Lower bounds for the conductor of $L$-functions

by

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Dedicated to Professor A. Schinzel on the occasion of his 75th birthday

1. Introduction. Our aim in this paper is to obtain lower bounds for the conductor of $L$-functions in a general setting. We therefore start with the definition of our framework, i.e. the Selberg class $S$ of $L$-functions: $F \in S$ if

(i) $F(s)$ is an absolutely convergent Dirichlet series for $\sigma > 1$,

\[ F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}; \]

(ii) $(s - 1)^m F(s)$ is an entire function of finite order for some $m \in \mathbb{N}$;

(iii) $F(s)$ satisfies a functional equation of the type $\Phi(s) = \omega \bar{\Phi}(1 - s)$, where $|\omega| = 1$, $\bar{f}(s) = f(\bar{s})$ and

\[ \Phi(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s) = \gamma(s) F(s), \]

say, with $Q > 0$, $r \geq 0$, $\lambda_j > 0$ and $\Re \mu_j \geq 0$;

(iv) $a(n) \ll n^\varepsilon$ for every $\varepsilon > 0$;

(v) $\log F(s)$ is a Dirichlet series with coefficients $b(n)$ satisfying $b(n) = 0$ unless $n = p^k$ with $p$ prime and $k \geq 1$, and $b(n) \ll n^\vartheta$ for some $\vartheta < 1/2$.

The extended Selberg class $S^\#$ is the larger class of the functions $F(s)$ satisfying (i)–(iii) above.

We refer to Selberg [15], Conrey–Ghosh [2] and to our survey papers [7], [4], [12], [13], [14] for a discussion of the basic properties of $S$ and $S^\#$. Here

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we recall that (v) implies an Euler product expansion of general type, i.e.

\begin{equation}
F(s) = \prod_p F_p(s), \quad F_p(s) = \sum_{m=0}^{\infty} \frac{a(p^m)}{p^{ms}}.
\end{equation}

Moreover, the degree and the conductor of \( F \in S^\# \) are defined respectively by

\[ d_F = 2 \sum_{j=1}^{r} \lambda_j, \quad q_F = (2\pi)^{d_F} Q^2 \prod_{j=1}^{r} \lambda_j^{2\lambda_j}. \]

Note that the real numbers \( d_F \) and \( q_F \) are invariants, i.e. they depend only on \( F(s) \) and not on the shape of the functional equation (which may be changed by means of suitable formulae for the \( \Gamma \) function). We refer to \([8]\) and \([9]\) for the invariant theory of the Selberg class. Note also that in the case of classical \( L \)-functions, the conductor \( q_F \) coincides with well known objects associated with the underlying structure of the \( L \)-functions. For example, \( q_F \) is the conductor of the primitive Dirichlet character \( \chi \) if \( F(s) = L(s, \chi) \), the level of the normalized newform \( f(z) \) if \( F(s) = L_f(s) \), the absolute value of the discriminant of the number field \( K \) if \( F(s) = \zeta_K(s) \), and so on.

From now on we assume that \( d_F > 0 \). Indeed, the structure of the degree 0 functions from both \( S \) and \( S^\# \) is quite well understood (see Conrey–Ghosh \([2]\) and, e.g., Kaczorowski–Molteni–Perelli \([5]\)); in this case sharper results than those presented below are easily obtained.

It turns out that \( q_F \in \mathbb{N} \) when \( F(s) \) is a classical \( L \)-function, and we expect that \( q_F \in \mathbb{N} \) for every \( F \in S \). This is mainly based on the expectation that \( S \) coincides with the class of automorphic \( L \)-functions. However, at present the classification of \( S \) is far from being complete, and the question if \( q_F \in \mathbb{N} \) is an interesting open problem. As usual, the situation is more complicated for \( S^\# \). Indeed, in this case \( q_F \) does not need to be an integer, as one can see from Hecke’s theory of (suitably normalized) Dirichlet series associated with \( G(\lambda) \)-modular forms; see e.g. Berndt–Knopp \([1]\). In fact, \( q_F = \lambda^2 \) if \( F(s) \) comes from \( G(\lambda) \). Nevertheless, we still expect a universal lower bound, say \( q_F \geq c_0 > 0 \), for all \( F \in S^\# \). Actually, since \( S^\# \) is a multiplicative semigroup and \( q_{FG} = q_F q_G \), if such a \( c_0 \) exists then \( c_0 = 1 \). We wish to thank Brian Conrey for pointing out that in the Hecke theory case, although a priori conductors can be arbitrary positive numbers, the spaces of modular forms are trivial when the conductor is \(< 1 \). We further note that the situation changes completely if generalized Dirichlet series are allowed. Indeed, in this case \( q_F \) can be arbitrarily small; see \([10]\).

