Products of binomial coefficients modulo $p^2$

by

ZHI-WEI SUN (Nanjing)

1. Introduction. As usual $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the ring of integers, the rational field, the real field and the complex field respectively. We also let $\mathbb{Z}^+ = \{1, 2, \ldots\}$ and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. For $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$, by $(a, n)$ we mean the greatest common divisor of $a$ and $n$. If $n$ is odd then the Jacobi symbol $\left(\frac{a}{n}\right)$ is defined in terms of Legendre symbols (see, e.g., [IR]). For $x \in \mathbb{R}$, $\lfloor x \rfloor$ and $\{x\}$ stand for the integral and the fractional parts of $x$ respectively. For a prime $p$ and an integer $a$ prime to $p$, the Fermat quotient $(a^{p-1} - 1)/p$ is denoted by $q_p(a)$. For an odd prime $p$ and $a \in \mathbb{Z}$, we define the Euler quotient

$$
\text{eq}_p(a) = \frac{a^{(p-1)/2} - (a/p)}{p}.
$$

The Gauss lemma used to prove the law of quadratic reciprocity is as follows:

GAUSS’S LEMMA. Let $n > 0$ be an odd integer and $a$ an integer prime to $n$. Then

$$
\left(\frac{a}{n}\right) = (-1)^{|S_n(a)|}
$$

where $S_n(a) = \left\{ k \in \mathbb{Z}^+ : \frac{k}{n} < \frac{1}{2} < \left\{ \frac{ka}{n} \right\} \right\}$.

Almost every textbook on number theory only contains Gauss’s Lemma with $n = p$ being an odd prime. The general version of Gauss’s Lemma was first published by M. Jenkins [J] in 1867 with an elementary proof; in the textbook [R] H. Rademacher supplied a proof using subtle properties of quadratic Gauss sums.

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For \( x \in \mathbb{R} \) let
\[
\begin{pmatrix} x \\ 0 \end{pmatrix} = 1 \quad \text{and} \quad \begin{pmatrix} x \\ n \end{pmatrix} = \frac{1}{n!} \prod_{j=0}^{n-1} (x - j) \quad \text{for } n = 1, 2, \ldots
\]

Recently A. Granville [G] obtained a congruence for \( \prod_{0 < k < n} \left( \frac{p-1}{[pk/n]} \right) \mod p^2 \) where \( p \) is an odd prime not dividing \( n \in \mathbb{Z}^+ \). With the help of Gauss’s Lemma, we are able to get the following more general result.

**Theorem 1.1.** Let \( m \in \mathbb{Z} \) and \( n \in \mathbb{Z}^+ \). Let \( p \) be an odd prime not dividing \( n \).

(i) If \( \delta \in \{0, 1\} \) then
\[
(1.3) \quad (-1)^{\frac{n-1}{2} - \left\lfloor \frac{n-\delta}{2} \right\rfloor} \prod_{0 < k \leq \left\lfloor \frac{n-\delta}{2} \right\rfloor} \left( \frac{pm-1}{[pk/n]} \right) \\
\equiv \begin{cases} 
\frac{n}{p} + pmn \text{eq}_p(n) \pmod{p^2} & \text{if } 2 \mid n, \\
\frac{2n}{p} + p\left( -1 \right)^{\delta(n/p)} 2 \text{eq}_p(2) + \left( \frac{2}{p} \right) n \text{eq}_p(n) \pmod{p^2} & \text{if } 2 \nmid n.
\end{cases}
\]

(ii) We have
\[
(1.4) \quad \sum_{k=0}^{n-1} (-1)^{k+(n-1)[pk/n]} \left( \frac{pm-1}{[pk/n]} \right) \\
\equiv \begin{cases} 
mn(1 - 2^{p-1}) \pmod{p^2} & \text{if } 2 \mid n, \\
1 \pmod{p^2} & \text{if } 2 \nmid n.
\end{cases}
\]

**Remark 1.1.** In (1.3) we use Euler quotients instead of Fermat quotients, this makes the congruence somewhat symmetric in the case \( 2 \mid n \).

