ACTA ARITHMETICA
117.1 (2005)

Theoretical and computational bounds for m-cycles of the $3n+1$-problem

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1. Introduction

1.1. *Generalities.* The $3n+1$-problem is a well known problem in elementary number theory. Let n be a natural number, and consider the sequence generated by iteration of $T(n) = \frac{1}{2}(3n+1)$ if n is odd, $T(n) = \frac{1}{2}n$ if n is even. Numerical verification indicates that for all natural numbers finally the cycle $\{1, 2\}$ appears. A proof of this so-called $3n+1$-conjecture is lacking so far, in spite of various approaches to the problem. See Lagarias [La] and Wirsching [Wi] for extensive overviews on the $3n+1$-problem.

We call a cycle an *m-cycle* if the numbers in it appear in m sequences, each consisting of an increasing subsequence of odd numbers, followed by a decreasing subsequence of even numbers. Let such an m-cycle contain in total K odd numbers and L even numbers. The cycle $\{1, 2\}$ is a 1-cycle, and in 1977 Steiner [St] proved that no other 1-cycles exist. Any m-cycle containing natural numbers greater than 2 is called *nontrivial*, and in 2004 Simons [Si] proved the nonexistence of nontrivial 2-cycles.

In this paper we generalize the approach of Steiner and Simons, and for an arbitrary value of m we derive the following:

(1) An upper bound for $\Lambda = (K+L)\log 2 - K\log 3$ that is exponential in K, following from estimates for m-cycles.
(2) A lower bound for Λ that is subexponential in K, following from transcendence theory.
(3) Upper bounds for K and L, and for the minimal element $x_{\min}$ of the cycle, following from comparing the upper and lower bounds for Λ. These upper bounds appear to be exponential in m.

2000 *Mathematics Subject Classification*: Primary 11B83.

We want to express our gratitude to Ray Steiner for useful remarks on the manuscript, and to Rob Tijdeman for acting as *trait d'union*.

Further, for "small" and "medium" values of m we derive

(4) Lower bounds for K and L, following from brute force computations and diophantine approximation techniques (continued fractions, lattice basis reduction) applied to $\log 3/\log 2$.

For "small" values of m (up to $m \leq 68$) the lower bound of (4) is larger than the upper bound of (3). For "medium" values of m (up to $m \leq 515\,620$) the diophantine approximation techniques lead to an improvement of the upper bounds for K, L and $x_{\min}$, which remain exponential in m.

1.2. *m-Cycles of the $3n+1$-problem.* We study periodic sequences $\{T^k(n)\}$, i.e. we do not consider unbounded sequences. Without loss of generality we are not interested in nonperiodic parts, so if necessary we replace n by some $T^k(n)$ which is in the periodic part. Hence we may assume that the sequence is purely periodic, i.e. there exists an integer $p \geq 1$ such that in the periodic sequence $\{n, T(n), T^2(n), \ldots, T^p(n), \ldots\}$ it is the case that $T^p(n) = n$ (we may take p minimal with this property, but for our arguments that is not essential). We consider only the period, i.e. $\{n, T(n), \ldots, T^{p-1}(n)\}$. We further may assume that $T^0(n) = n$ is at a local minimum in the sequence (not necessarily the global minimum).

Let there be m local minima in the periodic sequence, with indices $t_0, t_1, \ldots, t_{m-1}$ such that $0 = t_0 < t_1 < \cdots < t_{m-1} < p$. Then there are also m local maxima, say with indices $s_0, s_1, \ldots, s_{m-1}$. As each maximum lies in between two minima, we may assume $0 = t_0 < s_0 < t_1 < s_1 < \cdots < t_{m-1} < s_{m-1} \leq p-1$. We call such a periodic sequence an *m-cycle.* Define x_i, y_i as the values of the local minima and maxima, viz.

$$x_i = T^{t_i}(n), \qquad y_i = T^{s_i}(n).$$

We put $k_i = s_i - t_i$ for $i = 0, \ldots, m-1$, and $l_i = t_{i+1} - s_i$ for $i = 0, \ldots, m-2$, and $l_{m-1} = p + t_0 - s_{m-1}$. Further we put

$$K = \sum_{i=0}^{m-1} k_i, \qquad L = \sum_{i=0}^{m-1} l_i.$$

The sequence thus starts with an odd number x_0, increases in k_0 steps until an even number y_0 is encountered, then decreases in l_0 steps until an odd number x_1 is encountered, again increases in k_1 steps until an even number y_1 is encountered, etc.

We call the m-fold repeated 1-cycle $\{1, 2, 1, 2, \ldots, 1, 2\}$ the *trivial m-cycle.*

1.3. *Crandall's Lemma.* For a nontrivial m-cycle, put $x_{\min} = \min\{x_0, x_1, \ldots, x_{m-1}\}$. By X_0 we denote a lower bound for $x_{\min}$ that is known to be true at a certain point in time. At the moment of writing this (November 10, 2004) it is known from extensive distributed computations [Ro] that

$$x_{\min} > X_0 = 301 \cdot 2^{50} > 3.3889 \cdot 10^{17}.$$

The computation is going on. The rate of improvement presently is about $2^{48} \approx 3 \cdot 10^{14}$ per day ([1]).

Let $\delta = \log 3/\log 2 = 1.5849\ldots$. Throughout this paper this number δ plays a central role. Let its continued fraction expansion be given by $\delta = [a_0, a_1, a_2, \ldots] = [1, 1, 1, 2, 2, 3, 1, 5, 2, 23, 2, 2, 1, 1, 55, \ldots]$, with convergents $p_n/q_n = [a_0, a_1, a_2, \ldots, a_n]$ for $n = 0, 1, \ldots$.

Crandall [Cr] showed the following result.

Lemma 1 (Crandall, 1978). *If p_n/q_n is any convergent to δ with $n \geq 4$, then for a nontrivial m-cycle*,

$$K > \min\left\{q_n, \frac{2x_{\min}}{q_n + q_{n+1}}\right\}.$$

As a consequence, we have the following result.

Corollary 2.

$$K > 2.2564 \cdot 10^8.$$

Proof. This follows immediately from Lemma 1, the present value for X_0, and the fact that $q_{18} = 225\,644\,606$, $q_{19} = 397\,573\,379$. ∎

In this paper we generalize Crandall's Lemma for m-cycles, which in general results in sharper lower bounds for K.

1.4. *Known results on the nonexistence of cycles.* Steiner [St] proved in 1977 the nonexistence of nontrivial 1-cycles. He assumes the existence of a 1-cycle with k odd numbers and l even numbers, and proves the following partial results:

(1) An inequality for the ratio $(k + l)/k$.
(2) A numerical lower bound for k, from which it follows that $(k + l)/k$ must be a convergent from the continued fraction expansion of δ.
(3) An upper bound for k from a theorem of Baker [Ba] on linear forms in two logarithms.
(4) A (very effective) lower bound for the partial quotients in the continued fraction expansion of δ.

