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Lower bounds for the conductor of L-functions

by

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Dedicated to Professor A. Schinzel on the occasion of his 75th birthday

1. Introduction. Our aim in this paper is to obtain lower bounds for the conductor of L-functions in a general setting. We therefore start with the definition of our framework, i.e. the *Selberg class* $\mathcal{S}$ of L-functions: $F \in \mathcal{S}$ if

(i) $F(s)$ is an absolutely convergent Dirichlet series for $\sigma > 1$,

$$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s};$$

(ii) $(s-1)^m F(s)$ is an entire function of finite order for some $m \in \mathbb{N}$;
(iii) $F(s)$ satisfies a functional equation of the type $\Phi(s) = \omega \bar{\Phi}(1-s)$, where $|\omega| = 1$, $\bar{f}(s) = \overline{f(\bar{s})}$ and

$$\Phi(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s) = \gamma(s) F(s),$$

say, with $Q > 0$, $r \geq 0$, $\lambda_j > 0$ and $\Re \mu_j \geq 0$;
(iv) $a(n) \ll n^{\varepsilon}$ for every $\varepsilon > 0$;
(v) $\log F(s)$ is a Dirichlet series with coefficients $b(n)$ satisfying $b(n) = 0$ unless $n = p^k$ with p prime and $k \geq 1$, and $b(n) \ll n^{\vartheta}$ for some $\vartheta < 1/2$.

The *extended Selberg* class $\mathcal{S}^\sharp$ is the larger class of the functions $F(s)$ satisfying (i)–(iii) above.

We refer to Selberg [15], Conrey–Ghosh [2] and to our survey papers [7], [4], [12], [13], [14] for a discussion of the basic properties of $\mathcal{S}$ and $\mathcal{S}^\sharp$. Here

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we recall that (v) implies an Euler product expansion of general type, i.e.

$$F(s)=\prod_p F_p(s),\qquad F_p(s)=\sum_{m=0}^{\infty}\frac{a(p^m)}{p^{ms}}. \tag{1.1}$$

Moreover, the *degree* and the *conductor* of $F\in\mathcal{S}^\sharp$ are defined respectively by

$$d_F=2\sum_{j=1}^{r}\lambda_j,\qquad q_F=(2\pi)^{d_F}Q^2\prod_{j=1}^{r}\lambda_j^{2\lambda_j}.$$

Note that the real numbers d_F and q_F are invariants, i.e. they depend only on $F(s)$ and not on the shape of the functional equation (which may be changed by means of suitable formulae for the Γ function). We refer to [8] and [9] for the invariant theory of the Selberg class. Note also that in the case of classical L-functions, the conductor q_F coincides with well known objects associated with the underlying structure of the L-functions. For example, q_F is the conductor of the primitive Dirichlet character χ if $F(s)=L(s,\chi)$, the level of the normalized newform $f(z)$ if $F(s)=L_f(s)$, the absolute value of the discriminant of the number field K if $F(s)=\zeta_K(s)$, and so on.

From now on we assume that $d_F>0$. Indeed, the structure of the degree 0 functions from both $\mathcal{S}$ and $\mathcal{S}^\sharp$ is quite well understood (see Conrey–Ghosh [2] and, e.g., Kaczorowski–Molteni–Perelli [5]); in this case sharper results than those presented below are easily obtained.

It turns out that $q_F\in\mathbb{N}$ when $F(s)$ is a classical L-function, and we expect that $q_F\in\mathbb{N}$ for every $F\in\mathcal{S}$. This is mainly based on the expectation that $\mathcal{S}$ coincides with the class of automorphic L-functions. However, at present the classification of $\mathcal{S}$ is far from being complete, and the question if $q_F\in\mathbb{N}$ is an interesting open problem. As usual, the situation is more complicated for $\mathcal{S}^\sharp$. Indeed, in this case q_F does not need to be an integer, as one can see from Hecke's theory of (suitably normalized) Dirichlet series associated with $G(\lambda)$-modular forms; see e.g. Berndt–Knopp [1]. In fact, $q_F=\lambda^2$ if $F(s)$ comes from $G(\lambda)$. Nevertheless, we still expect a universal lower bound, say $q_F\ge c_0>0$, for all $F\in\mathcal{S}^\sharp$. Actually, since $\mathcal{S}^\sharp$ is a multiplicative semigroup and $q_{FG}=q_Fq_G$, if such a c_0 exists then $c_0=1$. We wish to thank Brian Conrey for pointing out that in the Hecke theory case, although a priori conductors can be arbitrary positive numbers, the spaces of modular forms are trivial when the conductor is <1. We further note that the situation changes completely if generalized Dirichlet series are allowed. Indeed, in this case q_F can be arbitrarily small; see [10].