Now we deduce Granville’s result from our Theorem 1.1.

**Corollary 1.1 (Granville [G]).** Let \( n \) be a positive integer and \( p \) an odd prime not dividing \( n \). Then
\[
(1.5) \quad \prod_{0 < k < n} \left( \frac{p-1}{[pk/n]} \right) \equiv (-1)^{\frac{n-1}{2} + (n-1)}(n^p - n + 1) \pmod{p^2}.
\]

**Proof.** Observe that
\[
(-1)^{\frac{n-1}{2} + (n-1)} \prod_{0 < k < n} \left( \frac{p-1}{[pk/n]} \right) \\
= \left( -1 \right)^{\frac{n-1}{2} \left( \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil \right)} \prod_{0 < k \leq \left\lfloor \frac{n-1}{2} \right\rfloor} \left( \frac{p-1}{[pk/n]} \right) \cdot \prod_{0 < k \leq \left\lfloor \frac{n}{2} \right\rfloor} \left( \frac{p-1}{[p(k-n)/n]} \right) \\
= (-1)^{\frac{n-1}{2} + (n-1)} \prod_{0 < k \leq \left\lfloor \frac{n-1}{2} \right\rfloor} \left( \frac{p-1}{[pk/n]} \right) \cdot (-1)^{\frac{n-1}{2} + (n-1)} \prod_{0 < k \leq \left\lfloor \frac{n}{2} \right\rfloor} \left( \frac{p-1}{[pk/n]} \right).
\]
Applying Theorem 1.1(i) with \( m = 1 \) and \( \delta = 0, 1 \), we then obtain
\[
(-1)^{\frac{p-1}{2}} (n-1) \prod_{0 < k < n} \left( \frac{p-1}{[pk/n]} \right) \equiv 1 + 2pn \left( \frac{n}{p} \right) \text{eq}_p(n) \pmod{p^2}.
\]
For any integer \( a \) prime to \( p \), clearly
\[
a^{p-1} - 1 = \left( a^{(p-1)/2} + \left( \frac{a}{p} \right) \right) \left( a^{(p-1)/2} - \left( \frac{a}{p} \right) \right) \equiv 2 \left( \frac{a}{p} \right) p \text{eq}_p(a) \pmod{p^2}.
\]
So (1.5) follows. ■

For \( a, n \in \mathbb{Z} \) with \( 0 \leq a < n \), we let
\[a(n) = a \mod n = a + n\mathbb{Z} = \{a + nx : x \in \mathbb{Z}\}.
\]
For a finite system \( A = \{a_s(n_s)\}_{s=1}^k \) of such residue classes, we define the covering function \( w_A : \mathbb{Z} \to \{0, 1, \ldots\} \) by
\[
w_A(x) = |\{1 \leq s \leq k : x \in a_s(n_s)\}|.
\]
When \( w_A(x) = m \) for all \( x \in \mathbb{Z} \), \( A \) is said to be an exact \( m \)-cover (of \( \mathbb{Z} \)). We also use the term disjoint cover instead of exact 1-cover. (See [S3] and [S4] for problems and results on covers of \( \mathbb{Z} \).) For two systems \( A = \{a_s(n_s)\} \) and \( B = \{b_t(m_t)\} \) of residue classes, if \( w_A = w_B \), then we say that \( A \) is covering equivalent to \( B \), and denote this by \( A \sim B \). For \( d, n \in \mathbb{Z}^+ \) and \( a \in \{0, 1, \ldots, d-1\} \), clearly
\[
\{a + jd(n)\}_{j=0}^{n-1} \sim \{a(d)\},
\]
in particular \( \{r(n)\}_{r=0}^{n-1} \sim \{0(1)\} \).

In this paper we will also prove the following extension of Corollary 1.1.