Numerical calculation of partial quotients then shows that the only 1-cycle that satisfies these conditions is the trivial one.

Crucial in Steiner's proof for the nonexistence of 1-cycles is (implicitly) the inequality $0 < (k + l)\log 2 - k \log 3 < 1/x_{\min}$. The right hand side is exponentially small in k, since the existence of k successive odd numbers

([1]) Personal communication by Eric Roosendaal.

starting with x_0 implies that $x_0 = a2^k - 1 \geq 2^k - 1$. Hence $(k+l)/k$ must be a convergent of δ for $k \geq 5$. This inequality has a "natural" generalization for 2-cycles in the form of $0 < (K+L)\log 2 - K\log 3 < 1/x_0 + 1/x_1$, however the convergent argument fails because k_0 or k_1 can be small even if K is large. As has been remarked by Lagarias [La], the result of Steiner's proof seems rather weak, considering the strength of the underlying number theory.

Simons [Si] proved in 2004 the nonexistence of nontrivial 2-cycles. By exploiting the average values of k_0 and k_1 he derived for the expression $1/x_0 + 1/x_1$ an effective upper bound of the form $ce^{-d(K+L)}$, where c and d are positive constants. He generalizes Steiner's approach and derives:

(1) A generalized inequality for the ratio $(K+L)/K$.
(2) A numerical lower bound for K, from which it follows that $(K+L)/K$ must be a convergent from the continued fraction expansion of δ.
(3) An upper bound for K by applying a theorem of Laurent, Mignotte and Nesterenko [LMN] on linear forms in two logarithms.
(4) A lower bound for the partial quotients in the continued fraction expansion of δ.

Steiner's original numerical verification finally shows that the only 2-cycle that satisfies these conditions is the trivial 2-cycle $\{1, 2, 1, 2\}$.

This approach however fails to prove the nonexistence of m-cycles for $m > 2$, because then the coefficient d in Simons's upper bound becomes negative, which makes the upper bound ineffective.

Many partial results on the (non)existence of cycles for the $3n+1$-problem, as well as for generalizations, have been conjectured and proved by applying a scala of theoretical methods (see Lagarias [La] and Wirsching [Wi]). In particular, using transcendence methods like we do, Brox [Br] showed that there are only finitely many m-cycles with $m < 2\log K$, and from that he derived the result that for each m there are only finitely many m-cycles (see Theorem 3(a) below).

1.5. *Lower and upper bounds on cycle elements and lengths.* In this paper we extend the cited results of Steiner [St] and Simons [Si], and prove that for each m there are only finitely many m-cycles. Indeed, extending the result of Brox [Br], for m-cycles we derive explicit upper bounds, depending only on m, for the values of K, L and $x_{\min}$.

By doing extensive computations we also derive new lower bounds for these values. Then, combining upper and lower bounds, we prove that there exist no nontrivial m-cycles for $m \leq 68$. For $69 \leq m \leq 72$ we give possible solutions, which will be excluded when exterior computations à la [Ro] lead to new values for X_0. For $m \geq 73$ we derive explicit lower and upper bounds for the cycle length and for the numbers in the cycle.

Our main result is

THEOREM 3 (Main Theorem). *For an m-cycle for the $3n+1$-problem, let $K, L, x_{\min}$ be defined as above.*

(a) (Brox) *For any m there are only finitely many m-cycles.*
(b) *For $1 \le m \le 68$ there do not exist nontrivial m-cycles.*
(c) *For $69 \le m \le 72$ the only possible nontrivial m-cycles satisfy $x_{\min} > 3.3889 \cdot 10^{17}$ and*

m	K	L	$x_{\min}$
69	5 750 934 602 875 680	3 364 081 086 781 987	$< 6.4877 \cdot 10^{17}$
70	5 750 934 602 875 680	3 364 081 086 781 987	$< 6.5817 \cdot 10^{17}$
71	5 750 934 602 875 680	3 364 081 086 781 987	$< 6.6758 \cdot 10^{17}$
	11 985 484 530 117 643	7 011 059 003 092 348	$< 7.0209 \cdot 10^{17}$
72	5 750 934 602 875 680	3 364 081 086 781 987	$< 6.7698 \cdot 10^{17}$
	11 985 484 530 117 643	7 011 059 003 092 348	$< 7.1198 \cdot 10^{17}$
	17 736 419 132 993 323	10 375 140 089 874 335	$< 3.4702 \cdot 10^{17}$
	18 220 034 457 359 606	10 658 036 919 402 709	$< 7.5079 \cdot 10^{17}$
	24 454 584 384 601 569	14 305 014 835 713 070	$< 7.9408 \cdot 10^{17}$

(d) *For $m \ge 73$ the possible nontrivial m-cycles satisfy $x_{\min} > 3.3889 \cdot 10^{17}$ and*

- *if* $73 \le m \le 90$ *then*

$$5.2673 \cdot 10^{15} < K < 1.3993 m\delta^m < e^{0.46057m+\log m+0.33593},$$
$$3.0811 \cdot 10^{15} < L < 0.81850 m\delta^m < e^{0.46057m+\log m-0.20028},$$
$$x_{\min} < 339.14 m^2\delta^m < e^{0.46057m+2\log m+5.8265},$$

- *if* $91 \le m \le 515\,619$ *then*

$$6.5470 \cdot 10^{10} < K < 1.4784 m\delta^m < e^{0.46057m+\log m+0.39095},$$
$$3.8297 \cdot 10^{10} < L < 0.86480 m\delta^m < e^{0.46057m+\log m-0.14525},$$
$$x_{\min} < 5.1825 \cdot 10^7 m^2\delta^m < e^{0.46057m+2\log m+17.764},$$

- *if* $515\,620 \le m \le 131\,993\,764$ *then*

$$2.2564 \cdot 10^8 < K < 15.109 m\delta^m < e^{0.46057m+\log m+2.7153},$$
$$1.3199 \cdot 10^8 < L < 8.8379 m\delta^m < e^{0.46057m+\log m+2.1791},$$
$$x_{\min} < e^{6.1260m},$$

- *if* $m \ge 131\,993\,765$ *then*

$$1.7095\, m < K < 15.108 m\delta^m < e^{0.46057m+\log m+2.7152},$$
$$m \le L < 8.8372 m\delta^m < e^{0.46057m+\log m+2.1790},$$
$$x_{\min} < e^{6.1255m}.$$

In the cases $91 \leq m \leq 131\,993\,764$ the lower bounds can be a bit more fine-tuned. The upper bounds in terms of powers of e are provided to ease comparison.