In order to state our results we first have to introduce and discuss several interesting invariants; again, we refer to [8] and [9] for a full account. For any integer $n\ge 0$ let $B_n(z)$ denote the nth Bernoulli polynomial. The

H-invariants of $F \in \mathcal{S}^\sharp$ are defined as

$$H_F(n) = 2\sum_{j=1}^r \frac{B_n(\mu_j)}{\lambda_j^{n-1}}, \qquad n = 0, 1, \ldots .$$

The interest of the H-invariants comes from the fact that if $F, G \in \mathcal{S}^\sharp$ have the same conductor, root number (see e.g. [9] for its definition) and all H-invariants, then they satisfy the same functional equation. Moreover, $H_F(0) = d_F$ and $H_{FG}(n) = H_F(n) + H_G(n)$. Another interesting invariant is the meromorphic function

$$K_F(z) = z\sum_{j=1}^r \frac{e^{z\mu_j/\lambda_j}}{e^{z/\lambda_j} - 1}.$$

$K_F(z)$ is related both to the H-invariants and to the poles ρ of the γ-factor $\gamma(s)$ in the functional equation of $F(s)$, thanks to the following expressions (valid in suitable regions of $\mathbb{C}$):

$$K_F(z) = \frac{1}{2}\sum_{n=0}^\infty \frac{H_F(n)}{n!} z^n = -z\sum_\rho e^{-\rho z},$$

where ρ runs over such poles.

Now we define three new invariants. For $F \in \mathcal{S}^\sharp$ let

$$H_F^* = \sup_{n\geq 1}\left(\frac{|H_F(n)|}{n!}\right)^{1/n}, \qquad H_F^\bullet = \limsup_{n\to\infty}\left(\frac{|H_F(n)|}{n!}\right)^{1/n}$$

and, if $d_F > 0$,

$$D_F = \max_{j=1,\ldots,r} \frac{|\Im \mu_j|}{\lambda_j}.$$

Clearly, H_F^* and $H_F^\bullet$ are invariants, and $H_F^* \geq H_F^\bullet$. Moreover, D_F is an invariant since

$$D_F = \max_\rho |\Im\rho|,$$

where ρ runs over the trivial zeros of $F(s)$. We have

Theorem 1. *Let $F \in \mathcal{S}^\sharp$ with $d_F > 0$. Then*

$$\frac{1}{\pi d_F} \leq H_F^\bullet \leq H_F^* < \infty.$$

From the proof of Theorem 1 (see (2.1) below), and the fact that $d_F \geq 1$ if $d_F \neq 0$ (see [6]), we have the upper bound

$$H_F^* \ll d_F \max_{j=1,\ldots,r} \frac{1+|\mu_j|}{\lambda_j}. \tag{1.2}$$

It is expected that every $F \in \mathcal{S}$ has an Euler product of polynomial type, i.e. for every prime p the shape of $F_p(s)$ in (1.1) is

$$F_p(s) = \prod_{j=1}^{\partial_p} \left(1 - \frac{\alpha_{j,p}}{p^s}\right)^{-1} \tag{1.3}$$

with $|\alpha_{j,p}| \le 1$ and $\partial_p \le d_F$. Our main result holds for functions satisfying (1.3); therefore we denote by $\mathcal{S}^*$ the subclass of $\mathcal{S}$ of the functions satisfying (1.3), and hence conjecturally $\mathcal{S}^* = \mathcal{S}$. We have

Theorem 2. *Let $F \in \mathcal{S}^*$ with $d_F > 0$. Then there exists an absolute constant $c_0 > 0$ such that*

$$(1 + H_F^* + D_F) q_F^{1/d_F} \ge c_0.$$

In accordance with a previous remark, the bound in Theorem 2 does not hold if generalized Dirichlet series are allowed. Theorem 2 provides at once a lower bound for q_F in terms of the other invariants d_F, H_F^* and D_F, hence relations between H_F^* and D_F would be of interest. For example, is it true that something like

$$D_F \ll H_F^*$$

holds? From (1.2) we see that if the μ_j are pure imaginary with modulus, say, ≥ 1, then $H_F^* \ll d_F D_F$. On the other hand, $D_F = 0$ if the μ_j are all real, and hence $H_F^* \ll D_F$ certainly does not hold in general.