**Theorem 1.2.** Let \( p \) be an odd prime. Let \( A = \{a_s(n_s)\}_{s=1}^k \) (\( 0 \leq a_s < n_s \)) and \( B = \{b_t(m_t)\}_{t=1}^l \) (\( 0 \leq b_t < m_t \)) be covering equivalent systems with the moduli \( n_s \) and \( m_t \) not divisible by \( p \) but dividing an integer \( N \). Then for any \( x \in [0, p) \) we have
\[
\prod_{s=1}^k \left( \frac{pN/n_s - 1}{[(x + pa_s)/n_s]} \right) / \prod_{t=1}^l \left( \frac{pN/m_t - 1}{[(x + pb_t)/m_t]} \right) \equiv (-1)^{(k-l)(p-1)/2} \left( 1 + pN \left( \sum_{s=1}^k \frac{q_p(n_s)}{n_s} - \sum_{t=1}^l \frac{q_p(m_t)}{m_t} \right) \right) \pmod{p^2}.
\]

**Remark 1.2.** Actually we may not require the integer \( N \) in Theorem 1.2 to be a common multiple of those moduli \( n_s \) and \( m_t \). For example \( N = 1 \) is allowed if we do not mind using \( x \not\in \mathbb{Z} \) in the notation \( \binom{x}{n} \).

**Corollary 1.2.** Let \( A = \{a_s(n_s)\}_{s=1}^k \) (\( 0 \leq a_s < n_s \)) be an exact \( m \)-cover of \( \mathbb{Z} \). Let \( N \) be the least common multiple of \( n_1, \ldots, n_k \) and \( p \) an
odd prime not dividing $N$. Then

\[(1.9) \prod_{s=1}^{k} \left( \frac{pN/n_s - 1}{p a_s/n_s} \right) \equiv (-1)^{(k-m)(p-1)/2} \left( 1 + pN \sum_{s=1}^{k} \frac{q_p(n_s)}{n_s} \right) \pmod{p^2}. \]

**Proof.** Let $B$ be the system consisting of $m$ copies of $0(1)$. Then $A \sim B$. Since $\left[ \frac{p0}{1} \right] = \frac{q_p(1)}{1} = 0$, Corollary 1.2 follows immediately from Theorem 1.2.

**Remark 1.3.** Applying Corollary 1.2 to the trivial disjoint cover $A = \{r(n)\}_{r=0}^{n-1}$ we then get Corollary 1.1 again.

In the next section we will give some examples of uniform maps the concept of which arose from our previous study of covering equivalence (cf. [S1] and [S2]). On the basis of Section 2, we prove Theorems 1.1 and 1.2 in Section 3.

2. Some uniform maps

**Definition 2.1.** Let $m$ be an integer and $M$ an additive abelian group. Let $f$ be a map from a subset of $\mathbb{C} \times \mathbb{C}$ into $M$. If for any ordered pair $(x, y)$ in the domain $\text{Dom}(f)$ of $f$ and each positive integer $n$ prime to $m$, we have

\[(2.1) \ \{ \left\langle \frac{x + mr}{ny}, ny \right\rangle : r = 0, 1, \ldots, n - 1 \} \subseteq \text{Dom}(f) \]

and

\[(2.2) \ \sum_{r=0}^{n-1} f \left( \frac{x + mr}{n}, ny \right) = f(x, y), \]

then we call $f$ an $m$-uniform map (into $M$).

The functional equation (2.2) with $m = 1$ was first introduced by the author in [S1] where he showed the following theorem in the case $m = 1$ by a complicated induction method.

**Theorem 2.1.** Let $m$ be an integer and $M$ a left $R$-module where $R$ is a ring with identity. Let $f$ be a map into $M$ with $\text{Dom}(f) \subseteq \mathbb{C} \times \mathbb{C}$ such that (2.1) holds for any $(x, y) \in \text{Dom}(f)$ and $n \in \mathbb{Z}^+$ with $(m, n) = 1$. Then the following two statements are equivalent:

(a) $f$ is an $m$-uniform map into $M$.

