ACTA ARITHMETICA
XCVII.1 (2001)

Exponential sums for $\mathrm{O}^-(2n,q)$ and their applications

by

Dae San Kim (Seoul)

1. Introduction. Let λ be a nontrivial additive character of the finite field $\mathbb{F}_q$, χ a multiplicative character of $\mathbb{F}_q$, and let r be a positive integer. Throughout this paper, we assume that q is a power of an odd prime. Then we consider the exponential sum

$$\sum_{w\in \mathrm{SO}^-(2n,q)} \lambda((\operatorname{tr} w)^r), \tag{1.1}$$

where $\mathrm{SO}^-(2n,q)$ is a special orthogonal group over $\mathbb{F}_q$ (cf. (2.3)) and $\operatorname{tr} w$ is the trace of w. Also, we consider

$$\sum_{w\in \mathrm{O}^-(2n,q)} \chi(\det w)\lambda((\operatorname{tr} w)^r), \tag{1.2}$$

where $\mathrm{O}^-(2n,q)$ is an orthogonal group over $\mathbb{F}_q$ (cf. (2.2)) and $\det w$ is the determinant of w.

The main purpose of this paper is to find explicit expressions for the sums (1.1) and (1.2). It turns out that (1.1) is a polynomial in q times

$$\sum_{\gamma\in\mathbb{F}_q} \lambda(\gamma^r) \tag{1.3}$$

plus another polynomial in q involving certain exponential sums (cf. (2.14), (2.15)). We mention in passing that, in [14], we gave O-estimates for two kinds of new exponential sums which include the above ones. On the other hand, the expression for (1.2) is that for (1.1) plus $\chi(-1)$ times a similar

2000 *Mathematics Subject Classification*: Primary 11T23, 11T24; Secondary 11L05, 20G40.

Key words and phrases: exponential sum, additive character, multiplicative character, orthogonal group, Bruhat decomposition, maximal parabolic subgroup.

This work was supported by grant No. 98-0701-01-01-3 from the Basic Research program of the KOSEF.

one corresponding to the subsum of (1.2) over $\mathrm{O}^-(2n,q) - \mathrm{SO}^-(2n,q) = \varrho\mathrm{SO}^-(2n,q)$ (cf. (2.12)).

In [9], the sums in (1.1) and (1.2) were studied for $r=1$ and the connection of the sum in (1.2) for χ trivial with Hodges' generalized Kloosterman sum over nonsingular symmetric matrices was also investigated ([5], [6]). As the sum in (1.3) vanishes for $r=1$, the polynomials involving (1.3) do not appear in that case. For $r=1$, similar sums for other classical groups over a finite field had been considered ([7]–[12], [15], [16]).

The sums in (1.1) and (1.2) may be viewed as generalizations to $\mathrm{O}^-(2n,q)$ and $\mathrm{SO}^-(2n,q)$ of the sum in (1.3) which was studied by several authors ([1]–[3]).

Another purpose of this paper is to find formulas for the number of elements w in $\mathrm{O}^-(2n,q)$ and $\mathrm{SO}^-(2n,q)$ with $(\operatorname{tr} w)^r = \beta$, for each $\beta \in \mathbb{F}_q$. We derive them from (5.2) based on a well known principle, though they could also be obtained from the expressions for (1.1) and (1.2) by specializing them to $r = q-1$ and $r=1$.

We now state the main results of this paper. The reader is referred to the next section for some notations here, to (4.34)–(4.39) for X_n, X'_n, Y_n, Y'_n, Z_n, Z'_n (cf. (2.14), (2.15)), and to the Remark just before Theorem 5.1 for $\widetilde{Y}_n$, $\widetilde{Y}'_n$, $\widetilde{Z}_n$, $\widetilde{Z}'_n$ (cf. (4.28), (4.29)).

THEOREM A. *The sum* $\sum_{w\in \mathrm{SO}^-(2n,q)} \lambda((\operatorname{tr} w)^r)$ *equals*

$$q^{n^2-n-1}\Big\{(q^n+1)\prod_{j=1}^{n-1}(q^{2j}-1) + (q-1)X_n + (q+1)X'_n\Big\}\sum_{\gamma\in\mathbb{F}_q}\lambda(\gamma^r) - q^{n^2-n-1}\{(q+1)Y'_n + Z_n\}.$$

THEOREM B. *The sum* $\sum_{w\in \mathrm{O}^-(2n,q)} \chi(\det w)\lambda((\operatorname{tr} w)^r)$ *equals*

$$q^{n^2-n-1}q^{(1-\chi(-1))/2} \times \Big\{(1+\chi(-1))(q^n+1)\prod_{j=1}^{n-1}(q^{2j}-1) + 2(X'_n - \chi(-1)X_n)\Big\}\sum_{\gamma\in\mathbb{F}_q}\lambda(\gamma^r)$$
$$- q^{n^2-n-1}\{(q+1)(Y'_n - \chi(-1)Y_n) - \chi(-1)(Z'_n - \chi(-1)Z_n)\},$$

where we understand that

$$q^{(1-\chi(-1))/2} = \begin{cases} 1 & \text{if } \chi(-1)=1, \\ q & \text{if } \chi(-1)=-1. \end{cases}$$

THEOREM C. *For each* $\beta\in\mathbb{F}_q$ *and each positive integer* r, *the number* $N_{\mathrm{O}^-(2n,q)}(\beta; r)$ *of* $w \in \mathrm{O}^-(2n,q)$ *with* $(\operatorname{tr} w)^r = \beta$ *is given by*

$$2N(y^r=\beta)q^{n^2-n-1}\Big\{(q^n+1)\prod_{j=1}^{n-1}(q^{2j}-1)+(X_n'-X_n)\Big\}$$
$$+q^{n^2-n-1}\{(\widetilde{Z}_n'-\widetilde{Z}_n)-(q+1)(\widetilde{Y}_n'-\widetilde{Y}_n)\},$$

where $N(y^r=\beta)$ *denotes the number of* y *in* $\mathbb{F}_q$ *with* $y^r=\beta$.

The above Theorems A, B and C are respectively stated below as Theorems 4.1, 4.2 and 5.1.

2. Preliminaries. In this section, we will fix some notations and collect from [9] some facts that will be used in what follows. Also, we refer to [4] and [17] for some elementary facts of the following.

Let $\mathbb{F}_q$ denote the finite field with q elements, $q=p^d$ (p an odd prime, d a positive integer).

In the following, $\operatorname{tr}A$ and $\det A$ denote respectively the trace of A and the determinant of A for a square matrix A, and ${}^tB$ denotes the transpose of B for any matrix B.