We can avoid the invariant D_F in lower bounds for q_F assuming that the functional equation of $F \in \mathcal{S}^*$ has the expected shape, i.e. if we can take all λ_j equal to $1/2$. In this case we have, as expected, $d_F \in \mathbb{N}$ and $F(s)$ has a γ-factor of the form

$$\gamma(s) = Q^s \prod_{j=1}^{d_F} \Gamma\left(\frac{s}{2} + \mu_j\right). \tag{1.4}$$

Note that if $F \in \mathcal{S}^*$ we expect that $\partial_p = d_F$ for almost all primes p. We refer to [8] and [11] for a discussion of these matters. We have

Theorem 3. *Let $F \in \mathcal{S}^*$ with $d_F > 0$ and suppose that $F(s)$ has a γ-factor of the form (1.4). Then there exists an absolute constant $c_1 > 0$ such that*

$$d_F H_F^* q_F^{1/d_F} \ge c_1.$$

Finally, we refer to Section 3 below for sharper results and computation of the above invariants in several special cases.

2. Proofs

Proof of Theorem 1. Since $H_F^\bullet \le H_F^*$, we prove that $H_F^* < \infty$ and $H_F^\bullet \ge 1/\pi d_F$. We refer to our paper [9] for several results needed in the proof. From (3) and (22) of Section 1.13 of the Bateman Project [3] we see that the Bernoulli polynomials $B_n(z)$ satisfy

$$|B_n(z)| \le \sum_{r=0}^{n} \binom{n}{r} |B_r|\,|z|^{n-r} \ll \frac{n!}{(2\pi)^n}(1+|z|)^n.$$

Therefore from the definition of the H-invariants we obtain

$$|H_F(n)| \ll \frac{n!}{(2\pi)^n} \sum_{j=1}^{r} \frac{(1+|\mu_j|)^n}{\lambda_j^{n-1}} \ll n! d_F \Big(\max_{j=1,\dots,r} \frac{1+|\mu_j|}{\lambda_j} \Big)^n, \tag{2.1}$$

hence $H_F^* < \infty$.

To prove the lower bound for $H_F^\bullet$, thanks to Theorem 2 of [9] we first write the γ-factor of $F(s)$ in the form

$$\gamma(s) = Q^s \prod_{j=1}^{h_F} \prod_{k=1}^{N_j} \Gamma(\lambda_j s + \mu_{j,k}),$$

where h_F is the γ-class number of $F(s)$ (see [8]), N_j are suitable positive integers, $\Re\mu_{j,k} \ge 0$ and different λ_j's are not $\mathbb{Q}$-equivalent (i.e. $\lambda_i/\lambda_j \notin \mathbb{Q}$ if $i \ne j$); note that these Q and λ_j are not necessarily equal to the Q and λ_j introduced in (iii) in Section 1. Then formula (2.3) of [9] becomes

$$\sum_{j=1}^{h_F} S_j(z) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{H_F(n)}{n!} z^n \tag{2.2}$$

with

$$S_j(z) = \frac{z}{e^{z/\lambda_j}-1} \sum_{k=1}^{N_j} e^{z\mu_{j,k}/\lambda_j} = \frac{z}{e^{z/\lambda_j}-1} \tilde{S}_j(z),$$

say. Note that $S_j(z)$ has poles at the points $z = 2\pi i m\lambda_j$ with $0 \ne m \in \mathbb{Z}$ such that $\tilde{S}_j(2\pi i m\lambda_j) \ne 0$. Denoting by m_j the integer $m \ne 0$ with smallest absolute value for which $\tilde{S}_j(2\pi i m\lambda_j) \ne 0$, we have $|m_j| \le N_j$. Indeed, if $\tilde{S}_j(2\pi i m\lambda_j) = 0$ for $m = 1,\dots,N_j$ (or $m = -1,\dots,-N_j$) then by (i) of Lemma 4.1 of [9] we deduce that $e^{2\pi i\mu_{j,k}} = 0$ for $k = 1,\dots,N_j$, a contradiction. Note also that the poles of distinct $S_j(z)$ are all distinct since the λ_j are not $\mathbb{Q}$-equivalent. Therefore, the left hand side of (2.2) is holomorphic in the disc

$$|z| < 2\pi \min_{1\le j\le h_F} |m_j|\lambda_j$$

and in no larger disc, and hence

$$H_F^\bullet = \limsup_{n\to\infty} \Big(\frac{|H_F(n)|}{n!} \Big)^{1/n} = \frac{1}{2\pi \min_{1\le j\le h_F} |m_j|\lambda_j}.$$

But $\min_{1\le j\le h_F}|m_j|\lambda_j \le \min_{1\le j\le h_F} N_j\lambda_j \le \sum_{j=1}^{h_F} N_j\lambda_j = d_F/2$, and the result follows. ∎