We expect that, with current technology and efforts, $m = 69$ to $m = 72$ will be completely solved within 5 years, that $m = 73$ up to $m = 75$ can be solved within 25 years, and that $m = 76$ and beyond may take a considerably larger effort to finish.

2. Conditions for the existence of an m-cycle

2.1. *The chain equation.* We first describe an m-cycle in more detail.

The existence of k_i successive odd numbers starting with x_i implies that $x_i \equiv -1 \pmod{2^{k_i}}$, i.e. $x_i = 2^{k_i} a_i - 1$ for some integer $a_i \geq 1$. Going up from a local minimum x_i to the next local maximum y_i is now expressed in the formula $y_i = 3^{k_i} a_i - 1$. Then going down to the next local minimum x_{i+1} is done by $y_i = 2^{l_i} x_{i+1}$ (with $x_m = x_0$). Putting this together we arrive at the *chain equation*

$$3^{k_i} a_i - 1 = 2^{k_{i+1}+l_i} a_{i+1} - 2^{l_i}$$

for $i = 0, 1, \ldots, m-1$. If we put $x_m = x_0$, i.e. $a_m = a_0$ and $k_m = k_0$, we impose the existence of an m-cycle.

Note that the chain equation with $m = 1$ is at the heart of Steiner's result [St]. From now on we take $m \geq 2$.

2.2. *Integral and rational solutions.* When we fix k_i, l_i in the chain equation, we have m linear equations in the m variables a_i. We get the *matrix equation*

$$M \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{m-1} \end{pmatrix} = \begin{pmatrix} 2^{l_0} - 1 \\ 2^{l_1} - 1 \\ \vdots \\ 2^{l_{m-1}} - 1 \end{pmatrix},$$

with the matrix M defined as

$$M = \begin{pmatrix} -3^{k_0} & 2^{k_1+l_0} & 0 & \cdots & 0 \\ 0 & -3^{k_1} & 2^{k_2+l_1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & -3^{k_{m-2}} & 2^{k_{m-1}+l_{m-2}} \\ 2^{k_0+l_{m-1}} & 0 & \cdots & 0 & -3^{k_{m-1}} \end{pmatrix}.$$

Put

$$\Delta = 2^{K+L} - 3^K.$$

Then $\det(M) = (-1)^{m-1}\Delta$. Applying linear algebra, in fact using an argument of Böhm and Sontacchi [BS], we proceed to find the inverse matrix M^{-1}. Put

$$\Delta M^{-1} = (m_{i,j})_{i,j=0,1,\ldots,m-1}, \qquad \alpha_{i,j} = \sum_{h=i+1}^{j^*} k_h, \qquad \beta_{i,j} = \sum_{h=i+1}^{j^*} l_{h-1},$$

where we take $j^* = j$ if $i \le j$, and $j^* = j + m$ if $i > j$, and where the indices of k_h, l_h are taken modulo m. Then we have $m_{i,j} = 2^{\alpha_{i,j}+\beta_{i,j}} 3^{K-k_i-\alpha_{i,j}}$, as is easily verified. In particular $m_{i,j} > 0$, and then the matrix equation and $a_i > 0$ imply that $\Delta > 0$.

It follows that m-cycles are in one-to-one correspondence with solutions k_i, l_i, a_i of the matrix equation, with k_i, l_i, a_i all positive integers. For a given combination of k_i and l_i at most one such solution exists (there is exactly one solution with a_i rational ([2]), but in most cases the a_i are not integral). It follows that when we know upper bounds for K and L, there are only finitely many rational solutions left that can in principle be enumerated.

3. Conditions on K and m from a linear form in logarithms

3.1. *Introducing Λ.* As in Steiner's and Simons's proofs [St], [Si], the basis of our final result is a linear form in logarithms of integers, that for large m-cycles turns out to be too small to be possible. This linear form is

$$\Lambda = (K+L)\log 2 - K\log 3.$$

3.2. *A first inequality for Λ*

LEMMA 4.

$$0 < \Lambda < \sum_{i=1}^{m} \frac{1}{x_i}.$$

Proof. Rewrite the chain equation as

$$\frac{2^{k_{i+1}+l_i}}{3^{k_i}} \frac{a_{i+1}}{a_i} = 1 + \frac{2^{l_i}-1}{3^{k_i}a_i}.$$

Taking the product over all $i = 0, 1, \ldots, m-1$, and using the cyclicity, we get

$$\frac{2^{K+L}}{3^K} = \prod_{i=0}^{m-1}\left(1 + \frac{2^{l_i}-1}{3^{k_i}a_i}\right).$$

([2]) Rational solutions with common denominator q correspond to solutions of the $3n+q$-problem.

We now apply $\log(1+x) < x$ to each term in the product, and thus obtain

$$0 < \Lambda = \sum_{i=0}^{m-1} \log\left(1 + \frac{2^{l_i}-1}{3^{k_i}a_i}\right) < \sum_{i=0}^{m-1} \frac{2^{l_i}-1}{3^{k_i}a_i}.$$

The result now follows from $3^{k_i}a_i = y_i + 1 > y_i = 2^{l_i}x_{i+1} > (2^{l_i}-1)x_{i+1}$, and the cyclicity. ∎

From Lemma 4 we immediately have

Corollary 5.

$$0 < \Lambda < \frac{m}{x_{\min}} \le \frac{m}{X_0}.$$

Proof. Use Lemma 4 together with $x_i \ge x_{\min} \ge X_0$. ∎

3.3. *Chaining.* Now we will show that all x_i are exponentially large in terms of K (or, equivalently, $K+L$). To do that, we first show that all x_i are of about the same size, by "chaining" them.

Put $b = (1+X_0^{-1})/2^{1/\delta}$. With the present value of X_0 we have $b = 0.64576\ldots$.

Lemma 6. *For all $i = 0, 1, \ldots, m-1$ we have $x_{i+1} < b^\delta x_i^\delta$.*

Proof. We have

$$x_{i+1} = \frac{y_i}{2^{l_i}} < \frac{y_i+1}{2^{l_i}} = \frac{3^{k_i}a_i}{2^{l_i}} \le \frac{3^{k_i}}{2}\,a_i = \frac{3^{k_i}}{2}\,\frac{x_i+1}{2^{k_i}} = \left(\frac{3}{2}\right)^{k_i}\frac{x_i+1}{2}$$
$$= \frac{1}{2}\,(2^{k_i})^{\delta-1}(x_i+1).$$

Now use $a_i \ge 1$ and $x_i \ge X_0$ to get

$$x_{i+1} < \frac{1}{2}\,(2^{k_i}a_i)^{\delta-1}(x_i+1) = \frac{1}{2}\,(x_i+1)^\delta$$
$$\le \frac{1}{2}\,(1+X_0^{-1})^\delta x_i^\delta = b^\delta x_i^\delta. \ ∎$$

Note that the inequality of Lemma 6 holds cyclically. As a result we can estimate all x_i in terms of one x_i of our choice, say x_0.

