On certain Diophantine systems with infinitely many parametric solutions and applications

by

MACIEJ ULAS (Kraków and Warszawa)

1. Introduction. Let \( f(x, y) = y^2 - x^n \), where \( n \) is an odd integer. In [4], we proved that for any quadruple \( a, b, c, d \) of distinct integers the set of rational parametric solutions of the system

\[
\frac{f(x_1, y_1)}{a} = \frac{f(x_2, y_2)}{b} = \frac{f(x_3, y_3)}{c} = \frac{f(x_4, y_4)}{d}
\]

is infinite. In the cited paper, this result was used to show that if \( C_i : y^2 = x^n + a_i \), where \( a_i \in \mathbb{Z} \setminus \{0\} \) are pairwise distinct, then there exists a polynomial \( D \in \mathbb{Z}[t] \) such that the \( \mathbb{Q}(t) \)-rank of the Jacobian variety \( \text{Jac}(C_i, D) \) is positive, where \( C_i, D : y^2 = x^n + a_i D(t) \) for \( i = 1, 2, 3, 4 \). Similar results were proved in [8, 9] and [3], where instead of \( f(x, y) \), we considered \( g(x, y) = (y^2 - x^3)/x \) and \( g(x, y) = (y^2 - x^5)/x \) respectively. In the light of these results, it is natural to ask what can be said about a general system of the form

\[
\frac{h(x_1, y_1)}{a_1} = \frac{h(x_2, y_2)}{a_2} = \cdots = \frac{h(x_k, y_k)}{a_k},
\]

where \( h \in \mathbb{Z}[x, y] \) and \( k \) is a fixed positive integer. In general, this is a difficult question. The most interesting but difficult case is that of a homogeneous form \( h \). It seems that the only pertinent results available concern the case where all \( a_i \) are equal and \( \deg h = 2, 3 \). In the case of \( \deg h = 2 \), the problem is related to the construction of a rational number \( A \) such that the curve \( C : h(x, y) = A \) is rational over \( \mathbb{Q} \). In the case of \( \deg h = 3 \), the problem is related to the construction of a rational number \( A \) such that the curve \( C : h(x, y) = A \) (i.e. \( h \) is an elliptic curve with nonzero discriminant) has infinitely many solutions in rationals. In this connection, we mention the work of Choudhry and Wróblewski [1], who showed that if \( h(x, y) = x^4 - y^4 \)

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then the system of equations
\[ h(x_1, y_1) = h(x_2, y_2) = h(x_3, y_3) \]
has infinitely many nontrivial solutions in integers (a nontrivial solution is a solution that cannot be obtained from another by multiplication by a nonzero integer).

In this paper, we investigate the question of solvability of the system \(^{(1)}\) with the additional condition \(\min\{\deg_x h, \deg_y h\} \geq 3\) and \(h\) of the form \(y^n - x^m\).

2. First result. In this section, we are interested in constructing rational parametric solutions of the system of Diophantine equations
\[
\begin{align*}
\frac{y_1^3 - x_1^m}{a} &= \frac{y_2^3 - x_2^m}{b} = \frac{y_3^3 - x_3^m}{c},
\end{align*}
\]
where \(a, b, c\) are pairwise distinct integers and \(m \in \mathbb{N}_+\) such that \((3, m) = 1\).

Before we state the main result of this section, let us recall what the torsion part of the curve \(E : y^2 = x^3 + q\) looks like for a fixed \(q \in \mathbb{Z}\) \(^{[7]}\). If \(q = 1\), then \(\text{Tors } E(\mathbb{Q}) \cong \mathbb{Z}/6\mathbb{Z}\). If \(q \neq 1\) and \(q\) is a square in \(\mathbb{Z}\), then \(\text{Tors } E(\mathbb{Q}) = \{O, (0, \sqrt{q}), (0, -\sqrt{q})\}\). If \(q = -432\), then \(\text{Tors } E(\mathbb{Q}) = \{O, (12, 36), (12, -36)\}\). If \(q \neq 1\) and \(q\) is a cube in \(\mathbb{Z}\), then \(\text{Tors } E(\mathbb{Q}) = \{O, (-\sqrt[3]{q}, 0)\}\). In the remaining cases, \(\text{Tors } E(\mathbb{Q}) = \{O\}\). Therefore, if \(E : y^2 = x^3 + Q\), where \(Q \in \mathbb{Z}[u, v, w] \setminus \mathbb{Z}\), is an elliptic curve defined over the field \(\mathbb{Q}(u, v, w)\) and if on \(E\) there exists a \(\mathbb{Q}(u, v, w)\)-rational point \(P = (x, y)\) with \(xy \neq 0\), then the order of \(P\) in the group \(E(\mathbb{Q}(u, v, w))\) is not finite provided that \(E\) is not isomorphic to an elliptic curve defined over \(\mathbb{Q}\). Thus, in that case the curve \(E\) over \(\mathbb{Q}(u, v, w)\) has a positive rank.