Proof of Theorem 2. We start with several preliminary lemmas. For $\tau,\mu\in\mathbb{C}$ we write

$$A(\tau)=\int_0^1 (e^{-\xi/\tau}-e^{-\xi})\,\frac{d\xi}{\xi}+\int_1^\infty e^{-\xi/\tau}\,\frac{d\xi}{\xi}$$

$$B(\tau,\mu)=\int_0^1\left(e^{-\xi}-\frac{\xi e^{-\xi(1+\mu/\tau)}}{\tau(1-e^{-\xi/\tau})}\right)\frac{d\xi}{\xi}$$

$$C(\tau,\mu)=\frac{1}{\tau}\int_1^\infty \frac{e^{-\xi(1+\mu/\tau)}}{1-e^{-\xi/\tau}}\,d\xi.$$

Clearly, $A(\tau)$ and $B(\tau,\mu)$ are absolutely convergent and holomorphic for $\Re\tau>0$, while $C(\tau,\mu)$ is absolutely convergent and holomorphic for $\Re\tau>0$ and $\Re\frac{\mu}{\tau}>-1$.

LEMMA 1. *For real $\tau>0$ we have*

$$A(\tau)=\log\tau+O(1)$$

with an absolute constant in the O-symbol.

Proof. Suppose first that $\tau\ge1$. Since $e^{-\xi/\tau}-e^{-\xi}=O(\xi)$ for $0\le\xi\le1$ we have

$$\int_0^1 (e^{-\xi/\tau}-e^{-\xi})\,\frac{d\xi}{\xi}\ll 1.$$

Moreover,

$$\begin{aligned}\int_1^\infty e^{-\xi/\tau}\,\frac{d\xi}{\xi}&=\int_1^\tau\frac{d\xi}{\xi}+\int_1^\tau (e^{-\xi/\tau}-1)\,\frac{d\xi}{\xi}+\int_\tau^\infty e^{-\xi/\tau}\,\frac{d\xi}{\xi}\\&=\log\tau+O\Big(\int_1^\tau\frac{\xi/\tau}{\xi}\,d\xi\Big)+O\Big(\int_1^\infty\frac{e^{-\xi}}{\xi}\,d\xi\Big)=\log\tau+O(1),\end{aligned}$$

as required. Let now $0<\tau<1$. Then

$$\int_1^\infty e^{-\xi/\tau}\,\frac{d\xi}{\xi}\le\int_1^\infty e^{-\xi}\,\frac{d\xi}{\xi}\ll 1.$$

Moreover, for $0<\xi<\tau$ we have $e^{-\xi/\tau}-e^{-\xi}=O(\xi/\tau)$, hence

$$\int_0^\tau (e^{-\xi/\tau}-e^{-\xi})\,\frac{d\xi}{\xi}\ll\frac{1}{\tau}\int_0^\tau d\xi\ll 1$$

and

$$\int_\tau^1 (e^{-\xi/\tau} - e^{-\xi})\,\frac{d\xi}{\xi} = \int_\tau^1 e^{-\xi/\tau}\,\frac{d\xi}{\xi} + \int_\tau^1 (1 - e^{-\xi})\,\frac{d\xi}{\xi} - \int_\tau^1 \frac{d\xi}{\xi} = O(1) + O(1) + \log\tau,$$

as required. ∎

Lemma 2. *For $\Re\tau > 0$, $|\tau| > 1/2\pi$ and $\mu \in \mathbb{C}$ we have*

$$B(\tau,\mu) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} B_n(\mu) c_n}{n!}\,\frac{1}{\tau^n} \quad \text{with} \quad c_n = \int_0^1 e^{-\xi}\xi^{n-1}\,d\xi.$$

Proof. By the substitution $\xi \mapsto \tau\xi$ we get

$$B(\tau,\mu) = \int_0^{1/\tau} \left(-\frac{\xi e^{-\xi(\tau+\mu)}}{1-e^{-\xi}} + e^{-\tau\xi} \right) \frac{d\xi}{\xi}. \tag{2.3}$$

By (2) and (12) of Sect. 1.13 of Bateman's Project [3], for $|\xi| \le 1/\tau$ $(< 2\pi)$ we have

$$\frac{\xi e^{-\xi\mu}}{1-e^{-\xi}} = \frac{\xi e^{\xi(1-\mu)}}{e^{\xi}-1} = \sum_{n=0}^{\infty} \frac{B_n(1-\mu)\xi^n}{n!} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n B_n(\mu)\xi^n}{n!}.$$

Hence the integral in (2.3) becomes

$$\int_0^{1/\tau} e^{-\tau\xi} \left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1} B_n(\mu)\xi^{n-1}}{n!} \right) d\xi = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} B_n(\mu)}{n!} \int_0^{1/\tau} e^{-\tau\xi}\xi^{n-1}\,d\xi,$$

and the result follows by the substitution $\xi \mapsto \xi/\tau$. ∎

Lemma 3. *For real $\tau > 0$ and $\Re\mu \ge 0$ we have*

$$C(\tau,\mu) = \frac{e^{-\mu/\tau}}{e(\tau+\mu)} + O(1)$$

with an absolute constant in the O-symbol.

