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Decomposing rational numbers

by

Håkan Lennerstad and Lars Lundberg (Karlskrona)

1. Introduction. We consider a basic number-theoretical question that does not appear to have been examined before. Suppose a white marbles and b black marbles are to be placed into c bags, with at least one marble in each bag, in such a way that the white-to-black ratio a_i/b_i in each bag is as close to a/b as possible. How large can we make the smallest such ratio? How small can we make the largest such ratio?

The problem has arisen from a computer science scenario. The results of this paper provide an important argument in [4], which extends the so-called 4/3-conjecture in computer science to a general setting. In [2] this conjecture is proven in the original version.

Formally, if $a \geq 0$, $b \geq 0$, and $1 \leq c \leq a+b$, then a *c-decomposition of* a/b is a collection P of ordered pairs $\{(a_1,b_1),\dots,(a_c,b_c)\}$ such that

$$a_i \geq 0, \quad b_i \geq 0, \quad a_i+b_i>0 \quad \text{for } 1 \leq i \leq c, \tag{1.1}$$

$$(a_1,b_1)+\dots+(a_c,b_c)=(a,b). \tag{1.2}$$

All variables in this paper are integers, and addition of ordered pairs is done componentwise. We define

$$\min P = \min(a_i/b_i,\dots,a_c/b_c), \qquad \max P = \max(a_i/b_i,\dots,a_c/b_c), \tag{1.3}$$

$$\left\lfloor \begin{matrix} a \\ b \end{matrix} \right\rfloor_c = \max_P \min P, \qquad \left\lceil \begin{matrix} a \\ b \end{matrix} \right\rceil_c = \min_P \max P, \tag{1.4}$$

where the outermost max and min are taken over all c-decompositions P of a/b. The quantities $\left\lfloor \begin{smallmatrix} a \\ b \end{smallmatrix} \right\rfloor_c$ and $\left\lceil \begin{smallmatrix} a \\ b \end{smallmatrix} \right\rceil_c$ can be termed "c-floor" and "c-ceiling" of a and b, respectively. Indeed,

$$\left\lfloor \begin{matrix} a \\ b \end{matrix} \right\rfloor_b = \left\lfloor \frac{a}{b} \right\rfloor \quad \text{and} \quad \left\lceil \begin{matrix} a \\ b \end{matrix} \right\rceil_b = \left\lceil \frac{a}{b} \right\rceil. \tag{1.5}$$

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The reader may verify, for example, that $\left\lfloor \frac{16}{7} \right\rfloor_4 = \frac{2}{1}$ and $\left\lceil \frac{16}{7} \right\rceil_4 = \frac{5}{2}$. The 4-decomposition $\{(2,1),(2,1),(7,3),(5,2)\}$ realizes both bounds simultaneously, and can be generated by the algorithm that is constructed upon the Stern–Brocot tree in Section 3. For the 4-decomposition $\{(2,1),(4,2),(5,2),(5,2)\}$ this is not true; it has the same minimum and maximum. However, in that decomposition, not all ratios are relatively prime.

The main goal of this paper is to give a constructive proof of the following results:

THEOREM 1. *Let $d = \gcd(a,b)$. If $d \le c \le a+b$, there is a unique c-decomposition such that*

$$\left\lfloor \begin{matrix} a \\ b \end{matrix} \right\rfloor_c = a_1/b_1 \le \cdots \le a_c/b_c = \left\lceil \begin{matrix} a \\ b \end{matrix} \right\rceil_c \tag{1.6}$$

and such that the number

$$\min\left\{ j : a_j/b_j = \left\lceil \begin{matrix} a \\ b \end{matrix} \right\rceil_c \right\} - \max\left\{ i : a_i/b_i = \left\lfloor \begin{matrix} a \\ b \end{matrix} \right\rfloor_c \right\} - 1 \tag{1.7}$$

of ratios that lie strictly between $\left\lfloor \frac{a}{b} \right\rfloor_c$ and $\left\lceil \frac{a}{b} \right\rceil_c$ is maximized. At most three distinct ordered pairs (a_i, b_i) occur in this decomposition, and a_i is relatively prime to b_i in each case. Such a c-decomposition can be computed in $O(\log \min(a,b))$ steps.

