On the Andrianov-type identity for power series attached to Jacobi forms and its application

by

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1. Introduction. The theory of Jacobi forms (that is, automorphic forms on the Jacobi group and its generalizations to higher degree) has been studied by several authors (cf. [7, 29, 20, 21, 11]). In particular, Shintani introduced the standard $L$-function attached to a Jacobi form of arbitrary degree, and afterward Murase derived in a series of papers [20, 21] its meromorphic continuation and functional equation by making use of its integral expression. Moreover, Murase and Sugano derived in [22] an expression of the standard $L$-function attached to a Jacobi form in terms of a power series generated by eigenvalues of Hecke operators. In this paper, we derive a local expression of the standard $L$-function attached to a Jacobi form in terms of a power series related to its Fourier coefficients. This can be regarded as an analogue of Andrianov’s identity in [1] for Siegel modular forms. As an application, we shall also prove a rationality theorem for a formal power series related to a polynomial appearing in the theory of local densities of quadratic forms, which is very similar to the result obtained in [6] by Böcherer and Sato.

Let us describe our main results precisely. Let $p$ be an arbitrary rational prime. For any non-zero element $a$ of the field $\mathbb{Q}_p$ of $p$-adic numbers, we put

$$
\chi_p(a) = \begin{cases} 
1 & \text{if } \mathbb{Q}_p(a^{1/2}) = \mathbb{Q}_p, \\
-1 & \text{if } \mathbb{Q}_p(a^{1/2})/\mathbb{Q}_p \text{ is unramified}, \\
0 & \text{if } \mathbb{Q}_p(a^{1/2})/\mathbb{Q}_p \text{ is ramified}.
\end{cases}
$$

Let $n$ be a positive even integer. For each non-degenerate half-integral symmetric matrix $B'$ of degree $n$ over the ring $\mathbb{Z}_p$ of $p$-adic integers, we define
the local Siegel series with complex parameter \( s \) by

\[
b_p(B'; s) := \sum_{R \in \text{Sym}_n(\mathbb{Q}_p)/\text{Sym}_n(\mathbb{Z}_p)} e_p(\text{tr}(-B'R)) \mu_p(R)^{-s},
\]

where \( \mu_p(R) = [\mathbb{Z}_p^n/R + \mathbb{Z}_p^n : \mathbb{Z}_p^n] \), and \( e_p \) is the standard additive character of \( \mathbb{Q}_p \). It is well-known that such singular series appear naturally in the study of Fourier coefficients of Siegel Eisenstein series of degree \( n \) and there exists a unique polynomial \( F_p(B'; X) \) in one variable \( X \) such that

\[
b_p(B'; s) = \frac{(1 - p^{-s}) \prod_{i=1}^{n/2} (1 - p^{2i-2s})}{1 - \xi_p(B')p^{n/2-s}} F_p(B'; p^{-s}),
\]

where \( \xi_p(B') = \chi_p((-1)^{n/2} \det(2B')) \) (cf. [18]). Let \( B \) be a non-degenerate symmetric matrix of degree \( n - 1 \) over a subring \( R \) of \( \mathbb{Z}_p \) satisfying the condition

\[
(1.1) \quad (B + t_r B'r_B)/4 \text{ is a half-integral symmetric matrix over } R \text{ for some } r_B \in R^{n-1}.
\]

Then we can associate \( B \) with a non-degenerate half-integral symmetric matrix

\[
B^{(1)} = \begin{pmatrix}
1 & r_B/2 \\
t_r B/2 & (B + t_r B'r_B)/4
\end{pmatrix}
\]

of degree \( n \) over \( R \). Here we easily see that the vector \( r_B \) is uniquely determined by \( B \) modulo \( 2R^{n-1} \), and therefore \( B^{(1)} \) is uniquely determined by \( B \) up to \( \text{GL}_n(R) \)-equivalence. For such a \( B \) over \( \mathbb{Z}_p \), we define a polynomial \( F_p^{(1)}(B; X) \) in \( X \) by

\[
F_p^{(1)}(B; X) := F_p(B^{(1)}; X)
\]

and put

\[
G_p^{(1)}(B; X) = \sum_{D \in \text{GL}_{n-1}(\mathbb{Z}_p)\setminus\text{M}_{n-1}(\mathbb{Z}_p) \cap \text{GL}_{n-1}(\mathbb{Q}_p)} \pi_p(D) F_p^{(1)}(B[D^{-1}]; X)(p^n X^2)^{\text{ord}_p(\det D)},
\]

where \( \pi_p(D) \) denotes the generalized local Möbius function, that is, \( \pi_p(D) = (-1)^i p^{(i-1)/2} \) or 0 according as \( D \in \text{GL}_{n-1}(\mathbb{Z}_p) \left(\frac{1}{p1_i} \right) \text{GL}_{n-1}(\mathbb{Z}_p) \) for some \( 0 \leq i \leq n - 1 \) or not. We note that these polynomials do not depend on the choice of \( r_B \). In addition, we also define a polynomial \( B_p^{(1)}(B; t) \) in one variable \( t \) by

\[
B_p^{(1)}(B; t) := \frac{(1 - \xi_p(B^{(1)}))p^{-(n-1)/2t}}{G_p^{(1)}(B; p^{-n+1/2t})} \prod_{i=1}^{n/2-1} (1 - p^{-2i+1/2})
\]

for some \( 0 \leq i \leq n - 1 \) or not.
On the other hand, for any positive even integers \( k \) and \( n \), let \( \phi \) be a Jacobi form of weight \( k \) and of index 1 with respect to the Jacobi modular group \( \Gamma_{n-1}^J \) of degree \( n-1 \), and \( \sigma(\phi) \) a Siegel modular form of weight \( k-1/2 \) with respect to the congruence subgroup \( \Gamma_0^{(n-1)}(4) \) of the Siegel modular group of degree \( n-1 \) corresponding to \( \phi \) under the Eichler–Zagier–Ibukiyama correspondence \( \sigma \) (cf. \$2.3 and 2.4 below). Let \( \mathbf{D}_p^{(n-1)}(\mathbb{Z}) \) be the set of all \( (n-1) \times (n-1) \) matrices with entries in \( \mathbb{Z} \) whose determinant is a power of \( p \). For each positive definite half-integral symmetric matrix \( B \) of degree \( n-1 \) over \( \mathbb{Z} \), we define a power series \( \widetilde{G}_{\phi,p}(B;t) \) in \( t \) by

\[
\widetilde{G}_{\phi,p}(B;t) := \sum_{D \in \text{GL}_{n-1}(\mathbb{Z}) \setminus \mathbf{D}_p^{(n-1)}(\mathbb{Z})} \pi_p(D)C_{\sigma(\phi)}(B[D^{-1}])(p^kt)^{\text{ord}_p(\det D)},
\]

where \( C_{\sigma(\phi)}(B) \) denotes the \( B \)th Fourier coefficient of \( \sigma(\phi) \). Then our first main result is the following:

**Theorem 1.1** (cf. Theorem 3.1 below). Suppose that \( \phi \) is a Hecke eigenform, that is, a common eigenfunction of all Hecke operators, whose Satake \( p \)-parameter is of the form \((\chi_\phi^{(1)}(p), \ldots, \chi_\phi^{(n-1)}(p))\) up to the action of the Weyl group. Then, for each positive definite half-integral symmetric matrix \( B \) of degree \( n-1 \) over \( \mathbb{Z} \) satisfying the condition (1.1), we have

\[
\frac{\mathbf{B}_p^{(1)}(B;p^{n-1/2}t)\widetilde{G}_{\phi,p}(B;t)}{\prod_{i=1}^{n-1}(1 - \chi_\phi^{(i)}(p)p^{n-1/2}t)(1 - \chi_\phi^{(i)}(p)^{-1}p^{n-1/2}t)} = \sum_{W \in \text{GL}_{n-1}(\mathbb{Z}) \setminus \mathbf{D}_p^{(n-1)}(\mathbb{Z})} C_{\sigma(\phi)}(B[W])p^{-(k-n-1)\text{ord}_p(\det W)}t^{\text{ord}_p(\det W)}.
\]

This can be regarded as an analogue of the so-called Andrianov identity, which was obtained in the study of standard \( L \)-functions attached to Siegel modular forms of integral weight (cf. \cite{1}, see also \cite{5}). We also note that the above identity for \( p \neq 2 \) can be derived from a similar result for Siegel modular forms of half-integral weight due to Shimura and Zhuravlev (cf. Corollary 5.2 in \cite{25}, see also Theorem 1.1 in \cite{28}). However, we cannot use their results to prove the above identity for \( p = 2 \).

Next, we explain an application of the above result to the rationality of a certain formal power series related to the polynomial \( F_p^{(1)}(B;X) \). For each non-degenerate half-integral symmetric matrix \( B \) of degree \( n-1 \) over \( \mathbb{Z}_p \) satisfying the condition (1.1), we define a Laurent polynomial \( \tilde{F}_p^{(1)}(B;X) \) in \( X \) by

\[
\tilde{F}_p^{(1)}(B;X) := X^{-\text{ord}_p((-1)^{n/2}\det(2B^{(1)})\sigma(B^{(1)})^{-1})/2}F_p^{(1)}(B;p^{-(n+1)/2}X),
\]
and put
\[ \tilde{G}_p^{(1)}(B; X, t) = \sum_{D \in \text{GL}_{n-1}(\mathbb{Z}_p) \setminus M_{n-1}(\mathbb{Z}_p) \cap \text{GL}_{n-1}(\mathbb{Q}_p)} \pi_p(D) \tilde{F}_p^{(1)}(B[D^{-1}]; X)t^{\text{ord}_p(\det D)}, \]
where \( \mathfrak{o}(B^{(1)}) \) is the discriminant of the field extension \( \mathbb{Q}_p(\sqrt{(-1)^{n/2} \det(2B^{(1)})})/\mathbb{Q}_p. \)
We note that the functional equation \( \tilde{F}_p^{(1)}(B; X) = \tilde{F}_p^{(1)}(B; X^{-1}) \) holds (cf. [12]). Thus \( \tilde{F}_p^{(1)}(B; X) \) is a polynomial in \( X + X^{-1} \), and therefore \( \tilde{G}_p^{(1)}(B; X, t) \) is a polynomial in \( X + X^{-1} \) and \( t \). Now we put
\[ R_p^{(1)}(B; X, t) = \sum_{W \in \text{GL}_{n-1}(\mathbb{Z}_p) \setminus M_{n-1}(\mathbb{Z}_p) \cap \text{GL}_{n-1}(\mathbb{Q}_p)} \tilde{F}_p^{(1)}(B[W]; X)t^{\text{ord}_p(\det W)}. \]
By applying Theorem 1.1 to the Jacobi Eisenstein series, we obtain the following:

**Theorem 1.2 (cf. Theorem 3.4 below).** Let \( n \) be a positive even integer. If \( B \) is a non-degenerate half-integral symmetric matrix of degree \( n - 1 \) over \( \mathbb{Z}_p \) satisfying the condition (1.1), then
\[ R_p^{(1)}(B; X, t) = \frac{B_p^{(1)}(B; p^{n/2-1}t) \tilde{G}_p^{(1)}(B; X, t)}{\prod_{j=1}^{n-1}(1 - p^{j-1}Xt)(1 - p^{j-1}X^{-1}t)}. \]

We note that Böcherer and Sato ([6]) obtained a similar identity for a half-integral symmetric matrix of degree \( n \). The above identity will play an important role in proving a conjecture on the period of the Ikeda lift proposed in [13] by Ikeda (cf. [16] [17]).

**Notation.** We denote by \( \mathbb{Z}, \mathbb{Q}, \mathbb{R} \) and \( \mathbb{C} \) the ring of rational integers, the field of rational numbers, the field of real numbers and the field of complex numbers, respectively. We put \( e(x) = \exp(2\pi \sqrt{-1}x) \) for any \( x \in \mathbb{C} \). For each rational prime \( p \), let \( \mathbb{Q}_p \) and \( \mathbb{Z}_p \) be the field of \( p \)-adic rational numbers and the ring of \( p \)-adic integers, respectively. We denote by \( \text{ord}_p \) the valuation of \( \mathbb{Q}_p \) normalized as \( \text{ord}_p(p) = 1 \), and by \( e_p \) the continuous additive character of \( \mathbb{Q}_p \) such that \( e_p(x) = e(x) \) for any \( x \in \mathbb{Q} \), which will be called the **standard additive character** of \( \mathbb{Q}_p \).

Let \( R \) be a commutative ring. We denote by \( R^\times \) the unit group of \( R \), and by \( M_{m,n}(R) \) the set of \( m \times n \) matrices with entries in \( R \). In particular, we write \( M_n(R) = M_{n,n}(R) \) and \( R^n = M_{1,n}(R) \). We denote by \( 1_n, 0_n \in M_n(R) \) the unit matrix and the zero matrix of degree \( n \), respectively. We put \( \text{GL}_n(R) = \{ U \in M_n(R) \mid \det U \in R^\times \} \), where \( \det U \) is the determinant of \( U \). For \( X \in M_{m,n}(R) \) and \( A \in M_m(R) \), we write \( A[X] = ^tXAX \in M_{m,n}(R) \).
For a subgroup \( \Gamma \) of degree \( n \). For any commutative ring \( \mathcal{R} \) of characteristic different from 2, let \( \operatorname{Sym}^*_n(\mathcal{R}) \) be the set of all half-integral symmetric matrices of degree \( n \) over \( \mathcal{R} \), that is,

\[
\operatorname{Sym}^*_n(\mathcal{R}) := \left\{ T = (t_{ij}) \in \operatorname{Sym}_n(\operatorname{Frac}(\mathcal{R})) \mid \begin{aligned}
& t_{ii} \in \mathcal{R} \quad (1 \leq i \leq n), \\
& 2t_{ij} \in \mathcal{R} \quad (1 \leq i \neq j \leq n)
\end{aligned} \right\},
\]

where \( \operatorname{Frac}(\mathcal{R}) \) is the field of fractions of \( \mathcal{R} \). In addition, for any subset \( S \) of \( \operatorname{Sym}_n(\mathcal{R}) \), we denote by \( S^\times \) the subset of \( S \) consisting of all non-degenerate elements in \( S \). In particular, if \( \mathcal{R} \) is a subring of \( \mathbb{R} \), we denote by \( S_{>0} \) (resp. \( S_{\geq 0} \)) the subset of \( S \) consisting of all positive definite (resp. semi-positive definite) matrices. For any commutative ring \( \mathcal{R} \), the group \( \operatorname{GL}_n(\mathcal{R}) \) acts on \( \operatorname{Sym}_n(\mathcal{R}) \) in the following way:

\[
\operatorname{GL}_n(\mathcal{R}) \times \operatorname{Sym}_n(\mathcal{R}) \ni (U, A) \mapsto A[U] \in \operatorname{Sym}_n(\mathcal{R}).
\]

For a subgroup \( G \) of \( \operatorname{GL}_n(\mathcal{R}) \), and a subset \( S \) of \( \operatorname{Sym}_n(\mathcal{R}) \) stable under the action of \( G \), we denote by \( S/G \) the set of \( G \)-orbits in \( S \). For a subring \( \mathcal{R}' \) of \( \mathcal{R} \) we define an equivalence relation on \( \operatorname{Sym}_n(\mathcal{R}) \) as follows: for any \( A_1, A_2 \in \operatorname{Sym}_n(\mathcal{R}) \),

\[
A_1 \sim_{\mathcal{R}'} A_2 \iff A_2 = A_1[U] \quad \text{for some} \ U \in \operatorname{GL}_n(\mathcal{R}').
\]

For square matrices \( X \in \operatorname{M}_m(\mathcal{R}) \) and \( Y \in \operatorname{M}_n(\mathcal{R}) \), we write \( X \perp Y = (X \ Y) \). In particular, we often write \( x \perp Y \) instead of \((x) \perp Y \) for any \( x \in \mathcal{R} \).