Let $\mathrm{GL}(n,q)$ denote the group of all invertible $n\times n$ matrices with entries in $\mathbb{F}_q$. The order of $\mathrm{GL}(n,q)$ equals

(2.1) $$g_n=\prod_{j=0}^{n-1}(q^n-q^j)=q^{\binom{n}{2}}\prod_{j=1}^{n}(q^j-1).$$

Throughout this paper, we let ε denote a fixed element in $\mathbb{F}_q^\times-\mathbb{F}_q^{\times 2}$. Then

(2.2) $$\mathrm{O}^-(2n,q)=\{w\in\mathrm{GL}(2n,q)\mid {}^twJ^-w=J^-\},$$

where

$$J^-=\begin{bmatrix}0&1_{n-1}&0&0\\1_{n-1}&0&0&0\\0&0&1&0\\0&0&0&-\varepsilon\end{bmatrix}.$$

Also,

(2.3) $$\mathrm{SO}^-(2n,q)=\{w\in\mathrm{O}^-(2n,q)\mid \det w=1\}$$

is a subgroup of index 2 in $\mathrm{O}^-(2n,q)$. It is well known that

(2.4) $$|\mathrm{O}^-(2n,q)|=2q^{n^2-n}(q^n+1)\prod_{j=1}^{n-1}(q^{2j}-1),$$

(2.5) $$|\mathrm{SO}^-(2n,q)|=q^{n^2-n}(q^n+1)\prod_{j=1}^{n-1}(q^{2j}-1).$$

Put

$$(2.6)\quad \mathrm{Q} = \mathrm{Q}(2n, q)$$
$$= \left\{ \begin{bmatrix} A & 0 & 0 \\ 0 & {}^tA^{-1} & 0 \\ 0 & 0 & i \end{bmatrix} \begin{bmatrix} 1_{n-1} & B & -{}^th\delta_\varepsilon \\ 0 & 1_{n-1} & 0 \\ 0 & h & 1_2 \end{bmatrix} \;\middle|\; \begin{array}{l} A \in \mathrm{GL}(n, q), \\ i \in \mathrm{SO}^-(2, q), \\ {}^tB + B + {}^th\delta_\varepsilon h = 0 \end{array} \right\},$$

where, for $\alpha \in \mathbb{F}_q^\times$, δ_α denotes the 2×2 matrix over $\mathbb{F}_q$

$$(2.7)\qquad \delta_\alpha = \begin{bmatrix} 1 & 0 \\ 0 & -\alpha \end{bmatrix}.$$

If we let i run over $\mathrm{O}^-(2, q)$ in (2.6), we get the maximal parabolic subgroup $\mathrm{P} = \mathrm{P}(2n, q)$ of $\mathrm{O}^-(2n, q)$. One now observes that

$$\mathrm{Q}(2n, q) = \mathrm{P}(2n, q) \cap \mathrm{SO}^-(2n, q)$$

is a subgroup of index 2 in $\mathrm{P}(2n, q)$.

In [9], it was noted that, starting from the Bruhat decomposition

$$\mathrm{O}^-(2n, q) = \coprod_{b=0}^{n-1} \mathrm{P}\sigma_b\mathrm{P},$$

one can obtain the following decompositions:

$$(2.8)\qquad \mathrm{SO}^-(2n, q) = \Big(\coprod_{0 \le b \le n-1,\, b \text{ even}} \mathrm{Q}\sigma_b(\mathrm{B}_b\backslash\mathrm{Q}) \Big) \amalg \Big(\coprod_{0 \le b \le n-1,\, b \text{ odd}} (\varrho\mathrm{Q})\sigma_b(\mathrm{B}_b\backslash\mathrm{Q}) \Big),$$

$$(2.9)\qquad \mathrm{O}^-(2n, q) = \Big(\coprod_{0 \le b \le n-1,\, b \text{ even}} \mathrm{Q}\sigma_b(\mathrm{B}_b\backslash\mathrm{Q}) \Big) \amalg \Big(\coprod_{0 \le b \le n-1,\, b \text{ odd}} (\varrho\mathrm{Q})\sigma_b(\mathrm{B}_b\backslash\mathrm{Q}) \Big) \amalg \Big(\coprod_{0 \le b \le n-1,\, b \text{ odd}} \mathrm{Q}\sigma_b(\mathrm{B}_b\backslash\mathrm{Q}) \Big) \amalg \Big(\coprod_{0 \le b \le n-1,\, b \text{ even}} (\varrho\mathrm{Q})\sigma_b(\mathrm{B}_b\backslash\mathrm{Q}) \Big),$$

where

$$(2.10)\qquad \mathrm{B}_b = \mathrm{B}_b(q) = \{ w \in \mathrm{Q}(2n, q) \mid \sigma_b w \sigma_b^{-1} \in \mathrm{P}(2n, q) \},$$

$$\sigma_b = \begin{bmatrix} 0 & 0 & 1_b & 0 & 0 \\ 0 & 1_{n-1-b} & 0 & 0 & 0 \\ 1_b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{n-1-b} & 0 \\ 0 & 0 & 0 & 0 & 1_2 \end{bmatrix}, \tag{2.11}$$

$$\varrho = \begin{bmatrix} 1_{n-1} & 0 & 0 & 0 \\ 0 & 1_{n-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{2.12}$$

From (3.12) and (3.21) of [9] (cf. (2.17)), we have

$$|\mathrm{B}_b(q)\backslash \mathrm{Q}(2n,q)| = \begin{bmatrix} n-1 \\ b \end{bmatrix}_q q^{b(b+3)/2}. \tag{2.13}$$

Let λ be a nontrivial additive character of $\mathbb{F}_q$, $a, b \in \mathbb{F}_q$, and let r be a positive integer. Then the exponential sum $MK_m(\lambda^r; a, b)$ is defined as

$$MK_m(\lambda^r; a, b) = \sum_{\gamma_1,\ldots,\gamma_m \in \mathbb{F}_q^\times} \lambda((a\gamma_1 + b\gamma_1^{-1} + \ldots + a\gamma_m + b\gamma_m^{-1})^r) \tag{2.14}$$

for $m \geq 1$, and

$$MK_0(\lambda^r; a, b) = 1. \tag{2.15}$$

Note that, in the special case of $r = 1$,

$$MK_m(\lambda^r; a, b) = K(\lambda; a, b)^m,$$

where $K(\lambda; a, b)$ is the usual Kloosterman sum given by

$$K(\lambda; a, b) = \sum_{\gamma \in \mathbb{F}_q^\times} \lambda(a\gamma + b\gamma^{-1}). \tag{2.16}$$

For integers n, b with $0 \leq b \leq n$, the q-*binomial coefficients* are defined by

$$\begin{bmatrix} n \\ b \end{bmatrix}_q = \prod_{j=0}^{b-1} (q^{n-j} - 1)/(q^{b-j} - 1). \tag{2.17}$$

Then, from the q-binomial theorem (cf. [13, (2.18)]), one obtains

$$\sum_{b=0}^{n-1} \begin{bmatrix} n-1 \\ b \end{bmatrix}_q q^{b(b+3)/2} = \prod_{j=2}^{n} (q^j + 1). \tag{2.18}$$

Finally, $[y]$ denotes the greatest integer $\leq y$, for a real number y.

3. Some propositions. In this section, we will consider two propositions which will be of use in the next section. The first one is about the Gauss sum over $\mathrm{SO}^-(2,q)$, which is a restatement of Proposition 4.5 of [9] with one minor modification and shows it is the negative of the usual Kloosterman sum. This improvement was observed by Prof. D. Wan to whom I wish to thank.

The second proposition is a generalization of Proposition 4.2 of [9].

PROPOSITION 3.1. *Let λ be a nontrivial additive character of $\mathbb{F}_q$. Then*

$$\sum_{w\in \mathrm{SO}^-(2,q)} \lambda(\mathrm{tr}\, w) = -K(\lambda;1,1), \tag{3.1}$$

$$\sum_{w\in \mathrm{SO}^-(2,q)} \lambda(\mathrm{tr}\, \delta_1 w) = q+1, \tag{3.2}$$

where $K(\lambda;1,1)$ is the ordinary Kloosterman sum as in (2.16), *and δ_1 is as in* (2.7).