The following approximation property with rationals with denominator restricted to $b-c+1$ underlines the generalization of the usual floor and ceiling functions, which correspond to the case $c = b$.

THEOREM 2. *For any non-negative integers a, b, c,*

$$\left\lfloor \begin{matrix} a \\ b \end{matrix} \right\rfloor_c = \max\left\{ \frac{c}{d} : \frac{c}{d} \le \frac{a}{b},\ c, d \in \mathbb{N},\ d \le b-c+1 \right\},$$

$$\left\lceil \begin{matrix} a \\ b \end{matrix} \right\rceil_c = \min\left\{ \frac{c}{d} : \frac{c}{d} \ge \frac{a}{b},\ c, d \in \mathbb{N},\ d \le b-c+1 \right\}.$$

2. Basic properties. Since $a/b = \sum_{i=1}^{c} (b_i/b)(a_i/b_i)$ is a convex combination of the ratios a_i/b_i, we always have

$$0 \le \left\lfloor \begin{matrix} a \\ b \end{matrix} \right\rfloor_c \le \frac{a}{b} \le \left\lceil \begin{matrix} a \\ b \end{matrix} \right\rceil_c \le \infty. \tag{2.1}$$

Furthermore, one sees easily that $0 = \left\lfloor \frac{a}{b} \right\rfloor_c$ if and only if $c > a$; $\left\lfloor \frac{a}{b} \right\rfloor_c = \frac{a}{b}$ if and only if $\frac{a}{b} = \left\lceil \frac{a}{b} \right\rceil_c$ if and only if $c \le \gcd(a,b)$; and $\left\lceil \frac{a}{b} \right\rceil_c = \infty$ if and only if $c > b$.

Given any c-decomposition with $\min(a_1/b_1, \ldots, a_c/b_c) = \left\lfloor \frac{a}{b} \right\rfloor_c$ and $c > 1$, we have

$$\left\lfloor \begin{matrix} a \\ b \end{matrix} \right\rfloor_{c-1} \geq \min\bigl(a_1/b_1, \ldots, a_{c-2}/b_{c-2}, (a_{c-1}+a_c)/(b_{c-1}+b_c)\bigr) \geq \left\lfloor \begin{matrix} a \\ b \end{matrix} \right\rfloor_{c}; \tag{2.2}$$

and a similar argument proves that

$$\left\lceil \begin{matrix} a \\ b \end{matrix} \right\rceil_{c-1} \leq \left\lceil \begin{matrix} a \\ b \end{matrix} \right\rceil_{c}. \tag{2.3}$$

The symmetry of the definitions (1.1)–(1.4) immediately implies the reciprocity laws

$$\left\lfloor \begin{matrix} b \\ a \end{matrix} \right\rfloor_{c} = 1\Big/\left\lceil \begin{matrix} a \\ b \end{matrix} \right\rceil_{c} \quad \text{and} \quad \left\lceil \begin{matrix} b \\ a \end{matrix} \right\rceil_{c} = 1\Big/\left\lfloor \begin{matrix} a \\ b \end{matrix} \right\rfloor_{c}. \tag{2.4}$$

These laws lead to an efficient evaluation procedure when combined with the following lemma:

Lemma 3. *The operations* $\left\lfloor \begin{smallmatrix} a \\ b \end{smallmatrix} \right\rfloor_c$ *and* $\left\lceil \begin{smallmatrix} a \\ b \end{smallmatrix} \right\rceil_c$ *satisfy*

$$\left\lfloor \begin{matrix} a \\ b \end{matrix} \right\rfloor_{c} = \begin{cases} \left\lfloor \dfrac{a}{b} \right\rfloor + \left\lfloor \begin{matrix} a \bmod b \\ b \end{matrix} \right\rfloor_{c} & \textit{if } 1 \leq c \leq b, \\ \left\lfloor \dfrac{a+b-c}{b} \right\rfloor & \textit{if } b < c \leq a+b, \end{cases} \tag{2.5}$$

$$\left\lceil \begin{matrix} a \\ b \end{matrix} \right\rceil_{c} = \begin{cases} \left\lfloor \dfrac{a}{b} \right\rfloor + \left\lceil \begin{matrix} a \bmod b \\ b \end{matrix} \right\rceil_{c} & \textit{if } 1 \leq c \leq b, \\ \infty & \textit{if } b \leq c \leq a+b. \end{cases} \tag{2.6}$$