We can simply write the diagonal matrix with entries \( x_1, \ldots, x_n \) in \( \mathcal{R} \) by \( x_1 \perp \cdots \perp x_n \).

# 2. Preliminaries

## 2.1. Siegel modular forms of integral weight.

Let \( G_n(\mathbb{R}) \) be the real symplectic group of degree \( n \), that is,

\[
G_n(\mathbb{R}) := \operatorname{Sp}_n(\mathbb{R}) = \{ M \in \operatorname{GL}_{2n}(\mathbb{R}) \mid \begin{bmatrix} M & J_n \\ J_n & M \end{bmatrix} = J_n \},
\]

where \( J_n = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix} \). For any \( S \in \operatorname{Sym}_n(\mathbb{R}) \) and \( A \in \operatorname{GL}_n(\mathbb{R}) \), we put \( n_n(S) = \begin{pmatrix} 1_n & S \\ 0_n & 1_n \end{pmatrix} \) and \( d_n(A) = \begin{pmatrix} A & 0_n \\ 0_n & \epsilon_A^{-1} \end{pmatrix} \), respectively. We easily see that the elements \( n_n(S) \), \( d_n(A) \) and \( J_n \) generate \( G_n(\mathbb{R}) \). The discrete subgroup \( \Gamma_n := \operatorname{Sp}_n(\mathbb{Z}) = G_n(\mathbb{R}) \cap \operatorname{M}_{2n}(\mathbb{Z}) \) of \( G_n(\mathbb{R}) \) is called the \textit{Siegel modular group} of degree \( n \). For any \( N \in \mathbb{Z}_{>0} \), we denote by \( \Gamma_0^{(n)}(N) \) the congruence subgroup of \( \Gamma_n \) defined by

\[
\Gamma_0^{(n)}(N) := \{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n \mid C \equiv 0_n \pmod{N} \}.
\]

We denote the Siegel upper half-space of degree \( n \) by \( \mathcal{H}_n \), that is,

\[
\mathcal{H}_n := \{ Z = X + \sqrt{-1} Y \in \operatorname{Sym}_n(\mathbb{C}) \mid Y > 0 \text{ (positive definite)} \}.
\]

\textit{Andrianov-type identity for Jacobi forms}
For any $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_n(\mathbb{R})$ and $Z \in \mathfrak{H}_n$, we easily see that $j(M, Z) := CZ + D \in \text{GL}_n(\mathbb{C})$ and we put $M(Z) := (AZ + B)(CZ + D)^{-1}$. As is well-known, this defines a transitive action of $G_n(\mathbb{R})$ on $\mathfrak{H}_n$.

For any $Z \in H_n$, we easily see that $j(M, Z) := CZ + D \in \text{GL}_n(\mathbb{C})$ and we put $M(Z) := (AZ + B)(CZ + D)^{-1}$. As is well-known, this defines a transitive action of $G_n(\mathbb{R})$ on $H_n$.

For any $k \in \mathbb{Z}$, a $C$-valued holomorphic function $F(Z)$ on $H_n$ is called a (holomorphic) Siegel modular form of degree $n$ and weight $k$ if it satisfies the following two conditions:

(i) $F(M(Z)) = \det(j(M, Z))^k F(Z)$ for any $M \in \Gamma_n$;
(ii) $F$ possesses a Fourier expansion of the form

$$F(Z) = \sum_{B \in \text{Sym}_n^+(\mathbb{Z})} A_F(B) e(\text{tr}(BZ)),$$

where $\text{tr}(*)$ denotes the trace of a matrix.

In particular, a Siegel modular form $F$ is called a cusp form if it satisfies the stronger condition $A_F(B) = 0$ unless $B > 0$ (positive definite). We denote by $M_k(\Gamma_n)$ and $S_k(\Gamma_n)$ the $\mathbb{C}$-vector spaces consisting of all Siegel modular forms and Siegel cusp forms of degree $n$ and weight $k$, respectively. For further details on Siegel modular forms of integral weight, see [1] or [8].

2.2. Review of the theory of Jacobi forms of higher degree. In this subsection, we introduce some basic facts on Jacobi forms of integral weight whose index is a scalar. For further details on Jacobi forms, see [7, 20, 21, 29].

2.2.1. Jacobi group and complex analytic Jacobi forms. Let $G_n = \text{Sp}_n(\mathbb{Q}) = \{ M \in \text{GL}_{2n}(\mathbb{Q}) \mid ^tMJ_nM = J_n \}$; we naturally identify $G_n$ with its image under the natural inclusion

$$G_n \ni M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto [M] := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & 1 & 0 \\ 0 & C & 0 & D \end{pmatrix} \in G_{n+1}. $$

We denote by $H_n$ the Heisenberg group consisting of all elements of the form

$$[(\lambda, \mu), \kappa] := \begin{pmatrix} 1 & 0 & \kappa & \mu \\ 0 & 1_n & t\mu & 0_n \\ 1 & 0 & 0 & 1_n \\ 0 & 1 & 0 & -t\lambda & 1_n \end{pmatrix}$$

for some $(\lambda, \mu) \in \mathbb{Q}^n \oplus \mathbb{Q}^n$ and $\kappa \in \mathbb{Q}$. Then

$$G^J_n := \{ [(\lambda, \mu), \kappa] \cdot [M] \in G_{n+1} \mid [(\lambda, \mu), \kappa] \in H_n, M \in G_n \}$$

is a $\mathbb{Q}$-algebraic subgroup of $G_{n+1}$; it is called the Jacobi group of degree $n$. We note that the Jacobi group $G^J_n$ is a semi-direct product $G_n \rtimes H_n$, and
forms a connected non-reductive $\mathbb{Q}$-algebraic group with the center

$$Z_n^J = \{[(0, 0), \kappa] \mid \kappa \in \mathbb{Q}\}.$$ 

It is easy to see the following:

**Lemma 2.1.** For each $[(\lambda, \mu), \kappa], [(\lambda', \mu'), \kappa'] \in H_n$, and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_n$, we have

(2.1) $[(\lambda, \mu), \kappa] \cdot [(\lambda', \mu'), \kappa'] = [(\lambda + \lambda', \mu + \mu'), \kappa + \kappa' + 2\lambda'\mu'],$

(2.2) $[(\lambda, \mu), \kappa] \cdot [M]$

$$= [M] \cdot [(\lambda A + \mu C, \lambda B + \mu D), \kappa + (\lambda A + \mu C)^t (\lambda B + \mu D) - \lambda^t \mu].$$

**Proof.** Since it is an easy calculation, we omit the proof. □

According to the action of $G_{n+1}(\mathbb{R}) = \text{Sp}_{n+1}(\mathbb{R})$ on the Siegel upper half-space $\mathcal{H}_n$, the group $G_n^J(\mathbb{R})$ of real points of $G_n^J$ naturally acts on the space $\mathcal{H}_n \times \mathbb{C}^n$ as follows. For each $g = [(\lambda, \mu), \kappa] \cdot [M] \in G_n^J(\mathbb{R})$ with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_n(\mathbb{R})$ and $(\tau, z) \in \mathcal{H}_n \times \mathbb{C}^n$, we put

$$g(\tau, z) := (M(\tau), z(C\tau + D)^{-1} + \lambda M(\tau) + \mu).$$

We easily see that this action is transitive and the stabilizer of the point $(\sqrt{-1} 1_n, 0) \in \mathcal{H}_n \times \mathbb{C}^n$ in $G_n^J(\mathbb{R})$ coincides with $Z_n^J(\mathbb{R}) \cdot K_\infty$, where $K_\infty$ is the stabilizer of $\sqrt{-1} 1_n \in \mathcal{H}_n$ in $G_n(\mathbb{R})$, that is,

$$K_\infty = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in G_n(\mathbb{R}) \mid A + \sqrt{-1} B \text{ is unitary} \right\}.$$ 

The map $g \mapsto g(\sqrt{-1} 1_n, 0)$ induces a diffeomorphism of the quotient $G_n^J(\mathbb{R})/(Z_n^J(\mathbb{R}) \cdot K_\infty)$ onto $\mathcal{H}_n \times \mathbb{C}^n$.

Let $l$ and $m$ be non-negative integers. For any $\mathbb{C}$-valued function $\phi(\tau, z)$ on $\mathcal{H}_n \times \mathbb{C}^n$, we define the action of $g \in G_n^J(\mathbb{R})$ on $\phi$ by

$$(\phi|_{l,m} g)(\tau, z) := J_{l,m}(g, (\tau, z))^{-1} \phi(g(\tau, z)), $$

where for $g = [(\lambda, \mu), \kappa] \cdot [M]$, we put

$$J_{l,m}(g, (\tau, z)) := \det(C\tau + D)^l$$

$$\times e(-m\kappa - m(\tau[l]_{\lambda} - 2m\lambda^t\tau z - m\lambda^t\mu + m((C\tau + D)^{-1} C)[t](z + \lambda \tau + \mu))).$$

It is easy to see that for any $g_i \in G_n^J(\mathbb{R})$ ($i = 1, 2$),

$$(\phi|_{l,m} g_1)|_{l,m} g_2 = \phi|_{l,m} (g_1 g_2).$$

In particular, it follows from Lemma 2.1 that for any $M, M' \in G_n(\mathbb{R})$ and $[(\lambda, \mu), \kappa], [(\lambda', \mu'), \kappa'] \in H_n(\mathbb{R})$, we have
\[
\begin{align*}
\phi|_{l,m}[M][l,m][M'] &= \phi|_{l,m}[MM'], \\
\phi|_{l,m}[(\lambda, \mu), \kappa][l,m][(\lambda', \mu'), \kappa'] &= \phi|_{l,m}[(\lambda + \lambda', \mu + \mu'), \kappa + \kappa' + 2\lambda'\mu'], \\
\phi|_{l,m}[M][l,m]
\begin{pmatrix}
\lambda, \mu \\
\lambda', \mu'
\end{pmatrix}
M
\begin{pmatrix}
on_m^n & 1_n^n \\
0_n^n & 0_n^n
\end{pmatrix}
M
\begin{pmatrix}
\lambda, \mu \\
\lambda'\mu'
\end{pmatrix}
 &= \phi|_{l,m}[(\lambda, \mu), \kappa] \cdot [M].
\end{align*}
\]

We also define a subgroup of \( G^J_n(\mathbb{R}) \) by \( \Gamma^J_n := \Gamma_n \ltimes H_n(\mathbb{Z}) \), where \( H_n(\mathbb{Z}) \) is a subgroup of \( H_n(\mathbb{R}) \) consisting of all elements with integral entries.

Let \( l \) and \( m \) be positive integers. A holomorphic function \( \phi(\tau, z) \) on \( H_n \times \mathbb{C}^n \) is called a (holomorphic) \emph{Jacobi form} of degree \( n \), weight \( l \) and index \( m \) if it satisfies the following two conditions:

(i) \( \phi|_{l,m}\gamma = \phi \) for any \( \gamma \in \Gamma^J_n \).

(ii) \( \phi \) possesses a Fourier expansion of the form

\[
\phi(\tau, z) = \sum_{T \in \text{Sym}^*_n(\mathbb{Z}), r \in \mathbb{Z}^n} c_\phi(T, r) e(\text{tr}(T\tau) + r^t z)
\]

with \( c_\phi(T, r) = 0 \) unless \( 4mT - trr \geq 0 \).

In particular, a Jacobi form \( \phi \) is called \emph{cuspidal} if it satisfies the stronger condition \( c_\phi(T, r) = 0 \) unless \( 4mT - trr > 0 \). We denote by \( J_{l,m}(\Gamma^J_n) \) and \( J^\text{cusp}_{l,m}(\Gamma^J_n) \) the \( \mathbb{C} \)-vector spaces consisting of all Jacobi forms and cuspidal Jacobi forms of degree \( n \), weight \( l \) and index \( m \), respectively.

As an important example of Jacobi form, we consider Fourier–Jacobi coefficients of Siegel modular forms of arbitrary degree \( n > 1 \). For any \( k \in \mathbb{Z} \), let \( F \in M_k(\Gamma_n) \) possess a Fourier expansion

\[ F(Z) = \sum_{B' \in \text{Sym}^*_n(\mathbb{Z}), r \in \mathbb{Z}^n} A_F(B') e(\text{tr}(B'Z)) \quad (Z \in \mathfrak{H}_n), \]

and we put

\[ Z = \begin{pmatrix} \tau' & z \\ t_z & \tau \end{pmatrix} \quad \text{with} \ \tau \in \mathfrak{H}_{n-1}, \ z \in \mathbb{C}^{n-1} \ \text{and} \ \tau' \in \mathfrak{H}_1. \]

Then we have the so-called Fourier–Jacobi expansion

\[ F\left( \begin{pmatrix} \tau' & z \\ t_z & \tau \end{pmatrix} \right) = \sum_{m=0}^{\infty} \phi_m(\tau, z) e(m\tau'), \]

where

\[ \phi_m(\tau, z) = \sum_{T \in \text{Sym}^*_{n-1}(\mathbb{Z}), r \in \mathbb{Z}^{n-1}} A_F\left( \begin{pmatrix} m & r/2 \\ T \end{pmatrix} \right) e(\text{tr}(T\tau) + r^t z). \]
We easily see that $\phi_m \in J_{k,m}(\Gamma_{n-1}^J)$ for each $m \in \mathbb{Z}_{>0}$. In particular, if $F \in S_k(\Gamma_n)$, then $\phi_m \in J_{k,m}^{\text{cusp}}(\Gamma_{n-1}^J)$.