Proof. In view of [9, Proposition 4.5], we only need to see that, for a multiplicative character ψ of $\mathbb{F}_q$ of order $q-1$,

$$\sum_{j=1}^{q-1} G(\psi^j,\lambda)^2 = (q-1)K(\lambda;1,1). \tag{3.3}$$

Here $G(\psi^j,\lambda) = \sum_{\alpha\in\mathbb{F}_q^\times}\psi^j(\alpha)\lambda(\alpha)$ is the usual Gauss sum. However, this follows from a simple change of order of summation and (5.13) of [17]. ∎

PROPOSITION 3.2. *Let λ be a nontrivial additive character of $\mathbb{F}_q$, $c\in\mathbb{F}_q$, r,b positive integers, and let Ω_b be the set of all $b\times b$ nonsingular symmetric matrices over $\mathbb{F}_q$. Then*

$$a_b(\lambda;r;c) := \sum_{B\in\Omega_b}\sum_{h\in\mathbb{F}_q^{2\times b}} \lambda((\mathrm{tr}\,\delta_\varepsilon hB\,{}^th + c)^r) \tag{3.4}$$

$$= q^{2b-1}s_b\sum_{\gamma\in\mathbb{F}_q}\lambda(\gamma^r) + q^{-1}a_b\sum_{\beta\in\mathbb{F}_q^\times}\lambda(c\beta)\sum_{\gamma\in\mathbb{F}_q}\lambda(\gamma^r-\beta\gamma) \tag{3.5}$$

$$= (q^{2b-1}s_b - q^{-1}a_b)\sum_{\gamma\in\mathbb{F}_q}\lambda(\gamma^r) + a_b\lambda(c^r), \tag{3.6}$$

where s_b is the number of all $b\times b$ nonsingular symmetric matrices over $\mathbb{F}_q$, δ_ε is as in (2.7), *and*

$$
(3.7)\qquad a_b := a_b(\lambda;1;0) = \sum_{B\in\Omega_b}\sum_{h\in\mathbb{F}_q^{2\times b}} \lambda(\operatorname{tr}\delta_\varepsilon hB\,{}^t h)
$$
$$
= \begin{cases} q^{b(b+6)/4}\displaystyle\prod_{j=1}^{b/2}(q^{2j-1}-1) & \textit{for } b \textit{ even},\\ -q^{(b^2+4b-1)/4}\displaystyle\prod_{j=1}^{(b+1)/2}(q^{2j-1}-1) & \textit{for } b \textit{ odd}\end{cases}
$$

(*cf.* [9, (4.6)]).

Remark. The independence of a_b from λ is clear from its definition or the expressions of it in the above.

Proof of Proposition 3.2. Put, for each $\gamma\in\mathbb{F}_q$,

$$
N_\gamma = |\{(B,h)\in\Omega_b\times\mathbb{F}_q^{2\times b} \mid \operatorname{tr}\delta_\varepsilon hB\,{}^t h + c = \gamma\}|.
$$

Then $a_b(\lambda;r;c) = \sum_{\gamma\in\mathbb{F}_q} N_\gamma\lambda(\gamma^r)$, with

$$
N_\gamma = q^{-1}\Big\{q^{2b}s_b + \sum_{\beta\in\mathbb{F}_q^\times}\lambda(-\gamma\beta)\sum_{B,h}\lambda((\operatorname{tr}\delta_\varepsilon hB\,{}^t h + c)\beta)\Big\}.
$$

So

$$
\begin{aligned}
a_b(\lambda;r;c) &= q^{2b-1}s_b\sum_{\gamma\in\mathbb{F}_q}\lambda(\gamma^r)\\
&\quad + q^{-1}\sum_{\beta\in\mathbb{F}_q^\times}\lambda(c\beta)\sum_{B,h}\lambda((\operatorname{tr}\delta_\varepsilon hB\,{}^t h)\beta)\sum_{\gamma\in\mathbb{F}_q}\lambda(\gamma^r-\beta\gamma)\\
&= q^{2b-1}s_b\sum_{\gamma\in\mathbb{F}_q}\lambda(\gamma^r) + q^{-1}a_b\sum_{\beta\in\mathbb{F}_q^\times}\lambda(c\beta)\sum_{\gamma\in\mathbb{F}_q}\lambda(\gamma^r-\beta\gamma),
\end{aligned}
$$

since, as was noted above, the sum over B, h is independent of $\beta\in\mathbb{F}_q^\times$. This shows (3.5) from which (3.6) follows by interchanging the order of summation in the second term of (3.5). ∎

4. Main theorems. In this section, we first consider the sum in (1.1)

$$
\sum_{w\in\mathrm{SO}^-(2n,q)}\lambda((\operatorname{tr} w)^r)
$$

for any nontrivial additive character λ of $\mathbb{F}_q$ and any positive integer r, and find an explicit expression for this by using the decomposition in (2.8).

The sum in (1.1) can be written, using (2.8), as

$$(4.1)\qquad \sum_{0\le b\le n-1,\, b\,\text{even}} |\mathrm{B}_b\backslash \mathrm{Q}| \sum_{w\in\mathrm{Q}} \lambda((\operatorname{tr} w\sigma_b)^r) + \sum_{0\le b\le n-1,\, b\,\text{odd}} |\mathrm{B}_b\backslash \mathrm{Q}| \sum_{w\in\mathrm{Q}} \lambda((\operatorname{tr} \varrho w\sigma_b)^r),$$

where $\mathrm{B}_b = \mathrm{B}_b(q)$, $\mathrm{Q} = \mathrm{Q}(2n, q)$, ϱ, σ_b are respectively as in (2.10), (2.6), (2.12), (2.11). Here one has to observe that, for each $u \in \mathrm{Q}$,

$$\sum_{w\in\mathrm{Q}} \lambda((\operatorname{tr} w\sigma_b u)^r) = \sum_{w\in\mathrm{Q}} \lambda((\operatorname{tr} uw\sigma_b)^r) = \sum_{w\in\mathrm{Q}} \lambda((\operatorname{tr} w\sigma_b)^r)$$

and $\varrho^{-1}u\varrho \in \mathrm{Q}$. Write $w \in \mathrm{Q}$ (cf. (2.6)) as

$$w = \begin{bmatrix} A & 0 & 0 \\ 0 & {}^tA^{-1} & 0 \\ 0 & 0 & i \end{bmatrix} \begin{bmatrix} 1_{n-1} & B & -{}^th\delta_\varepsilon \\ 0 & 1_{n-1} & 0 \\ 0 & h & 1_2 \end{bmatrix},$$

with

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad {}^tA^{-1} = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & -{}^tB_{21} - {}^th_1\delta_\varepsilon h_2 \\ B_{21} & B_{22} \end{bmatrix},$$
$$h = [\,h_1 \quad h_2\,],$$

$$(4.2)\qquad {}^tB_{11} + B_{11} + {}^th_1\delta_\varepsilon h_1 = 0, \qquad {}^tB_{22} + B_{22} + {}^th_2\delta_\varepsilon h_2 = 0.$$

Note that, together with $B_{12} + {}^tB_{21} + {}^th_1\delta_\varepsilon h_2 = 0$ (i.e., denoting the upper right block of B by B_{12}), the conditions in (4.2) are equivalent to ${}^tB + B + {}^th\delta_\varepsilon h = 0$. Here A_{11}, A_{12}, A_{21}, A_{22} are respectively of sizes $b \times b$, $b \times (n-1-b)$, $(n-1-b) \times b$, $(n-1-b) \times (n-1-b)$, similarly for ${}^tA^{-1}$, B, and h_1 is of size $2 \times b$. Then

$$(4.3)\qquad \sum_{w\in\mathrm{Q}} \lambda((\operatorname{tr} w\sigma_b)^r)$$

$$(4.4)\qquad = \sum \lambda((\operatorname{tr} A_{11}B_{11} + \operatorname{tr} A_{12}B_{21} + \operatorname{tr} A_{22} + \operatorname{tr} E_{22} + \operatorname{tr} i)^r)$$

and

$$(4.5)\qquad \sum_{w\in\mathrm{Q}} \lambda((\operatorname{tr} \varrho w\sigma_b)^r)$$

$$(4.6)\qquad = \sum \lambda((\operatorname{tr} A_{11}B_{11} + \operatorname{tr} A_{12}B_{21} + \operatorname{tr} A_{22} + \operatorname{tr} E_{22} + \operatorname{tr} \delta_1 i)^r),$$

where the sums in (4.4) and (4.6) are respectively over A, B_{11}, B_{21}, B_{22}, h, i subject to the conditions in (4.2).