In order to state our results we first have to introduce and discuss several interesting invariants; again, we refer to \([8]\) and \([9]\) for a full account. For any integer \( n \geq 0 \) let \( B_n(z) \) denote the \( n \)th Bernoulli polynomial. The
$H$-invariants of $F \in \mathcal{S}^\sharp$ are defined as

$$H_F(n) = 2 \sum_{j=1}^{r} \frac{B_n(\mu_j)}{\lambda_j^{n-1}}, \quad n = 0, 1, \ldots.$$ 

The interest of the $H$-invariants comes from the fact that if $F, G \in \mathcal{S}^\sharp$ have the same conductor, root number (see e.g. [9] for its definition) and all $H$-invariants, then they satisfy the same functional equation. Moreover, $H_F(0) = d_F$ and $H_{FG}(n) = H_F(n) + H_G(n)$. Another interesting invariant is the meromorphic function

$$K_F(z) = z \sum_{j=1}^{r} e^{z\mu_j/\lambda_j} e^{z/\lambda_j - 1}.$$ 

$K_F(z)$ is related both to the $H$-invariants and to the poles $\rho$ of the $\gamma$-factor $\gamma(s)$ in the functional equation of $F(s)$, thanks to the following expressions (valid in suitable regions of $\mathbb{C}$):

$$K_F(z) = 1 \frac{1}{2} \sum_{n=0}^{\infty} \frac{H_F(n)}{n!} z^n = -z \sum_{\rho} e^{-\rho z},$$

where $\rho$ runs over such poles.

Now we define three new invariants. For $F \in \mathcal{S}^\sharp$ let

$$H^*_F = \sup_{n \geq 1} \left( \frac{|H_F(n)|}{n!} \right)^{1/n}, \quad H^*_F = \limsup_{n \to \infty} \left( \frac{|H_F(n)|}{n!} \right)^{1/n}$$

and, if $d_F > 0$,

$$D_F = \max_{j=1, \ldots, r} \frac{|\Im \mu_j|}{\lambda_j}.$$ 

Clearly, $H^*_F$ and $H^*_F$ are invariants, and $H^*_F \geq H^*_F$. Moreover, $D_F$ is an invariant since

$$D_F = \max_{\rho} |\Im \rho|,$$

where $\rho$ runs over the trivial zeros of $F(s)$. We have

**Theorem 1.** Let $F \in \mathcal{S}^\sharp$ with $d_F > 0$. Then

$$\frac{1}{\pi d_F} \leq H^*_F \leq H^*_F < \infty.$$ 

From the proof of Theorem 1 (see (2.1) below), and the fact that $d_F \geq 1$ if $d_F \neq 0$ (see [6]), we have the upper bound

$$H^*_F \ll d_F \max_{j=1, \ldots, r} \frac{1 + |\mu_j|}{\lambda_j}.$$
It is expected that every $F \in \mathcal{S}$ has an Euler product of polynomial type, i.e. for every prime $p$ the shape of $F_p(s)$ in (1.1) is
\begin{equation}
F_p(s) = \prod_{j=1}^{\partial_p} \left( 1 - \frac{\alpha_{j,p}}{p^s} \right)^{-1}
\end{equation}
with $|\alpha_{j,p}| \leq 1$ and $\partial_p \leq d_F$. Our main result holds for functions satisfying (1.3); therefore we denote by $\mathcal{S}^*$ the subclass of $\mathcal{S}$ of the functions satisfying (1.3), and hence conjecturally $\mathcal{S}^* = \mathcal{S}$. We have