As another example, if $k$ is an even integer such that $k > n + 1$, for each $m \in \mathbb{Z}_{>0}$, we define the Jacobi Eisenstein series of degree $n - 1$, weight $k$ and index $m$ by

$$
\mathcal{E}_{k,m}^{(n-1)}(\tau, z) := \sum_{\gamma \in P_n^J \cap \Gamma_{n-1}^J \setminus \Gamma_{n-1}^J} J_{k,m}(\gamma, (\tau, z))^{-1} \quad (\tau \in \mathbb{H}_{n-1}, \ z \in \mathbb{C}^{n-1}),
$$

where

$$
P_n^J := \left\{ [(\lambda, \mu), \kappa] \cdot \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \right\} \in G_{n-1}^J \mid C = 0_{n-1}, \ \lambda = 0 \right\}.
$$

We easily see that the right-hand side of the above definition is absolutely convergent and $\mathcal{E}_{k,m}^{(n-1)} \in J_{k,m}(\Gamma_{n-1}^J)$. Moreover, Böcherer ([4]) showed that for any $m \in \mathbb{Z}_{>0}$, there exists a certain relation between $\mathcal{E}_{k,m}^{(n-1)}$ and the $m$th coefficient $e_{k,m}^{(n-1)}$ of the above Fourier–Jacobi expansion of the Siegel Eisenstein series $E_k^{(n)} \in M_k(\Gamma_n)$. In particular, when $m = 1$, we have $\mathcal{E}_{k,1}^{(n-1)} = e_{k,1}^{(n-1)}$.

For later use, we give an explicit formula for the Fourier coefficients of $e_{k,1}^{(n-1)}$ in case $n$ is even. Let $k$ be a positive even integer such that $k > n + 1$. The Siegel Eisenstein series $E_k^{(n)}$ of weight $k$ with respect to $\Gamma_n$ is defined by

$$
E_k^{(n)}(Z) = \sum_{(C,D)} \det(CZ + D)^{-k} \quad (Z \in \mathbb{H}_n)
$$

where $(C, D)$ runs through a complete set of representatives of the equivalence classes of coprime symmetric pairs of size $n$. For each positive definite half-integral symmetric matrix $B'$ of degree $n$, we denote by $\mathfrak{d}(B')$ the discriminant of the field extension $\mathbb{Q}(\sqrt{(-1)^{n/2} \det(2B')/\mathfrak{d}(B')})/\mathbb{Q}$ and put $f(B') = \sqrt{(-1)^n/2 \det(2B')/\mathfrak{d}(B')}$. It is well-known that $f(B') \in \mathbb{Z}_{>0}$. Furthermore, we denote by $\chi_B'$ the Kronecker character corresponding to the above field extension. For each $B' \in \text{Sym}^*_{n}(\mathbb{Z})_{>0}$, the $B'$th Fourier coefficient $A_k^{(n)}(B')$ of $E_k^{(n)}$ is described as

$$
A_k^{(n)}(B') = \xi(n, k)L(1 - k/2 + n/2, \chi_{B'})f(B')^{k-(n+1)/2} \times \prod_{p|f(B')} \tilde{F}_p(B'; p^{k-(n+1)/2}),
$$

where

$$
\xi(n, k) = 2^{n/2} \zeta(1 - k)^{-1} \prod_{i=1}^{n/2} \zeta(1 + 2i - 2k)^{-1},
$$

and $L(\cdot, \chi_B')$ is the standard $L$-function of $\chi_B'$.
$L(s, \chi_{B'})$ denotes the Dirichlet $L$-function associated with $\chi_{B'}$, and
\[
F_p(B'; X) = X^{-\ord_p(f(B'))}F_p(B'; p^{-(n+1)/2}X).
\]
We note that if $B \in \text{Sym}^*_n(\mathbb{Z}) > 0$ satisfies condition (1.1), then $F_p^{(1)}(B; X) = F_p(B^{(1)}; X)$. Thus we have

**Proposition 2.1.** Under the same assumption as above, let $e^{(n-1)}_{k,1}$ possess a Fourier expansion
\[
e^{(n-1)}_{k,1}(\tau, z) = \sum_{T \in \text{Sym}^*_n(\mathbb{Z}), r \in \mathbb{Z}^{n-1}} c^{(n-1)}_{k,1}(T, r) e^{(tr(T\tau) + r^t z)}.
\]
Then, for each $T \in \text{Sym}^*_n(\mathbb{Z})$ such that $B_T = 4T - t^r r > 0$ with $r \in \mathbb{Z}^{n-1}$, we have
\[
c^{(n-1)}_{k,1}(T, r) = \xi(n, k)L(1 - k + n/2, \chi_{B_T^{(1)}})(B_T^{(1)})^{k-(n+1)/2} \prod_{p|f(B_T^{(1)})} F_p^{(1)}(B_T^{(1)}; p^{k-(n+1)/2}),
\]
where
\[
B_T^{(1)} = \begin{pmatrix}
1 & r/2 \\
t_r/2 & (B_T + t^r r)/4
\end{pmatrix} = \begin{pmatrix}
1 & r/2 \\
t_r/2 & T
\end{pmatrix} \in \text{Sym}^*_n(\mathbb{Z}) > 0.
\]

**Proof.** Since
\[
c^{(n-1)}_{k,1}(T, r) = A^{(n)}_k(B_T^{(1)}),
\]
the assertion immediately follows from (2.4). 

Returning to the general theory of Jacobi forms, we now consider the action of Hecke operators on Jacobi forms. Let $M \in \text{Sp}_n(\mathbb{Q})$ and decompose the double coset $\Gamma_n^J M \Gamma_n^J$ into disjoint right cosets:
\[
\Gamma_n^J M \Gamma_n^J = \bigsqcup_{i=1}^d \Gamma_n^J g_i,
\]
where $d$ is the number of right cosets, that is, $d = [\Gamma_n^J M \Gamma_n^J : \Gamma_n^J]$. Then, for any $\phi \in J_{l,m}(\Gamma_n^J)$, we define the action of the double coset $\Gamma_n^J M \Gamma_n^J$ on $\phi$ by
\[
\phi|_{l,m} \Gamma_n^J M \Gamma_n^J := \sum_{i=1}^d \phi|_{l,m} g_i,
\]
where the summation on the right-hand side is well-defined. We easily see that for any $\gamma \in \Gamma_n^J$,
\[
(\phi|_{l,m} \Gamma_n^J M \Gamma_n^J)|_{l,m} \gamma = \phi|_{l,m} \Gamma_n^J M \Gamma_n^J,
\]
that is, $\phi|_{l,m} \Gamma_n J M \Gamma_n J \in J_{l,m}(\Gamma_n J)$. Moreover, if $\phi \in J_{l,m}^{\text{cusp}}(\Gamma_n J)$, we have $\phi|_{l,m} \Gamma_n J M \Gamma_n J \in J_{l,m}^{\text{cusp}}(\Gamma_n J)$. Here we note that each double coset $\Gamma_n J M \Gamma_n J$ with $M \in G_n(\mathbb{Q})$ contains a unique representative of the form

$$d_n(\delta_1 \perp \cdots \perp \delta_n) = (\delta_1 \perp \cdots \perp \delta_n) \perp (\delta_1^{-1} \perp \cdots \perp \delta_n^{-1})$$

with $0 < \delta_1 | \cdots | \delta_n$. Moreover, let $D = \delta_1 \perp \cdots \perp \delta_n$ and $D' = \delta_1' \perp \cdots \perp \delta_n'$ be two diagonal matrices with $0 < \delta_1 | \cdots | \delta_n$, $0 < \delta_1' | \cdots | \delta_n'$. We easily see that if $(\delta_n, \delta_n') = 1$, then for any $\phi \in J_{l,m}(\Gamma_n J)$,

$$\phi|_{l,m} \Gamma_n J d_n(DD') \Gamma_n J = \phi|_{l,m} \Gamma_n J d_n(D) \Gamma_n J|_{l,m} \Gamma_n J d_n(D') \Gamma_n J.$$

A Jacobi form $\phi \in J_{l,1}(\Gamma_n J)$ is called a Hecke eigenform if it is a common eigenfunction of all actions of double cosets $\Gamma_n J M \Gamma_n J$ with $M \in G_n(\mathbb{Q})$, that is, for any $M \in G_n(\mathbb{Q})$, the equation

$$\phi|_{l,m} \Gamma_n J M \Gamma_n J = \lambda_\phi(M) \phi$$

holds with some $\lambda_\phi(M) \in \mathbb{C}$. We easily see from the above argument that $\phi$ is a Hecke eigenform if and only if it satisfies for any rational prime $p$ and $D = p^{\alpha_1} \perp \cdots \perp p^{\alpha_n} \in D_p^{(n)}(\mathbb{Z})$ with $0 \leq \alpha_1 \leq \cdots \leq \alpha_n$,

$$\phi|_{l,m} \Gamma_n J d_n(D) \Gamma_n J = \lambda_\phi(D) \phi$$

with $\lambda_\phi(D) \in \mathbb{C}$.

### 2.2.2. Jacobi forms on the adele group.

Let $\mathbb{A}$ be the adele ring of $\mathbb{Q}$ and let $\Psi_\mathbb{A}$ be the character of $\mathbb{Q} \setminus \mathbb{A}$ such that $\Psi_\mathbb{A}(x_\infty) = e(x_\infty)$ for any $x_\infty \in \mathbb{R}$. In addition, for each $m \in \mathbb{Z}$, we put $\Psi_\mathbb{A}^m(\kappa) = \Psi_\mathbb{A}(m \kappa)$ for any $\kappa \in \mathbb{A}$. We denote by $G_n^J(\mathbb{A})$ the adele group of the Jacobi group $G_n^J$ defined in the previous subsection. It follows from the strong approximation theorem for $G_n^J$ that

$$G_n^J(\mathbb{A}) = G_n^J(\mathbb{Q})G_n^J(\mathbb{R})K_n^J,$$

where $K_n^J := \prod_{p < \infty} G_n^J(\mathbb{Z}_p)$.

Let $l$ and $m$ be positive integers. A $\mathbb{C}$-valued function $f$ on $G_n^J(\mathbb{A})$ is called a Jacobi form of weight $l$ and index $m$ if it satisfies the following two conditions:

(i) The transformation formula

$$f([(0, 0), \kappa], \gamma g k_{\infty} k_{\text{fin}}) = \det(j(k_{\infty}, \sqrt{-1} 1_n))^{-l} \Psi_\mathbb{A}^m(\kappa) f(g)$$

holds for any $\kappa \in \mathbb{A}$, $\gamma \in G_n^J(\mathbb{Q})$, $g \in G_n^J(\mathbb{A})$, $k_{\infty} \in K_{\infty}$ and $k_{\text{fin}} \in K_n^J$.

(ii) For any $(\tau, z) \in \mathfrak{h}_n \times \mathbb{C}^n$, we fix an element $g_{\infty} \in G_n^J(\mathbb{R})$ such that $g_{\infty}(\sqrt{-1} 1_n, 0) = (\tau, z)$ and put

$$\Phi_f(\tau, z) := J_{l,m}(g_{\infty}, (\sqrt{-1} 1_n, 0)) f(g_{\infty}),$$

(2.5)
with the factor of automorphy \( J_{l,m} : G^J_n(\mathbb{R}) \times (\mathfrak{H}_n \times \mathbb{C}^n) \to \mathbb{C} \) defined in §2.2.1. Here we easily see that the value \( \Phi_f \) does not depend on the choice of \( g_\infty \). Then the function \( \Phi_f \) is holomorphic on \( \mathfrak{H}_n \times \mathbb{C}^n \).

In particular, a Jacobi form \( f \) is called cuspidal if it satisfies the further condition that

\[
|\det(\text{Im}(\tau))|^{1/2} \exp(-2m\pi \text{tr}(\text{Im}(\tau))^{-1}[\text{Im}(z)])| \Phi_f(\tau, z) |
\]

is bounded on \( \mathfrak{H}_n \times \mathbb{C}^n \). We denote by \( J_{l,m}(G^J_n(\mathbb{A})) \) and \( \text{J}^{\text{cusp}}_{l,m}(G^J_n(\mathbb{A})) \) the \( \mathbb{C} \)-vector spaces of the Jacobi forms and cuspidal Jacobi forms of weight \( l \) and index \( m \) on the group \( G^J_n(\mathbb{A}) \), respectively.

It is easy to see that for each \( f \in J_{l,m}(G^J_n(\mathbb{A})) \), the associated function \( \Phi_f \) is an element of \( J_{l,m}(\Gamma^J_n) \). In particular, if \( f \in \text{J}^{\text{cusp}}_{l,m}(G^J_n(\mathbb{A})) \), then \( \Phi_f \in \text{J}^{\text{cusp}}_{l,m}(\Gamma^J_n) \). Furthermore we have

**Lemma 2.2.** The map \( J_{l,m}(G^J_n(\mathbb{A})) \ni f \mapsto \Phi_f \in J_{l,m}(\Gamma^J_n) \) induces \( \mathbb{C} \)-linear isomorphisms \( J_{l,m}(G^J_n(\mathbb{A})) \cong J_{l,m}(\Gamma^J_n) \) and \( \text{J}^{\text{cusp}}_{l,m}(G^J_n(\mathbb{A})) \cong \text{J}^{\text{cusp}}_{l,m}(\Gamma^J_n) \).

**Proof.** Since it is straightforward, we omit the proof. \( \blacksquare \)

### 2.3. Standard L-functions attached to Jacobi forms

In this subsection we study Shintani’s standard \( L \)-functions attached to Jacobi forms. In particular, we derive an explicit formula for the standard \( L \)-function attached to the Jacobi Eisenstein series of index 1. It might be given in a classical way, but here we treat it adelically.

Let \( p \) be an arbitrary rational prime. For simplicity, we write \( G^J_p, G_p, K^J_p, K_p, Z^J_p \) instead of \( G^J_n(\mathbb{Q}_p), G_n(\mathbb{Q}_p), G^J_n(\mathbb{Z}_p), G_n(\mathbb{Z}_p) \) and \( Z^J_n(\mathbb{Q}_p) \), respectively. We denote by \( \Psi_p \) and \( | \ast |_p \) the restriction of \( \Psi_\mathbb{A} \) to \( \mathbb{Q}_p \) and the \( p \)-adic valuation of \( \mathbb{Q}_p \) normalized as \( |p|_p = p^{-1} \), respectively. Let \( \mathcal{H}_p = \mathcal{H}(G^J_p, K^J_p; \Psi_p) \) be the \( \mathbb{C} \)-algebra consisting of \( \mathbb{C} \)-valued functions \( \varphi \) on \( G^J_p \) satisfying the following two conditions:

(i) The equation

\[
\varphi(\{(0,0), \kappa|kgk'\}) = \Psi_p(\kappa)\varphi(g)
\]

holds for any \( \kappa \in \mathbb{Q}_p, k, k' \in K^J_p \) and \( g \in G^J_p \).