(b) Whenever

\[(2.3) \ \sum_{1 \leq s \leq k, \ x \in a_s(n_s)} \lambda_s = \sum_{1 \leq t \leq l, \ x \in b_t(m_t)} \mu_t \quad \text{for all } x \in \mathbb{Z} \]
(with \( \lambda_s, \mu_t \in R, a_s, n_s, b_t, m_t \in \mathbb{Z}, 0 \leq a_s < n_s, 0 \leq b_t < m_t \) and \((n_sm_t, m_t) = 1\)), we have

\[
\binom{x + ma_s}{n_s} \binom{y + mb_t}{m_t} = \binom{x + ma_s}{n_s} \binom{y + mb_t}{m_t} \quad \text{for } \langle x, y \rangle \in \text{Dom}(f).
\]

**Proof.** Since \( \{r(n)\}_{r=0}^{n-1} \sim \{0(1)\} \) for all \( n \in \mathbb{Z}^+ \), (b) implies (a).

Now we show (b) under the condition (a). Suppose that (2.3) holds. Let \( N \) be the least common multiple of those moduli \( n_s \) and \( m_t \). If \( \langle x, y \rangle \in \text{Dom}(f) \), then

\[
\sum_{s=1}^{k} \lambda_s \binom{x + ma_s}{n_s} = \sum_{s=1}^{k} \lambda_s \sum_{j=0}^{N/n_s - 1} \binom{(x + ma_s)/n_s + jm}{N/n_s} \binom{y}{n_s} = \sum_{s=1}^{k} \sum_{r=0}^{N-1} \binom{x + mr}{N} \binom{y}{n_s} = \sum_{s=1}^{k} \sum_{1 \leq r \leq k} \binom{x + mr}{N} \binom{y}{n_s} = \sum_{t=1}^{l} \mu_t \binom{x + mb_t}{m_t} \binom{y}{m_t}.
\]

**Proposition 2.1.** (i) Let \( m \in \mathbb{Z} \). Then the function \([ ]_m : \mathbb{R} \times \mathbb{R} \to \mathbb{Q}\) given by

\[
[x, y]_m = [x] + \frac{1 - m}{2}
\]

is an \( m \)-uniform map into the rational field \( \mathbb{Q} \).

(ii) For each \( m \geq 0, 1, \ldots \) the functions \( b_m : \mathbb{C} \times \mathbb{C}^* \to \mathbb{C} \) and \( e_m : \mathbb{C} \times \mathbb{Z} \to \mathbb{C} \) given by

\[
b_m(x, y) = y^{m-1} B_m(x)
\]

and

\[
e_m(x, y) = \begin{cases} e^{\pi ixy} y^m E_m(x) & \text{if } y \text{ is odd}, \\ -\frac{2}{m+1} e^{\pi ixy} y^m B_{m+1}(x) & \text{if } y \text{ is even}, \end{cases}
\]

are \( 1 \)-uniform maps into the complex field \( \mathbb{C} \), where \( B_m(x) \) and \( E_m(x) \) are the \( m \)th Bernoulli polynomial and the \( m \)th Euler polynomial respectively.

**Proof.** Let \( n \) be any positive integer.

(i) If \((m, n) = 1\) then
\[
\sum_{r=0}^{n-1} \left( \left\lfloor \frac{x + mr}{n} \right\rfloor + \frac{1 - m}{2} \right)
= \sum_{r=0}^{n-1} \left( \frac{x + mr}{n} + \frac{1 - m}{2} \right) - \left\{ \frac{x + mr}{n} \right\} \\
= x + m \sum_{r=0}^{n-1} \left( \frac{r}{n} - \frac{1}{2} \right) - \sum_{r=0}^{n-1} \left( \left\{ \frac{x}{n} + \left\lfloor \frac{x}{n} \right\rfloor + mr \right\} - \frac{1}{2} \right) \\
= x - \frac{m}{2} - \sum_{s=0}^{n-1} \left( \left\{ \frac{x}{n} + s \right\} - \frac{1}{2} \right) = x - \frac{m}{2} - \left( \left\{ \frac{x}{n} \right\} - \frac{1}{2} \right) = \lceil x \rceil + \frac{1 - m}{2}.
\]