Consider the sum in (4.4) first for the case $1 \le b \le n-2$ so that A_{12} does appear. We separate the sum into two subsums, with $A_{12} \ne 0$ and with $A_{12} = 0$; the latter will be further divided into two subsums, with A_{11}

symmetric or not. That is, we write the sum in (4.4) as

$$(4.7) \qquad \sum_{A_{12}\neq 0} \cdots + \sum_{A_{12}=0,\, A_{11} \text{ not symmetric}} \cdots + \sum_{A_{12}=0,\, A_{11} \text{ symmetric}} \cdots$$

Before we move on, we recall from [9] the following with one correction. Put

$$(4.8) \qquad A_{11} = (\alpha_{ij}), \quad B_{11} = (\beta_{ij}), \quad h_1 = \begin{bmatrix} h_{11} & h_{12} & \dots & h_{1b} \\ h_{21} & h_{22} & \dots & h_{2b} \end{bmatrix}.$$

Then the first condition in (4.2) is equivalent to

$$(4.9) \qquad \begin{aligned} &\beta_{ii} = \tfrac{1}{2}(h_{2i}^2\varepsilon - h_{1i}^2) && \text{for } 1 \le i \le b, \\ &\beta_{ij} + \beta_{ji} = h_{2i}h_{2j}\varepsilon - h_{1i}h_{1j} && \text{for } 1 \le i < j \le b. \end{aligned}$$

In particular, for each given h_1,

$$(4.10) \qquad |\{B_{11} \mid {}^tB_{11} + B_{11} + {}^th_1\delta_\varepsilon h_1 = 0\}| = q^{\binom{b}{2}}.$$

Similarly, for each given h_2,

$$(4.11) \qquad |\{B_{22} \mid {}^tB_{22} + B_{22} + {}^th_2\delta_\varepsilon h_2 = 0\}| = q^{\binom{n-1-b}{2}}.$$

Also, using the relations in (4.9), one shows that

$$(4.12) \qquad \begin{aligned} \operatorname{tr} A_{11}B_{11} = &-\tfrac{1}{2}\operatorname{tr}\delta_\varepsilon h_1 A_{11}{}^th_1 \\ &+ \sum_{1\le i<j\le b} (\alpha_{ji} - \alpha_{ij})\big\{\beta_{ij} + \tfrac{1}{2}(h_{1i}h_{1j} - \varepsilon h_{2i}h_{2j})\big\}. \end{aligned}$$

Here we remark that in [9, p. 359] we neglected to put $\frac{1}{2}(h_{1i}h_{1j} - \varepsilon h_{2i}h_{2j})$ for the expression of $\operatorname{tr} A_{11}B_{11}$ in (4.12). However, the rest of the computations there goes through without any change.

The first sum in (4.7), by (4.11), is

$$(4.13) \qquad q^{(n-1-b)(n+2-b)/2} \sum_{A \text{ with } A_{12}\neq 0,\, i, h_1, B_{11}} \sum_{B_{21}} ((\operatorname{tr} A_{11}B_{11} + \operatorname{tr} A_{12}B_{21} + \operatorname{tr} A_{22} + \operatorname{tr} E_{22} + \operatorname{tr} i)^r).$$

Fix A with $A_{12} \neq 0$, i, h_1, B_{11}. Write $A_{12} = (\mu_{ij})$, $B_{21} = (\nu_{ij})$. Then $\mu_{kl} \neq 0$ for some k, l ($1 \le k \le b$, $1 \le l \le n-1-b$).

Noting that, for $a \in \mathbb{F}_q^\times$ and $b \in \mathbb{F}_q$,

$$\sum_{\gamma\in\mathbb{F}_q} \lambda((a\gamma + b)^r) = \sum_{\gamma\in\mathbb{F}_q} \lambda(\gamma^r),$$

we see that the inner sum of (4.13) equals

$$\sum_{\text{all } \nu_{ji} \text{ with } (j,i)\neq(l,k)} \sum_{\nu_{lk}} \lambda((\mu_{kl}\nu_{lk}+\ldots)^r) = q^{b(n-1-b)-1} \sum_{\gamma\in\mathbb{F}_q} \lambda(\gamma^r). \tag{4.14}$$

Combining (4.13) and (4.14), and using (4.10) and (2.5) with $n=1$, we see that the first sum in (4.7) equals

$$(q+1)q^{(n^2+n-4)/2}(g_{n-1}-g_b g_{n-1-b} q^{b(n-1-b)}) \sum_{\gamma\in\mathbb{F}_q} \lambda(\gamma^r). \tag{4.15}$$

The subsum of (4.4) with $A_{12}=0$ is

$$\begin{aligned}\sum_{A_{21},B_{21},B_{22},h_2} \sum_{A_{11},A_{22},B_{11},h_1,i} &\lambda((\operatorname{tr} A_{11}B_{11}+\operatorname{tr} A_{22}+\operatorname{tr} A_{22}^{-1}+\operatorname{tr} i)^r) \\ &= q^{\binom{n-1-b}{2}+2(b+1)(n-1-b)} \\ &\quad\times \sum_{A_{11},A_{22},B_{11},h_1,i} \lambda((\operatorname{tr} A_{11}B_{11}+\operatorname{tr} A_{22}+\operatorname{tr} A_{22}^{-1}+\operatorname{tr} i)^r).\end{aligned} \tag{4.16}$$

The subsum of the sum in (4.16) with A_{11} not symmetric is

$$\sum_{A_{11} \text{not symmetric},\, A_{22},h_1,i} \sum_{B_{11}} \lambda((\operatorname{tr} A_{11}B_{11}+\operatorname{tr} A_{22}+\operatorname{tr} A_{22}^{-1}+\operatorname{tr} i)^r). \tag{4.17}$$

Since $A_{11}=(\alpha_{ij})$ is not symmetric, $\alpha_{ts}-\alpha_{st}\neq 0$, for some s,t with $1\le s<t\le b$. By the same argument as in the case of (4.13) and in view of (4.10) and (4.12), we see that the inner sum in (4.17) is

$$q^{\binom{b}{2}-1} \sum_{\gamma\in\mathbb{F}_q} \lambda(\gamma^r). \tag{4.18}$$

Combining (4.16)–(4.18) shows that the middle sum in (4.7) is

$$(q+1)q^{(n^2+n-4)/2+b(n-b-1)} g_{n-1-b}(g_b-s_b) \sum_{\gamma\in\mathbb{F}_q} \lambda(\gamma^r), \tag{4.19}$$

where s_b denotes the number of all $b\times b$ nonsingular symmetric matrices over $\mathbb{F}_q$ for each positive integer b.