(ii) The function \( \varphi \) is compactly supported modulo \( Z^J_p \).

We note that \( \mathcal{H}_p \) forms a \( \mathbb{C} \)-algebra with the convolution product

\[
(\varphi_1 \ast \varphi_2)(g) := \int_{Z^J_p \backslash G^J_p} \varphi_1(gx^{-1})\varphi_2(x) \, dx \quad (\varphi_1, \varphi_2 \in \mathcal{H}_p),
\]

where \( dx \) is a Haar measure on \( Z^J_p \backslash G^J_p \) normalized by \( \int_{Z^J_p \backslash G^J_p} \, dx = 1 \).

The algebra \( \mathcal{H}_p \) is called the **Hecke algebra** of \( (G^J_p, K^J_p) \) with respect to the additive character \( \Psi_p \).
We put
\[ N_p^J := \{[[0, \mu], 0]d_n(A)n_n(S) \in G_p^J \mid \mu \in \mathbb{Q}_p^n, A \in U_n, S \in \text{Sym}_n(\mathbb{Q}_p)\}, \]
\[ T_p = T(\mathbb{Q}_p) := \{d_n(t_1 \perp \cdots \perp t_n) \in G_p \mid t_i \in \mathbb{Q}_p^\times\} \]
and \( T(\mathbb{Z}_p) := T_p \cap K_p \), where \( U_n \subset \text{GL}_n(\mathbb{Q}_p) \) is the group of upper unipotent matrices. We fix Haar measures \( d\mathbf{n} \) and \( d\mathbf{t} \) on \( N_p^J \) and \( T_p \) respectively normalized by
\[ \int_{N_p^J \cap K_p^J} d\mathbf{n} = 1 \quad \text{and} \quad \int_{T(\mathbb{Z}_p)} d\mathbf{t} = 1. \]
We define the module \( \delta_{N_p^J}(t) \) of \( t \in T_p \) to be the ratio \( d(t\mathbf{n}^{-1})/d\mathbf{n} \). For any \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n \), we put
\[ \pi_\alpha = p^{\alpha_1} \perp \cdots \perp p^{\alpha_n} \in \text{GL}_n(\mathbb{Q}_p). \]
We easily see that
\[ \delta_{N_p^J}(\pi_\alpha) = p^{-\sum_{i=1}^n (2n+3-2i)\alpha_i}. \]

Let \( X_0(T_p) \) be the group of unramified characters of \( T_p \), that is,
\[ X_0(T_p) := \{\chi \in \text{Hom}(T_p, \mathbb{C}^\times) \mid \chi \text{ is trivial on } T(\mathbb{Z}_p)\}. \]
In particular, if \( n = 1 \), then \( X_0(T_p) \) coincides with the group \( X_0(\mathbb{Q}_p^\times) \) consisting of all unramified characters of \( \mathbb{Q}_p^\times \). For any \( \chi \in X_0(T_p) \) and \( \varphi \in \mathcal{H}_p \), we define the \textit{zonal spherical function} \( \widehat{\omega}_\chi(\varphi) \) by
\[ \widehat{\omega}_\chi(\varphi) := \sum_{\alpha \in \mathbb{Z}^n} \chi^{-1}(d_n(\pi_\alpha))\tilde{\varphi}(d_n(\pi_\alpha)), \]
where
\[ \tilde{\varphi}(t) := \delta_{N_p^J}(t)^{-1/2} \int_{N_p^J} \varphi(t) d\mathbf{n} \quad (t \in T_p). \]
It was shown by Murase that the map \( \varphi \mapsto \widehat{\omega}_\chi(\varphi) \) gives a \( \mathbb{C} \)-algebra homomorphism of \( \mathcal{H}_p \) to \( \mathbb{C} \) and that every \( \mathbb{C} \)-algebra homomorphism of \( \mathcal{H}_p \) to \( \mathbb{C} \) is given by \( \varphi \mapsto \widehat{\omega}_\chi(\varphi) \) for some \( \chi \in X_0(T_p) \) (cf. Proposition 4.10 and Theorem 4.15 in [20]).

On the other hand, for any \( \chi \in X_0(T_p) \), let \( \phi_\chi \) be the \( \mathbb{C} \)-valued function on \( G_p^J \) defined by
\[ \phi_\chi([[0, \mu], 0]n[t[(\lambda, 0), 0]k]) = \Psi_p(\kappa)(\chi \delta_{N_p^J}^{-1/2}(t) \text{ char}_{\mathbb{Z}_p^n}(\lambda)) \]
for any \( \kappa \in \mathbb{Q}_p, n \in N_p^J, t \in T_p, \lambda \in \mathbb{Q}_p^n \) and \( k \in K_p^J \), where we denote by \( \text{char}_{\mathbb{Z}_p^n} \) the characteristic function of \( \mathbb{Z}_p^n \). Here we note that each \( \chi \in X_0(T_p) \) can be written in the form
\[ \chi(d_n(t_1 \perp \cdots \perp t_n)) = \chi^{(1)}(t_1) \cdots \chi^{(n)}(t_n), \]
with uniquely determined \( n \) unramified characters \( \chi^{(1)}, \ldots, \chi^{(n)} \in X_0(\mathbb{Q}_p^\times) \). In that case, we simply write \( \chi = (\chi^{(1)}, \ldots, \chi^{(n)}) \). For each \( \chi = (\chi^{(1)}, \ldots, \chi^{(n)}) \in X_0(T_p) \), we easily see that

\[
\phi_\chi([(0, 0), \kappa]|_{\mathbb{Q}_p^\times})(\lambda, \varphi) = \Psi_p(\kappa) \prod_{i=1}^n \chi^{(i)}(t_i)|_{\mathbb{Q}_p^\times}^{(2n+3-2i)/2} \text{char}_{\mathbb{Q}_p^\times}(\lambda)
\]

for any \( \kappa \in \mathbb{Q}_p^\times, n \in \mathbb{N}_p^J, t = d_n(t_1 \perp \cdots \perp t_n) \in T_p, \lambda \in \mathbb{Q}_{p^n}^\times \) and \( k \in K_p^J \).

For each rational prime \( p \), we define the action of the Hecke algebra \( H_p \) on the space \( J_{l,1}(G_n^J(A)) \) as follows: for any \( f \in J_{l,1}(G_n^J(A)) \) and \( \varphi \in H_p \),

\[
(f \ast \varphi)(g) := \int_{Z_p^J \setminus G_n^J} f(gx^{-1})\varphi(x) \, dx \quad (g \in G_n^J(A)).
\]

A Jacobi form \( f \in J_{l,1}(G_n^J(A)) \) is called a Hecke eigenform if it is a common eigenfunction of all elements of \( \bigotimes_p H_p \), that is, for any rational prime \( p \) and \( \varphi \in H_p \), the equation

\[
f \ast \varphi = \lambda f(\varphi)
\]

holds with some \( \lambda_f(\varphi) \in \mathbb{C} \). Since, for each \( p \), the map \( \lambda_f : H_p \rightarrow \mathbb{C} \) gives a \( \mathbb{C} \)-algebra homomorphism of \( H_p \) to \( \mathbb{C} \), it determines a \( \chi_f \in X_0(T_p) \) such that

\[
\lambda_f(\varphi) = \hat{\omega}_\chi_f(\varphi)
\]

for any \( \varphi \in H_p \). Then the Satake \( p \)-parameter of \( f \) is defined to be the orbit of \( \chi_f = (\chi_f^{(1)}, \ldots, \chi_f^{(n)}) \) in \( X_0(T_p) \) under the action of the Weyl group \( W_n \) of type \( C_n \) isomorphic to the semi-direct product of \( S_n \) and \( \{\pm 1\}^n \). We also call the vector \( (\chi_f^{(1)}(p), \ldots, \chi_f^{(n)}(p)) \in (\mathbb{C}^\times)^n/W_n \) the Satake \( p \)-parameter of \( f \). Then, for a Hecke eigenform \( f \in J_{l,1}(G_n^J(A)) \), we define the standard \( L \)-function attached to \( \phi \) by

\[
L(s, f, \text{St}) := \prod_p \prod_{i=1}^n \{(1 - \chi_f^{(i)}(p)p^{-s})(1 - \chi_f^{(i)}(p)^{-1}p^{-s})\}^{-1},
\]

which was introduced by Shintani in his unpublished paper, and afterwards was studied by Murase (cf. [20, 21]).

By Lemma 2.2, for each \( f \in J_{l,1}(G_n^J(A)) \), we obtain the associated element \( \Phi_f \in J_{l,1}^{\text{cusp}}(\Gamma_n^J) \). Then we easily have the following relation between the action of the Hecke algebra \( H_p \) on \( f \) and the operation \( \Phi_f|_{l,1} \Gamma_n^J M \Gamma_n^J \) for some \( M \in G_n(\mathbb{Z}[p^{-1}]) \):

**Lemma 2.3.** Let \( f \in J_{l,1}(G_n^J(A)) \). For any \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n \) with \( 0 \leq \alpha_1 \leq \cdots \leq \alpha_n \), we have

\[
\Phi_{f \ast \varphi_\alpha} = \Phi_f|_{l,1} \Gamma_n^J d_n(\pi_\alpha) \Gamma_n^J.
\]
Here $\varphi_\alpha$ is the element of $\mathcal{H}_p$ defined by

$$
\varphi_\alpha(g) = \begin{cases} 
\Psi_p(\kappa) & \text{if } g \in \mathbb{Z}_p^J K_p^J d_n(\pi_\alpha) K_p^J \text{ and } g = [(0, 0), \kappa]k d_n(\pi_\alpha) k', \\
0 & \text{otherwise},
\end{cases}
$$

where $\kappa \in \mathbb{Q}_p$ and $k, k' \in K_p^J$. In particular, if $f$ is a Hecke eigenform, then $\Phi_f$ is also a Hecke eigenform in the sense of §2.2.1.

Let $\phi \in J_{l,1}(\Gamma_n^J)$ be the Hecke eigenform corresponding to a Hecke eigenform $f \in J_{l,1}(G_n^J(\mathbb{A}))$ via the mapping defined in (2.5), that is, $\phi = \Phi_f$. By Lemma 2.3, we naturally define the standard $L$-function attached to $\phi$ as $L(s, \phi, \text{St}) := L(s, f, \text{St})$, that is,

$$
L(s, \phi, \text{St}) := \prod_{p<\infty} \prod_{i=1}^n \{(1 - \chi_\phi^{(i)}(p)p^{-s})(1 - \chi_\phi^{(i)}(p)^{-1}p^{-s})\}^{-1},
$$

where we put $\chi_\phi^{(i)}(p) = \chi_\phi^{(i)}(p)$ for $i = 1, \ldots, n$.

If $\phi$ is a cuspidal Hecke eigenform, the following analytic properties of $L(s, \phi, \text{St})$ have been shown by Murase ([21]):

**Lemma 2.4** (cf. [21]). If $\phi \in J_{l,1}^{\text{cusp}}(\Gamma_n^J)$ is a Hecke eigenform, then the standard $L$-function $L(s, \phi, \text{St})$ has a meromorphic continuation to the entire complex plane $\mathbb{C}$. More precisely, the function

$$
L^*(s, \phi, \text{St}) = \prod_{i=1}^n \Gamma_{\mathbb{C}}(s + l - 1/2 - i)L(s, \phi, \text{St}),
$$

with $\Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s}\Gamma(s)$, is meromorphic on $\mathbb{C}$ and satisfies the functional equation

$$
L^*(1 - s, \phi, \text{St}) = \varepsilon_n L^*(s, \phi, \text{St}),
$$

where

$$
\varepsilon_n = \begin{cases} 
-1 & \text{if } n \equiv 1, 2 \pmod{4}, \\
1 & \text{otherwise}.
\end{cases}
$$

**Remark.** Murase derived similar properties for the standard $L$-functions attached to more general cuspidal Jacobi forms whose index is a matrix.

In the rest of this subsection we consider the standard $L$-function attached to the Jacobi Eisenstein series $\mathcal{E}_{l,1}^{(n)} \in J_{l,1}(G_n^J(\mathbb{A}))$.

For any quasi-character $\xi : \mathbb{Q}^\times \setminus \mathbb{A}^\times \to \mathbb{C}^\times$, we define a $\mathbb{C}$-valued function $\tilde{\phi}_\xi$ on $G_n^J(\mathbb{A})$ by

$$
\tilde{\phi}_\xi([(0, \mu), \kappa]g[(\lambda, 0), 0]k_\infty k_{\text{fin}}) = \xi(\det(A))\varphi_0(\lambda)j(k_\infty, \sqrt{-1}1_n)^{-l}
$$

for any $\kappa \in \mathbb{A}$, $g = (\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}) \in G_n^J(\mathbb{A})$, $k_\infty \in K_\infty$ and $k_{\text{fin}} \in K_{\text{fin}}^J$, where $\varphi_0 = \prod_{v} \varphi_{0,v}$,

$$
\varphi_{0,v}(\lambda) = \begin{cases} 
\text{char}_{\mathbb{Z}_p}(\lambda) & \text{if } v = p < \infty, \\
\exp(-2\pi\lambda^t\lambda) & \text{if } v = \infty.
\end{cases}
$$
Then we define the Eisenstein series $E_\xi$ on $G_n^J(\mathbb{A})$ associated with $\xi$ by

$$E_\xi(g) := \sum_{\gamma \in P_n^I(\mathbb{Q}) \backslash G_n^I(\mathbb{Q})} \tilde{\phi}_\xi(\gamma g) \quad (g \in G_n^J(\mathbb{A})).$$

In particular, we denote by $E_{l,1}^{(n)}$ the Eisenstein series on $G_n^J(\mathbb{A})$ associated with a special character $\xi_l(x) = |x|_A^l$ ($x \in \mathbb{A}^\times$). We easily see that $E_{l,1}^{(n)}$ is an element of $J_{l,1}(G_n^J(\mathbb{A}))$ and corresponds to the Jacobi Eisenstein series $E_{l,1}^{(n)} \in J_{l,1}(\Gamma_n^J)$ in the same manner as in Lemma 2.2. Hence we also call $E_{l,1}^{(n)}$ the Jacobi Eisenstein series of weight $l$ and index 1. Then we have

**Proposition 2.2.** The Jacobi Eisenstein series $E_{l,1}^{(n)} \in J_{l,1}(G_n^J(\mathbb{A}))$ is a Hecke eigenform, that is, for any $\varphi \in \mathfrak{H}_p$,

$$E_{l,1}^{(n)} \ast \varphi = \lambda_\xi(\varphi)E_{l,1}^{(n)}$$

with $\lambda_\xi(\varphi) \in \mathbb{C}^\times$. Moreover, the Satake $p$-parameter of $E_{l,1}^{(n)}$ is taken to be of the form

$$(p^{l-(n+1)+i-1/2})_{1 \leq i \leq n}$$

up to the action of the Weyl group $W_n$.

