) + 4.94\cdot 10^5B_1B_2 + 0.084KL|S|B_1B_2\frac{P^2+P}{(\log P)^3}}{\log|\alpha| - \frac{1}{2}\log r}\right)^2.$$

In the case when $|\alpha| > |\beta|$, we assume that $X > 10^8$. By the second half of Lemma 2.6 we get

$$X\big(\log|\alpha| - \tfrac{1}{2}\log r\big) - \log 2 < 2KL\log 2 + 2KL(B_1+B_2)\sqrt{X}$$
$$+ KL|S|\left(6144\left(B_1B_2\frac{P^2+P}{(\log P)^3}(\log X)^2 + 4O\log P\right)\right).$$

If we also assume that

$$\sqrt{X} > \frac{(2KL+1)\log 2 + 4KL|S|O\log P}{KL(B_1+B_2)},$$

then

$$X\big(\log|\alpha| - \tfrac{1}{2}\log r\big)$$
$$< 3KL(B_1+B_2)\sqrt{X} + 6144KL|S|B_1B_2\frac{P^2+P}{(\log P)^3}(\log X)^2.$$

Since the function

$$X \mapsto \frac{(\log X)^2}{\sqrt{X}}$$

is decreasing for $X > 10^8$, we obtain

$$X < \left(KL\frac{3(B_1+B_2)+209|S|B_1B_2(P^2+P)/(\log P)^3}{\log|\alpha|-\frac{1}{2}\log r}\right)^2.$$

This finishes the proof of Theorem 1.1. ∎

4. An example. This section is devoted to the resolution of the Diophantine equation (1.11). First, let us note that the sequence $\{u_n\}_{n\geq 0}$ is strictly increasing and therefore $u_{n_i} = u_{n_j}$ if and only if $n_i = n_j$. By canceling eventually equal terms u_{n_i} and u_{n_j} we arrive at a Diophantine equation of the form

$$u_{n_1}^{x_1}u_{n_2}^{x_2}u_{n_3}^{x_3} = u_{n_4}^{x_4}u_{n_5}^{x_5},$$

with $0 \leq x_i \leq i$ for $i = 1,2,3,4,5$. Moreover, since a cancellation can only occur if $n_i = n_j$, the occurrence of an exponent $x_i = 0$ implies that there exists an index $j \in \{1,2,3,4,5\}$ with $j \neq i$ such that $n_i = n_j$. The only possibilities that $x_i = x_j = 0$ are when $n_1 = n_3 = n_4$ or $n_2 = n_3 = n_5$. Hence, we obtain the two equations $u_x^2 = u_y^5$ and $u_x = u_y^4$ respectively. But neither equation has a solution, as is shown in the next lemma.

LEMMA 4.1. *The equations $u_x^2 = u_y^5$ and $u_x = u_y^4$ have no solution.*

Proof. The above equations imply that u_x or u_y is a square. Since $u_n \equiv 3 \pmod 4$ for all $n \geq 2$, it follows that either $x \leq 1$ or $y \leq 1$, and a quick computation finishes the job. ∎

Therefore we shall consider the following problem:

Find all solutions to

$$u_{n_1}^{x_1}u_{n_2}^{x_2}u_{n_3}^{x_3} = u_{n_4}^{x_4}u_{n_5}^{x_5} \tag{4.1}$$

such that $0 \leq x_i \leq i$ for $i = 1,2,3,4,5$ and $x_i = 0$ implies that there exists an $j \in \{1,2,3,4,5\}$ with $j \neq i$, $n_i = n_j$ and $x_j \neq 0$.

For this purpose, we define the two sets

$$N_{123} = \{n_i : i = 1,2,3,\ x_i > 0\}, \qquad N_{45} = \{n_i : i = 4,5,\ x_i > 0\}.$$

By canceling equal factors in (4.1), we may assume that $N_{123} \cap N_{45} = \emptyset$. First, we observe that $\alpha = 2$, $\beta = 1$, $c = 1$ and $d = 3$, and $u_n > 2^n$ for all $n \geq 0$. Since $S = \{2,3\}$ and $\gcd(6, u_n) = 1$, unless $n = 0$, we may assume that $D_{n,m} = \gcd(u_n, u_m)$.