(ii) Let \(m\) be a nonnegative integer. Raabe’s identity states that
\[
\sum_{r=0}^{n-1} B_m \left( z + \frac{r}{n} \right) = n^{1-m} B_m(nz).
\]
Another known identity (cf. \([B]\)) asserts that
\[
E_m(nz) = \begin{cases} 
\frac{n^m}{m+1} \sum_{r=0}^{n-1} (-1)^r B_{m+1} \left( z + \frac{r}{n} \right) & \text{if } 2 \nmid n, \\
- \frac{2n^m \sum_{r=0}^{n-1} (-1)^r B_{m+1} \left( z + \frac{r}{n} \right)}{m+1} & \text{if } 2 \mid n.
\end{cases}
\]

By these two identities we can easily check that
\[
\sum_{r=0}^{n-1} b_m \left( \frac{x + r}{n}, ny \right) = b_m(x, y) \quad \text{for } x \in \mathbb{C} \text{ and } y \in \mathbb{C}^* \\
\text{and} \\
\sum_{r=0}^{n-1} e_m \left( \frac{x + r}{n}, ny \right) = e_m(x, y) \quad \text{for } x \in \mathbb{C} \text{ and } y \in \mathbb{Z}.
\]

Remark 2.1. In \([S1]\) the author briefly mentioned the basic things for Proposition 2.1. For more examples of 1-uniform maps, the reader is referred to \([S5]\).

Corollary 2.1. Let \(p\) be an odd prime and \(n > 0\) an even integer prime to \(p\). Then
\[
\sum_{r=0}^{n-1} (-1)^r B_{p-1} \left( \frac{r}{n} \right) \equiv -n q_p(2) \pmod{p}.
\]
Proof. By Proposition 2.1,
\[
\frac{2n^{p-2}}{1-p} \sum_{r=0}^{n-1} (-1)^r \binom{r}{n} B_{p-1} \left( \frac{r}{n} \right) = \sum_{r=0}^{n-1} e_{p-2} \left( \frac{r}{n}, n \right) = e_{p-2}(0,1)
\]
does not depend on the value of the positive even integer \(n\). So
\[
n^{p-2} \sum_{r=0}^{n-1} (-1)^r \binom{r}{n} B_{p-1} \left( \frac{r}{n} \right) = 2^{p-2} \left( 2B_{p-1} - \sum_{r=0}^{2-1} B_{p-1} \left( \frac{r}{2} \right) \right) = 2^{p-1} B_{p-1} - B_{p-1}.
\]
Since
\[
pB_{p-1} \equiv \sum_{r=1}^{p-1} r^{p-1} \equiv -1 \pmod{p}
\]
(see, e.g., [IR]), (2.10) follows at once. \[\blacksquare\]

**Proposition 2.2.** Let \(p\) be an odd prime. For \(x \geq 0\) and \(m \in \mathbb{Z} \setminus p\mathbb{Z}\) let
\[
q(x, m) = \frac{q_p(m)}{m} + \sum_{0 < j \leq [x]} \frac{1}{jm}.
\]
Then the function \(q(x, m) = q(x, m) \pmod{p}\) is a \(p\)-uniform map into the finite field \(\mathbb{Z}/p\mathbb{Z}\).