**Proof.** For any quasi-character $\xi$ of $\mathbb{Q}^\times \backslash \mathbb{A}^\times$, we take a $\chi = (\chi^{(1)}, \ldots, \chi^{(n)}) \in X_0(T_p)$ such that

$$\chi^{(i)}(t_i) = \xi(t_i)t_i^{-2n^2+2i}/2 \quad (t_i \in \mathbb{Q}_p^\times)$$

for each $1 \leq i \leq n$. Then, by (2.6) and the definition of $\tilde{\phi}_\xi$, we have $\tilde{\phi}_\xi = \phi_\chi$. Therefore it suffices to prove that for any $\varphi \in \mathfrak{H}_p$ and $\lambda \in \mathbb{Q}_p^n$, (2.8)

$$(\phi_\chi \ast \varphi)([(\lambda, 0), 0]) = c \cdot \mathrm{char}_{\mathbb{Z}_p^n}(\lambda)$$

with some $c \in \mathbb{C}^\times$. Indeed, if $\lambda \notin \mathbb{Z}_p^n$, there exists $0 \neq \mu \in \mathbb{Z}_p^n$ such that $\Psi_p(\lambda \mu) \neq 1$. Thus we have

$$(\phi_\chi \ast \varphi)([(\lambda, 0), 0]) = (\phi_\chi \ast \varphi)([(\lambda, 0), 0] \cdot [(0, \mu), 0])$$

$$= (\phi_\chi \ast \varphi)([(\lambda, \mu), \lambda \mu])$$

$$= (\phi_\chi \ast \varphi)([(0, \mu), \lambda \mu] \cdot [(\lambda, 0), 0])$$

$$= \Psi_p(\lambda \mu)(\phi_\chi \ast \varphi)([(\lambda, 0), 0]),$$

and $(\phi_\chi \ast \varphi)([(\lambda, 0), 0]) = 0$. Now we have proved that the Eisenstein series $E_\xi$ is a Hecke eigenform. Moreover, it follows from (2.8) that

$$c = (\phi_\chi \ast \varphi)(1_{G_p^J}) = \int_{\mathbb{Z}_p^J \backslash G_p^J} \phi_\chi(g)\varphi(g^{-1})\,dg = \hat{\omega}_\chi(\varphi)$$
and hence the eigenvalue $\lambda_\varepsilon(\varphi)$ coincides with the zonal spherical function $\tilde{\omega}_\chi(\varphi)$. Therefore it follows from (2.7) that

$$\chi^{(i)}(t_i) = \xi_i(t_i)|t_i|^{{(2n+3-2i)}/2} = |t_i|^{{l-(2n+3-2i)}/2}$$

for each $i$. By substituting $t_i = p$, we obtain $\chi^{(i)}(p) = p^{-l+(2n+3-2i)/2}$ and complete the proof. ■

By Proposition 2.2, we obtain the following conclusion:

**Corollary.** Let $l$ be a positive even integer such that $l > n + 2$. Then

$$L(s, \varepsilon_{l,1}^{(n)}, \text{St}) = L(s, \varepsilon_{l,1}^{(n)}, \text{St}) = \prod_{i=1}^{n} \zeta(s - l + 1/2 + i)\zeta(s + l - 1/2 - i).$$

In particular, $L(s, \varepsilon_{l,1}^{(n)}, \text{St})$ and $L(s, \varepsilon_{l,1}^{(n)}, \text{St})$ converge absolutely for $\Re(s) > l - n - 1/2$. In addition, they have meromorphic continuations to the entire complex plane $\mathbb{C}$ and satisfy functional equations under $s \mapsto 1 - s$.

**Remark.** Let $k$ and $n$ be positive even integers such that $k > n + 1$. As mentioned in §2.1, $\varepsilon_{k,1}^{(n-1)}$ coincides with the first Fourier–Jacobi coefficient $e_{k,1}^{(n-1)}$ of the Siegel Eisenstein series $E_k^{(n)} \in M_k(\Gamma_n)$ of degree $n$ and weight $k$. Thus it follows from the Corollary to Proposition 2.2 that

$$L(s, e_{l,1}^{(n)}, \text{St}) = \prod_{p} \prod_{i=1}^{n-1} \{(1 - p^{-k-(n+1)/2}p^{-s+i-n/2}) \{1 - (p^{-k-(n+1)/2}p^{-s+i-n/2})\}^{-1}$$

$$= \prod_{i=1}^{n-1} L(s + k - 1/2 - i, E_{2k-n}^{(1)}),$$

where $E_{2k-n}^{(1)} \in M_{2k-n}(\Gamma_1)$. Moreover, replacing $e_{k,1}^{(n-1)}$ by the first Fourier–Jacobi coefficient $\phi_1 \in J_{k,1}^{\text{cusp}}(\Gamma_{n-1})$ of a Siegel cusp form $F \in S_k(\Gamma_n)$ which is connected to a normalized Hecke eigenform $f \in S_{2k-n}(\Gamma_1)$ via a lifting procedure due to Ikeda (cf. [12]), we also obtain a similar explicit formula for the standard $L$-function attached to $\phi_1$ (cf. [10]).

2.4. Eichler–Zagier–Ibukiyama correspondence between Jacobi forms and Siegel modular forms of half-integral weight. For later use, we recall that there exists a natural $\mathbb{C}$-linear correspondence from the space of Jacobi forms of even integral weight and of index 1 into that of Siegel modular forms of half-integral weight.
For any \((\tau, z) \in \mathfrak{H}_n \times \mathbb{C}^n\) and \((r_1, r_2) \in \mathbb{Q}^n \oplus \mathbb{Q}^n\), we define the \textit{theta series} of characteristic \((r_1, r_2)\) by
\[
\theta_{(r_1, r_2)}(\tau, z) = \theta_{(r_1, r_2)}^{(n)}(\tau, z) := \sum_{\lambda \in \mathbb{Z}^n} e((\tau/2)[(\lambda + r_1)] + (\lambda + r_1)^t(z + r_2)).
\]
In particular, for any \(r \in \mathbb{Z}^n\), we put \(\theta_r(\tau, z) = \theta_{(r/2, 0)}^{(n)}(2\tau, 2z)\). We note that the function \(\theta_r(\tau, z)\) depends only on \(r \mod 2\mathbb{Z}^n\). For a fixed \(\tau \in \mathfrak{H}_n\), it is known that \((\theta_{r}(\tau, z))_{r \in \mathbb{Z}^n/2\mathbb{Z}^n}\) forms a basis of the \(\mathbb{C}\)-vector space \(\Theta_{\tau}^{(n)}\) consisting of all \(\mathbb{C}\)-valued holomorphic functions \(\theta(z)\) on \(\mathbb{C}^n\) which satisfy
\[
\theta(z + \lambda \tau + \mu) = e(-\text{tr}(\tau^t[\lambda] + 2^t\lambda z))\theta(z)
\]
for any \(\lambda, \mu \in \mathbb{Z}^n\).

For any \(\tau \in \mathfrak{H}_n\), we put
\[
\theta(\tau) = \theta_{(0, 0)}^{(n)}(\tau) := \theta_{(0, 0)}^{(n)}(2\tau, 0) = \sum_{\lambda \in \mathbb{Z}^n} e(\tau^t[\lambda]).
\]
Then, for any \(M = \begin{pmatrix} A & B \\ \tilde{C} & \tilde{D} \end{pmatrix} \in I_0^{(n)}(4)\), we define \textit{Shimura’s factor of automorphy} by
\[
J(M, \tau) = J^{(n)}(M, \tau) := \frac{\theta^{(n)}(M \langle \tau \rangle)}{\theta^{(n)}(\tau)}.
\]
As is well-known,
\[
J(M, \tau)^2 = (-1)^{(\det D - 1)/2} \det(C\tau + D).
\]

For any \(l \in \mathbb{Z}\), a holomorphic function \(F(\tau)\) on \(\mathfrak{H}_n\) is called a \textit{Siegel modular form} of degree \(n\) and weight \(l - 1/2\) if it satisfies the following two conditions:

(i) \(F(M \langle \tau \rangle) = J(M, \tau)^{2l-1}F(\tau)\) for any \(M \in I_0^{(n)}(4)\).

(ii) For any \(M = \begin{pmatrix} A & B \\ \tilde{C} & \tilde{D} \end{pmatrix} \in \Gamma_n\), the function \(\det(C\tau + D)^{-l+1/2}F(M \langle \tau \rangle)\) possesses a Fourier expansion of the form
\[
det(C\tau + D)^{-l+1/2}F(M \langle \tau \rangle) = \sum_{B \in \text{Sym}_n^+(\mathbb{Z}) \geq 0} C_{F,M}(B)e(\text{tr}(B\tau)/4),
\]
where \(\det(C\tau + D)^{-l+1/2}\) is an appropriately defined single-valued function of \(\tau\). We note that such a \(F\) possesses a usual Fourier expansion
\[
F(\tau) = \sum_{B \in \text{Sym}_n^+(\mathbb{Z}) \geq 0} C_F(B)e(\text{tr}(B\tau)).
\]
In particular, a Siegel modular form \(F\) is called a \textit{cusp form} if it satisfies the stronger condition \(C_{F,M}(B) = 0\) unless \(B > 0\) (positive definite). We denote by \(M_{l-1/2}(I_0^{(n)}(4))\) and \(S_{l-1/2}(I_0^{(n)}(4))\) the \(\mathbb{C}\)-vector spaces of
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Siegel modular forms and Siegel cusp forms of degree \( n \) and weight \( l - 1/2 \), respectively.

We now define the generalized Kohnen plus space \( M_{l-1/2}^+(\Gamma_0(4)) \) to consist of all elements \( F \in M_{l-1/2}(\Gamma_0(4)) \) whose Fourier coefficients \( C_F(B) \) satisfy the condition

\[
C_F(B) = 0 \text{ unless } B \equiv (-1)^{l+1} t r_B r_B \mod 4 \text{Sym}_n^*(\mathbb{Z}) \text{ for some } r_B \in \mathbb{Z}^{n-1},
\]

and put \( S_{l-1/2}^+(\Gamma_0(4)) := M_{l-1/2}^+(\Gamma_0(4)) \cap S_{l-1/2}(\Gamma_0(4)) \). These spaces were introduced by Kohnen [19] for \( n = 1 \), and by Ibukiyama [11] for general \( n \).

Now, we recall an important fact that if \( l \) is even, then there exists a \( \mathbb{C} \)-linear isomorphism between the space \( J_{l,1}(\Gamma_n) \) of Jacobi forms of index 1 and the generalized Kohnen plus space \( M_{l-1/2}^+(\Gamma_0(4)) \) defined as follows.

Let \( \phi \in J_{l,1}(\Gamma_n) \) possess a Fourier expansion of the form

\[
\phi(\tau, z) = \sum_{T \in \text{Sym}_n^*(\mathbb{Z}), r \in \mathbb{Z}^n} c_\phi(T, r)e(\text{tr}(T\tau) + r^t z).
\]

Since, for each \( \tau \in \mathcal{H}_n \), \( \phi(\tau, z) \) belongs to the space \( \Theta_{\tau}^{(n)} \), generated by \( (\theta_r(\tau, z))_{r \in \mathbb{Z}^n/2\mathbb{Z}^n} \), \( \phi \) can be expressed as

\[
\phi(\tau, z) = \sum_{r \in \mathbb{Z}^n/2\mathbb{Z}^n} h_r(\tau)\theta_r(\tau, z)
\]

with some \( 2^n \) holomorphic functions \( (h_r(\tau))_{r \in \mathbb{Z}^n/2\mathbb{Z}^n} \) on \( \mathcal{H}_n \) whose Fourier expansion is of the form

\[
h_r(\tau) = \sum_{T \in \text{Sym}_n^*(\mathbb{Z}), 4T - trr \geq 0} c_\phi(T, r)e(\text{tr}((T - trr/4)\tau)).
\]

Then we put

\[
\sigma(\phi)(\tau) = \sum_{r \in \mathbb{Z}^n/2\mathbb{Z}^n} h_r(4\tau).
\]

The following statement was shown by Eichler and Zagier [7] in the case \( n = 1 \) and by Ibukiyama for general \( n \):

**Proposition 2.3** (cf. Theorems 1, 2 in [11]). If \( l \) is even, then the map \( \phi \mapsto \sigma(\phi) \) gives a \( \mathbb{C} \)-linear isomorphism

\[
J_{l,1}(\Gamma_n) \cong M_{l-1/2}^+(\Gamma_0(4)),
\]

which is compatible with the actions of Hecke operators. Furthermore, its
restriction to the space $J^\text{cusp}_{l,1}(\Gamma_n^J)$ also induces a $\mathbb{C}$-linear isomorphism

$$J^\text{cusp}_{l,1}(\Gamma_n^J) \cong S^+_{l-1/2}(\Gamma_0^{(n)}(4)).$$

We call it the Eichler–Zagier–Ibukiyama correspondence.

**Remark.** When $l$ is odd, the space $J_{l,1}(\Gamma_n^J)$ is not isomorphic to the Kohnen plus space $M^+_{l-1/2}(\Gamma_0^{(n)}(4))$. However, we note that a similar claim is also valid for the space $J_{l,1}^{\text{skew}}(\Gamma_n^J)$ of skew holomorphic Jacobi forms, which was shown by Skoruppa ([26, 27]) in the case $n = 1$ and by Arakawa ([2]) and Hayashida ([9]) for general $n$.