Proof. We have already observed that the second line of (2.6) is obvious. A c-decomposition in which $a_i > 1$ and $b_i = 0$ for some i can be ignored when using (2.1) to evaluate $\left\lfloor \begin{smallmatrix} a \\ b \end{smallmatrix} \right\rfloor_c$ or $\left\lceil \begin{smallmatrix} a \\ b \end{smallmatrix} \right\rceil_c$, because the decomposition will also have $b_j > 0$ for some j; decreasing a_i by 1 and increasing a_j by 1 will yield another c-decomposition in which the min is no lower and the max is no higher. Therefore if $c \geq b$ we can restrict consideration to c-decompositions in which exactly $c-b$ of the pairs (a_i, b_i) are $(1,0)$. Deleting these pairs proves that

$$\left\lfloor \begin{matrix} a \\ b \end{matrix} \right\rfloor_{c} = \left\lfloor \begin{matrix} a-(c-b) \\ b \end{matrix} \right\rfloor_{b} \quad \text{when } c \geq b,$$

and the second line of (2.5) follows from (1.5).

Assume now that $0 < c \leq b$ and that $a = qb + (a \bmod b)$. A c-decomposition that achieves $\left\lfloor \begin{smallmatrix} a \\ b \end{smallmatrix} \right\rfloor_c$ or $\left\lceil \begin{smallmatrix} a \\ b \end{smallmatrix} \right\rceil_c$ in (2.1) can be assumed to have $b_i > 0$ for all i; for if $b_i = 0$ we could find a j with $b_j > 1$, and obtain a decomposition that is at least as balanced by increasing b_i and decreasing b_j. We can also assume that $a_i \geq qb_i$ for all i. For if $a_i < qb_i$, there will be a j with $a_j > qb_j$, since $a \geq qb$; increasing a_i and decreasing a_j does not decrease the min or increase the max in (2.1). This observation proves the first lines of (2.5) and (2.6), because $q = \lfloor a/b \rfloor$. ∎

Using (2.4), (2.5), and (2.6) we find for example that $\left\lceil \frac{16}{7} \right\rceil_4 = 2 + \left\lceil \frac{2}{7} \right\rceil_4$, $\left\lceil \frac{2}{7} \right\rceil_4 = 1/\left\lfloor \frac{7}{2} \right\rfloor_4$, and $\left\lfloor \frac{7}{2} \right\rfloor_4 = \left\lfloor \frac{5}{2} \right\rfloor = 2$, confirming the value $\left\lceil \frac{16}{7} \right\rceil_4 = \frac{5}{2}$ claimed above. In general these formulas yield evaluation procedures analogous to Euclid's algorithm, by which we can compute $\left\lfloor \frac{a}{b} \right\rfloor_c$ and $\left\lceil \frac{a}{b} \right\rceil_c$ by doing at most $O(\log \min(a,b))$ arithmetic operations on positive integers that do not exceed $a+b$.