Let us consider the quantity $D_{n,m} = \gcd(u_n, u_m)$ a little more closely. Due to inequality (2.9) we immediately obtain

$$\gcd(u_n, u_m) \leq 2^{mu+nv} + 3^{u+v} < 2^{2\sqrt{X}} + 3^{2\sqrt{X}} \leq 3^{2\sqrt{X}}\left(1 + e^{(2\log 2 - 2\log 3)\sqrt{X}}\right)$$
$$< 1.027 \cdot 3^{2\sqrt{X}}$$

provided $X = \max\{n, m\} \geq 20$.

Let $n_i \in N_{123}$ and write $\max\{n_i \in N_{123}\} = X_{123}$. Then we have

$$u_{n_i} \mid \prod_{n_j \in N_{45}} \gcd(u_{n_i}, u_{n_j})^j.$$

The bound for the greatest common divisor now is

$$u_{n_i} \leq 1.28 \cdot 3^{18\sqrt{X_{123}}};$$

hence,

$$X_{123} \log 2 < \log(1.28) + 18 \log 3 \sqrt{X_{123}}.$$

This inequality yields $X_{123} \leq 814$. Now, we assume $n_i \in N_{45}$ and write $\max\{n_i \in N_{45}\} = X_{45}$; hence,

$$u_{n_i} \mid \prod_{n_j \in N_{123}} \gcd(u_{n_i}, u_{n_j})^j,$$

and therefore

$$u_{n_i} \leq 1.18 \cdot 3^{12\sqrt{X_{45}}}.$$

This yields the upper bound $X_{45} \leq 362$.

Let us write $G_{45}(n) = \max\{\gcd(u_j, u_n) : 0 \leq j \leq 814,\ j \neq n\}$. Next, we compute for all $0 \leq n \leq 362$ the value $G_{45}(n)$, and use the inequality $u_{n_i} \leq G_{45}(n_i)^6$ for $n_i \in N_{45}$ to decide which n are possible solutions. By a quick computer search we deduce

$$N_{45} \subseteq \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 17, 19, 20, 21, 22, 23, 27, 29,$$
$$31, 33, 34, 36, 42, 43, 44, 49, 51, 63, 68, 80\}.$$

Now let us write $G_{123}(n) = \max\{\gcd(u_j, u_n) : 1 \leq j \leq 80,\ j \neq n\}$. We note that $u_{n_i} \leq G_{123}(n_i)^9$ for $n_i \in N_{123}$ and find by a computer search that

$$N_{123} \subseteq \{1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 13, 14, 17, 19, 20, 21, 22, 23,$$
$$29, 33, 37, 38, 41, 44, 49, 53, 58, 60\}.$$

Next, we observe that

$$u_{n_i} \mid \prod_{n_j \in N_{123}} u_{n_j}^j$$

for $n_i \in N_{45}$ and all admissible triples (n_1, n_2, n_3). By a computer search, we further exclude several possibilities for the set N_{45} and we are left with

$$N_{45} \subseteq \{1, 2, 3, 4, 5, 8, 9, 10\}.$$

Similarly we have

$$u_{n_i} \mid \prod_{n_j \in N_{45}} u_{n_j}^j$$

for $n_i \in N_{123}$ and all admissible pairs (n_4, n_5). That is, $N_{123} \subseteq \{1, 2, 5\}$. Since $u_1 = 5, u_2 = 7$ and $u_5 = 35$, and since $u_3 = 11$, $u_4 = 19$, $u_8 = 37 \cdot 7$, $u_9 = 5 \cdot 103$, $u_{10} = 13 \cdot 79$ have other prime divisors than 5 and 7, we deduce that also $N_{45} \subseteq \{1, 2, 5\}$. Now, it is easy to conclude that the only solutions to (1.11) are $(n_1, n_2, n_3, n_4, n_5) = (1, 2, 5, 1, 2)$ and $(2, 1, 5, 2, 1)$.

Remark 4.2. We want to emphasize that the computer searches described above took altogether less than one minute on a common PC.