Proof. We have

$$\begin{aligned}
\frac{1}{\tau}\int_1^{\infty} \frac{e^{-\xi(1+\mu/\tau)}}{1-e^{-\xi/\tau}}\,d\xi &= \frac{1}{\tau}\sum_{k=0}^{\infty}\int_1^{\infty} e^{-\xi(1+(\mu+k)/\tau)}\,d\xi = \frac{1}{\tau}\sum_{k=0}^{\infty} \frac{e^{-(1+(\mu+k)/\tau)}}{1+(\mu+k)/\tau} \\
&= \frac{e^{-\mu/\tau}}{e(\tau+\mu)} + O\left(\sum_{k=1}^{\infty} \frac{e^{-k/\tau}}{\tau+k} \right) \\
&= \frac{e^{-\mu/\tau}}{e(\tau+\mu)} + O\left(\frac{1}{\tau}\sum_{1\le k<\tau} 1 + \tau \sum_{k\ge\max(1,\tau)} \frac{1}{k^2} \right) \\
&= \frac{e^{-\mu/\tau}}{e(\tau+\mu)} + O(1),
\end{aligned}$$

as required. ∎

Lemma 4. *Let $c_2 > 0$ be the unique solution of the equation $\xi + \arctan \xi = \pi/2$. Then there exists an absolute constant $c_3 \geq 0$ such that for real $\tau > 0$ and $\mu \in \mathbb{C}$ satisfying $\Re\mu \geq 0$ and $|\Im\mu| \leq c_2\tau$ we have*

$$\Re C(\tau, \mu) \geq -c_3.$$

Proof. By Lemma 3 it suffices to show that $\Re\frac{e^{-\mu/\tau}}{\tau+\mu} \geq 0$ for τ and μ subject to the above conditions. But

$$\Re\frac{e^{-\mu/\tau}}{\tau+\mu} = \frac{1}{\tau}\frac{e^{-\Re\mu/\tau}}{|1+\mu/\tau|}\cos\left(\frac{|\Im\mu|}{\tau} + \arctan\frac{|\Im\mu/\tau|}{1+\Re\mu/\tau}\right) \geq 0$$

since

$$0 \leq \frac{|\Im\mu|}{\tau} + \arctan\frac{|\Im\mu/\tau|}{1+\Re\mu/\tau} \leq \frac{|\Im\mu|}{\tau} + \arctan\frac{|\Im\mu|}{\tau} \leq \frac{\pi}{2},$$

and the result follows at once. ∎

As usual we write

$$\psi(s) = \frac{\Gamma'}{\Gamma}(s).$$

Thanks to equation (17) of Section 1.7.2 of the Bateman Project [3], for $\lambda > 0$, $\Re\mu \geq 0$, real $s > 0$ and $\Re\frac{\mu}{\lambda s} > -1$ we have

$$\psi(\lambda s + \mu) = A(\lambda s) + B(\lambda s, \mu) - C(\lambda s, \mu). \tag{2.4}$$

Indeed, by a change of variable we see that

$$A(\lambda s) + B(\lambda s, \mu) - C(\lambda s, \mu) = \int_0^\infty \left(e^{-\xi} - \frac{\xi e^{-\xi(\lambda s+\mu)}}{1-e^{-\xi}}\right)\frac{d\xi}{\xi},$$

and the last integral equals $\psi(\lambda s + \mu)$ by the above-mentioned equation in [3].

Lemma 5. *Let $F \in \mathcal{S}^\sharp$ with $d_F > 0$ and c_2 be as in Lemma 4. Then for real $s \geq \max(2H_F^*, D_F/c_2)$ we have*

$$\Re\sum_{j=1}^r (\lambda_j\psi(\lambda_j s + \mu_j) - \lambda_j \log\lambda_j) \leq \frac{1}{2}d_F\log s + O(d_F)$$

with an absolute constant in the O-symbol.