We easily see by the definition that the Fourier expansion of $\sigma(\phi)$ can be expressed in terms of Fourier coefficients of $\phi$ as

$$\sigma(\phi)(\tau) = \sum_{B \in \text{Sym}_n(\mathbb{Z}) \geq 0} c_\phi((B + t_{BR}B)/4, r_B)e(\text{tr}(B\tau)),$$

where $r_B$ denotes an element of $\mathbb{Z}^n$ such that $B + t_{BR}B \in 4\text{Sym}_n^*(\mathbb{Z})$. We note that $r_B$ is uniquely determined by $B$ modulo $2\mathbb{Z}^n$, and furthermore $c_\phi((B + t_{BR}B)/4, r_B)$ does not depend on the choice of the representative of $r_B$ mod $2\mathbb{Z}^n$. Moreover, if $\phi$ coincides with the first Fourier–Jacobi coefficient of a Siegel modular form $F \in M_1(\Gamma_{n+1})$, we have

$$\sigma(\phi)(\tau) = \sum_{B \in \text{Sym}_n(\mathbb{Z}) \geq 0} A_F(B(1))e(\text{tr}(B\tau)),$$

where $B(1) \in \text{Sym}_{n+1}^*(\mathbb{Z})$ denotes the matrix defined in §1, and $A_F(B(1))$ is the $B(1)$th Fourier coefficient of $F$. In particular, let $n$ and $k$ be positive even integers such that $k > n + 1$ and take $\phi = e_{k,1}^{(n-1)} \in J_{k,1}(\Gamma_{n-1}^J)$. Then we have the following explicit formula for the Fourier coefficients of the associated form $\sigma(e_{k,1}^{(n-1)}) \in M^+_{k-1/2}(\Gamma_0^{(n-1)}(4))$:

**Proposition 2.4.** Under the same assumption as in Proposition 2.1, let $\sigma(e_{k,1}^{(n-1)})$ possess a Fourier expansion

$$\sigma(e_{k,1}^{(n-1)})(\tau) = \sum_{B \in \text{Sym}_n(\mathbb{Z}) \geq 0} C_{k-1/2}^{(n-1)}(B)e(\text{tr}(B\tau)).$$

Then, for each $B \in \text{Sym}_{n-1}^*(\mathbb{Z}) \geq 0$ satisfying the condition (1.1), we have

$$C_{k-1/2}^{(n-1)}(B) = \xi(n, k)L(1 - k + n/2, \chi_B(1))f(B(1))^{k-(n+1)/2} \prod_{p|f(B(1))} \tilde{F}_p^{(1)}(B; p^{k-(n+1)/2}).$$

**Proof.** If $B = 4T - tx$ with $T \in \text{Sym}_{n-1}^*(\mathbb{Z})$ and $r \in \mathbb{Z}^{n-1}$, we have $C_{k-1/2}^{(n-1)}(B) = c_{k,1}^{(n-1)}(T, r)$. Thus the assertion follows from Proposition 2.1. $lacksquare$
3. Andrianov-type identity for power series attached to Jacobi forms. Throughout this section, let \( n \) and \( k \) be positive even integers such that \( k > n + 1 \), and fix a rational prime \( p \). For a subring \( R \) of \( \mathbb{Z}_p \), we denote by \( \text{Sym}_{n-1}(R)^{(1)} \) the subset of \( \text{Sym}_{n-1}(R)^\times \) consisting of all elements which satisfy the condition (1.1) in §1:

\[
\text{Sym}_{n-1}(R)^{(1)} = \{ B \in \text{Sym}_{n-1}(R)^\times \mid B + t_B t_R B \in 4\text{Sym}^*_{n-1}(R) \text{ for some } r_B \in R^{n-1} \}. 
\]

As mentioned in §1, with each \( B \in \text{Sym}_{n-1}(R)^{(1)} \) we can associate an element

\[
B^{(1)} = \left( \frac{1}{t_B/2} \frac{r_B/2}{(B + t_T B r_B)/4} \right) \in \text{Sym}^*_n(R)^\times.
\]

For \( B \in \text{Sym}_{n-1}(\mathbb{Z}_p)^{(1)} \), we define a modified local Siegel series \( b_p^{(1)}(B; t) \) as follows. For each \( R \in \text{Sym}_{n-1}(\mathbb{Z}_p[p^{-1}]) \) and \( r \in \mathbb{Z}_p^{n-1} \), if \( R \in p^{-1}\text{Sym}_{n-1}(\mathbb{Z}_p) \) with \( l \geq 0 \), we put

\[
\omega(R; r) = p^{-(n-1)l} \mu_p(R)^{1/2} \sum_{x \in \mathbb{Z}_p^{n-1}/p^l \mathbb{Z}_p^{n-1}} e_p(-R[t]x + rR^t x/2 + xR^t r/2),
\]

where \( \mu_p(R) = [\mathbb{Z}_p^{n-1}R + \mathbb{Z}_p^{n-1} : \mathbb{Z}_p^{n-1}] \), and the right-hand side does not depend on the choice of \( l \). Suppose that \( B \in \text{Sym}_{n-1}(\mathbb{Q}_p) \) is of the form \( B = 4T - t_T r \) with \( T \in \text{Sym}_{n-1}(\mathbb{Q}_p) \) and \( r \in \mathbb{Z}_p^{n-1} \). We put

\[
b_p^{(1)}(B; t) = \sum_{R \in \text{Sym}_{n-1}(\mathbb{Q}_p[p^{-1}])/\text{Sym}_{n-1}(\mathbb{Z}_p)} \omega(R; r)e_p(-\text{tr}(TR)) t^\text{ord}_p(\mu_p(R)).
\]

We note that this series coincides with \( \alpha_1(B, t) \) of \( [23] \) if \( p \neq 2 \) and \( r = 0 \). As will be shown later, the above definition does not depend on the choice of \( T \) and \( r \) (cf. Proposition 3.1 below).

On the other hand, if \( m > 1 \), for each \( S \in \text{Sym}^*_m(\mathbb{Z}_p), T \in \text{Sym}_{n-1}(\mathbb{Q}_p), r \in \mathbb{Z}_p^{n-1} \) and \( e \in \mathbb{Z}_{\geq 0} \), we put

\[
\mathcal{A}_e(S, T, r)
\]

\[
:= \left\{ X \in M_{m,n-1}(\mathbb{Z}_p)/p^e M_{m,n-1}(\mathbb{Z}_p) \mid (-1 \perp S)[X] + t_r x_1/2 + t_r x_1 r/2 - T \in p^e \text{Sym}^*_m(\mathbb{Z}_p) \right\},
\]

where \( x_1 \in \mathbb{Z}_p^{n-1} \) denotes the first row of \( X \). We easily check that it is well-defined. Furthermore, if both \( S \) and \( (t_r/2)_T \) are non-degenerate, then \( p^{e(-m(n-1)+n(n-1)/2)} \# \mathcal{A}_e(S, T, r) \) has the same value for each

\[
e \geq \text{ord}_p \left( \det \left( \begin{array}{c} 1 \\ t_r/2 \\ T \end{array} \right) \right);
\]
this value will be denoted by $\alpha_p^{(1)}(S, T, r)$. We note that $\alpha_p^{(1)}(S, T, r)$ coincides with the usual local density $\alpha_p(-1 \perp S, T)$ if $r = 0$. Then we obtain the following lemmas:

**Lemma 3.1.** Suppose that $B \in \text{Sym}_{n-1}(\mathbb{Q}_p)^\times$ is of the form $B = 4T - trr$ with $T \in \text{Sym}_{n-1}(\mathbb{Q}_p)$ and $r \in \mathbb{Z}_p^{n-1}$. Then

$$b_p^{(1)}(B; p^{-k+1/2}) = \alpha_p(H_{k-1}, T, r),$$

where

$$H_{k-1} = H \perp \cdots \perp H_{k-1} \text{ with } H = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \in \text{Sym}_2(\mathbb{Z}_p).$$

In particular, $b_p^{(1)}(B; t) = 0$ unless $B \in \text{Sym}_{n-1}(\mathbb{Z}_p)^{(1)}$.

**Proof.** By Lemma 3.4 of [24], we have

$$b_p^{(1)}(B; p^{-k+1/2}) = \sum_{R \in \text{Sym}_{n-1}(\mathbb{Z}_p[p^{-1}])} \sum_{x \in \mathbb{Z}_p^{n-1}/p^{l}\mathbb{Z}_p^{n-1}} e_p(-R[t]x + rR[t]x/2 + xR[t]r/2) \times p^{-(k-1)\text{ord}_{p}(R)}p^{-(n-1)l}e_p(-\text{tr}(TR))$$

$$= \sum_{R} \sum_{x} e_p(-R[t]x + rR[t]x/2 + xR[t]r/2)p^{-(n-1)l}e_p(-\text{tr}(TR))p^{-2l(k-1)n}$$

$$\times \sum_{Y \in \text{M}_{2k-2, n-1}(\mathbb{Z}_p)/p^{l}\text{M}_{2k-2, n-1}(\mathbb{Z}_p)} e_p(\text{tr}(H_{k-1}[Y]R))$$

$$= \sum_{R} \sum_{x} \sum_{Y} e_p(\text{tr}((-ttx + H_{k-1}[Y] + ttx/2 + ttx/2 - T)R))p^{-l(2k-1)(n-1)}$$

$$= \#A_{l}(H_{k-1}, T, r)p^{-l(2k-1)(n-1)-n(n-1)/2}.$$ 

Thus the assertion holds. ■

**Lemma 3.2.** Suppose that $B \in \text{Sym}_{n-1}(\mathbb{Q}_p)^\times$ is of the form $B = 4T - trr$ with $T \in \text{Sym}_{n-1}(\mathbb{Q}_p)$ and $r \in \mathbb{Z}_p^{n-1}$. Then

$$\alpha_p(H_k, B^{(1)}) = (1 - p^{-k})\alpha_p(H_{k-1}, T, r).$$

**Proof.** The proof is similar to that of Proposition 2.4 in [14]; we give a sketch. For each $\xi = (\xi_i) \in \mathbb{Z}_p^{2k}$, we put

$$\mathcal{A}_{\xi}(H_k, B^{(1)}) = \{X \in \text{M}_{2k,n}(\mathbb{Z}_p)/p^{e}\text{M}_{2k,n}(\mathbb{Z}_p) \mid H_k[X] - B^{(1)} \in p^{e}\text{Sym}_n^{\ast}(\mathbb{Z}_p)\}$$

and

$$\mathcal{A}_{\xi}(H_k, B^{(1)}; \xi) = \{X = (x_{ij}) \in \mathcal{A}_{\xi}(H_k, B^{(1)}) \mid x_{i1} \equiv \xi_i \pmod{p^{e}} \text{ for } 1 \leq i \leq 2k\}.$$
We easily see that $\mathcal{A}_e(H_k, B^{(1)}; \xi) \neq \emptyset$ only if $\xi \in \mathcal{A}_e(H_k, 1)$. Fix such a $\xi$. Then $\xi \not\equiv 0 \pmod{p\mathbb{Z}_p^k}$. Thus by Lemma 2.3 in [14], we can take $U \in \text{GL}_{2k}(\mathbb{Z}_p)$ and $K \in \text{Sym}_{2k-2}^*(\mathbb{Z}_p)$ such that
\[
\begin{pmatrix}
1 & 1/2 \\
1/2 & 0
\end{pmatrix} \perp K = H_k[U]; \quad (i) \quad K \sim_{\mathbb{Z}_p} H_{k-1}; \quad (ii) \quad U^{-1}\xi = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.
\]
For each $X \in \mathcal{A}_e(H_k, B^{(1)}; \xi)$, we write $X = (t\xi \mid Y)$ with $Y \in M_{2k,n-1}(\mathbb{Z}_p)$, and
\[
Y = \begin{pmatrix} y_1 \\ y_2 \\ Y_3 \end{pmatrix}
\]
with $y_1, y_2 \in \mathbb{Z}_p^{n-1}$ and $Y_3 \in M_{2k-2,n-1}(\mathbb{Z}_p)$.
Then, by an easy calculation, we have
\[
y_1 + y_2/2 - r/2 \in p^n\mathbb{Z}_p^{n-1}
\]
and
\[
-t^t y_1 y_1 + K[Y_3] + t^t y_1 y_2/2 + t^t y_2 y_1/2 - T \in p^n\text{Sym}^*_n(\mathbb{Z}_p).
\]
Thus we have
\[
-t^t y_1 y_1 + K[Y_3] + t^r y_1 r/2 + t^t y_1 r/2 - T \in p^n\text{Sym}^*_n(\mathbb{Z}_p),
\]
that is, $(y_1, Y_3) \in \mathcal{A}_e(H_{k-1}, T, r)$. Moreover, we easily see that $Y \mapsto (y_1, Y_3)$ induces a bijection between $\mathcal{A}_e(H_k, B^{(1)}; \xi)$ and $\mathcal{A}_e(H_{k-1}, T, r)$. Thus
\[
p^n(-2kn+n(n+1)/2)\#\mathcal{A}_e(H_k, B^{(1)})
= p^n(-2k+1)\#\mathcal{A}_e(H_k, 1)p^n(-2k-1(n-1)+n(n-1)/2)\#\mathcal{A}_e(H_{k-1}, T, r)
= \alpha_p(H_k, 1)\alpha_p(H_{k-1}, T, r) = (1 - p^{-k})\alpha_p(H_{k-1}, T, r).
\]
Hence the assertion holds. ■

Now, by combining Lemmas 3.1 and 3.2, we obtain the following:

**Proposition 3.1.** For each $B \in \text{Sym}_{n-1}(\mathbb{Z}_p)^{(1)}$ and $s \in \mathbb{C}$, we have
\[
b_p^{(1)}(B; p^{-s+1/2}) = (1 - p^{-s})^{-1} b_p(B^{(1)}; s).
\]

**Proof.** It is well-known that for each $B' \in \text{Sym}_{n}(\mathbb{Z}_p)^\times$ with $n < 2k$, the Siegel series $b_p(B'; s)$ in §1 satisfies the equation
\[
b_p(B'; k) = \alpha_p(H_k, B').
\]
Hence, by Lemmas 3.1 and 3.2, we have
\[
b_p^{(1)}(B; p^{-k+1/2}) = (1 - p^{-k})^{-1} b_p(B^{(1)}; k)
\]
for infinitely many $k$, and hence the assertion follows. ■
Remark. The definition of the series $b_p^{(1)}(B; t)$ for $B = 4T - \ell r r$ with $T \in \text{Sym}_{n-1}(\mathbb{Q}_p)$ and $r \in \mathbb{Z}_p^{n-1}$ does not depend on the choice of $T$ and $r$. Indeed, if $T \in \text{Sym}^*_n(\mathbb{Z}_p)$, the vector $r$ is uniquely determined by $B$ modulo $2\mathbb{Z}_p^{n-1}$, and the matrix $\left( \frac{1}{r} \frac{r}{T} \right)$ is uniquely determined by $B$ up to $\text{GL}_n(\mathbb{Z}_p)$-equivalence. Thus, by Proposition 3.1, $b_p^{(1)}(B; t)$ is uniquely determined by $B$. If $T \not\in \text{Sym}^*_n(\mathbb{Z}_p)$, we have $b_p^{(1)}(B; t) = 0$. Furthermore, if $B = 4T' - \ell r' r'$ is another expression, then $T'$ does not belong to $\text{Sym}^*_n(\mathbb{Z}_p)$ either. This proves that $b_p^{(1)}(B; t)$ is well-defined.