The subsum of the sum in (4.16) with A_{11} symmetric, by (4.12), is

$$\begin{aligned}\sum_{h_1}\sum_{B_{11}} \sum_{A_{22},i,\, A_{11} \text{ symmetric}} &\lambda\Big(\big(-\tfrac{1}{2}\operatorname{tr}\delta_\varepsilon h_1 A_{11}{}^t h_1+\operatorname{tr} A_{22}+\operatorname{tr} A_{22}^{-1}+\operatorname{tr} i\big)^r\Big) \\ = q^{\binom{b}{2}} \sum_{A_{22},h_1,i,\, A_{11} \text{ symmetric}} &\lambda((\operatorname{tr}\delta_\varepsilon h_1 A_{11}{}^t h_1+\operatorname{tr} A_{22}+\operatorname{tr} A_{22}^{-1}+\operatorname{tr} i)^r).\end{aligned} \tag{4.20}$$

Combining (4.16) and (4.20), we see that the last sum in (4.7) is

$$(4.21)\qquad q^{(n^2+n-2)/2+b(n-b-3)} \times \sum_{A_{22},i} \sum_{A_{11}\text{ symmetric},\, h_1} \lambda((\operatorname{tr}\delta_\varepsilon h_1 A_{11}{}^t h_1 + \operatorname{tr} A_{22} + \operatorname{tr} A_{22}^{-1} + \operatorname{tr} i)^r).$$

For each fixed A_{22}, i, from (3.4) and (3.5) the inner sum of (4.21) is

$$(4.22)\qquad \sum_{A_{11}\text{ symmetric},\, h_1} \lambda((\operatorname{tr}\delta_\varepsilon h_1 A_{11}{}^t h_1 + \operatorname{tr} A_{22} + \operatorname{tr} A_{22}^{-1} + \operatorname{tr} i)^r)$$
$$= a_b(\lambda; r; \operatorname{tr} A_{22} + \operatorname{tr} A_{22}^{-1} + \operatorname{tr} i)$$
$$= q^{2b-1} s_b \sum_{\gamma\in\mathbb{F}_q} \lambda(\gamma^r) + q^{-1} a_b \sum_{\beta\in\mathbb{F}_q^\times} \lambda(\beta(\operatorname{tr} A_{22} + \operatorname{tr} A_{22}^{-1}))\lambda(\beta \operatorname{tr} i) \sum_{\gamma\in\mathbb{F}_q} \lambda(\gamma^r - \beta\gamma),$$

where a_b is as in (3.7).

By summing (4.22) over A_{22}, i, and from (3.1), we see that the double sum in (4.21) is

$$(4.23)\qquad (q+1)q^{2b-1} g_{n-b-1} s_b \sum_{\gamma\in\mathbb{F}_q} \lambda(\gamma^r) - q^{-1} a_b \sum_{\beta\in\mathbb{F}_q^\times} K_{\mathrm{GL}(n-b-1,q)}(\lambda;\beta,\beta) K(\lambda;\beta,\beta) \sum_{\gamma\in\mathbb{F}_q} \lambda(\gamma^r - \beta\gamma),$$

where, for $a, b \in \mathbb{F}_q$,

$$(4.24)\qquad K_{\mathrm{GL}(t,q)}(\lambda; a, b) = \sum_{w\in \mathrm{GL}(t,q)} \lambda(a \operatorname{tr} w + b \operatorname{tr} w^{-1}).$$

From the explicit expression of (4.24) in [8, (4.19)], (4.23) can be written as

$$(4.25)\qquad (q+1)q^{2b-1} g_{n-b-1} s_b \sum_{\gamma\in\mathbb{F}_q} \lambda(\gamma^r) - q^{(n-3-b)(n-b)/2-1} a_b \sum_{l=1}^{[(n-b+1)/2]} q^l \sum \prod_{\nu=1}^{l-1} (q^{j_\nu - 2\nu} - 1) \times \sum_{\gamma\in\mathbb{F}_q} \Big(\sum_{\beta\in\mathbb{F}_q^\times} K(\lambda;\beta,\beta)^{n-b+2-2l} \lambda(-\beta\gamma)\Big) \lambda(\gamma^r),$$

where the unspecified sum runs over all integers

$$(4.26)\qquad j_1, \ldots, j_{l-1} \quad \text{satisfying} \quad 2l-1 \le j_{l-1} \le \ldots \le j_1 \le n-b$$

and it is 1 for $l = 1$ by our convention.

As was noted in [13, (5.3)], for $\gamma \in \mathbb{F}_q$ and m a nonnegative integer, we have

$$\sum_{\beta \in \mathbb{F}_q^\times} \lambda(-\gamma\beta) K(\lambda; \beta, \beta)^m = q\delta(m, q; \gamma) - (q-1)^m, \tag{4.27}$$

where, for $m \geq 1$,

$$\delta(m, q; \gamma) = |\{(\alpha_1, \ldots, \alpha_m) \in (\mathbb{F}_q^\times)^m \mid \alpha_1 + \alpha_1^{-1} + \ldots + \alpha_m + \alpha_m^{-1} = \gamma\}| \tag{4.28}$$

and

$$\delta(0, q; \gamma) = \begin{cases} 1 & \text{if } \gamma = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.29}$$

From (4.27), it is easily seen that

$$\sum_{\gamma \in \mathbb{F}_q} \Big(\sum_{\beta \in \mathbb{F}_q^\times} K(\lambda; \beta, \beta)^m \lambda(-\beta\gamma) \Big) \lambda(\gamma^r) = qMK_m(\lambda^r; 1, 1) - (q-1)^m \sum_{\gamma \in \mathbb{F}_q} \lambda(\gamma^r), \tag{4.30}$$

where $MK_m(\lambda^r; a, b)$ is as in (2.14) and (2.15).

Substituting the expression in (4.30) into (4.25), we see that the double sum in (4.21) is

$$\begin{aligned} &\Big\{ (q+1) q^{2b-1} g_{n-b-1} s_b + q^{(n-b-3)(n-b)/2} a_b \\ &\times \sum_{l=1}^{[(n-b+1)/2]} q^{l-1} (q-1)^{n-b+2-2l} \sum \prod_{\nu=1}^{l-1} (q^{j_\nu - 2\nu} - 1) \Big\} \sum_{\gamma \in \mathbb{F}_q} \lambda(\gamma^r) \\ &- q^{(n-b-3)(n-b)/2} a_b \sum_{l=1}^{[(n-b+1)/2]} q^l MK_{n-b+2-2l}(\lambda^r; 1, 1) \sum \prod_{\nu=1}^{l-1} (q^{j_\nu - 2\nu} - 1), \end{aligned} \tag{4.31}$$

where both the unspecified sums run over the same set of integers as in (4.26) and they are 1 for $l = 1$.

Adding up (4.15), (4.19), and (4.21) with the expression of the double sum there in (4.31), and from (2.1), we find that, for each $1 \leq b \leq n-2$, the sum in (4.3) is given by

$$\begin{aligned} &\sum_{w \in \mathrm{Q}} \lambda((\operatorname{tr} w\sigma_b)^r) \\ &= q^{n^2-n-1} \Big\{ (q+1) \prod_{j=1}^{n-1} (q^j - 1) + q^{-b(b+3)/2} a_b \\ &\quad \times \sum_{l=1}^{[(n-b+1)/2]} q^{l-1} (q-1)^{n-b+2-2l} \sum \prod_{\nu=1}^{l-1} (q^{j_\nu - 2\nu} - 1) \Big\} \sum_{\gamma \in \mathbb{F}_q} \lambda(\gamma^r) \end{aligned} \tag{4.32}$$

$$- q^{n^2-n-1}q^{-b(b+3)/2}a_b \times \sum_{l=1}^{[(n-b+1)/2]} q^l MK_{n-b+2-2l}(\lambda^r;1,1)\sum\prod_{\nu=1}^{l-1}(q^{j_\nu-2\nu}-1),$$

where both the unspecified sums are as in (4.31) above.