3. The Stern–Brocot tree. Relation (2.2) above derives the inequality $\left\lfloor \frac{a}{b} \right\rfloor_c \le \left\lfloor \frac{a}{b} \right\rfloor_{c-1}$ by using the "mediant" operation (see [3])

$$\text{mediant}\left(\frac{a}{b}, \frac{a'}{b'}\right) = \frac{a+a'}{b+b'} \tag{3.1}$$

to obtain a $(c-1)$-decomposition from a c-decomposition. In the course of proving the main theorem, we will see that it is always possible to begin with the unique $(a+b)$-decomposition, which consists of a pairs $(1,0)$ and b pairs $(0,1)$, and to combine mediants judiciously one by one, thereby obtaining decompositions into $a+b-1, a+b-2, \ldots, d = \gcd(a,b)$ pairs that are simultaneously optimal for $\left\lfloor \frac{a}{b} \right\rfloor_c$ and $\left\lceil \frac{a}{b} \right\rceil_c$. We can also go the other way, starting with the d-decomposition that contains d pairs $(a/d, b/d)$ and judiciously applying "inverse mediant" operations to get optimum decompositions of sizes $d+1, d+2, \ldots, a+b$.

The mediant operation is the basis of an important construction of the positive rational numbers known as the Stern–Brocot tree (see Stern [5], Brocot [1], and the exposition in [3]). The first few levels of this structure, which is really a directed acyclic graph rather than a tree, are shown in Figure 1.

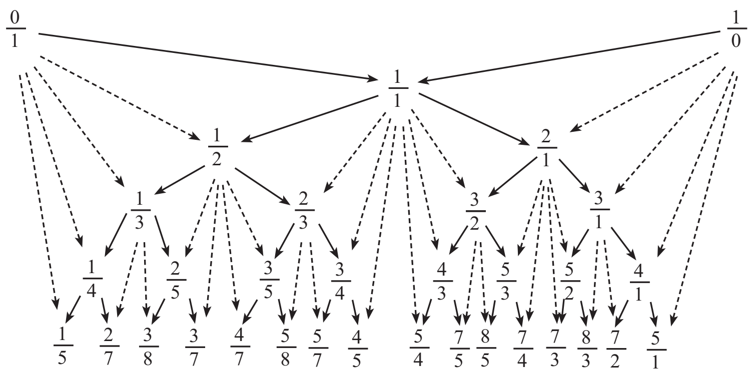

Fig. 1. The Stern–Brocot tree—a directed graph

Every positive rational $\frac{a}{b}$ arises as the mediant of two "parents": If $\frac{a}{b} = \frac{1}{1}$, the parents are $\frac{0}{1}$ and $\frac{1}{0}$, which are parentless; otherwise $\frac{a}{b}$ has a "close parent" indicated by a solid arrow and a "distant parent" indicated by a dashed arrow. The solid arrows, except for the two at the top, form an ordered binary tree rooted at $\frac{1}{1}$, and the symmetric ordering of vertices in this tree agrees with the usual ordering of rational numbers. Dashed arrows go from each vertex to elements on the right branch below its left child and to elements on the left branch below its right child.

The decompositions of a/b that will turn out to be uniquely optimal in our main theorem can be defined for decreasing $c = a + b, \dots, 1$ in a very simple way. For this we only need to apply the mediant operation on pairs already in the set. We start by defining $P_{a+b}\binom{a}{b} = \{b \times (0,1), a \times (1,0)\}$. From this decomposition of two distinct pairs we proceed by merging pairs by the mediant operation for the same pair of pairs, all the time producing a third pair, until one of the pairs runs out. We thus have $P_{a+b-k}\binom{a}{b} = \{(b-k) \times (0,1), k \times (1,1), (a-k) \times (1,0)\}$ for $k = 0, \dots, \min(a,b) - 1$. Then if for $k = \min(a,b)$ we obtain only one pair (if $a = b$), we are done. Otherwise we obtain two pairs, and the same procedure begins for this pair of pairs, until one of them runs out, again finally producing one or two distinct pairs. The procedure is either finished or starts again with the new pair of pairs. The procedures end with $P_c\binom{a}{b} = \{(c-1) \times (a/d, b/d), (a/d, b/d)\}$ for $c = d, \dots, 1$. Here $d = \gcd(a,b)$ as usual.