Now we are ready to prove the following theorem.

**Theorem 2.1.** Let \(a, b, c \in \mathbb{Z} \setminus \{0\}\) and suppose that \((3, m) = 1\). Then the system \(^{(2)}\) has infinitely many rational three-parameter solutions.

**Proof.** We consider the variety \(U\) defined by \(^{(2)}\). Since we are looking for parametric solutions, we are interested in nontrivial points on \(U\), i.e. points which satisfy \(y_i^3 \neq x_i^m\) and \(x_iy_i \neq 0\), for \(i = 1, 2, 3\). Let \(\alpha, \beta \in \mathbb{Z}\) be such that \(m\beta - 3\alpha = 1\). Note that this is the only place where we need the condition \((3, m) = 1\). Put
\[
\begin{align*}
x_1 &= u^3T^\beta, & x_2 &= v^3T^\beta, & x_3 &= w^3T^\beta, \\
y_1 &= pT^\alpha, & y_2 &= qT^\alpha, & y_3 &= rT^\alpha,
\end{align*}
\]
where \(u, v, w\) are rational parameters and \(p, q, r, T\) are variables. Now, if \(T = (bp^3 - aq^3)/(bu^3m - av^3m)\), then the first equation defining the variety \(U\) is satisfied. On the other hand, if \(T = (cq^3 - br^3)/(cv^3m - bw^3m)\), then the second equation defining \(U\) is satisfied. From the above, after some necessary
simplifications, we can see that in order to find $\mathbb{Q}(u, v, w)$-rational points on our variety we must show that the Diophantine equation

\begin{equation}
Ap^3 + Bq^3 + Cr^3 = 0,
\end{equation}

where

\begin{equation}
A = bw^{3m} - cv^{3m}, \quad B = cu^{3m} - aw^{3m}, \quad C = av^{3m} - bu^{3m},
\end{equation}

has infinitely many nontrivial $\mathbb{Q}(u, v, w)$-rational solutions. From a geometric viewpoint, equation (4) defines a cubic curve $C$ over the field $\mathbb{Q}(u, v, w)$. This is an elliptic curve with the $\mathbb{Q}(u, v, w)$-rational point $P = [u^m : v^m : w^m]$. Doubling the point $P$ on the curve $C$, we find that $2P = [p', q', r']$, where

\begin{align*}
p' &= -u^m(cu^{3m}v^{3m} + bu^{3m}w^{3m} - 2av^{3m}w^{3m}), \\
q' &= -v^m(cu^{3m}v^{3m} - 2bu^{3m}w^{3m} + av^{3m}w^{3m}), \\
r' &= w^m(2cu^{3m}v^{3m} - bu^{3m}w^{3m} - av^{3m}w^{3m}).
\end{align*}

In this case, the value of $T = T(u, v, w)$ is given by $T = -G(u^m, v^m, w^m)$, where

\begin{align*}
G(u, v, w) &= c^3u^9v^9 + 3bc^2u^9v^6w^3 + 3ac^2u^6v^9w^3 + 3b^2cu^9v^3w^6 \\
&\quad - 21abcu^6v^6w^6 + 3a^2cu^3v^9w^6 + b^3u^9w^9 + 3ab^2u^6v^3w^9 \\
&\quad + 3a^2bu^3v^6w^9 + a^3v^9w^9.
\end{align*}

Using these quantities, we find that a solution of the system (2) has the form

\begin{align*}
x_1 &= u^3T(u, v, w)\alpha, \\
x_2 &= v^3T(u, v, w)\alpha, \\
x_3 &= w^3T(u, v, w)\alpha.
\end{align*}

By a standard argument, we find that $C$ is birationally equivalent to the elliptic curve with Weierstrass equation

\begin{equation}
\end{equation}

The mapping $\psi : E \to C$ is given by

\begin{align*}
\psi(X, Y, Z) &= (-4ABCXYZ, -4ABC(BY^3 - CZ^3), AX^3),
\end{align*}

where $A, B, C$ are given by (5).