One can check that, even for $b=0$ and $b=n-1$, the sum in (4.3) is given by the same expression as in (4.32) with the convention $a_0=1$. Here the details are left to the reader. Also, observe that $a_0=1$ is natural in view of the formula in (3.7).

The sum in (4.5) can be treated, first considering the cases $1\le b\le n-2$ and then the extreme cases $b=0$ and $b=n-1$, just as that in (4.3). Here we only note that $\operatorname{tr}\delta_1 i=0$ in (4.6) and hence the sum there is

$$(q+1)\sum\lambda((\operatorname{tr}A_{11}B_{11}+\operatorname{tr}A_{12}B_{21}+\operatorname{tr}A_{22}+\operatorname{tr}E_{22})^r).$$

The sum in (4.5) is given by

$$\begin{aligned}(4.33)\quad &\sum_{w\in Q}\lambda((\operatorname{tr}\varrho w\sigma_b)^r)\\ &= (q+1)q^{n^2-n-1}\Big\{\prod_{j=1}^{n-1}(q^j-1)-q^{-b(b+3)/2}a_b\\ &\quad\times\sum_{l=1}^{[(n-b+1)/2]}q^{l-1}(q-1)^{n-b+1-2l}\sum\prod_{\nu=1}^{l-1}(q^{j_\nu-2\nu}-1)\Big\}\sum_{\gamma\in\mathbb{F}_q}\lambda(\gamma^r)\\ &\quad+(q+1)q^{n^2-n-1}q^{-b(b+3)/2}a_b\\ &\quad\times\sum_{l=1}^{[(n-b+1)/2]}q^l MK_{n-b+1-2l}(\lambda^r;1,1)\sum\prod_{\nu=1}^{l-1}(q^{j_\nu-2\nu}-1),\end{aligned}$$

where both the unspecified sums run over the same set of integers as in (4.26) and they are 1 for $l=1$.

To express our final results as neatly as possible, we will introduce the following notations. Here one is referred to (2.13), (2.18), and (3.7). In the following, we observe that X_n, X_n' are independent of λ, while the rest do depend on it. Put

$$\begin{aligned}(4.34)\quad X_n &= X_n(q)\\ &= \sum_{0\le b\le n-1,\, b\text{ even}} a_b\begin{bmatrix} n-1\\ b\end{bmatrix}_q\\ &\quad\times\sum_{l=1}^{[(n-b+1)/2]}q^{l-1}(q-1)^{n-b+1-2l}\sum\prod_{\nu=1}^{l-1}(q^{j_\nu-2\nu}-1)\end{aligned}$$

$$= \sum_{b=0}^{[(n-1)/2]} q^{b(b+3)} \begin{bmatrix} n-1 \\ 2b \end{bmatrix}_q \prod_{j=1}^{b} (q^{2j-1}-1)$$

$$\times \sum_{l=1}^{[(n-2b+1)/2]} q^{l-1}(q-1)^{n-2b+1-2l} \sum \prod_{\nu=1}^{l-1} (q^{j_\nu - 2\nu}-1),$$

(4.35) $$X'_n = X'_n(q)$$

$$= - \sum_{0 \le b \le n-1,\, b\ \mathrm{odd}} a_b \begin{bmatrix} n-1 \\ b \end{bmatrix}_q$$

$$\times \sum_{l=1}^{[(n-b+1)/2]} q^{l-1}(q-1)^{n-b+1-2l} \sum \prod_{\nu=1}^{l-1} (q^{j_\nu - 2\nu}-1)$$

$$= \sum_{b=0}^{[(n-2)/2]} q^{b(b+3)} \begin{bmatrix} n-1 \\ 2b+1 \end{bmatrix}_q \prod_{j=1}^{b+1} (q^{2j-1}-1)$$

$$\times \sum_{l=1}^{[(n-2b)/2]} q^{l}(q-1)^{n-2b-2l} \sum \prod_{\nu=1}^{l-1} (q^{j_\nu - 2\nu}-1),$$

(4.36) $$Y_n = Y_n(q, \lambda)$$

$$= \sum_{0 \le b \le n-1,\, b\ \mathrm{even}} a_b \begin{bmatrix} n-1 \\ b \end{bmatrix}_q$$

$$\times \sum_{l=1}^{[(n-b+1)/2]} q^{l} MK_{n-b+1-2l}(\lambda^r; 1, 1) \sum \prod_{\nu=1}^{l-1} (q^{j_\nu - 2\nu}-1)$$

$$= \sum_{b=0}^{[(n-1)/2]} q^{b(b+3)} \begin{bmatrix} n-1 \\ 2b \end{bmatrix}_q \prod_{j=1}^{b} (q^{2j-1}-1)$$

$$\times \sum_{l=1}^{[(n-2b+1)/2]} q^{l} MK_{n-2b+1-2l}(\lambda^r; 1, 1) \sum \prod_{\nu=1}^{l-1} (q^{j_\nu - 2\nu}-1),$$

(4.37) $$Y'_n = Y'_n(q, \lambda)$$

$$= - \sum_{0 \le b \le n-1,\, b\ \mathrm{odd}} a_b \begin{bmatrix} n-1 \\ b \end{bmatrix}_q$$

$$\times \sum_{l=1}^{[(n-b+1)/2]} q^{l} MK_{n-b+1-2l}(\lambda^r; 1, 1) \sum \prod_{\nu=1}^{l-1} (q^{j_\nu - 2\nu}-1)$$

$$= \sum_{b=0}^{[(n-2)/2]} q^{b(b+3)} \begin{bmatrix} n-1 \\ 2b+1 \end{bmatrix}_q \prod_{j=1}^{b+1} (q^{2j-1}-1)$$

$$\times \sum_{l=1}^{[(n-2b)/2]} q^{l+1} MK_{n-2b-2l}(\lambda^r;1,1) \sum \prod_{\nu=1}^{l-1} (q^{j_\nu-2\nu}-1),$$

(4.38) $$Z_n = Z_n(q,\lambda)$$

$$= \sum_{0\le b\le n-1,\, b\text{ even}} a_b \begin{bmatrix} n-1 \\ b \end{bmatrix}_q$$

$$\times \sum_{l=1}^{[(n-b+1)/2]} q^{l} MK_{n-b+2-2l}(\lambda^r;1,1) \sum \prod_{\nu=1}^{l-1} (q^{j_\nu-2\nu}-1)$$

$$= \sum_{b=0}^{[(n-1)/2]} q^{b(b+3)} \begin{bmatrix} n-1 \\ 2b \end{bmatrix}_q \prod_{j=1}^{b} (q^{2j-1}-1)$$

$$\times \sum_{l=1}^{[(n-2b+1)/2]} q^{l} MK_{n-2b+2-2l}(\lambda^r;1,1) \sum \prod_{\nu=1}^{l-1} (q^{j_\nu-2\nu}-1),$$

(4.39) $$Z'_n = Z'_n(q,\lambda)$$

$$= -\sum_{0\le b\le n-1,\, b\text{ odd}} a_b \begin{bmatrix} n-1 \\ b \end{bmatrix}_q$$

$$\times \sum_{l=1}^{[(n-b+1)/2]} q^{l} MK_{n-b+2-2l}(\lambda^r;1,1) \sum \prod_{\nu=1}^{l-1} (q^{j_\nu-2\nu}-1)$$

$$= \sum_{b=0}^{[(n-2)/2]} q^{b(b+3)} \begin{bmatrix} n-1 \\ 2b+1 \end{bmatrix}_q \prod_{j=1}^{b+1} (q^{2j-1}-1)$$

$$\times \sum_{l=1}^{[(n-2b)/2]} q^{l+1} MK_{n-2b+1-2l}(\lambda^r;1,1) \sum \prod_{\nu=1}^{l-1} (q^{j_\nu-2\nu}-1).$$

In the above, all the unspecified sums appearing in X_n, Y_n, and Z_n run over the set of integers $j_1,\dots,j_{l-1}$ satisfying $2l-1 \le j_{l-1} \le \dots \le j_1 \le n-2b$, while all those appearing in X'_n, Y'_n, and Z'_n run over the set of integers $j_1,\dots,j_{l-1}$ satisfying $2l-1 \le j_{l-1} \le \dots \le j_1 \le n-2b-1$. In addition, all the unspecified sums are 1 for $l=1$.