Proof. By (2.4) we rewrite the left hand side as

$$\Re\sum_{j=1}^r(\lambda_j A(\lambda_j s) - \lambda_j\log\lambda_j) + \Re\sum_{j=1}^r \lambda_j B(\lambda_j s, \mu_j) - \Re\sum_{j=1}^r \lambda_j C(\lambda_j s, \mu_j)$$
$$= S_1 + S_2 - S_3,$$

say. Thanks to Lemma 1 we have

$$S_1 = \frac{1}{2}d_F\log s + O(d_F).$$

Moreover, for $s > \frac{1}{2\pi}\max_{1\le j\le r}\frac{1}{\lambda_j}$ from Lemma 2 and the definition of the $H_F(n)$'s we get

$$\sum_{j=1}^{r}\lambda_j B(\lambda_j s,\mu_j) = -\sum_{j=1}^{r}\lambda_j\sum_{n=1}^{\infty}\frac{(-1)^n B_n(\mu_j)c_n}{n!\lambda_j^n}\,\frac{1}{s^n}$$
$$= -\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n H_F(n)c_n}{n!}\,\frac{1}{s^n}.$$

Since $|c_n|\le 1$, the last series is certainly absolutely convergent for $s\ge 2H_F^*$ and we have

$$|S_2| \le \Big|\sum_{j=1}^{r}\lambda_j B(\lambda_j s,\mu_j)\Big| \ll \sum_{n=1}^{\infty}\left(\frac{H_F^*}{s}\right)^n \ll 1.$$

Finally, from Lemma 4, for $s\ge D_F/c_2$ we obtain

$$S_3 \ge -\frac{1}{2}d_F c_3,$$

and the result follows. ∎

Let m_F denote the order of the pole of $F(s)$ at $s=1$, with the convention that $-m_F$ is the order of zero if $F(1)=0$, and write

$$\xi(s) = s^{m_F}(1-s)^{m_F}Q^s\prod_{j=1}^{r}\Gamma(\lambda_j s+\mu_j)F(s).$$

Then $\xi(s)$ is entire and non-vanishing at $s=0$, $s=1$ and satisfies $\xi(s)=\omega\bar{\xi}(1-s)$.

Lemma 6. *For $F\in\mathcal{S}$ with $d_F>0$ and $\sigma>1$ we have*

$$\Re\frac{\xi'}{\xi}(s)\ge 0.$$

Proof. By Hadamard's theory we observe that

$$\xi_F(s) = e^{As+B}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho}$$

where $\rho=\beta+i\gamma$ runs over the zeros of $F(s)$, and hence

$$\frac{\xi_F'}{\xi_F}(s) = A+\sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right).$$

On the other hand the functional equation gives

$$\frac{\xi_F'}{\xi_F}(s) = -\overline{\frac{\xi_F'}{\xi_F}(1-\bar{s})},$$

therefore

$$A+\sum_\rho\Big(\frac{1}{s-\rho}+\frac{1}{\rho}\Big)=-\bar{A}-\sum_\rho\Big(\frac{1}{1-s-\bar{\rho}}+\frac{1}{\bar{\rho}}\Big).$$

Since ρ is a zero of $F(s)$ if and only if $1-\bar{\rho}$ is also a zero, the sums involving $s-\rho$ and $1-s-\bar{\rho}$ cancel, thus giving

$$\Re A=-\sum_\rho \Re\frac{1}{\rho}.$$

Consequently, for $\sigma>1$,

$$\Re\frac{\xi_F'}{\xi_F}(s)=\Re A+\sum_\rho\Big(\frac{\sigma-\beta}{|s-\rho|^2}+\Re\frac{1}{\rho}\Big)=\sum_\rho\frac{\sigma-\beta}{|s-\rho|^2}\ge 0,$$

and the result follows. ∎

The proof of Theorem 2 is now easy. From Lemma 6, for real $s>1$ we have

$$\frac{m_F}{s}+\frac{m_F}{s-1}+\log Q+\Re\sum_{j=1}^r\lambda_j\psi(\lambda_j s+\mu_j)+\Re\frac{F'}{F}(s)\ge 0.$$

Moreover, since $F\in\mathcal{S}^*$ with $d_F>0$, comparing with the Riemann zeta function we immediately see that $m_F\le d_F$ and $(F'/F)(s)=O(d_F)$ for $s\ge 2$. Hence, recalling the definition of q_F, for real $s\ge 2$ we get

$$\frac{1}{2}\log q_F+\Re\sum_{j=1}^r(\lambda_j\psi(\lambda_j s+\mu_j)-\lambda_j\log\lambda_j)+O(d_F)\ge 0.$$

With the notation of Lemma 5, choosing $s=\max(2H_F^*,D_F/c_2,2)$ and applying Lemma 5 we obtain

$$\frac{1}{2}\log q_F+\frac{1}{2}d_F\log s+O(d_F)\ge 0.$$

Hence for some constant $c_4>0$ we have

$$q_F^{1/d_F}s\ge c_4,$$

and now Theorem 2 follows immediately upon recalling the fact that $s=\max(2H_F^*,D_F/c_2,2)$. ∎