3.4. *Another inequality for Λ.* Put

$$c_m = 2^{\frac{m}{\delta}\frac{\delta-1}{\delta^m-1}}\, b^{\frac{\delta}{\delta-1}-\frac{m}{\delta^m-1}}.$$

With the present value of X_0 we find that c_m decreases from $c_2 = 0.76479\ldots$ to $b^{\delta/(\delta-1)} = 0.30576\ldots$. When we also let X_0 tend to infinity, we get $c_m \to 2^{-1/(\delta-1)} = 0.30576\ldots$. Anyway, $c_m < 0.30577$ for $m \ge 30$.

As a consequence of Lemmas 4 and 6 we can now estimate Λ in terms of its coefficients, i.e. K.

Lemma 7.

$$0 < \Lambda < m c_m 2^{-\frac{\delta-1}{\delta^m-1}K}.$$

Proof. Lemma 6 implies $x_i < b^{\delta+\delta^2+\cdots+\delta^i} x_0^{\delta^i}$ for $i = 1, \ldots, m-1$. Hence

$$\prod_{i=0}^{m-1} x_i < b^{(m-1)\delta+(m-2)\delta^2+\cdots+\delta^{m-1}} x_0^{1+\delta+\delta^2+\cdots+\delta^{m-1}} = b^{\frac{\delta}{\delta-1}\left(\frac{\delta^m-1}{\delta-1}-m\right)} x_0^{\frac{\delta^m-1}{\delta-1}}.$$

On the other hand, also

$$\prod_{i=0}^{m-1} x_i = \prod_{i=0}^{m-1} \frac{x_i+1}{1+x_i^{-1}} \geq (1+X_0^{-1})^{-m} \prod_{i=0}^{m-1} 2^{k_i} a_i \geq (1+X_0^{-1})^{-m} 2^K,$$

where we simply estimated $a_i \geq 1$. Hence

$$x_0^{-\frac{\delta^m-1}{\delta-1}} < (1+X_0^{-1})^m b^{\frac{\delta}{\delta-1}\left(\frac{\delta^m-1}{\delta-1}-m\right)} 2^{-K} = c_m^{\frac{\delta^m-1}{\delta-1}} 2^{-K}.$$

Now we choose $x_0 = x_{\min}$, which we can do because of the cyclicity. Corollary 5 to Lemma 4 then shows that

$$0 < \Lambda < \frac{m}{x_{\min}} < mc_m 2^{-\frac{\delta-1}{\delta^m-1}K}. \ \blacksquare$$

Though strictly speaking the inequality of Lemma 7 depends on the value of X_0, this dependence is negligible.

4. Conditions on K and m from continued fractions

4.1. *A useful lemma.* A consequence of Corollary 5 to Lemma 4 is we have sharp lower and upper bounds for the ratios $(K+L)/K$, $(K+L)/L$ and K/L. This is useful not only in this section but also further in this paper.

LEMMA 8.

$$\delta K < K + L < 1.000001\delta K,$$
$$0.999999\,\frac{\delta}{\delta-1}\,L < K + L < \frac{\delta}{\delta-1}\,L,$$
$$0.999999\,\frac{1}{\delta-1}\,L < K < \frac{1}{\delta-1}\,L.$$

Proof. By Corollary 5 and the present value of X_0 we have

$$0 < K + L - K\delta < \frac{m}{X_0 \log 2} \leq \frac{K}{X_0 \log 2} < 10^{-17}K,$$

and the inequalities readily follow. $\blacksquare$

4.2. *Continued fraction results.* Recall that we denote by p_n/q_n the nth convergent to δ. Continued fraction theory shows that convergents are best approximations, i.e. any other approximation with smaller denominator is worse. Further necessary and sufficient inequalities for convergents are available. Indeed, we have the following results, the proofs of which can be found in many introductory texts on number theory (see e.g. [HW, Chapter 10]).

Lemma 9. (a) *If p/q is a rational approximation to δ satisfying $|p-q\delta| < 1/(2q)$, then p/q is a convergent.*

(b) $|p_n - q_n\delta| > \dfrac{1}{q_n + q_{n+1}} > \dfrac{1}{(a_{n+1}+2)q_n}$.

(c) *If p/q is a rational approximation to δ, and if $q \le q_n$, then $|p-q\delta| \ge |p_n - q_n\delta|$.*

(d) *If n is odd then $p_n - q_n\delta > 0$; if n is even then $p_n - q_n\delta < 0$.*

4.3. *A generalization of Crandall's Lemma to m-cycles.* With Corollary 5 we now can derive a result like Crandall's Lemma 1, which gives a lower bound for K that depends on m.

Lemma 10. *If $q_n + q_{n+1} \le (\log 2)X_0/m$, then $K > q_n$.*

Proof. Assume $K \le q_n$. By Lemma 9(c), (b),

$$\Lambda = (\log 2)|(K+L) - K\delta| \ge (\log 2)|p_n - q_n\delta| > \frac{\log 2}{q_n + q_{n+1}} \ge \frac{m}{X_0},$$

which contradicts Corollary 5. ∎

Applying this with for each m the maximal n that satisfies the condition, leads to the following result.

Corollary 11.

$$\begin{aligned}
2 \le m \le 19 &\;\Rightarrow\; K > q_{31} > 5.7509 \cdot 10^{15},\\
20 \le m \le 37 &\;\Rightarrow\; K > q_{30} > 4.8361 \cdot 10^{14},\\
38 \le m \le 256 &\;\Rightarrow\; K > q_{29} > 4.3116 \cdot 10^{14},\\
257 \le m \le 485 &\;\Rightarrow\; K > q_{28} > 5.2449 \cdot 10^{13},\\
486 \le m \le 3\,669 &\;\Rightarrow\; K > q_{27} > 1.1571 \cdot 10^{13},\\
3\,670 \le m \le 13\,245 &\;\Rightarrow\; K > q_{26} > 6.1624 \cdot 10^{12},\\
13\,246 \le m \le 20\,299 &\;\Rightarrow\; K > q_{25} > 5.4093 \cdot 10^{12},\\
20\,300 \le m \le 38\,118 &\;\Rightarrow\; K > q_{24} > 7.5311 \cdot 10^{11},\\
38\,119 \le m \le 263\,748 &\;\Rightarrow\; K > q_{23} > 1.3752 \cdot 10^{11},\\
263\,749 \le m \le 1\,157\,173 &\;\Rightarrow\; K > q_{22} > 6.5470 \cdot 10^{10},\\
1\,157\,174 \le m \le 3\,259\,965 &\;\Rightarrow\; K > q_{21} > 6.5868 \cdot 10^{9},\\
3\,259\,966 \le m \le 18\,386\,313 &\;\Rightarrow\; K > q_{20} > 6.1892 \cdot 10^{9},\\
18\,386\,314 \le m \le 35\,662\,848 &\;\Rightarrow\; K > q_{19} > 3.9757 \cdot 10^{8},\\
35\,662\,849 \le m \le 131\,993\,764 &\;\Rightarrow\; K > q_{18} > 2.2564 \cdot 10^{8},\\
m \ge 131\,993\,765 &\;\Rightarrow\; K > 1.7095\, m.
\end{aligned}$$

Proof. All lines but the last two follow immediately from Lemma 10. The one but last line is Corollary 2 to Crandall's original Lemma 1. The last line follows by Lemma 8 from the trivial observation that $L \ge m$. ∎

Note that the bounds for m in Corollary 11 depend heavily on the value of X_0. Also note that Corollary 11 implies $K > 5.2424 \cdot 10^{15}/m$).