The next theorem now follows from (4.1), (2.13), (2.18), (4.32)–(4.35), (4.37), and (4.38).

THEOREM 4.1. *For any nontrivial additive character* λ *of* $\mathbb{F}_q$ *and any positive integer* r, *the exponential sum over* $\mathrm{SO}^-(2n,q)$

$$\sum_{w\in \mathrm{SO}^-(2n,q)} \lambda((\mathrm{tr}\, w)^r)$$

is given by

$$(4.40)\quad q^{n^2-n-1}\Big\{(q^n+1)\prod_{j=1}^{n-1}(q^{2j}-1)+(q-1)X_n+(q+1)X'_n\Big\} \times \sum_{\gamma\in\mathbb{F}_q}\lambda(\gamma^r) - q^{n^2-n-1}\{(q+1)Y'_n+Z_n\},$$

where $X_n = X_n(q)$, $X'_n = X'_n(q)$, $Y'_n = Y'_n(q,\lambda)$, *and* $Z_n = Z_n(q,\lambda)$ *are respectively as in* (4.34), (4.35), (4.37), *and* (4.38).

REMARK. As is well known [17, Theorem 5.30],

$$\sum_{\gamma\in\mathbb{F}_q}\lambda(\gamma^r)=\sum_{j=1}^{e-1}G(\psi^j,\lambda),$$

where ψ is a multiplicative character of $\mathbb{F}_q$ of order $e=(r,q-1)$ and $G(\psi^j,\lambda)$ is the usual Gauss sum given by

$$G(\psi^j,\lambda)=\sum_{\gamma\in\mathbb{F}_q^\times}\psi^j(\gamma)\lambda(\gamma).$$

Let χ be a multiplicative character of $\mathbb{F}_q$, λ a nontrivial additive character of $\mathbb{F}_q$, and let r be any positive integer. We next want to consider the sum in (1.2)

$$\sum_{w\in \mathrm{O}^-(2n,q)} \chi(\det w)\lambda((\mathrm{tr}\, w)^r)$$

and to find an explicit expression for it.

From the decompositions in (2.8) and (2.9), we see that the above sum is $\sum_{w\in\mathrm{SO}^-(2n,q)}\lambda((\mathrm{tr}\,w)^r)$ plus

$$(4.41)\quad \chi(-1)\Big\{\sum_{0\le b\le n-1,\, b\,\mathrm{odd}}|\mathrm{B}_b\backslash \mathrm{Q}|\sum_{w\in\mathrm{Q}}\lambda((\mathrm{tr}\,w\sigma_b)^r) + \sum_{0\le b\le n-1,\, b\,\mathrm{even}}|\mathrm{B}_b\backslash \mathrm{Q}|\sum_{w\in\mathrm{Q}}\lambda((\mathrm{tr}\,\varrho w\sigma_b)^r)\Big\}.$$

We now obtain the following expression for (4.41) from (2.13), (2.18), (4.32)–(4.36), and (4.39):

$$(4.42)\qquad \chi(-1)q^{n^2-n-1}\Big\{(q^n+1)\prod_{j=1}^{n-1}(q^{2j}-1)-(q+1)X_n-(q-1)X'_n\Big\} \times\sum_{\gamma\in\mathbb{F}_q}\lambda(\gamma^r)+\chi(-1)q^{n^2-n-1}\{(q+1)Y_n+Z'_n\}.$$

Adding up (4.40) and (4.42) and considering $\chi(-1)=1$ and $\chi(-1)=-1$ separately, we get the following result.

THEOREM 4.2. *Let* χ *be a multiplicative character of* $\mathbb{F}_q$, λ *a nontrivial additive character of* $\mathbb{F}_q$, *and let* r *be a positive integer. Then the exponential sum over* $\mathrm{O}^-(2n,q)$

$$\sum_{w\in\mathrm{O}^-(2n,q)}\chi(\det w)\lambda((\mathrm{tr}\,w)^r)$$

is given by

$$(4.43)\qquad q^{n^2-n-1}q^{(1-\chi(-1))/2} \times\Big\{(1+\chi(-1))(q^n+1)\prod_{j=1}^{n-1}(q^{2j}-1)+2(X'_n-\chi(-1)X_n)\Big\}\sum_{\gamma\in\mathbb{F}_q}\lambda(\gamma^r) - q^{n^2-n-1}\{(q+1)(Y'_n-\chi(-1)Y_n)-\chi(-1)(Z'_n-\chi(-1)Z_n)\},$$

where

$$q^{(1-\chi(-1))/2}=\begin{cases}1 & \text{if } \chi(-1)=1,\\ q & \text{if } \chi(-1)=-1,\end{cases}$$

and $X_n=X_n(q)$, $X'_n=X'_n(q)$, $Y_n=Y_n(q,\lambda)$, $Y'_n=Y'_n(q,\lambda)$, $Z_n=Z_n(q,\lambda)$, $Z'_n=Z'_n(q,\lambda)$ *are respectively given by* (4.34)–(4.39).

5. Application to certain countings. If $\mathrm{G}(q)$ is one of finite classical groups over $\mathbb{F}_q$, then, for each $\beta\in\mathbb{F}_q$ and each positive integer r, we put

$$(5.1)\qquad N_{\mathrm{G}(q)}(\beta;r)=|\{w\in\mathrm{G}(q)\mid(\mathrm{tr}\,w)^r=\beta\}|.$$

As applications of the results in Section 4, we derive formulas for (5.1) with $\mathrm{G}(q)=\mathrm{O}^-(2n,q)$ and $\mathrm{SO}^-(2n,q)$. First, we recall the necessary things from [14].

For a nontrivial additive character λ of $\mathbb{F}_q$, a nonnegative integer m, and with β, r as above, we have

$$(5.2)\qquad N_{\mathrm{G}(q)}(\beta;r)=q^{-1}|\mathrm{G}(q)|+q^{-1}\sum_{\alpha\in\mathbb{F}_q^\times}\lambda(-\beta\alpha)\sum_{w\in\mathrm{G}(q)}\lambda(\alpha(\mathrm{tr}\,w)^r),$$

$$(5.3)\qquad \sum_{\alpha\in\mathbb{F}_q^\times}\lambda(-\beta\alpha)\sum_{\gamma\in\mathbb{F}_q}\lambda(\alpha\gamma^r)=q\{N(y^r=\beta)-1\},$$

$$(5.4)\qquad \sum_{\alpha\in\mathbb{F}_q^\times}\lambda(-\beta\alpha)\Big\{\sum_{\gamma_1,\ldots,\gamma_m\in\mathbb{F}_q^\times}\lambda(\alpha(\gamma_1+\gamma_1^{-1}+\ldots+\gamma_m+\gamma_m^{-1})^r)\Big\} = q\sum_{y^r=\beta}\delta(m,q;y)-(q-1)^m,$$

where

$$(5.5)\qquad N(y^r=\beta)=|\{y\in\mathbb{F}_q \mid y^r=\beta\}|,$$

$\delta(m,q;y)$ is as in (4.28) and (4.29), and the sum in (5.4) is over all $y\in\mathbb{F}_q$ with $y^r=\beta$. Note that (4.27) is the $r=1$ case of (5.4).