Now we will see that the curve $E$ has positive rank over the field $\mathbb{Q}(u, v, w)$. Note that on $E$, we have the point $S = (X, Y, 1)$, where

\begin{align*}
X &= -\frac{4BCq'r'}{p'^2}, \\
Y &= -\frac{4BC(Bq'^3 - Cr'^3)}{p'^3}.
\end{align*}

Here $p', q', r'$ are defined by (6). Note also that $XY \neq 0$ in $\mathbb{Q}(u, v, w)$ and $16A^2B^2C^2$ is not of the form $a' F(u, v, w)^6$ for $a' \in \mathbb{Z}, F \in \mathbb{Z}[u, v, w]$. By the remark at the beginning of this section, the point $S$ is of infinite order in
the group $E(\mathbb{Q}(u, v, w))$. This shows that the set of rational three-parameter solution of the system \[2\] is infinite. ■

**Remark 2.2** (see also \[2\]). If $C$ is a curve of genus $g \geq 2$ and if there exists a morphism from $C$ to an elliptic curve $E$, then $\text{Jac}(C)$, the Jacobian variety of $C$, is isogenous to $A \times E$, where $A$ is an abelian variety of dimension $g - 1$. In particular, if the rank of $E$ is positive then so is the rank of $\text{Jac}(C)$.

As an application of our result, we prove the following theorem.

**Theorem 2.3.** Let $a_i \in \mathbb{Z} \setminus \{0\}$, $i = 1, 2, 3$. Suppose that $a_i \neq a_j$ for $i \neq j$, and let $m \in \mathbb{N}^+$ be such that $(3, m) = 1$. Consider the superelliptic curves defined by

$$C_i : y^3 = x^{2m} + a_i, \quad i = 1, 2, 3.$$  

Then there exists a polynomial $D \in \mathbb{Z}[u, v, w]$ such that the Jacobian variety associated with the curve $C_{i,D}: y^3 = x^{2m} + a_i D(u, v, w)$ has a positive rank over $\mathbb{Q}(u, v, w)$ for $i = 1, 2, 3$.

**Proof.** First, note that from Theorem 2.1 we can deduce the existence of a polynomial $D \in \mathbb{Z}[u, v, w]$ such that the set of $\mathbb{Q}(u, v, w)$-rational points on the curve $C_{i,D}$ is nonempty for $i = 1, 2, 3$. (In fact, infinitely many such polynomials exist.) Second, on each curve $C_{i,D}$, we have a point $P_i = (x_i, y_i)$ satisfying $x_i y_i \neq 0$ for $i = 1, 2, 3$. Moreover, one has the morphism:

$$\varphi_i : C_{i,D} \ni (x, y) \mapsto (y, x^m) \in E_i : Y^2 = X^3 - a_i D(u, v, w).$$

On the curve $E_i$, we have a $\mathbb{Q}(u, v, w)$-rational point $Q_i = (y_i, x_i^m)$ for $i = 1, 2, 3$. As $-a_i D(u, v, w)$ is not of the form $a' F(u, v, w)^6$ for $a' \in \mathbb{Z}$ and $F \in \mathbb{Z}(u, v, w)$, we deduce that $E_i$ is not birationally equivalent to an elliptic curve defined over $\mathbb{Q}$. As the coordinates of the point $Q_i$ are nonzero, $Q_i$ is of infinite order on the curve $E_i$ for $i = 1, 2, 3$. Using now Remark 2.2, we deduce that the point $P_i = (x_i, y_i)$ which lies on the curve $C_{i,D}$ for $i = 1, 2, 3$ corresponds to the divisor of infinite order in the group $\text{Jac}(C_{i,D})(\mathbb{Q}(u, v, w))$. Thus the rank of $\text{Jac}(C_{i,D})(\mathbb{Q}(u, v, w))$ is positive for $i = 1, 2, 3$. ■