Now we put

$$\tilde{b}_p^{(1)}(B; t) := \sum_{D \in \text{GL}_n(\mathbb{Z}_p) \setminus D_p^{(n-1)}(\mathbb{Z}_p)} \pi_p(D) b_p^{(1)}(B[D^{-1}]; t) (p^{n-1} t^2)^{\text{ord}_p(\det D)}.$$

Then, by Proposition 3.1, we obtain the following rationality theorem for the polynomial $B_p^{(1)}(B; t)$ defined in §1:

**Proposition 3.2.** For each $B \in \text{Sym}_{n-1}(\mathbb{Z}_p)^{(1)}$, we have

$$B_p^{(1)}(B; p^{n-1/2} t) \tilde{b}_p^{(1)}(B; p^{1/2} t) = \prod_{i=1}^{n-1} (1 - p^{2i} t^2).$$

Next, we study the standard $L$-function attached to a Hecke eigenform and some power series related to it. For a Hecke eigenform $\phi \in J_{k,1}^{\text{cusp}}(\Gamma_{n-1}^J)$, and $D \in D_p^{(n-1)}(\mathbb{Z})$, let

$$\phi|_{k,1} \Gamma_{n-1}^J d_{n-1}(D) \Gamma_{n-1}^J = \lambda_\phi(D) \phi$$

with $\lambda_\phi(D) \in \mathbb{C}$. Then we define a power series $Z_p(t, \phi)$ by

$$Z_p(t, \phi) := \sum_{D \in \text{ED}_p^{(n-1)}(\mathbb{Z})} \lambda_\phi(D) t^{\text{ord}_p(\det D)},$$

where $\text{ED}_p^{(n-1)}(\mathbb{Z})$ denotes the set of all elementary divisors of the form $p^{\alpha_1} \perp \cdots \perp p^{\alpha_{n-1}}$ with $0 \leq \alpha_1 \leq \cdots \leq \alpha_{n-1}$. The following statement is shown by Murase and Sugano:

**Proposition 3.3** (cf. Lemma 6.5 in [22], see also Theorem 5.5 in [3]). Let $\phi \in J_{k,1}^{\text{cusp}}(\Gamma_{n-1}^J)$ be a Hecke eigenform whose Satake $p$-parameter is of the form $(\chi_\phi^{(1)}(p), \ldots, \chi_\phi^{(n-1)}(p)) \in (\mathbb{C}^\times)^{n-1}/W_{n-1}$. Then

$$Z_p(t, \phi) = \prod_{i=1}^{n-1} \frac{1 - p^{2i} t^2}{(1 - \chi_\phi^{(i)}(p)p^{n-1/2} t)(1 - \chi_\phi^{(i)}(p^{-1} p^{n-1/2} t))}.$$
Let
\[ \mathcal{A}_p^{(n-1)} := \left\{ \begin{pmatrix} V \\ W \end{pmatrix} \in M_{2n-2,n-1}(\mathbb{Z}) \ \bigg| \ V, W \in D_p^{(n-1)}(\mathbb{Z}), \ \gcd(V, W) = 1 \right\}, \]
where \( \gcd(V, W) = 1 \) means that \( V \) and \( W \) are coprime to each other. For each \( \left( \begin{array}{c} V \\ W \end{array} \right) \in \mathcal{A}_p^{(n-1)} \), \( R \in \text{Sym}_{n-1}(\mathbb{Z}[p^{-1}]) \) and \( (\lambda_1, \lambda_2) \in \mathbb{Z}^{n-1} \oplus \mathbb{Z}^{n-1} \), we put
\[ M_{V,W,R} := \begin{pmatrix} tW^{-1}tV & tW^{-1}RV^{-1} \\ 0_{n-1} & WV^{-1} \end{pmatrix} \in G_{n-1}(\mathbb{Z}[p^{-1}]) \]
and
\[ [\lambda_1, \lambda_2] := [(\lambda_1, \lambda_2), \lambda_1 t \lambda_2] = \begin{pmatrix} 1 & \lambda_1 & 0 & \lambda_2 \\ 0 & -t \lambda_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -t \lambda_2 \end{pmatrix} \in H_{n-1}(\mathbb{Z}). \]
By combining Lemma 2.1 and some easy calculation (cf. [5]), we obtain the following:

**Lemma 3.3.** We have
\[ \Gamma_{n-1}^J G_{n-1}(\mathbb{Z}[p^{-1}]) \Gamma_{n-1}^J = \bigcup_{D \in \mathbf{ED}_p^{(n-1)}(\mathbb{Z})} \bigcup_{(V,W) \in \mathcal{A}_p^{(n-1)}} \Gamma_{n-1}^J d_{n-1}(D) \Gamma_{n-1}^J \]
\[ = \bigcup_{(V,W) \in \mathcal{A}_p^{(n-1)}} \bigcup_{R \in \text{Sym}_{n-1}(\mathbb{Z}[p^{-1}])} \bigcup_{(\lambda_1, \lambda_2) \in \mathbb{Z}^{n-1} \oplus \mathbb{Z}^{n-1}} \Gamma_{n-1}^J [M_{V,W,R}] \cdot [\lambda_1, \lambda_2], \]
where \( \left( \begin{array}{c} V \\ W \end{array} \right), R \) and \( (\lambda_1, \lambda_2) \) run respectively over
- \( (\mathbf{1}_{n-1} \perp \text{GL}_{n-1}(\mathbb{Z})) \setminus \mathcal{A}_p^{(n-1)} / \text{GL}_{n-1}(\mathbb{Z}), \)
- \( \text{Sym}_{n-1}(\mathbb{Z}[p^{-1}]) \setminus tW \text{Sym}_{n-1}(\mathbb{Z}) W, \)
and
- \( (\mathbb{Z}^{n-1} \oplus \mathbb{Z}^{n-1}) + (\mathbb{Z}^{n-1} \oplus \mathbb{Z}^{n-1}) M_{V,W,R} / (\mathbb{Z}^{n-1} \oplus \mathbb{Z}^{n-1}) M_{V,W,R}. \)

Furthermore, if \( M_{V,W,R} \in \Gamma_{n-1}^J d_{n-1}(D) \Gamma_{n-1}^J \) with \( D \in \mathbf{ED}_p^{(n-1)}(\mathbb{Z}) \), we have \( \text{ord}_p(\det D) = \text{ord}_p(\det V \det W \mu_p(R)) \).

Therefore, we get the following explicit formula for the actions of Hecke operators:

**Corollary.** For each \( \phi \in J_{k-1}(\Gamma_{n-1}^J) \), we have
\[ \sum_{D \in \mathbf{ED}_p^{(n-1)}(\mathbb{Z})} (\phi |_{k-1} \Gamma_{n-1}^J d_{n-1}(D) \Gamma_{n-1}^J)(\tau, z) = \sum_{(V,W)} \sum_{R} p^{(-2n+3)\delta_{V,W,R}} \det V^{k-1} \]
\[ \times \det W^{-k} \sum_{(\lambda_1, \lambda_2) \in (\mathbb{Z}^{n-1} \oplus \mathbb{Z}^{n-1})/p^\delta} e(\tau^{t} \lambda_1 + 2\lambda_1 t z) \times \phi(\tau[VW^{-1}] + R[W^{-1}], (z + \lambda_1 \tau + \lambda_2)VW^{-1}), \]

where \( \left( \frac{V}{W} \right) \) and \( R \) run over the sets stated in Lemma 3.3, and \( \delta_{V,W,R} = \text{ord}_p(\det V \det W \mu_p(R)). \)

Proof. For each \( \left( \frac{V}{W} \right) \in J_{p}^{(n-1)} \) and \( R \in \text{Sym}_{n-1}(\mathbb{Z}[p^{-1}]) \), we have
\[ \Gamma_{n-1}^J M_{V,W,R} \Gamma_{n-1}^J = \Gamma_{n-1}^J d_{n-1}(D) \Gamma_{n-1}^J \]
for some \( D = p^{\alpha_1} \cdots p^{\alpha_{n-1}} \in \text{ED}_p^{(n-1)}(\mathbb{Z}). \) Then we have
\[
(\mathbb{Z}^{n-1} \oplus \mathbb{Z}^{n-1}) + (\mathbb{Z}^{n-1} \oplus \mathbb{Z}^{n-1}) M_{V,W,R}/(\mathbb{Z}^{n-1} \oplus \mathbb{Z}^{n-1}) M_{V,W,R}
\simeq (\mathbb{Z}^{n-1} \oplus \mathbb{Z}^{n-1}) + (\mathbb{Z}^{n-1} \oplus \mathbb{Z}^{n-1}) d_{n-1}(D)/(\mathbb{Z}^{n-1} \oplus \mathbb{Z}^{n-1}) d_{n-1}(D)
\simeq \mathbb{Z}^{n-1}/\mathbb{Z}^{n-1} D.
\]
It follows from Lemma 3.3 that \( \#(\mathbb{Z}^{n-1}/\mathbb{Z}^{n-1} D) = p^{\delta_{V,W,R}} \) and \( \alpha_1, \ldots, \alpha_{n-1} \leq \delta_{V,W,R}. \) Thus we have a natural surjection
\[ \pi : (\mathbb{Z}^{n-1} \oplus \mathbb{Z}^{n-1})/p^{\delta_{V,W,R}}(\mathbb{Z}^{n-1} t V \oplus \mathbb{Z}^{n-1}) \to \mathbb{Z}^{n-1}/\mathbb{Z}^{n-1} D, \]
and \( \#\ker(\pi) = p^{(2n-3)\delta_{V,W,R}} \det V. \) Thus the assertion holds. \( \blacksquare \)

By the above corollary, we obtain the following conclusion:

**Proposition 3.4:** Suppose that \( \phi \in J_{k,1}(\Gamma_{n-1}^J) \) is a Hecke eigenform and the associated form \( \sigma(\phi) \in M_{k-1/2}^+(\mathbb{Z}[p^{-1}](4)) \) under the Eichler–Zagier–Ibukiyama correspondence possesses a Fourier expansion
\[ \sigma(\phi)(\tau) = \sum_{B \in \text{Sym}_{n-1}(\mathbb{Z}_p)_{>0}} C_{\sigma(\phi)}(B) e(\text{tr}(B \tau)). \]
Then, for each \( B \in \text{Sym}_{n-1}(\mathbb{Z})_{>0}^{(1)} \), we have
\[
\prod_{i=1}^{n-1} \frac{1-p^{2i} t^2}{(1-\chi_\phi^{(i)}(p)p^{n-1/2}t)(1-\chi_\phi^{(i)}(p)^{-1}p^{n-1/2}t)} C_{\sigma(\phi)}(B) = \sum_{\left( \frac{V}{W} \right)} b_p^{(1)}(B[V^{-1}; t]) C_{\sigma(\phi)}(B[V^{-1}[W]) \times p^{-(k-n-1) \text{ord}_p(\det W)} p^{k \text{ord}_p(\det V)} t^{\text{ord}_p(\det V \det W)}, \]
where \( \left( \frac{V}{W} \right) \) runs over the set stated in Lemma 3.3.

**Proof.** We put
\[ \Lambda_p(t) = \sum_{D \in \text{ED}_p^{(n-1)}(\mathbb{Z})} \Gamma_{n-1}^J d_{n-1}(D) \Gamma_{n-1}^J t^{\text{ord}_p(\det D)}. \]
By the Corollary to Lemma 3.3, we have

\[
(\phi \vert_{k,1} \Lambda_p(t))(\tau, z) = \sum_T \sum_r c_\phi(T, r) \\
\times \sum_{(V W) \in (1_{n-1} \perp GL_n(\mathbb{Z})) \setminus \mathcal{A}_{n-1, GL_n(\mathbb{Z})}} \begin{pmatrix} V \\ W \end{pmatrix}^{(k-1) \text{ord}_p(\det V) - k \text{ord}_p(\det W) \ell \text{ord}_p(\det V \det W)} \\
\times \mathbf{e}(\text{tr}(T^t W^{-1} t V \tau + t (r^t W^{-1} t V) z)) \\
\times \sum_{R \in \text{Sym}_{n-1}^{(Z[p^{-1}]})/tW\text{Sym}_{n-1}(\mathbb{Z})W} \mathbf{e}(\text{tr}(T^t W^{-1} R)) \ell \text{ord}_p(\mu_p(R)) \\
\times \sum_{\lambda_1 \in \mathbb{Z}^{n-1}/p^{(2n-3)\delta_{V,W,R}}} p^{(-2n-3)\delta_{V,W,R}} \mathbf{e}(\text{tr}(\lambda_1 z + t (r^t W^{-1} t V + \lambda_1) \lambda_1 \tau)) \\
\times \sum_{\lambda_2 \in \mathbb{Z}^{n-1}/p^{3\delta_{V,W,R}}} \mathbf{e}(\text{tr}(t (r^t W^{-1} t V + \lambda_1) \lambda_2)).
\]

Since

\[
\sum_{\lambda_2 \in \mathbb{Z}^{n-1}/p^{3\delta_{V,W,R}}} \mathbf{e}(\text{tr}(t (r^t W^{-1} t V + \lambda_1) \lambda_2)) = \begin{cases} p^{(n-1)\delta_{V,W,R}} & \text{if } r^t W^{-1} \in \mathbb{Z}^{n-1}, \\
0 & \text{otherwise,} \end{cases}
\]

and

\[
\sum_{R \in \text{Sym}_{n-1}^{(Z[p^{-1}]})/tW\text{Sym}_{n-1}(\mathbb{Z})W} \mathbf{e}(\text{tr}(T^t W^{-1} R)) \ell \text{ord}_p(\mu_p(R)) = \begin{cases} (\det W)^n \sum_{R \in \text{Sym}_{n-1}^{(Z[p^{-1}]})/\text{Sym}_{n-1}(\mathbb{Z})} \mathbf{e}(\text{tr}(T^t W^{-1} R)) \ell \text{ord}_p(\mu_p(R)) & \text{if } T^t W^{-1} \in \text{Sym}_{n-1}^*(\mathbb{Z}), \\
0 & \text{otherwise,} \end{cases}
\]

we have

\[
(\phi \vert_{k,1} \Lambda_p(t))(\tau, z) = \sum_T \sum_r \sum_{(V W)} p^{k \text{ord}_p(\det V) + (-k + n + 1) \text{ord}_p(\det W) \ell \text{ord}_p(\det V \det W)} \\
\times \sum_{R \in \text{Sym}_{n-1}^{(Z[p^{-1}]})/\text{Sym}_{n-1}(\mathbb{Z})} \mathbf{e}(\text{tr}(TR))(pt)^{\text{ord}_p(\mu_p(R))}
\]
For a fixed $r_0 \in \mathbb{Z}^{n-1}$, we put

$$\mathcal{S}_1(r_0) = \{ \lambda_1 \in \mathbb{Z}^{n-1}/p^{\delta_{V,W,R}}\mathbb{Z}^{n-1} \mid 2\lambda_1 \equiv r_0 \mod \mathbb{Z}^{n-1} \},$$

and

$$\mathcal{S}_2(r_0) = \{ r \in \mathbb{Z}^{n-1}/p^{\delta_{V,W,R}}\mathbb{Z}^{n-1} \mid r^tV \equiv r_0 \mod 2\mathbb{Z}^{n-1} \}.$$