**Proof.** Let \(m \in \mathbb{Z} \setminus p\mathbb{Z}\) and \(n \in \mathbb{Z}^+ \setminus p\mathbb{Z}\). Since
\[
q_p(mn) = \frac{m^{p-1} - 1}{p} + m^{p-1} \frac{n^{p-1} - 1}{p} \equiv q_p(m) + q_p(n) \pmod{p},
\]
for \(x \geq 0\) the congruence
\[
\sum_{k=0}^{n-1} q \left( \frac{x + pk}{n}, nm \right) \equiv q(x, m) \pmod{p}
\]
is equivalent to
\[
q_p(n) \equiv \sum_{0 < j \leq [x]} \frac{1}{j} - \frac{1}{n} \sum_{k=0}^{n-1} \sum_{0 < j \leq [(x+pk)/n]} \frac{1}{j} \pmod{p}.
\]
Now it suffices to show (2.12) for all \(x = 0, 1, \ldots\)

By pp. 125–126 of [GS] we have
\[
B_{p-1} \left( \left\{ \frac{pk}{n} \right\} \right) - B_{p-1} \equiv - \sum_{0 < j \leq [pk/n]} \frac{1}{j} \pmod{p}
\]
for \(k = 0, 1, \ldots, n - 1\).
Observe that
\[
\sum_{k=0}^{n-1} \left( B_{p-1} \left( \left\{ \frac{pk}{n} \right\} \right) - B_{p-1} \right) = \sum_{r=0}^{n-1} B_{p-1} \left( \frac{r}{n} \right) - nB_{p-1} = n^{2-p}B_{p-1} - nB_{p-1} = n \frac{1-n^{p-1}}{p} (pB_{p-1}) \equiv nq_p(n) \pmod{p}.
\]
Thus (2.12) holds for \( x = 0 \).

Let \( r \in \mathbb{Z}^+ \). Assume (2.12) for \( x = r-1 \). Denote by \( k_0 \) the unique integer \( k \in [0, n) \) such that \( r + pk \equiv 0 \pmod{n} \).

Clearly \( p \mid r \) if and only if \( p \) divides \( j_0 = (r + pk_0)/n \). For \( k \in \{0, 1, \ldots, n-1\} \), we have
\[
\left\lfloor \frac{r + pk}{n} \right\rfloor = \left\lfloor \frac{r - 1 + pk}{n} \right\rfloor + \begin{cases} 1 & \text{if } k = k_0, \\ 0 & \text{otherwise}. \end{cases}
\]
If \( p \nmid r \), then
\[
\frac{1}{r} - \frac{1}{n} \cdot \frac{1}{j_0} = \frac{1}{r} - \frac{1}{r + pk_0} \equiv 0 \pmod{p}.
\]
Thus
\[
\sum_{0<j\leq r \atop p\mid j} \frac{1}{j} - \frac{1}{n} \sum_{k=0}^{n-1} \sum_{0<j\leq \lfloor (r+pk)/n \rfloor \atop p\mid j} \frac{1}{j} = \frac{1}{n} \sum_{0<j\leq r-1 \atop p\mid j} \frac{1}{j} - \frac{1}{n} \sum_{k=0}^{n-1} \sum_{0<j\leq \lfloor (r-1+pk)/n \rfloor \atop p\mid j} \frac{1}{j} \equiv q_p(n) \pmod{p}.
\]
This concludes the induction step. We are done.

3. Proofs of Theorems 1.1 and 1.2

**Lemma 3.1.** (i) Let \( a \in \mathbb{Z} \), \( n \in \mathbb{Z}^+ \) and \( (2a, n) = 1 \). Then
\[(3.1) \quad |S_n(a)| \equiv \sum_{0<k\leq n/2} \left\lfloor \frac{ka}{n} \right\rfloor + \frac{n^2 - 1}{8} (a - 1) \pmod{2}.
\]
(ii) Let \( m, n \in \mathbb{Z}^+ \) and \( (m, n) = 1 \). Then for \( \delta \in \{0, 1\} \) we have
\[(3.2) \quad \sum_{0<k\leq (n-\delta)/2} \left\lfloor \frac{km}{n} \right\rfloor + \sum_{0<k\leq (m-\delta)/2} \left\lfloor \frac{kn}{m} \right\rfloor = \left\lfloor \frac{m-\delta}{2} \right\rfloor \left\lfloor \frac{n-\delta}{2} \right\rfloor.
\]
The above lemma is well known and usually stated in textbooks with \( a, m, n \) being odd primes.
Lemma 3.2. Let \( k, m, n \in \mathbb{Z} \) and \( 0 \leq k < n \). Let \( p \) be an odd prime not dividing \( n \). Then

\[
(-1)^{[pk/n]} \binom{pm-1}{[pk/n]} \equiv 1 + pm \left( B_{p-1} \left( \left\{ \frac{pk}{n} \right\} \right) - B_{p-1} \right) \pmod{p^2}.
\]