It is also possible to define $P_c\binom{a}{b}$ by increasing $c = 1, \dots, a + b$. Given $P_c\binom{a}{b}$ we then define $P_{c+1}\binom{a}{b}$ by removing a pair (a_i, b_i) from P_c having the greatest sum $a_i + b_i$ and replacing it by its two parents, for which we need to be familiar with the parents according to the Stern–Brocot tree. If follows by reversing the decreasing c order that if there are three distinct pairs in $P_c\binom{a}{b}$, the parents to (a_i, b_i) are the other two pairs, and if $P_c\binom{a}{b}$ contains only two pairs, the parents are (a_j, b_j) and $(a_i - a_j, b_i - b_j)$. Only in the first step, from $c = 1$ to $c = 2$, the pair is not given by the set itself. Therefore, decreasing c is a simpler algorithm to generate $P_c\binom{a}{b}$ than increasing c.

Here is the beginning of $P_c\binom{a}{b}$-constructions for decreasing c, if $a = 16$ and $b = 7$:

$$\begin{aligned}
P_{23}\left(\tfrac{16}{7}\right) &= \{7 \times (0,1), 16 \times (1,0)\},\\
P_{22}\left(\tfrac{16}{7}\right) &= \{6 \times (0,1), (1,1), 15 \times (1,0)\},\\
P_{21}\left(\tfrac{16}{7}\right) &= \{5 \times (0,1), 2 \times (1,1), 14 \times (1,0)\},\\
P_{20}\left(\tfrac{16}{7}\right) &= \{4 \times (0,1), 3 \times (1,1), 13 \times (1,0)\},\\
P_{19}\left(\tfrac{16}{7}\right) &= \{3 \times (0,1), 4 \times (1,1), 12 \times (1,0)\},\\
P_{18}\left(\tfrac{16}{7}\right) &= \{2 \times (0,1), 5 \times (1,1), 11 \times (1,0)\},\\
P_{17}\left(\tfrac{16}{7}\right) &= \{1 \times (0,1), 6 \times (1,1), 10 \times (1,0)\},\\
P_{16}\left(\tfrac{16}{7}\right) &= \{7 \times (1,1), 9 \times (1,0)\},\\
P_{15}\left(\tfrac{16}{7}\right) &= \{6 \times (1,1), (2,1), 8 \times (1,0)\}.
\end{aligned}$$

Similarly, if $a = 32$ and $b = 14$, the decompositions would be $P_1\binom{32}{14} = \{(32,14)\}$, $P_2\binom{32}{14} = \{(16,7),(16,7)\}, \ldots, P_{46}\binom{32}{14} = \{14\times(0,1),\ 32\times(1,0)\}$. The Stern–Brocot structure guarantees that $P_c\binom{a}{b}$ will always have at most three distinct pairs (a_i, b_i), and that a_i is relatively prime to b_i except in the case $c < \gcd(a,b)$.

4. Proof of the theorems. To prove Theorem 1, we will show that any c-distribution of a/b that differs from $P_c\binom{a}{b}$ can be changed to a better one. Our proof relies on a binary operation here called the *Stern–Brocot operation*, or SBO. This operation applies to two pairs of integers $(a,b),(a',b')$ with $\frac{a}{b} \le \frac{a'}{b'}$ and produces two new pairs $(A,B),(A',B')$ with $\frac{A}{B} \le \frac{A'}{B'}$. The operation is defined as follows: Let $d = \gcd(a+a', b+b')$. If $d = 1$, then $\frac{A}{B}$ and $\frac{A'}{B'}$ are the parents of $\frac{a+a'}{b+b'}$ in the Stern–Brocot graph, where $\frac{A}{B} < \frac{A'}{B'}$. If $d > 1$, then $(A,B) = ((a+a')/d, (b+b')/d)$ and $(A',B') = ((a+a')(d-1)/d, (b+b')(d-1)/d)$. Notice that $(a,b)+(a',b') = (A,B)+(A',B')$ in all cases.