5. Application of transcendence theory. Now that we know that Λ is exponentially small in terms of its coefficients, we can invoke deep but explicit results from transcendence theory, which tell us that linear forms in logarithms of integers cannot be too small in terms of their coefficients. Since Steiner published his paper [St] with essentially the same idea of applying transcendence theory to prove the nonexistence of nontrivial 1-cycles, transcendence theory has made substantial progress. For general linear forms $x \log a + y \log b$ with $x, y \in \mathbb{Z}$ and $a, b \in \mathbb{N}$ the best results today are the result of Laurent, Mignotte and Nesterenko [LMN] for small x, y, and for large x, y that of Matveev [Ma] (see also Nesterenko [Ne]). For our specific case $x \log 2 + y \log 3$ however, the result of Rhin [Rh] is best. From it we derive the following estimate.

Lemma 12.

$$\Lambda > e^{-13.3(0.46057+\log K)}.$$

Proof. We apply the Proposition on p. 160 of [Rh] with $u_0 = 0$, $H = u_1 = K + L$, and $u_2 = -K$. Together with Lemma 8 the result follows. ∎

As a consequence of Lemma 12 we can also estimate the global minimum of an m-cycle in terms of K.

Corollary 13.

$$x_{\min} < m e^{13.3(0.46057+\log K)}.$$

Proof. Apply Corollary 5 and Lemma 12. ∎

6. Upper bounds for K, L and $x_{\min}$

6.1. *Initial upper bound.* Clearly Lemmas 7 and 12 are contradictory when K is large enough. In other words, they provide an upper bound for K, and then by Lemma 8 and Corollary 13 also for L and $x_{\min}$.

Lemma 14. *Let $x = K_1(m)$ be the largest solution of*

$$e^{-13.3(0.46057+\log x)} = m c_m 2^{-\frac{\delta-1}{\delta^m-1}x}.$$

Then

$$K < K_1(m).$$

Proof. Immediate from Lemmas 7 and 12 and the definition of $K_1(m)$. ∎

Let $k_1(m) = K_1(m)/(m\delta^m)$. Note that $k_1(m)$ is a decreasing function that for increasing m tends to the constant

$$\frac{13.3 \log \delta}{\log 2(\delta-1)} = 15.107\ldots$$

as $m \to \infty$. Indeed:

m	$k_1(m) <$	$K_1(m) <$	m	$k_1(m) <$	$K_1(m) <$
2	61.456	$3.0877 \cdot 10^{2}$	10 000	15.150	$2.3460 \cdot 10^{2\,005}$
10	35.843	$3.5858 \cdot 10^{4}$	100 000	15.113	$1.1981 \cdot 10^{20\,008}$
27	23.710	$1.6100 \cdot 10^{8}$	515 619	15.109	$3.0475 \cdot 10^{103\,140}$
28	23.435	$2.6155 \cdot 10^{8}$	515 620	15.109	$4.8301 \cdot 10^{103\,140}$
57	19.533	$2.8037 \cdot 10^{14}$	1 000 000	15.108	$1.4816 \cdot 10^{200\,026}$
58	19.466	$4.5060 \cdot 10^{14}$	10 000 000	15.108	$1.2429 \cdot 10^{2\,000\,198}$
100	17.798	$1.7876 \cdot 10^{23}$	100 000 000	15.108	$2.1457 \cdot 10^{20\,001\,908}$
511	15.739	$1.3035 \cdot 10^{106}$	131 993 764	15.108	$7.2853 \cdot 10^{26\,401\,268}$
512	15.738	$2.0698 \cdot 10^{106}$	131 993 765	15.108	$1.1547 \cdot 10^{26\,401\,269}$
1 000	15.453	$1.6144 \cdot 10^{204}$	1 000 000 000	15.108	$5.0463 \cdot 10^{200\,019\,001}$

The dependence of $K_1(m)$ on X_0 is negligible.

For each m we now have proved Theorem 3(a), that there are only finitely many m-cycles. Moreover, we can derive explicit upper and lower bounds for K, L and $x_{\min}$, as stated in Theorem 3(d). For $m \leq 515\,619$ we can obtain better results in the next section.

Proof of Theorem 3(d) for $m \geq 515\,620$. The upper bound for K follows from Lemma 14, and the observation that k_1 is a decreasing function of m. The lower bound for K follows by Corollary 11. The bounds for L and $x_{\min}$ follow by combining this with Lemma 8 and Corollary 13. ∎

Moreover, we can combine Lemma 14 with the generalized Crandall Lemma 10, to prove that for small m there are no solutions at all. The next lemma proves a part of Theorem 3(b).

LEMMA 15. *There are no nontrivial m-cycles for $2 \leq m \leq 57$.*

Proof. By Lemma 14, for $m \leq 57$ we have

$$K < K_1(m) \leq K_1(57) < 2.8037 \cdot 10^{14}.$$

This contradicts the inequality $K > 4.3116 \cdot 10^{14}$ from Corollary 11 to the generalized Crandall Lemma 10. ∎

Note that Corollary 2 to Crandall's original Lemma 1 gives a result only for $m \leq 27$.

The maximum value of m in Lemma 15 does depend heavily on the value of X_0, in the sense that any substantial improvement of the value of X_0 immediately leads to an improvement of the upper bound for m for which the proof works.

6.2. *Reduced upper bound.* Next we use, like Steiner [St], a continued fraction argument to find a better upper bound for K. We computed the continued fraction of δ up to $a_{200\,001}$. Using Mathematica 5.0 on a 2GHz Pentium 4 personal computer this computation took about 45 seconds.

Let $x = J_2(m)$ be the largest solution of

$$mc_m 2^{-\frac{\delta-1}{\delta^m-1}x} = \frac{\log 2}{2x}.$$

Lemmas 7 and 9(a) imply that $(K+L)/K$ is a convergent to δ whenever $K > J_2(m)$. Note that only convergents with odd index are of interest, as the sign of $p_n - q_n\delta$ alternates (Lemma 9(d)), and we have $\Lambda > 0$.