Proof. For any \( l \in \{0, 1, \ldots, p-1\} \),

\[
(-1)^l \binom{pm-1}{l} = \prod_{0 < j \leq l} \left( 1 - p \frac{m}{j} \right) \equiv 1 - pm \sum_{0 < j \leq l} \frac{1}{j} \pmod{p^2}.
\]

Combining this with (2.13) we then obtain (3.3).

Proof of Theorem 1.1. As \( p - 1 \) is even, we have \( B_{p-1}(1-x) = B_{p-1}(x) \).

(i) Let \( l = \lfloor (n - \delta)/2 \rfloor \) and \( \varepsilon_n = (1 + (-1)^n)/2 \). By Lemma 3.2,

\[
\prod_{0 < k \leq l} (-1)^{[pk/n]} \binom{pm-1}{[pk/n]} \equiv 1 + pm \sum_{0 < k \leq l} \left( B_{p-1} \left( \left\{ \frac{pk}{n} \right\} \right) - B_{p-1} \right) \pmod{p^2}.
\]

Observe that

\[
2 \sum_{0 < k \leq l} \left( B_{p-1} \left( \left\{ \frac{pk}{n} \right\} \right) - B_{p-1} \right) - \varepsilon_n(-1)\delta \left( B_{p-1} \left( \frac{1}{2} \right) - B_{p-1} \right) = \sum_{0 < k \leq l} \left( B_{p-1} \left( \left\{ \frac{pk}{n} \right\} \right) + B_{p-1} \left( \left\{ \frac{p(n-k)}{n} \right\} \right) - 2B_{p-1} \right)
\]

\[
- \varepsilon_n(-1)\delta \left( B_{p-1} \left( \left\{ \frac{p}{2} \right\} \right) - B_{p-1} \right)
\]

\[
= \sum_{k=0}^{n-1} \left( B_{p-1} \left( \left\{ \frac{pk}{n} \right\} \right) - B_{p-1} \right) \equiv nq_p(n) \pmod{p}
\]

where the last step is taken as in the proof of Proposition 2.2. By Corollary 2.1, \( B_{p-1}(1/2) - B_{p-1} \equiv 2q_p(2) \pmod{p} \). Recall that \( q_p(a) \equiv 2 \left( \frac{a}{p} \right) \pmod{p} \) for any \( a \in \mathbb{Z} \) with \((a, p) = 1\). So

\[
\sum_{0 < k \leq l} \left( B_{p-1} \left( \left\{ \frac{pk}{n} \right\} \right) - B_{p-1} \right) \equiv n \left( \frac{n}{p} \right) \pmod{p} \]

By Lemma 3.1 and Gauss’s Lemma,

\[
(-1)^{\sum_{0 < k \leq l} [pk/n]} = (-1)^{l(p-1)/2 - \sum_{0 < k < p/2} [nk/p]} = (-1)^{l(p-1)/2} \left( \frac{n}{p} \right) \left( \frac{2}{p} \right)^{n-1}.
\]
Therefore
\[
(-1)^{l(p-1)/2} \left( \frac{n}{p} \right) \left( \frac{2}{p} \right)^{n-1} \prod_{0 < k \leq l} \left( \frac{pm - 1}{[pk/n]} \right)
\]
\[
= \prod_{0 < k \leq l} (-1)^{[pk/n]} \left( \frac{pm - 1}{[pk/n]} \right)
\]
\[
\equiv 1 + pm \left( n \left( \frac{n}{p} \right) \text{eq}_p(n) + \varepsilon_n(-1)^{\delta} \left( \frac{2}{p} \right) \text{eq}_p(2) \right) \pmod{p^2}
\]
and hence (1.3) follows.