Lemma 4. *Suppose $\frac{a}{b} < \frac{a'}{b'}$ and suppose that SBO maps the pair (a,b), (a',b') to the pair $(A,B),(A',B')$. If $a'b - b'a = 1$ then $(A,B) = (a,b)$ and $(A',B') = (a',b')$; otherwise $\frac{a}{b} < \frac{A}{B}$ and $\frac{A'}{B'} < \frac{a'}{b'}$.*

Proof. We use the fact that $\frac{a}{b}$ and $\frac{a'}{b'}$ are parents of $\frac{a+a'}{b+b'}$ if and only if $a'b - b'a = 1$. If $d > 1$ in the definition of SBO, then clearly $\frac{a}{b} < \frac{A}{B} = \frac{a+a'}{b+b'} = \frac{A'}{B'} < \frac{a'}{b'}$. Otherwise, if $d = 1$ and $a'b - b'a \ne 1$, all rational numbers between $\frac{A}{B}$ and $\frac{A'}{B'}$ have denominators $\ge B + B' = b + b'$. The fractions $\frac{a}{b}$ and $\frac{a'}{b'}$ have smaller denominators, and must lie outside this closed interval because they are not parents of $\frac{a+a'}{b+b'}$. ■

Consider now any c-decomposition $P = \{(a_1,b_1),\ldots,(a_c,b_c)\}$ of a/b. Let (a',b') be a pair (a_i,b_i) with $a'/b' = \min P$, and let (a'',b'') be a pair with $a''/b'' = \max P$. We assume that $c > \gcd(a,b)$, so that $a'/b' < a''/b''$. The decomposition P is said to be *invariant* if the SBO operation produces no change when applied to (a',b') and (a_i,b_i) for all i with $a'/b' < a_i/b_i$, or when applied to (a_j,b_j) and (a'',b'') for all j with $a_j/b_j < a''/b''$.

Every SBO operation on (a',b') and (a_i,b_i) either makes no change, or increases $\min P$, or decreases the number of occurrences of $\min P$; similarly, every SBO operation on (a_j,b_j) and (a'',b'') either makes no change, decreases $\max P$, or decreases the number of occurrences of $\max P$. We must eventually reach a c-decomposition that is invariant, because only finitely many c-decompositions exist and the quantity

$$(\max P - \min P,\ \max\{i : a_i/b_i = \min P\} - \min\{j : a_j/b_j = \max P\})$$

cannot be lexicographically decreased forever.

When P is invariant, the lemma tells us that $a_i b' - b_i a' = 1$ for all i with $a_i/b_i > a'/b'$, and $a'' a_j - b'' a_j = 1$ for all j with $a_j/b_j < a''/b''$. In particular, a_i must be relatively prime to b_i for $1 \le i \le c$.

Suppose $\frac{a'}{b'} < \frac{a_i}{b_i} < \frac{a''}{b''}$ for some i. The simultaneous equations $a_i b' - b_i a' = a'' b' - b'' a' = a'' b_i - b'' a_i = 1$ and relative primality imply that $a_i = a' + a''$ and $b_i = b' + b''$.

Therefore the decomposition P can be invariant only if it has the form $\{x \times (a', b'), y \times (a' + a'', b' + b''), z \times (a'', b'')\}$ for some integers x, y, and z. The determinant of the linear system

$$a'x + (a' + a'')y + a''z = a, \qquad b'x + (b' + b'')y + b''z = b, \qquad x + y + z = c$$

is $a'b'' - b'a'' = -1$; so the values of x, y, and z are uniquely determined, and there is at most one invariant c-decomposition.

Notice that the decomposition $P_c\binom{a}{b}$ defined in Section 3 is invariant. Therefore if we begin applying SBO reduction to any c-decomposition P, the process eventually converges to $P_c\binom{a}{b}$. Consequently, $P_c\binom{a}{b}$ realizes both $\left\lfloor \frac{a}{b} \right\rfloor_c$ and $\left\lceil \frac{a}{b} \right\rceil_c$. We have proved the main theorem, and shown that it characterizes $P_c\binom{a}{b}$.