5. Comments. Our result is very general modulo the condition that α/β and c/d are multiplicatively independent. However, the case when they are multiplicatively dependent is quite easy. Suppose first that $\mathbb{K} = \mathbb{Q}$. Since α/β and c/d are multiplicatively dependent, it follows that there exists some rational number $\rho = a/b$ with coprime integers a and b and coprime integers $u > 0$ and v such that $\alpha/\beta = \varepsilon\rho^u$ and $c/d = \eta\rho^v$ for some $\varepsilon, \eta \in \{\pm 1\}$. But then it is easy to see that $u_n = v_n(a^{un+v} \pm b^{un+v})$, where v_n is some rational number whose prime factors are in S and which depends on n. But, for the rational case, Birkhoff and Vandiver [3] showed that the Lucas sequences of general terms $(a^n - b^n)/(a - b)$ and $a^n + b^n$ with a and b integers have primitive divisors for $n > 6$. Moreover, Carmichael [5] showed that a primitive divisor p of the nth term of a Lucas sequence with rational roots satisfies $p \equiv 1 \pmod{n}$. Combining these two results implies, in view of equation (1.5), that $X \le \max\{6 + |v|, P - 1 + |v|\}$.

The case when $\mathbb{K}$ is quadratic is similar. In this case, since α/β and c/d are multiplicatively dependent, it follows that there exist $\rho \in \mathbb{K}$, coprime integers $u > 0$ and v, and roots of unity ε and η in $\mathbb{K}$ such that $\alpha/\beta = \varepsilon\rho^u$ and $c/d = \eta\rho^v$. Let σ be the only nontrivial Galois automorphism of $\mathbb{K}$. Note that $\sigma(\alpha) = \beta$ and $\sigma(c) = d$. Since ε and η are roots of unity, we get $\varepsilon\sigma(\varepsilon) = \eta\sigma(\eta) = 1$. Hence,

$$1 = (\alpha/\beta)(\beta/\alpha) = (\varepsilon\sigma(\varepsilon))(\rho\sigma(\rho))^u,$$

and therefore $(\rho\sigma(\rho))^u = 1$. Similarly,

$$1 = (c/d)(d/c) = (\eta\sigma(\eta))(\rho\sigma(\rho))^v,$$

hence $(\rho\sigma(\rho))^v = 1$. Since u and v are coprime, we get $\rho\sigma(\rho) = 1$, and by Hilbert's Theorem 90 we deduce that $\rho = \gamma/\delta$, where $\sigma(\gamma) = \delta$. We may certainly assume that γ is an algebraic integer. We easily deduce that $u_n = v_n L_{un+v}$, where v_n is a rational number whose numerator and denominator consist only of primes from S, and L_n is the nth term of one of the three Lucas sequences of general form $(\gamma^n - \delta^n)/(\gamma - \delta)$, or $\gamma^n + \delta^n$, or $(\eta\gamma^n + \overline{\eta}\delta^n)$, where η is a primitive root of unity of order 3, with the last case occurring only when $\mathbb{K} = \mathbb{Q}(\sqrt{-3})$. The existence of primitive divisors now yields a similar result as in the rational case. Schinzel [19] showed that, for an effectively computable constant n_0, sequences of the form $a^n - b^n$ have primitive divisors for $n > n_0$. However, using the most general version of the Primitive Divisor Theorem due to Bilu, Hanrot and Voutier [2], as well as the fact that for a general Lucas sequence a primitive divisor p of its nth term satisfies $p \equiv \pm 1 \pmod{n}$, we deduce that $X \le \max\{30+|v|, P+1+|v|\}$. We give no further details.

Acknowledgments. We thank the referee for a careful reading of the paper and for comments and remarks which improved its quality. F.L. was supported in part by Projects PAPIIT IN104512, CONACyT Mexico–France 193539, CONACyT Mexico–India 163787, and a Marcos Moshinsky Fellowship. V.Z. was supported by the Austrian Science Fund (FWF) under the project P 24801-N26.

References

[1] A. Bérczes and V. Ziegler, *On geometric progressions on Pell equations and Lucas sequences*, Glas. Mat. Ser. III 48 (2013), 1–22.

[2] Yu. Bilu, G. Hanrot, and P. M. Voutier, *Existence of primitive divisors of Lucas and Lehmer numbers*, J. Reine Angew. Math. 539 (2001), 75–122.

[3] G. D. Birkhoff and H. S. Vandiver, *On the integral divisors of* $a^n - b^n$, Ann. of Math. (2) 5 (1904), 173–180.