(ii) Write \( S \) for the left hand side of (1.4) and set
\[
S' = \sum_{r=0}^{n-1} (-1)^r B_{p-1} \left( \frac{r}{n} \right).
\]

By Lemma 3.2,
\[
S \equiv \sum_{k=0}^{n-1} (-1)^{\{pk\}_n} \left( 1 + pm \left( B_{p-1} \left( \frac{\{pk\}_n}{n} \right) - B_{p-1} \right) \right)
\]
\[
\equiv (1 - pm B_{p-1}) \Delta + pm S' \pmod{p^2}
\]
where
\[
\{pk\}_n = n \left\{ \frac{pk}{n} \right\} = pk - n \left\lfloor \frac{pk}{n} \right\rfloor \quad \text{and} \quad \Delta = \sum_{r=0}^{n-1} (-1)^r = \frac{1 - (-1)^n}{2}.
\]
If \( 2 \nmid n \), then \( S' = B_{p-1} \) since
\[
(-1)^{n-r} B_{p-1} \left( \frac{n-r}{n} \right) = (-1)^r B_{p-1} \left( \frac{r}{n} \right),
\]
therefore \( S \equiv 1 \pmod{p^2} \). When \( 2 \mid n \) we may apply Corollary 2.1. This concludes the proof. \( \blacksquare \)

**Proof of Theorem 1.2.** Since \( A \sim B \), by Theorem 2.1 and Proposition 2.1 we have
\[
\sum_{s=1}^{k} \left( \left\lfloor \frac{x + pa_s}{n_s} \right\rfloor + \frac{1-p}{2} \right) = \sum_{t=1}^{l} \left( \left\lfloor \frac{x + pb_t}{m_t} \right\rfloor + \frac{1-p}{2} \right).
\]
So (1.8) is equivalent to the following
\[
P_A = \prod_{s=1}^{k} (-1)^{(x+pa_s)/n_s} \left( \frac{pN/n_s - 1}{[(x+pa_s)/n_s]} \right) \cdot \left( 1 - pN \sum_{s=1}^{k} \frac{q_p(n_s)}{n_s} \right)
\]
\[
\equiv P_B = \prod_{t=1}^{l} (-1)^{(x+pb_t)/m_t} \left( \frac{pN/m_t - 1}{[(x+pb_t)/m_t]} \right) \times \left( 1 - pN \sum_{t=1}^{l} \frac{q_p(m_t)}{m_t} \right) \pmod{p^2}.
\]
By (3.4) we have
\[
P_A \equiv \prod_{s=1}^k \left(1 - p \frac{N}{n_s} \sum_{0 < j \leq [(x+pa_s)/n_s]} \frac{1}{j} \right) \left(1 - pN \frac{qp(n_s)}{n_s} \right)
\]
\[
\equiv \prod_{s=1}^k \left(1 - pN \frac{q(n_s)}{n_s} + \sum_{0 < j \leq [(x+pa_s)/n_s]} \frac{1}{j} \right)
\]
\[
\equiv \prod_{s=1}^k \left(1 - pNq \frac{x+pa_s}{n_s}, n_s \right)
\]
\[
\equiv 1 - pN \sum_{s=1}^k q \left(\frac{x+pa_s}{n_s}, n_s \right) \pmod{p^2};
\]
similarly
\[
P_B \equiv 1 - pN \sum_{i=1}^l q \left(\frac{x+pb_i}{m_i}, m_i \right) \pmod{p^2}.
\]

In view of Theorem 2.1 and Proposition 2.2, \(P_A \equiv P_B \pmod{p^2}\). We are done. \(\blacksquare\)

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Department of Mathematics
Nanjing University
Nanjing 210093
People’s Republic of China
E-mail: zwsun@nju.edu.cn

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