3. A generalization of Theorem 2.1. In this section we will prove the following theorem.

**Theorem 3.1.** Let $a, b, c \in \mathbb{Z} \setminus \{0\}$. Let $m \in \mathbb{N}^+$ be such that $(3, m) = 1$ and let $0 < k < m$. Then the system

$$y_1^3 - x_1^m = \frac{y_2^3 - x_2^m}{ax_1^k} = \frac{y_3^3 - x_3^m}{bx_2^k} = \frac{y_3^3 - x_3^m}{cx_3^k}$$  

has infinitely many rational parametric solutions depending on three parameters.
Proof. The method of proof is similar to that for Theorem 2.1. We consider the variety \( V \) defined by (7). We are interested in nontrivial points on \( V \), i.e., points which satisfy \( y_i^n \neq x_i^m \) and \( x_iy_i \neq 0 \) for \( i = 1, 2, 3 \). Take \( \alpha, \beta \in \mathbb{Z} \) such that \( m\beta - 3\alpha = 1 \). Put
\[
\begin{align*}
x_1 &= u^3T^\beta, \\
x_2 &= v^3T^\beta, \\
x_3 &= w^3T^\beta, \\
y_1 &= pu^3T^\alpha, \\
y_2 &= qv^3T^\alpha, \\
y_3 &= rw^3T^\alpha,
\end{align*}
\]
where \( u, v, w \) are rational parameters and \( p, q, r, T \) are variables.

Now, note that if \( T = (bp^3 - aq^3)/(bu^3(m-k) - av^3(m-k)) \) then the first equation defining the variety \( V \) is satisfied. On the other hand, if \( T = (cq^3 - br^3)/(cv^3(m-k) - bw^3(m-k)) \) then the second equation defining \( V \) is satisfied. Consequently, to finish the proof, it is enough to show that the set of \( \mathbb{Q}(u, v, w) \)-rational points on the curve \( C' : A'p^3 + B'q^3 + C'r^3 = 0 \), where
\[
\begin{align*}
A' &= bw^3(m-k) - cv^3(m-k), \\
B' &= cu^3(m-k) - aw^3(m-k), \\
C' &= av^3(m-k) - bu^3(m-k),
\end{align*}
\]
is infinite. But this is obvious, as \( C' \) is obtained from the curve \( C \) in the proof of Theorem 2.1 where instead of \( m \) we take \( m-k \). As we have proved that \( C \), for any given nonzero \( m \) (not necessarily satisfying the condition \( (3, m) = 1 \)) has infinitely many \( \mathbb{Q}(u, v, w) \)-rational points, the same is true for \( C' \). ■

4. Parametric solutions of the Diophantine equation \( (y_1^n - x_1^{2n})/a = (y_2^n - x_2^{2n})/b \), \( n \) odd. Before stating our result, let us recall what the torsion part of the curve \( E : y^2 = x^3 + px \), with a fixed \( p \in \mathbb{Z} \setminus \{0\} \), looks like (see [7, p. 311]). If \( p = 4 \), then \( \text{Tors}(\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z} \). If \(-p\) is a square in \( \mathbb{Z} \), then \( \text{Tors}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \), and finally if \( p \) does not satisfy any of these conditions, then \( \text{Tors}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \). As an immediate consequence, we deduce that if \( P \in \mathbb{Z}[t] \setminus \mathbb{Z} \) and \( \mathcal{E} : y^2 = x^3 + Px \) is an elliptic curve defined over \( \mathbb{Q}(t) \) with a \( \mathbb{Q}(t) \)-rational point \( P = (x, y) \) satisfying \( xy \neq 0 \), then the order of \( P \) in the group \( \mathcal{E}(\mathbb{Q}(t)) \) is not finite provided that \( \mathcal{E} \) is not isomorphic to an elliptic curve defined over \( \mathbb{Q} \). Thus, in that case the curve \( \mathcal{E} \) over \( \mathbb{Q}(t) \) has a positive rank.

Now we are ready to prove the following theorem.

**Theorem 4.1.** Let \( a, b \in \mathbb{Z} \setminus \{0\} \) and \( n \) be an odd integer. Then the Diophantine equation
\[
\frac{y_1^n - x_1^{2n}}{a} = \frac{y_2^n - x_2^{2n}}{b}
\]
has infinitely many rational parametric solutions.