Proof of Theorem 3. Let $F\in\mathcal{S}^*$ with $d_F>0$ and suppose that $F(s)$ has a γ-factor of the form (1.4); in particular, $d_F\in\mathbb{N}$. We use again formula (2.3) of [9], which in this case reads

$$\frac{z}{e^{2z}-1}\sum_{j=1}^{d_F}e^{2z\mu_j}=\frac{1}{2}\sum_{n=0}^\infty\frac{H_F(n)}{n!}z^n. \tag{2.5}$$

The right hand side of (2.5) converges absolutely for $|z| \leq 1/2H_F^*$, and for such z we have

$$\sum_{n=0}^{\infty} \frac{|H_F(n)|}{n!}|z|^n \leq d_F + \sum_{n=1}^{\infty} \frac{1}{2^n} = d_F + 1. \tag{2.6}$$

Choosing $z = z_l^{\pm} = \pm il/4H_F^* d_F$ with $d_F \leq l \leq 2d_F$ we have $|z_l^{\pm}| \leq 1/2H_F^*$ and hence from (2.5) and (2.6) we obtain

$$\Big|\sum_{j=1}^{d_F} (e^{\pm i\mu_j/2H_F^* d_F})^l\Big| \leq \left|\frac{e^{2z_l^{\pm}} - 1}{z_l^{\pm}}\right| \frac{1}{2}(d_F + 1) \ll d_F + 1 \tag{2.7}$$

for every $d_F \leq l \leq 2d_F$. By Turán's Second Main Theorem (see Theorem 8.1 of Turán [16] with $b_j = 1$, $m = 0$ and $n = d_F$), there exists l_0 in the above range such that the left hand side of (2.7) is

$$\geq \max_{1\leq j\leq r} |e^{\pm i\mu_j/2H_F^* d_F}|^{l_0} c_5^{-d_F} \tag{2.8}$$

with a suitable absolute constant $c_5 > 0$. Note that the max in (2.8) is due to a normalization in the above-cited Theorem 8.1. Recalling the definition of D_F, since $\lambda_j = 1/2$ for every j we can choose the signs $\pm$ in such a way that

$$\max_{1\leq j\leq r} |e^{\pm i\mu_j/2H_F^* d_F}| = e^{D_F/4H_F^* d_F}. \tag{2.9}$$

Therefore, from (2.7)–(2.9) we deduce that

$$e^{D_F l_0/4H_F^* d_F} \ll c_5^{d_F}(d_F + 1) \leq c_6^{d_F}$$

with some absolute constant $c_6 > 0$, and hence

$$D_F \ll H_F^* d_F.$$

The result now follows from Theorems 1 and 2. ∎

3. Special cases. In this section we collect further results and problems, and consider several special cases. Suppose first that the γ-factor of $F \in \mathcal{S}^\sharp$ has the form

$$\gamma(s) = Q^s \Gamma(\lambda s + \mu)^m, \qquad m \in \mathbb{N}.$$

Specializing formula (2.3) of [9] and arguing similarly to (2.5)–(2.7) above, choosing $z = \varepsilon i/2H_F^*$ with $\varepsilon = -\operatorname{sgn} \Im\mu$ we obtain

$$e^{D_F/2H_F^*} \leq 2\frac{H_F^*}{m}(d_F + 1) \leq 2H_F^*(d_F + 1),$$

and hence

$$(1 + d_F)H_F^* \geq \frac{1}{2} \quad \text{and} \quad D_F \leq 2H_F^* \log(2H_F^*(d_F + 1)). \tag{3.1}$$

If, in addition, we have $\lambda \le 1$ then $(d_F+1)/m \le (2m+1)/m \le 3$, thus $e^{D_F/2H_F^*} \le 6H_F^*$ and therefore

$$H_F^* \ge \frac{1}{6} \quad \text{and} \quad D_F \le 2H_F^* \log(6H_F^*).$$

Suppose now that the γ-factor of $F(s)$ has the form

$$\gamma(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda s + \mu_j), \qquad \mu_j = i\kappa_j \text{ with } \kappa_j \in \mathbb{R}.$$

By an analogous argument, with $\varepsilon = \pm 1$, we obtain

$$\frac{1}{2H_F^*}\Big|\sum_{j=1}^{r} e^{-\varepsilon\kappa_j/2\lambda H_F^*}\Big| \le d_F + 1.$$

Choosing $\varepsilon = -\operatorname{sgn}\max_j \kappa_j$ we obtain

$$e^{D_F/2H_F^*} \le 2(d_F+1)H_F^*,$$

and inequalities (3.1) follow in this case as well.