[4] Y. Bugeaud et M. Laurent, *Minoration effective de la distance p-adique entre puissances de nombres algébriques*, J. Number Theory 61 (1996), 311–342.

[5] R. D. Carmichael, *On the numerical factors of the arithmetic forms* $\alpha^n \pm \beta^n$, Ann. of Math. (2) 15 (1913/14), 30–48.

[6] G. Everest, A. van der Poorten, I. Shparlinski, and T. Ward, *Recurrence Sequences*, Math. Surveys Monogr. 104, Amer. Math. Soc., Providence, RI, 2003.

[7] P. Kiss, *Zero terms in second-order linear recurrences*, Math. Sem. Notes Kobe Univ. 7 (1979), 145–152.

[8] M. Laurent, *Équations exponentielles polynômes et suites récurrentes linéaires*, Astérisque 147–148 (1987), 121–139.

[9] C. Lech, *A note on recurring series*, Ark. Mat. 2 (1953), 417–421.

[10] D. J. Lewis and J. Turk, *Repetitiveness in binary recurrences*, J. Reine Angew. Math. 356 (1985), 19–48.

[11] F. Luca, *Divisibility properties of binary recurrent sequences*, Indag. Math. (N.S.) 12 (2001), 353–367.

[12] F. Luca, *Arithmetic properties of members of a binary recurrent sequence*, Acta Arith. 109 (2003), 81–107.

[13] F. Luca, *On the greatest common divisor of two Cullen numbers*, Abh. Math. Sem. Univ. Hamburg 73 (2003), 253–270.

[14] K. Mahler, *Eine arithmetische Eigenschaft der Taylor-Koeffizienten rationaler Funktionen*, Proc. Akad. Wet. Amsterdam 38 (1935), 51–60.

[15] E. M. Matveev, *An explicit lower bound for a homogeneous rational linear form in logarithms of algebraic numbers. II*, Izv. Ross. Akad. Nauk Ser. Mat. 64 (2000), no. 6, 125–180 (in Russian); English transl.: Izv. Math. 64 (2000), 1217–1269.

[16] M. Mignotte, *A note on linear recursive sequences*, J. Austral. Math. Soc. 20 (1975), 242–244.

[17] M. Mignotte, *Détermination des répetitions d'une certaine suite récurrente linéaire*, Publ. Math. Debrecen 33 (1986), 297–306.

[18] B. M. Phong, *On generalized Lehmer sequences*, Acta Math. Hungar. 57 (1991), 201–211.

[19] A. Schinzel, *Primitive divisors of the expression $A^n - B^n$ in algebraic number fields*, J. Reine Angew. Math. 268/269 (1974), 27–33.

[20] H. P. Schlickewei and W. M. Schmidt, *Equations $au_n^l = bu_m^k$ satisfied by members of recurrence sequences*, Proc. Amer. Math. Soc. 118 (1993), 1043–1051.

[21] H. P. Schlickewei and W. M. Schmidt, *Linear equations in members of recurrence sequences*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 20 (1993), 219–246.

[22] H. P. Schlickewei and W. M. Schmidt, *The number of solutions of polynomial-exponential equations*, Compos. Math. 120 (2000), 193–225.

[23] T. N. Shorey, *Linear forms in members of a binary recursive sequence*, Acta Arith. 43 (1984), 317–331.

[24] T. N. Shorey and R. Tijdeman, *Exponential Diophantine Equations*, Cambridge Tracts in Math. 87, Cambridge Univ. Press, Cambridge, 1986.

[25] Th. Skolem, *Einige Sätze über $\mathfrak{p}$-adische Potenzreihen mit Anwendung auf gewisse exponentielle Gleichungen*, Math. Ann. 111 (1935), 399–424.

[26] K. Yu, *p-adic logarithmic forms and group varieties. I*, J. Reine Angew. Math. 502 (1998), 29–92.

Florian Luca
Mathematical Institute, UNAM
Mexico, DF, 04510, Mexico
E-mail: fluca@matmor.unam.mx

Volker Ziegler
Johann Radon Institute for
Computational and Applied Mathematics (RICAM)
Austrian Academy of Sciences
Altenbergerstr. 69
A-4040 Linz, Austria
E-mail: volker.ziegler@ricam.oeaw.ac.at

Received on 13.12.2012
and in revised form on 11.6.2013 (7287)