**Proof.** We consider the hypersurface given by (9). Note that instead of (9) we can consider the system
\[
b(y_1^n - x_1^n) = aU(y_2^n - x_2^n), \quad U(y_1^n + x_1^n) = y_2^n + x_2^n,
\]
Thus we get the equation of a hyperelliptic quartic curve (defined over $F$) of the form

$$x_1 = T, \quad y_1 = T^{(n-1)/2}, \quad x_2 = -t^2T, \quad y_2 = qT^{(n-1)/2},$$

where $t$ is a rational parameter and $q, T$ are variables. From the first equation in (10), we get

$$T = \frac{b - aq^2U}{b + at^{2n}U},$$

and then from the second equation in (10),

$$(b + 2at^{2n}U + aU^2)q^2 = bt^{2n} + 2bU + at^{2n}U^2.$$ 

Thus we get the equation of a hyperelliptic quartic curve (defined over $\mathbb{Q}(t)$) of the form

$$C_{a,b} : V^2 = (bt^{2n} + 2bU + at^{2n}U^2)(b + 2at^{2n}U + aU^2),$$

where $V = q(b + 2at^{2n}U + aU^2)$. Note that $C_{a,b}$ has the $\mathbb{Q}(t)$-rational point $Q_{a,b} = (0, bt^n)$. Treating $Q_{a,b}$ as a point at infinity on the curve $C_{a,b}$ and using the method described in [6, p. 77], we see that $C_{a,b}$ is birationally equivalent to the elliptic curve with the Weierstrass equation

$$E_{a,b} : Y^2 = X^3 + 4ab(at^{4n} - b)^2X.$$ 

A mapping from $E_{a,b}$ to $C_{a,b}$ is given by $\varphi(X, Y) = (U, V)$, where

$$U = \frac{2b(b^2 - 2abt^{4n} + a^2t^{8n} - t^{2n}X)}{(at^{4n} + b)X - t^nY},$$

$$V = \frac{t^{3n}X^3 - (at^{4n} - b)^2(f_1X^2 + f_2X + f_3 + f_4Y)}{((at^{4n} + b)X - t^nY)^2},$$

and

$$f_1 = 3t^n, \quad f_2 = 4abt^{3n}, \quad f_3 = 4abt^n(at^{4n} - b)^2, \quad f_4 = -2(at^{4n} + b).$$

In order to show that the rank of $E_{a,b}$ is positive, we notice that the point $P_{a,b} = (X^2, XY)$, where

$$X = \frac{b^2 - 6abt^{4n} + a^2t^{8n}}{2t^n(b + at^{4n})}, \quad Y = X^2 + \frac{8abt^{2n}(at^{4n} - b)^2}{(at^{4n} + b)^2},$$

lies on our curve. As $4ab(at^{4n} - b)^2$ is not of the form $a't^4F(t)^4$ for $a' \in \mathbb{Z}$ and $F \in \mathbb{Z}[t]$, we deduce that $E_{a,b}$ is not birationally equivalent to an elliptic curve defined over $\mathbb{Q}$. Invoking now the remark from the beginning of this section we deduce that $P_{a,b}$ is of infinite order in the group $E_{a,b}(\mathbb{Q}(t))$. Now it is an easy task to obtain the statement of our theorem. First, for $m = 2, 3, \ldots$, we calculate $mP_{a,b}$ on the curve $E_{a,b}$. Next, we calculate the corresponding point $(U, V)$ on $C_{a,b}$ and from the equation $V = q(b + 2at^{2n}U + aU^2)$ we get
the value of $q$. Then we calculate the value of $T = T(t)$ and the values of $x_i, y_i$, $i = 1, 2$, given by (11) which give solutions of our equation.

For example, if $n = 3$, then the point $P_{a,b}$ leads (after necessary simplifications) to the solution of the equation $(y_4^4 - x_6^6)/a = (y_2^4 - x_2^6)/b$ of the form

$$
x_1 = t^3(-3a^2 + 6abt^{12} + b^2t^{24})y_2,
$$
$$
y_1 = t^2y_2,
$$
$$
x_2 = (-a^2 - 6abt^{12} + 3b^2t^{24})y_2,
$$
$$
y_2 = a^4 - 28a^3bt^{12} + 6a^2b^2t^{24} - 28ab^3t^{36} + b^4t^{48}. \quad \blacksquare
$$

Remark 4.2. With the use of Theorem 4.1, we can easily prove that the set of rational parametric solutions of the Diophantine equation

$$
x_1^4 + x_2^4 = y_1^{2n} + y_2^{2n},
$$

where $n$ is an odd integer, is infinite. Indeed, just take $b = -a$ in the solution obtained in Theorem 4.1. Thus our method shows that the set of integers which are simultaneously representable as a sum of two fourth powers and two $2m$th powers is infinite. This result is related to the Diophantine problem called equal sums of unlike powers investigated by Lander in [5]. The method presented by Lander cannot be used in order to construct integer solutions to the above Diophantine equation.