In view of Lemma 9(b), of particular interest are partial quotients a_n that are champions in the sense that $a_k < a_n$ whenever $k < n$, for even k, n.

For each m let $n(m)$ be the index n of the smallest champion for which $K_1(m) < q_{n-1}$. Then we define $A(m) = \max\{a_0, a_2, \dots, a_{n(m)-2}\}$. Clearly $A(m)$ is the champion before $n(m)$. Indeed, we have

	$n(m)$	$A(m)$	$K_1(m) <$	$q_{n(m)-1} >$
$58 \le m \le 511$	218	55	$1.3035 \cdot 10^{106}$	$1.3133 \cdot 10^{106}$
$512 \le m \le 551$	230	100	$1.4043 \cdot 10^{114}$	$1.7807 \cdot 10^{114}$
$552 \le m \le 816$	330	964	$2.0804 \cdot 10^{167}$	$2.6341 \cdot 10^{167}$
$817 \le m \le 1\,340$	528	2 436	$2.1843 \cdot 10^{272}$	$2.3119 \cdot 10^{272}$
$1\,341 \le m \le 7\,009$	2 764	3 308	$9.1127 \cdot 10^{1\,406}$	$1.2197 \cdot 10^{1\,407}$
$7\,010 \le m \le 11\,143$	4 312	4 878	$1.0938 \cdot 10^{2\,234}$	$1.4973 \cdot 10^{2\,234}$
$11\,144 \le m \le 54\,234$	21 150	8 228	$5.5425 \cdot 10^{10\,853}$	$8.2400 \cdot 10^{10\,853}$
$54\,235 \le m \le 315\,502$	122 416	59 599	$1.1752 \cdot 10^{63\,113}$	$1.2759 \cdot 10^{63\,113}$
$315\,503 \le m \le 515\,619$	200 002	104 733	$3.0475 \cdot 10^{103\,140}$	$3.5522 \cdot 10^{103\,140}$

For $m \ge 515\,620$ we can in principle find corresponding values for $A(m)$ when we compute more partial quotients.

With the part of the continued fraction that we have at our disposal, we can derive a sharper upper bound for all $m \le 515\,619$, as follows. Let $x = K_2(m)$ be the largest solution of

$$mc_m 2^{-\frac{\delta-1}{\delta^m-1}x} = \frac{\log 2}{(A(m)+2)x}.$$

Note that $j_2(m) = J_2(m)/(m\delta^m)$ is a bounded function that tends to $\log\delta/((\delta-1)\log 2) = 1.1358\dots$ as $m \to \infty$. Also put $k_2(m) = K_2(m)/(m\delta^m)$. As we do not know how $A(m)$ will grow when m increases, we do not know the exact behaviour of $k_2(m)$, but most probably it will also tend to $\log\delta/((\delta-1)\log 2) = 1.1358\dots$ as $m \to \infty$. Indeed:

m	$j_2(m) <$	$J_2(m) <$	$k_2(m) <$	$K_2(m) <$
58	1.4930	$3.4560 \cdot 10^{13}$	1.6394	$3.7949 \cdot 10^{13}$
63	1.4705	$3.6983 \cdot 10^{14}$	1.6051	$4.0367 \cdot 10^{14}$
64	1.4664	$5.9379 \cdot 10^{14}$	1.5988	$6.4742 \cdot 10^{14}$
90	1.3881	$1.2542 \cdot 10^{20}$	1.4817	$1.3388 \cdot 10^{20}$
91	1.3859	$2.0068 \cdot 10^{20}$	1.4784	$2.1408 \cdot 10^{20}$
100	1.3677	$1.3737 \cdot 10^{22}$	1.4518	$1.4582 \cdot 10^{22}$
511	1.1964	$9.9079 \cdot 10^{104}$	1.2126	$1.0043 \cdot 10^{105}$
512	1.1963	$1.5734 \cdot 10^{105}$	1.2153	$1.5984 \cdot 10^{105}$
551	1.1927	$1.0669 \cdot 10^{113}$	1.2103	$1.0827 \cdot 10^{113}$
552	1.1926	$1.6940 \cdot 10^{113}$	1.2203	$1.7333 \cdot 10^{113}$
816	1.1766	$1.5769 \cdot 10^{166}$	1.1953	$1.6020 \cdot 10^{166}$
817	1.1765	$2.5023 \cdot 10^{166}$	1.1980	$2.5480 \cdot 10^{166}$
1 000	1.1701	$1.2224 \cdot 10^{203}$	1.1876	$1.2407 \cdot 10^{203}$
1 340	1.1625	$1.6517 \cdot 10^{271}$	1.1756	$1.6703 \cdot 10^{271}$
1 341	1.1625	$2.6198 \cdot 10^{271}$	1.1761	$2.6506 \cdot 10^{271}$
7 009	1.1422	$6.8625 \cdot 10^{1\,405}$	1.1448	$6.8782 \cdot 10^{1\,405}$
7 010	1.1422	$1.0879 \cdot 10^{1\,406}$	1.1449	$1.0905 \cdot 10^{1\,406}$
10 000	1.1405	$1.7660 \cdot 10^{2\,004}$	1.1424	$1.7690 \cdot 10^{2\,004}$
11 143	1.1401	$8.2326 \cdot 10^{2\,232}$	1.1418	$8.2450 \cdot 10^{2\,232}$
11 144	1.1401	$1.3050 \cdot 10^{2\,233}$	1.1419	$1.3071 \cdot 10^{2\,233}$
54 234	1.1369	$4.1685 \cdot 10^{10\,852}$	1.1373	$4.1698 \cdot 10^{10\,852}$
54 235	1.1369	$6.6069 \cdot 10^{10\,852}$	1.1374	$6.6097 \cdot 10^{10\,852}$
100 000	1.1365	$9.0096 \cdot 10^{20\,006}$	1.1368	$9.0116 \cdot 10^{20\,006}$
315 502	1.1361	$8.8364 \cdot 10^{63\,111}$	1.1362	$8.8371 \cdot 10^{63\,111}$
315 503	1.1361	$1.4006 \cdot 10^{63\,112}$	1.1362	$1.4007 \cdot 10^{63\,112}$
515 619	1.1361	$2.2915 \cdot 10^{103\,139}$	1.1361	$2.2916 \cdot 10^{103\,139}$

The next lemma now gives a better upper bound for K in the range for which the continued fraction argument can be applied, given the part of the continued fraction we have computed.

LEMMA 16. *For* $58 \le m \le 515\,619$ *a possible* m*-cycle satisfies* $K < K_2(m)$.