If we are interested in $P_c\binom{a}{b}$ for a specific c, successive incrementing or decrementing c as described in Section 3 are not the fastest ways. By using the reciprocity, $P_c\binom{a}{b}$ can be computed with $O(\log \min(a, b))$ arithmetic operations. Clearly $P_c\binom{b}{a}$ can be obtained from $P_c\binom{a}{b}$ by simply changing each (a_i, b_i) to (b_i, a_i); and we have

$$P_c\binom{a}{b} = \begin{cases} \left\lfloor \frac{a}{b} \right\rfloor + P_c\binom{a \bmod b}{b}, & \gcd(a,b) \le c \le b, \\ \{(b - (a+b-c) \bmod b) \times (\lfloor (a+b-c)/b \rfloor, 1), & \\ \quad (a + b - c \bmod b) \times (\lceil (a+b-c)/b \rceil, 1), & \\ \quad (c - b) \times (1, 0)\}, & b < c \le a + b. \end{cases}$$

By repeated reciprocity we can by the first case above reduce $P_c\binom{a}{b}$ to $P_{c'}\binom{a'}{b'}$ where $b' < c' \le a' + b'$, which is the second case, in at most $\log_2 \min(a, b)$ steps. For example, $P_4\binom{16}{7} = 2 + P_4\binom{2}{7}$ and $P_4\binom{7}{2} = \{(2, 1), (3, 1), (1, 0), (1, 0)\}$; hence

$$P_4\binom{16}{7} = 2 + \{(0,1), (0,1), (1,3), (1,2)\} = \{(2,1), (2,1), (7,3), (5,2)\}$$

using $A + \frac{a}{b} = \frac{a + Ab}{b}$ in the form $A + (a, b) = (a + Ab, b)$. Each reduction step requires $O(1)$ arithmetic operations, since each set P_c has at most three distinct elements. Theorem 1 is proved.

Theorem 2 follows from the structure of the Stern–Brocot tree – all ratios with smaller denominators are at earlier generations (see [3]).

5. Additional properties. The structure of $P_c\binom{a}{b}$ also yields further identities that are satisfied by the functions $\left\lfloor {a \atop b} \right\rfloor_c$ and $\left\lceil {a \atop b} \right\rceil_c$:

$$\left\lfloor \begin{matrix} da \\ db \end{matrix} \right\rfloor_c = \left\lfloor \begin{matrix} a \\ b \end{matrix} \right\rfloor_{\lceil c/d \rceil} \qquad \text{and} \qquad \left\lceil \begin{matrix} da \\ db \end{matrix} \right\rceil_c = \left\lceil \begin{matrix} a \\ b \end{matrix} \right\rceil_{\lceil c/d \rceil},$$

$$\left\lfloor \begin{matrix} a \\ b \end{matrix} \right\rfloor_c + \left\lceil \begin{matrix} b-a \\ b \end{matrix} \right\rceil_c = 1 \qquad \text{if } 0 < a < b \text{ and } c \le b.$$

We thus have the extended cancellation rules $\left\lfloor {da \atop db} \right\rfloor_{dc} = \left\lfloor {a \atop b} \right\rfloor_c$ and $\left\lceil {da \atop db} \right\rceil_{dc} = \left\lceil {a \atop b} \right\rceil_c$. There is also a natural generalization to negative values of a and/or b, with $\gcd(|a|,|b|) \le c \le |a| + |b|$, if we simply negate the a_i and/or b_i components of $P_c\binom{a}{b}$ when a and/or b is negated. Then

$$\left\lfloor \begin{matrix} -a \\ b \end{matrix} \right\rfloor_c = \left\lfloor \begin{matrix} a \\ -b \end{matrix} \right\rfloor_c = -\left\lceil \begin{matrix} a \\ b \end{matrix} \right\rceil_c \qquad \text{and} \qquad \left\lceil \begin{matrix} -a \\ b \end{matrix} \right\rceil_c = \left\lceil \begin{matrix} a \\ -b \end{matrix} \right\rceil_c = -\left\lfloor \begin{matrix} a \\ b \end{matrix} \right\rfloor_c.$$

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Håkan Lennerstad, Lars Lundberg
School of Engineering
Blekinge Institute of Technology
S-371 79 Karlskrona, Sweden
E-mail: {Hakan.Lennerstad, Lars.Lundberg}@bth.se

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