In particular, if $n = 3$, then we get the following solution of the equation $x_1^4 + x_2^4 = y_1^6 + y_2^6$:

$$
x_1 = (-1 + 6t^{12} + 3t^{24})y_2,
$$
$$
x_2 = t^3(-3 - 6t^{12} + t^{24})y_2,
$$
$$
y_1 = t^2y_2,
$$
$$
y_2 = 1 + 28t^{12} + 6t^{24} + 28t^{36} + t^{48},
$$

which has not been obtained before.

From Theorem 4.1 we deduce an interesting result similar to Theorem 2.3.

Theorem 4.3. Let $a_1, a_2 \in \mathbb{Z} \setminus \{0\}$ be such that $a_1 \neq a_2$ and let $m \in \mathbb{N}_+$ be an odd integer. Consider the superelliptic curves defined by the equations

$$
C_i : y^4 = x^{2m} + a_i, \quad i = 1, 2.
$$

Then there exists a polynomial $D \in \mathbb{Z}[t]$ such that the Jacobian variety associated with the curve $C_{i,D} : y^4 = x^{2m} + a_iD(t)$ has a positive rank over $\mathbb{Q}(t)$ for $i = 1, 2$.

Proof. From Theorem 4.1 we can deduce the existence of a polynomial (in fact, there are infinitely many such polynomials) $D \in \mathbb{Z}[t]$ such that
the set of $\mathbb{Q}(t)$-rational points on the curve $C_{i,D}: y^4 = x^{2m} + a_i D(t)$ is nonempty for $i = 1, 2$. Moreover, on each curve $C_{i,D}$, we get a point $P_i = (x_i, y_i)$ satisfying $x_i y_i \neq 0$ for $i = 1, 2, 3$. Note the existence of the following morphism:

$$\varphi_i: C_{i,D} \ni (x, y) \mapsto (y, x^m) \in E'_i: Y^2 = X^4 - a_i D(t).$$

The curve $E'_i$ over $\mathbb{Q}(t)$ is birationally equivalent to the elliptic curve whose Weierstrass equation is

$$E_i: Y^2 = X^3 - 4a_i D(t)X.$$

The mapping $\chi_i: E_i \to E'_i$ is given by

$$\chi_i(X, Y) = \left(\frac{Y}{2X}, \frac{X^2 - 4a_i D(t)}{4X}\right),$$

and its inverse is given by

$$\chi_i^{-1}(X, Y) = (2(X^2 + Y), 4X(X^2 + Y)).$$

First of all, note that the curve $E_i$ is not birationally equivalent to one defined over $\mathbb{Q}$. This follows from the fact that $-4a_i D(t)$ is not of the form $a'F(t)^4$ for $a' \in \mathbb{Z}$ and $F \in \mathbb{Z}[t]$. Next, on $E'_i$, we have a $\mathbb{Q}(t)$-rational point $Q_i = (y_i, x_i^m)$ for $i = 1, 2$. Thus the point $(X, Y) = \chi_i^{-1}(Q_i)$ lies on $E_i$ and one can easily check that $XY \neq 0$. From the remark at the beginning of this section we deduce that the point $\chi_i^{-1}(Q_i)$ is of infinite order on $E_i$, and the same holds for the point $Q_i$ on $E'_i$. Now, using Remark 2.2 we conclude that the rank of $\text{Jac}(C_{i,D})(\mathbb{Q}(t))$ is positive for $i = 1, 2$. ■

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**References**


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Maciej Ulas
Institute of Mathematics
Jagiellonian University
Łojasiewicza 6
30-348 Kraków, Poland
and
Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-956 Warszawa, Poland
E-mail: Maciej.Ulas@im.uj.edu.pl

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