Proof. Note that $J_2(m) < K_2(m)$ for all m, so we may assume $K > J_2(m)$. Then by the definition of J_2, Lemma 7 implies $\Lambda < \log 2/(2K)$, and so Lemma 9(a) shows that $(K+L)/K = p_n/q_n$ for some odd n. From Lemma 14 we have $K < K_1(m) < q_{n(m)-1}$, implying $n \le n(m) - 2$. Hence $a_{n+1} \le A(m)$. The definition of K_2 and Lemmas 9(b) and 7 then imply $K < K_2(m)$. ∎

There are only 68 values of m between 58 and 515 619 for which there is a convergent p_n/q_n with odd index and $J_2(m) < q_n < K_2(m)$. In all other cases $q_n > J_2(m)$ implies $q_n > K_2(m)$.

As a consequence of Lemma 16 we can now prove a further part of Theorem 3(d), giving improved upper bounds for K, L and $x_{\min}$ in the case of $91 \le m \le 515\,619$. For $m \le 90$ we will achieve even better results in the next section.

Proof of Theorem 3(d) for $91 \le m \le 515\,619$. The upper bound for K follows from Lemma 16, and the observation that $k_2(m) < 1.4784$ (note that $k_2(m)$ is a decreasing function, except when $A(m)$ jumps to the next champion). The lower bound for K follow by Corollary 11. The bounds for L follow by combining this with Lemma 8.

For $x_{\min}$ we can now find a rather sharp upper bound, as follows. Let n be the index such that $q_{n-1} < K \le q_n$. Since

$$K < K_2(m) \le K_2(515\,619) < 2.2916 \cdot 10^{103\,139} < 7.5013 \cdot 10^{103\,139} < q_{199\,998}$$

we have $n \le 199\,998$. From the continued fraction we find that for this range

$$(a_n + 1)(a_{n+1} + 2) \le 24\,298\,288.$$

Now Corollary 5 and Lemma 9(c),(b) show that

$$\begin{aligned} x_{\min} &< \frac{m}{\Lambda} < \frac{m}{(\log 2)|p_n - q_n\delta|} < \frac{m(a_{n+1}+2)q_n}{\log 2} \\ &< \frac{m(a_{n+1}+2)(a_n+1)q_{n-1}}{\log 2} < \frac{m(a_{n+1}+2)(a_n+1)K}{\log 2} < \frac{24\,298\,288}{\log 2}\, mK, \end{aligned}$$

where we used $q_n = a_n q_{n-1} + q_{n-2} < (a_n+1)q_{n-1}$. The bound for $x_{\min}$ now follows at once. ∎

The improvement we reached here for the bound of $x_{\min}$ is substantial. Though the improvement of the bounds for K and L is to some extent marginal (we cannot improve on the dominating term δ^m), the improvement becomes significant for small m, when we combine Lemma 16 with the generalized Crandall Lemma 10, to get a result improving on Lemma 15. The next lemma proves another part of Theorem 3(b).

Lemma 17. *There are no nontrivial m-cycles for* $2 \le m \le 63$.

Proof. By Lemma 16, for $m \le 63$ we have $K < K_2(m) \le K_2(63) < 4.0367 \cdot 10^{14}$. This contradicts $K > 4.3116 \cdot 10^{14}$ from Corollary 11 to Lemma 10. ∎

The maximum value of m in Lemma 17 does depend on the value of X_0, in the sense that any substantial improvement of the value of X_0 leads to an improvement of the upper bound for m for which the proof works.

7. Elimination of small solutions

7.1. *Convergents.* The classical approach now is to show, based on Lemma 9(a) as well as on refinements for e.g. "secondary convergents", that possible solutions in certain ranges are (secondary) convergents, and then to check all such (secondary) convergents for Corollary 5. Instead we will use an approximation lattice method, which can be seen as a more powerful variant of the continued fraction method. See [dW, Section 1.4 and Chapter 3] for some background on approximation lattices.

7.2. *Lattices.* We look for solutions in an "approximation lattice", as follows.

For a vector $\mathbf{x} = \binom{x_1}{x_2}$ we define the norm $\|\mathbf{x}\|$ as $\|\mathbf{x}\| = \max\{|x_1|, |x_2|\}$. Put

$$C = \left\lfloor \frac{X_0}{m} \frac{\log 2}{2} K_2(m) \right\rfloor, \qquad \Gamma = \begin{pmatrix} 1 & 0 \\ [C\delta] & C \end{pmatrix},$$

where $[\cdot]$ stands for rounding towards the nearest integer. Then we look at the lattice of the $\mathbb{Z}$-linear combinations of the columns of Γ. For a solution K, L satisfying Corollary 5, Lemma 7 and Lemma 16 we look at the lattice point

$$\mathbf{x} = \Gamma \begin{pmatrix} -K \\ K+L \end{pmatrix} = \begin{pmatrix} -K \\ \Lambda_0 \end{pmatrix},$$

where $\Lambda_0 = (K+L)C - K[C\delta]$ is an approximation of $(C/\log 2)\Lambda$. Indeed, by Lemma 16 we have

$$\left| \Lambda_0 - \frac{C}{\log 2} \Lambda \right| \le K \, |C\delta - [C\delta]| \le \frac{1}{2} K_2(m).$$

It follows by Corollary 5 and the definition of C that

$$|\Lambda_0| \le \frac{1}{2} K_2(m) + \frac{C}{\log 2} |\Lambda| < \frac{1}{2} K_2(m) + \frac{C}{\log 2} \frac{m}{X_0} \le K_2(m).$$

It follows that we only have to search for lattice points with norm at most $K_2(m)$.

To achieve this efficiently, we compute (by the Euclidean, i.e. continued fraction, algorithm) a reduced basis of the lattice. Let Γ_{red} be a matrix with this reduced basis as columns. The lattice point $\mathbf{x}$ can be expressed in the reduced basis as $\mathbf{x} = \Gamma_{\mathrm{red}}\mathbf{z}$ for some $\mathbf{z} \in \mathbb{Z}^2$. Due to the reducedness of the lattice basis, the number of points to be searched is approximately $K_2(m)^2/C \approx (m/X_0)K_2(m)$. A brute force search for points $\mathbf{z} \in \mathbb{Z}^2$ such that $\|\Gamma_{\mathrm{red}}\mathbf{z}\| \le K_2(m)$ is therefore efficient when $K_2(m)$ is not much larger than X_0/m. Each lattice point found can be checked for fulfilling Corollary 5 and Lemma 7.

Doing this search for $64 \le m \le 90$ we found the following results.