For each $\alpha\in\mathbb{F}_q^\times$, $\widetilde{\lambda}(u)=\lambda(\alpha u)$ is a nontrivial additive character of $\mathbb{F}_q$. So the explicit expression of $\sum_{w\in \mathrm{O}^-(2n,q)}\lambda(\alpha(\operatorname{tr} w)^r)$ is given by (4.43) with λ replaced by $\widetilde{\lambda}$ and with χ trivial, i.e., with $\sum_{\gamma\in\mathbb{F}_q}\lambda(\gamma^r)$ replaced by $\sum_{\gamma\in\mathbb{F}_q}\lambda(\alpha\gamma^r)$ and $MK_m(\lambda^r;1,1)$ for various values of m in Y_n', Y_n, Z_n', Z_n replaced by the sum in curly brackets in (5.4) for the same corresponding values of m; that of $\sum_{w\in \mathrm{SO}^-(2n,q)}\lambda(\alpha(\operatorname{tr} w)^r)$ is given by (4.40) with the same replacements as for $\mathrm{O}^-(2n,q)$.

Now, these observations together with (5.2)–(5.4), (2.4), (2.5), (4.40), and (4.43) yield the following theorems. In order to state them, we need to introduce some notations.

Remark. By $\widetilde{Y}_n$, $\widetilde{Y}_n'$, $\widetilde{Z}_n$, $\widetilde{Z}_n'$ we denote Y_n, Y_n', Z_n, Z_n' respectively with $MK_m(\lambda^r;1,1)$ replaced by $\sum_{y^r=\beta}\delta(m,q;y)$. Here m is respectively equal to $n-2b+1-2l$, $n-2b-2l$, $n-2b+2-2l$, $n-2b+1-2l$, and Y_n, Y_n', Z_n, Z_n' are as in (4.36)–(4.39).

Theorem 5.1. *For each $\beta\in\mathbb{F}_q$ and each positive integer r, the quantity $N_{\mathrm{O}^-(2n,q)}(\beta;r)$ defined by* (5.1), *with* $\mathrm{G}(q)=\mathrm{O}^-(2n,q)$, *is given by*

$$(5.6)\qquad 2N(y^r=\beta)q^{n^2-n-1}\Big\{(q^n+1)\prod_{j=1}^{n-1}(q^{2j}-1)+(X_n'-X_n)\Big\} + q^{n^2-n-1}\{(\widetilde{Z}_n'-\widetilde{Z}_n)-(q+1)(\widetilde{Y}_n'-\widetilde{Y}_n)\},$$

where $N(y^r=\beta)$ is as in (5.5), *and one is referred to* (4.34) *and* (4.35) *for X_n', X_n and to the above remark for $\widetilde{Z}_n'$, $\widetilde{Z}_n$, $\widetilde{Y}_n'$, $\widetilde{Y}_n$.*

Theorem 5.2. *For each $\beta\in\mathbb{F}_q$ and each positive integer r, the quantity $N_{\mathrm{SO}^-(2n,q)}(\beta;r)$ defined by* (5.1), *with* $\mathrm{G}(q)=\mathrm{SO}^-(2n,q)$, *is given by*

$$(5.7)\qquad N(y^r=\beta)q^{n^2-n-1}\Big\{(q^n+1)\prod_{j=1}^{n-1}(q^{2j}-1)+(q+1)X_n'+(q-1)X_n\Big\} - q^{n^2-n-1}\{(q+1)\widetilde{Y}_n'+\widetilde{Z}_n\}.$$

REMARK. 1. For a finite classical group $G(q)$ over $\mathbb{F}_q$, we put, for brevity,

$$N_{G(q)}(\beta) := N_{G(q)}(\beta; 1) = |\{w \in G(q) \mid \operatorname{tr} w = \beta\}|. \tag{5.8}$$

Then formulas for $N_{O^-(2n,q)}(\beta)$ and $N_{SO^-(2n,q)}(\beta)$ can be obtained from (5.6) and (5.7) respectively by setting $r = 1$, which amounts to replacing $N(y^r = \beta)$ by 1 and $\sum_{y^r=\beta} \delta(m, q; y)$ for various m by $\delta(m, q; \beta)$.

Conversely, we see that the reversed ways are possible by noting that

$$N_{G(q)}(\beta; r) = \sum_{y^r=\beta} N_{G(q)}(y).$$

2. As was stated in [17, (5.70)], $N(y^r = \beta)$ appearing in the above theorems can be expressed as

$$N(y^r = \beta) = \sum_{j=0}^{e-1} \psi^j(\beta),$$

where ψ is any multiplicative character of $\mathbb{F}_q$ of order $e = (r, q-1)$.

References

[1] B. C. Berndt and R. J. Evans, *Sums of Gauss, Jacobi, and Jacobsthal*, J. Number Theory 11 (1979), 349–398.

[2] —, —, *Sums of Gauss, Eisenstein, Jacobi, Jacobsthal, and Brewer*, Illinois J. Math. 23 (1979), 374–437.

[3] —, —, *The determination of Gauss sums*, Bull. Amer. Math. Soc. (N.S.) 5 (1981), 107–129.

[4] B. C. Berndt, R. J. Evans and K. S. Williams, *Gauss and Jacobi Sums*, Canad. Math. Soc. Ser. Monographs Adv. Texts 21, Wiley, New York, 1998.

[5] J. H. Hodges, *Exponential sums for symmetric matrices in a finite field*, Math. Nachr. 14 (1955), 331–339.

[6] —, *Weighted partitions for symmetric matrices in a finite field*, Math. Z. 66 (1956), 13–24.

[7] D. S. Kim, *Gauss sums for general and special linear groups over a finite field*, Arch. Math. (Basel) 69 (1997), 297–304.

[8] —, *Gauss sums for symplectic groups over a finite field*, Monatsh. Math. 126 (1998), 55–71.

[9] —, *Gauss sums for* $O^-(2n, q)$, Acta Arith. 80 (1997), 343–365.

[10] —, *Gauss sums for* $O(2n+1, q)$, Finite Fields Appl. 4 (1998), 62–86.

[11] —, *Gauss sums for* $U(2n, q^2)$, Glasgow Math. J. 40 (1998), 79–95.

[12] —, *Gauss sums for* $U(2n+1, q^2)$, J. Korean Math. Soc. 34 (1997), 871–894.

[13] —, *Exponential sums for symplectic groups and their applications*, Acta Arith. 88 (1999), 155–171.

[14] —, *Exponential sums for* $O^+(2n, q)$ *and their applications*, Acta Math. Hungar., to appear.

[15] D. S. Kim and I.-S. Lee, *Gauss sums for* $O^+(2n, q)$, Acta Arith. 78 (1996), 75–89.

[16] D. S. Kim and Y. H. Park, *Gauss sums for orthogonal groups over a finite field of characteristic two*, ibid. 82 (1997), 331–357.

[17] R. Lidl and H. Niederreiter, *Finite Fields*, Encyclopedia Math. Appl. 20, Cambridge Univ. Press, Cambridge, 1987.

Department of Mathematics
Sogang University
Seoul 121-742, Korea
E-mail: dskim@ccs.sogang.ac.kr

Received on 30.8.1999
and in revised form on 17.8.2000 (3680)