A subset $\mathcal{F}$ of S^* is called an *H-family* if for every $F, G \in \mathcal{F}$ we have $H_F(n) = H_G(n)$ for all $n \ge 0$. For example, the set of the Dedekind zeta functions associated with all fields with given signature (r_1, r_2) is an H-family. We have

COROLLARY. *Given an H-family $\mathcal{F}$ there exists a constant $c(\mathcal{F}) > 0$ such that for every $F \in \mathcal{F}$,*

$$q_F \ge c(\mathcal{F}).$$

Proof. Clearly, H_F^* and d_F are constant for $F \in \mathcal{F}$. Let $F, G \in \mathcal{F}$ and $\gamma(s)$, $\gamma'(s)$ be γ-factors of $F(s)$ and $G(s)$, respectively. From p. 99 of [9] we know that $\gamma(s)$ and $\gamma'(s)$ have the same poles. But the poles ρ of $\gamma(s)$ (resp. $\gamma'(s)$) coincide, apart possibly from $\rho = 0$, with the trivial zeros of $F(s)$ (resp. $G(s)$), therefore

$$D_F = \max_{\rho} |\Im\rho| = D_G.$$

Hence D_F is also constant, and the result follows from Theorem 2. ∎

Now we compute $H_F^\bullet$ and H_F^* for $F(s) = \zeta(s)^k$ ($k \ge 1$ integer) and $\zeta_K(s)$. Recalling the definition of $H_F(n)$, for $n \ge 0$ we have

$$H_\zeta(n) = 2^n B_n(0) = 2^n B_n = \begin{cases} (-1)^{n/2-1}2^{n+1}n!(2\pi)^{-n}\zeta(n), & 2 \mid n, \\ 0, & 2 \nmid n,\ n > 1, \\ 1, & n = 1. \end{cases}$$

Writing the even n as $n = 2m \ge 2$ we get

$$\left(\frac{|H_\zeta(2m)|}{(2m)!}\right)^{1/2m} = \frac{(2\zeta(2m))^{1/2m}}{\pi},$$

hence

$$H_\zeta^\bullet = \limsup_{m\to\infty} \frac{(2\zeta(2m))^{1/2m}}{\pi} = \frac{1}{\pi}.$$

By Theorem 1 we have $d_F H_F^\bullet \geq 1/\pi$, therefore the infimum of $d_F H_F^\bullet$ for $F \in \mathcal{S}^\sharp$ is $H_\zeta^\bullet$. Moreover,

$$(2\zeta(2m))^{1/2m} = \exp\left(\frac{1}{2m}\sum_p\sum_{k=1}^\infty \frac{1}{kp^{2mk}} + \frac{\log 2}{2m}\right) \leq (2\zeta(2))^{1/2} = \frac{\pi}{\sqrt{3}}$$

since the argument of the exponential is decreasing in m, and $|H_\zeta(1)| = |2B_1| = 1$. Hence

$$H_\zeta^* = 1.$$

Since for integers $n, k \geq 1$ we have

$$H_{\zeta^k}(n) = kH_\zeta(n),$$

the above results also give

$$H_{\zeta^k}^\bullet = \frac{1}{\pi}, \qquad H_{\zeta^k}^* = k.$$

Consider now $\zeta_K(s)$ with K of signature (r_1, r_2). We have

$$H_{\zeta_K}(n) = (r_1 2^n + 2r_2)B_n,$$

and writing $n = 2m$ we deduce as before that

$$\left(\frac{|H_{\zeta_K}(2m)|}{(2m)!}\right)^{1/2m} = (r_1 2^{2m} + 2r_2)^{1/2m}\frac{(2\zeta(2m))^{1/2m}}{2\pi},$$

hence

$$H_{\zeta_K}^\bullet = \frac{1}{\pi}.$$

Moreover,

$$\left(\frac{|H_{\zeta_K}(2m)|}{(2m)!}\right)^{1/2m} \leq \frac{(4r_1 + 2r_2)^{1/2}}{2\pi}(2\zeta(2))^{1/2} = \sqrt{\frac{2r_1 + r_2}{6}}$$

since $(a2^\xi + b)^{1/\xi}$ is decreasing. Finally, $H_{\zeta_K}(1) = r_1 + r_2$ and therefore

$$H_{\zeta_K}^* = r_1 + r_2.$$

We conclude with two problems.

Problem 1. Does there exist a function Φ such that

$$d_F \leq \Phi(H_F^*)$$

(i.e. the degree is controlled by the invariant H_F^*)? We believe that there exists an absolute constant $c > 0$ such that $d_F \ll (H_F^*)^c$, or even $d_F \ll (H_F^*)^{1+\varepsilon}$.

Problem 2. Is it true that the infimum of $d_F H_F^*$ is H_ζ^*? Here the infimum is over $\mathcal{S}^\sharp$, or $\mathcal{S}$, or $\mathcal{S}^*$. We know that this is true with $H_F^\bullet$ in place of H_F^*.

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