LEMMA 18. (a) *With $64 \le m \le 68$ there are no nontrivial m-cycles.*
(b) *With $69 \le m \le 72$ the only possible nontrivial m-cycles satisfy*

m	K	L	*falls when* $X_0 \ge$	*year*
69, 70, 71, 72	5 750 934 602 875 680	3 364 081 086 781 987	$\{577, 585, 593, 602\} \cdot 2^{50}$	2007
71, 72	11 985 484 530 117 643	7 011 059 003 092 348	$\{624, 633\} \cdot 2^{50}$	2008
72	17 736 419 132 993 323	10 375 140 089 874 335	$309 \cdot 2^{50}$	2004
72	18 220 034 457 359 606	10 658 036 919 402 709	$667 \cdot 2^{50}$	2008
72	24 454 584 384 601 569	14 305 014 835 713 070	$706 \cdot 2^{50}$	2008

(c) *With $73 \le m \le 90$ the only possible nontrivial m-cycles satisfy*

m	# *solutions*	$K \ge$	$K \le$	*falls when* $X_0 \ge$	*year*
73	8	5 750 934 602 875 680	36 923 684 239 085 495	$809 \cdot 2^{50}$	2010
74	15	5 267 319 278 509 397	61 861 883 948 053 347	$1\,109 \cdot 2^{50}$	2013
75	24	5 267 319 278 509 397	99 269 183 511 505 125	$2\,385 \cdot 2^{50}$	2027
76	36	5 267 319 278 509 397	154 896 517 532 316 509	$37\,642 \cdot 2^{50}$	2413
77	54	5 267 319 278 509 397	254 649 316 368 187 917	$38\,137 \cdot 2^{50}$	2419
78	82	5 267 319 278 509 397	397 560 349 370 386 783	$456\,718 \cdot 2^{50}$	7006
79	135	5 267 319 278 509 397	658 444 215 665 816 663	$462\,573 \cdot 2^{50}$	7070
80	213	5 267 319 278 509 397	1 049 770 015 108 961 483	$468\,429 \cdot 2^{50}$	7134

m	# *solutions*	$K \ge$	$K \le$	m	# *solutions*	$K \ge$	$K \le$
81	332	$5.2673 \cdot 10^{15}$	$1.5716 \cdot 10^{18}$	86	3 507	$5.2673 \cdot 10^{15}$	$1.7102 \cdot 10^{19}$
82	535	$5.2673 \cdot 10^{15}$	$2.5159 \cdot 10^{18}$	87	5 623	$5.2673 \cdot 10^{15}$	$2.8240 \cdot 10^{19}$
83	845	$5.2673 \cdot 10^{15}$	$4.6403 \cdot 10^{18}$	88	9 017	$5.2673 \cdot 10^{15}$	$4.6006 \cdot 10^{19}$
84	1 370	$5.2673 \cdot 10^{15}$	$6.6281 \cdot 10^{18}$	89	14 457	$5.2673 \cdot 10^{15}$	$7.7693 \cdot 10^{19}$
85	2 190	$5.2673 \cdot 10^{15}$	$1.0604 \cdot 10^{19}$	90	23 181	$5.2673 \cdot 10^{15}$	$1.1442 \cdot 10^{20}$

For $69 \le m \le 72$ we give with each solution the minimal value for X_0 that has to be reached to show that the solution does not correspond to an m-cycle. We also give the year in which this is expected to happen when the current rate of checking 2^{48} values for $x_{\min}$ per day is continued.

For $73 \le m \le 90$ we only give for each m the number of solutions found, and the minimal and maximal K. For $m \le 80$ we also give the minimal value for X_0 that has to be reached to show that the solutions do not correspond to an m-cycle, and the year in which this is expected to happen.

The entire computation for the proof of Lemma 18 took less than 2.5 minutes.

As a result we can now complete the proof of Theorem 3.

Proof of Theorem 3(b), (c) and (d) for $m \leq 90$. (b) follows at once from Lemma 18(a). The values in (c) and upper bound in (d) for K follow from Lemma 18. The lower bound for K follows by Lemma 18. The bounds for L follow by combining this with Lemma 8. In (c) the upper bound for $x_{\min}$ is derived from Corollary 5.

In (d) the upper bound for $x_{\min}$ is derived as in the above proof of the other part of Theorem 3(d), as follows. Let n be the index such that $q_{n-1} < K \leq q_n$. Since

$$K < K_2(m) \leq K_2(90) < 1.3388 \cdot 10^{20} < 2.0563 \cdot 10^{20} < q_{43}$$

we have $n \leq 43$. From the continued fraction we find that $(a_n+1)(a_{n+1}+2) \leq 168$. Hence $x_{\min} < (168/\log 2)mK$. ■

8. Conclusion. In Theorem 3, it is of interest to note that the border $m = 68/69$ between (b) and (c) is directly related to the size of the lower bound X_0 for $x_{\min}$ that comes from external brute force computations (see [Ro]). We indicated above when this border would be crossed, assuming that these computations continued at the present speed, and no new ideas emerged. We expect, as indicated in Lemma 18, that $m = 69$ to $m = 72$ will be completely solved within 5 years, that $m = 73$ up to $m = 75$ can be solved within 25 years, and that $m = 76$ and beyond will take a considerably larger effort to finish.

The border $m = 72/73$ between Theorem 3(c) and (d) depends only on the amount of space one is willing to spend on listing candidate solutions that cannot yet be ruled out.

The border $m = 90/91$ in Theorem 3(d) depends directly on X_0, but it depends also on the amount of computation one is willing to spend on finding small lattice points. As the difference in the upper bounds between (c) and (d) is in the constants only, we see so far no need to do more extensive computations.

The border $m = 515\,619/515\,620$ in Theorem 3(d) is directly related to how far one wishes to compute the continued fraction expansion of δ. Note that the difference in the upper bounds is substantial.

The border $m = 131\,993\,764/131\,993\,765$ in Theorem 3(d) was introduced only because it has some impact on the lower bounds for K and L. It is therefore not of much significance.

It seems that with the present techniques one cannot go much further. To show the nonexistence of nontrivial m-cycles for essentially larger ranges of m an entirely new idea seems to be needed.

Note that if certain values for K and L cannot be excluded by the approximation lattice method, they can in principle be analyzed for an integer solution of the matrix equation. Without any knowledge on k_i and l_i such an analysis is very inefficient. Computational and heuristic evidence indicates that for given m and K the worst case values for k_i and l_i can be estimated, and then for any m an efficient analysis of possible m-cycles may become feasible. We believe that Lemma 7 is almost sharp for the worst case, in the sense that at best we expect that the factor mc_m may be improved to a constant, but the exponential dependence on m in the term $2^{-\frac{\delta-1}{\delta^m-1}K}$ seems unavoidable. Given the state of affairs in transcendence theory, we think that further improvement to Theorem 3 should come from sharpening the lower bounds for K. We leave this for future research.

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Received on 4.12.2003
and in revised form on 10.11.2004 (4673)