For each $\lambda_1 \in \mathcal{S}_1(r_0)$, the map $\lambda_1 \mapsto (2\lambda_1 - r_0)^tV$ induces a bijection between $\mathcal{S}_1(r_0)$ and $\mathcal{S}_2(r_0)$. Thus we have

$$(\phi|_{k,1} \Lambda_p(t))(\tau, z) = \sum_{r_0} \sum_{r \in \mathcal{S}_2(r_0)} e(\text{tr}(T^r)) (pt)^{\text{ord}_p(\mu_p(R))} p^{-(n-1)\delta_{V,W,R}}$$

$$\times \sum_{(V^r)} \sum_{(V^r)} c_\phi(T^r, r^tW) e(\text{tr}(t_0 r_0 z)) e(\text{tr}((T^r[V^1] + (t_0 r_0 - t(r^tV)(r^tV))/4)\tau))$$

$$= \sum_{T_0} \sum_{r_0} e(\text{tr}(T_0 \tau + t_0 r_0 z))$$

$$\times \sum_{(V^r)} \sum_{(V^r)} p^{k\text{ord}_p(\det V) - (k-n-1)\text{ord}_p(\det W)} p^{-(n-1)\delta_{V,W,R}}$$

$$\times c_\phi((T_0 - t_0 r_0/4)[t^1V^{-1}][tV] + (t_0 r_0/4)[tV], r^tW)$$

$$\times \sum_{R \in \text{Sym}_{n-1}(\mathbb{Z}[p^{-1}])/\text{Sym}_{n-1}(\mathbb{Z})} e(\text{tr}(((T_0 - t_0 r_0/4)[t^1V^{-1}] + (t_0 r_0/4)R) pt)^{\text{ord}_p(\mu_p(R))}).$$

Then, for a fixed $r \in \mathbb{Z}^{n-1}/2\mathbb{Z}^{n-1}$, the map $$(r + 2\mathbb{Z}^{n-1}) + 2p^{\delta_{V,W,R}}\mathbb{Z}^{n-1}/2p^{\delta_{V,W,R}}\mathbb{Z}^{n-1} \ni r + 2u \mapsto u \in \mathbb{Z}^{n-1}/p^{\delta_{V,W,R}}\mathbb{Z}^{n-1}$$
is a bijection, and we have

$$c_\phi((T_0 - t_0 r_0/4)[t^1V^{-1}][tV] + (t(r + 2u)(r + 2u)/4)[tW], (r + 2u)^tW)$$

$$= c_\phi((T_0 - t_0 r_0/4)[t^1V^{-1}][tV] + (t_0 r_0/4)[tW], r^tW).$$

Thus we have

$$(\phi|_{k,1} \Lambda_p(t))(\tau, z) = \sum_{T_0} \sum_{r_0} e(\text{tr}(T_0 \tau + t_0 r_0 z))$$

$$\times \sum_{(V^r)} \sum_{(V^r)} p^{k\text{ord}_p(\det V) - (k-n-1)\text{ord}_p(\det W)} p^{-(n-1)\delta_{V,W,R}}$$
We easily see that for an element \( r \in \mathbb{Z}^{n-1}/2\mathbb{Z}^{n-1} \), the sum

\[
\sum_{R \in \text{Sym}_{n-1}(\mathbb{Z}[p^{-1}])} (pt)^{\text{ord}_p(\mu_p(R))} p^{-(n-1)} \delta_{V,W,R}
\]

\[
\times \sum_{r \in \mathbb{Z}^{n-1}/2\mathbb{Z}^{n-1}} c_\phi((T_0 - t r_0 r_0/4)^t[V^{-1}][tW] + (t r r/4)[tW], r^t W)
\]

\[
\times \sum_{u \in \mathbb{Z}^{n-1}/p^u \mathbb{Z}^{n-1}} e(\text{tr}(((T_0 - t r_0 r_0/4)^t[V^{-1}] + t r r/4 + t u u + t u r/2 + t r u/2)R))
\]

equals \( b_p^{(1)}((4T_0 - t r_0 r_0)^t[V^{-1}]; t) \) or 0 according as \( (T_0 - t r_0 r_0/4)^t[V^{-1}] + t r r/4 \in \text{Sym}^*_{n-1}(\mathbb{Z}) \) (that is, \( (4T_0 - t r_0 r_0)^t[V^{-1}] \in \text{Sym}_{n-1}(\mathbb{Z}(1)) \)) or not. In the former case, such a vector \( r \) is uniquely determined by \( T_0, r_0, \) and \( V \), which will be denoted by \( r_1 \equiv r_1(T_0, r_0, V) \). Furthermore, we have

\[
(4T_0 - t r_0 r_0)^t[V^{-1}] + t r_1 r_1)^t[V] = (4T_0 - t r_0 r_0) + t (r_1^t V)r_1^t V \in 4\text{Sym}^*_{n-1}(\mathbb{Z}_p),
\]

and \( r_1^t V \equiv r_0 \mod 2\mathbb{Z}^{n-1} \) in that case. Thus

\[
(\phi|_{k,1} L_p(t))(\tau, z) = \sum_{T_0} \sum_{r_0} e(\text{tr}(T_0^t \tau + t r_0 z)) \sum_{V,W} p^{k \text{ord}_p(\det V) - (k-n-1) \text{ord}_p(\det W)}
\]

\[
\times e^{\text{ord}_p(\det V \det W)} b_p^{(1)}((4T_0 - t r_0 r_0)^t[V^{-1}]; t)
\]

\[
\times c_\phi((T_0 - t r_0 r_0/4)^t[V^{-1}][tW] + (t r_1 r_1/4)[tW], r_1^t W).
\]

Now we take an element \( B \in \text{Sym}_{n-1}(\mathbb{Z}(1)) \) so that \( B = 4T_0 - t r_0 r_0 \) with \( T_0 \in \text{Sym}^*_{n-1}(\mathbb{Z}) \) and \( r_0 \in \mathbb{Z}^{n-1} \). Then we have

\[
c_\phi(T_0, r_0) = C_{\sigma(\phi)}(B),
\]

\[
c_\phi((T_0 - t r_0 r_0/4)^t[V^{-1}][tW] + (t r_1 r_1/4)[tW], r_1^t W) = C_{\sigma(\phi)}(B[tV^{-1}][tW]),
\]

and

\[
b_p^{(1)}((4T_0 - t r_0 r_0)^t[V^{-1}]; t) = b_p^{(1)}(B[tV^{-1}]; t).
\]

Since \( \phi|_{k,1} L_p(t) = Z_p(t, \phi) \phi \), the assertion follows immediately from Proposition 3.3. ■
For each $B \in \text{Sym}_{n-1}(\mathbb{Z})_{>0}^{(1)}$, let $\tilde{G}_{\phi, p}(B; t)$ be the polynomial in $t$ defined in §1. Then, by making use of the same argument as in [5] combined with Propositions 3.2 and 3.4, we obtain the following:

**Theorem 3.1.** Let $n$ and $k$ be positive even integers such that $k > n + 1$. Suppose that $\phi \in J_{k,1}(\Gamma_{n-1}^{f})$ is a Hecke eigenform whose Satake $p$-parameter is of the form $(\chi_{\phi}^{(1)}(p), \ldots, \chi_{\phi}^{(n-1)}(p)) \in (\mathbb{C}^{*})^{n-1}/W_{n-1}$. Then, for each $B \in \text{Sym}_{n-1}(\mathbb{Z})_{>0}^{(1)}$, we have

$$\frac{B_{p}^{(1)}(B; p^{n-1/2}t)\tilde{G}_{\phi, p}(B; t)}{\prod_{i=1}^{n-1}(1 - \chi_{\phi}^{(i)}(p)p^{n-1/2}t)(1 - \chi_{\phi}^{(i)}(p)^{-1}p^{n-1/2}t)} = \sum_{W \in \text{GL}_{n-1}(\mathbb{Z}) \setminus D_{p}^{(n-1)}(\mathbb{Z})} C_{\sigma(\phi)}(B[W])(\text{det } W)^{-s-k+3/2} \text{ ord}_{p}(\text{det } W),$$

if $W_{\phi} \in \text{GL}_{n-1}(\mathbb{Z}) \cap \text{GL}_{n-1}(\mathbb{Q})$. For each $D \in \text{M}_{n-1}(\mathbb{Z}) \cap \text{GL}_{n-1}(\mathbb{Q})$, we define the generalized global Möbius function $\pi(D)$ as $\prod_{p} \pi(D)$, where $\pi(D)$ is the local Möbius function defined in §1. We easily see that this is a finite product. For each $B \in \text{Sym}_{n-1}(\mathbb{Z})_{>0}^{(1)}$, we put

$$\tilde{H}_{\phi}(B; s) = \sum_{D \in \text{GL}_{n-1}(\mathbb{Z}) \setminus \text{M}_{n-1}(\mathbb{Z}) \cap \text{GL}_{n-1}(\mathbb{Q})} \pi(D)C_{\sigma(\phi)}(B[D^{-1}]) \text{ det } D^{-s+k} \quad (s \in \mathbb{C}),$$

which is a finite sum, and $\tilde{H}_{\phi}(B; s) = \prod_{p} \tilde{G}_{\phi, p}(B; p^{-s})$. In addition, we also put $B_{p}^{(1)}(B; s) = \prod_{p} B_{p}^{(1)}(B; p^{-s})$. Then Theorem 3.1 can be restated globally as follows:

**Theorem 3.2.** Under the same situation as above, we have

$$B_{p}^{(1)}(B; s)L(s, \phi, St)\tilde{H}_{\phi}(B; s + n - 1/2) = \sum_{W \in \text{GL}_{n-1}(\mathbb{Z}) \setminus \text{M}_{n-1}(\mathbb{Z}) \cap \text{GL}_{n-1}(\mathbb{Q})} C_{\sigma(\phi)}(B[W])(\text{det } W)^{-s-k+3/2},$$

Moreover, by applying Theorem 3.1 to the Jacobi Eisenstein series $e_{k,1}^{(n-1)} = e_{k,1}^{(n-1)} \in J_{k,1}(\Gamma_{n-1}^{f})$, we obtain the following conclusion:

**Theorem 3.3.** Let $n$ and $k$ be as above. For each $B \in \text{Sym}_{n-1}(\mathbb{Z})_{>0}^{(1)}$,

$$\frac{B_{p}^{(1)}(B; p^{n-1/2}t)\tilde{G}_{p}^{(1)}(B; p^{k-(n+1)/2}, p^{(n+1)/2}t)}{\prod_{i=1}^{n-1}(1 - p^{j-1}p^{k-(n+1)/2}p^{(n+1)/2}t)(1 - p^{j-1}p^{k-(n+1)/2}p^{(n+1)/2}t)} = \sum_{W \in \text{GL}_{n-1}(\mathbb{Z}) \setminus D_{p}^{(n-1)}(\mathbb{Z})} \tilde{F}_{p}^{(1)}(B[W]; p^{k-(n+1)/2})(p^{(n+1)/2}t)^{\text{ord}_{p}(\text{det } W)},$$

where $\tilde{F}_{p}^{(1)}(B; X)$ and $\tilde{G}_{p}^{(1)}(B; X, t)$ are polynomials defined in §1.
Proof. Suppose that $B \in \text{Sym}_{n-1}(\mathbb{Z})_{>0}$. The $B$th Fourier coefficient of $\sigma(e^{(n-1)}) \in M_{k-\frac{1}{2}}^+(\Gamma_0^{(n-1)}(4))$ is expressed as

$$\xi(n,k)L(1-k/2+n/2,\chi_{B(1)})\mathcal{f}(B(1))^{k-(n+1)/2} \prod_{p|B(1)} \tilde{F}_p^{(1)}(B;p^{k-(n+1)/2})$$

(cf. Proposition 2.4). Thus the assertion follows from Theorem 3.1 and the Corollary to Proposition 2.2. Moreover, we easily see that it can be extended to any $B \in \text{Sym}_{n-1}(\mathbb{Z}_p)_{(1)}$.

For each $B \in \text{Sym}_{n-1}(\mathbb{Z}_p)_{(1)}$, let $R_p^{(1)}(B;X,t)$ be the formal power series in $X + X^{-1}$ and $t$ defined in §1. Eventually, we obtain the rationality for $R_p^{(1)}(B;X,t)$ as follows:

**Theorem 3.4.** Let $n$ be a positive even integer. For $B \in \text{Sym}_{n-1}(\mathbb{Z}_p)_{(1)}$, we have

$$R_p^{(1)}(B;X,t) = \frac{\mathcal{B}_p^{(1)}(B;p^{n/2-1}t)\tilde{G}_p^{(1)}(B;X,t)}{\prod_{j=1}^{n-1}(1-p^jXt)(1-p^jX^{-1}t)}.$$

**Proof.** We write both sides of the above equation as power series in $t$ as

$$R_p^{(1)}(B;X,t) = \sum_{i=1}^{\infty} A_i(X)t^i,$$

and

$$\frac{\mathcal{B}_p^{(1)}(B;p^{n/2-1}t)\tilde{G}_p^{(1)}(B;X,t)}{\prod_{j=1}^{n-1}(1-p^jXt)(1-p^jX^{-1}t)} = \sum_{i=1}^{\infty} B_i(X)t^i,$$

where for each $i$, $A_i(X)$ and $B_i(X)$ are polynomials in $X + X^{-1}$. Then, by Theorem 3.3,

$$A_i(p^{k-(n+1)/2}) = B_i(p^{k-(n+1)/2})$$

for infinitely many $k$. Thus $A_i(X) = B_i(X)$ for each $i$, completing the proof.

**Remark.** For a given pair of positive even integers $n$ and $k$ as in Theorem 3.1, let $f \in S_{2k-n}(\Gamma_1)$ be a Hecke eigenform, which possesses a Fourier expansion

$$f(z) = \sum_{N=1}^{\infty} a_f(N)e(Nz) \quad (z \in \mathcal{H})$$

normalized by $a_f(1) = 1$. For each rational prime $p$, we denote by $\alpha_p$ the Satake $p$-parameter of $f$, that is, an algebraic number determined by the condition $\alpha_p + \alpha_p^{-1} = a_f(p)p^{-k+(n+1)/2}$ uniquely up to inversion. By substituting $X = \alpha_p$ in the main identity of Theorem 3.4, we can also derive a
result similar to Theorem 3.3 for a power series related to the first Fourier–Jacobi coefficient of a Siegel cusp form $F \in S_k(\Gamma_0^1)$ which is connected to $f$ under Ikeda’s lifting procedure (cf. [12]). We note that it will play an important role in a proof of Ikeda’s conjecture on the period of such an $F$, which was proposed in [13] (cf. [16] [17]).

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References


Andrianov-type identity for Jacobi forms


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