**Theorem 2.** Let $F \in \mathcal{S}^*$ with $d_F > 0$. Then there exists an absolute constant $c_0 > 0$ such that
\begin{equation}
(1 + H_F^* + D_F)q_F^{1/d_F} \geq c_0.
\end{equation}

In accordance with a previous remark, the bound in Theorem 2 does not hold if generalized Dirichlet series are allowed. Theorem 2 provides at once a lower bound for $q_F$ in terms of the other invariants $d_F$, $H_F^*$, and $D_F$, hence relations between $H_F^*$ and $D_F$ would be of interest. For example, is it true that something like
\begin{equation}
D_F \ll H_F^*
\end{equation}
holds? From (1.2) we see that if the $\mu_j$ are pure imaginary with modulus, say, $\geq 1$, then $H_F^* \ll d_F D_F$. On the other hand, $D_F = 0$ if the $\mu_j$ are all real, and hence $H_F^* \ll D_F$ certainly does not hold in general.

We can avoid the invariant $D_F$ in lower bounds for $q_F$ assuming that the functional equation of $F \in \mathcal{S}^*$ has the expected shape, i.e. if we can take all $\lambda_j$ equal to $1/2$. In this case we have, as expected, $d_F \in \mathbb{N}$ and $F(s)$ has a $\gamma$-factor of the form
\begin{equation}
\gamma(s) = Q^s \prod_{j=1}^{d_F} \Gamma \left( \frac{s}{2} + \mu_j \right).
\end{equation}
Note that if $F \in \mathcal{S}^*$ we expect that $\partial_p = d_F$ for almost all primes $p$. We refer to [8] and [11] for a discussion of these matters. We have

**Theorem 3.** Let $F \in \mathcal{S}^*$ with $d_F > 0$ and suppose that $F(s)$ has a $\gamma$-factor of the form (1.4). Then there exists an absolute constant $c_1 > 0$ such that
\begin{equation}
d_F H_F^* q_F^{1/d_F} \geq c_1.
\end{equation}

Finally, we refer to Section 3 below for sharper results and computation of the above invariants in several special cases.

2. Proofs

**Proof of Theorem 1.** Since $H_F^* \leq H_F^*$, we prove that $H_F^* < \infty$ and $H_F^* \geq 1/\pi d_F$. We refer to our paper [9] for several results needed in the
proof. From (3) and (22) of Section 1.13 of the Bateman Project [3] we see that the Bernoulli polynomials $B_n(z)$ satisfy

$$|B_n(z)| \leq \sum_{r=0}^{n} \binom{n}{r} |B_r| |z|^{n-r} \ll \frac{n!}{(2\pi)^n} (1 + |z|)^n.$$  

Therefore from the definition of the $H$-invariants we obtain

$$(2.1) \quad |H_F(n)| \ll \frac{n!}{(2\pi)^n} \sum_{j=1}^{r} \frac{(1 + |\mu_j|)^n}{\lambda_{j}^{n-1}} \ll n! d_F \left( \max_{j=1,\ldots,r} \frac{1 + |\mu_j|}{\lambda_j} \right)^n,$$

hence $H_F^* \ll \infty$.

To prove the lower bound for $H_F^*$, thanks to Theorem 2 of [9] we first write the $\gamma$-factor of $F(s)$ in the form

$$\gamma(s) = Q^s \prod_{j=1}^{h_F} \prod_{k=1}^{N_j} \Gamma(\lambda_j s + \mu_{j,k}),$$

where $h_F$ is the $\gamma$-class number of $F(s)$ (see [8]), $N_j$ are suitable positive integers, $\Re \mu_{j,k} \geq 0$ and different $\lambda_j$'s are not $\mathbb{Q}$-equivalent (i.e. $\lambda_i/\lambda_j \notin \mathbb{Q}$ if $i \neq j$); note that these $Q$ and $\lambda_j$ are not necessarily equal to the $Q$ and $\lambda_j$ introduced in (iii) in Section 1. Then formula (2.3) of [9] becomes

$$(2.2) \quad \sum_{j=1}^{h_F} S_j(z) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{H_F(n)}{n!} z^n$$

with

$$S_j(z) = \frac{z}{e^{z/\lambda_j} - 1} \sum_{k=1}^{N_j} e^{z\mu_{j,k}/\lambda_j} = \frac{z}{e^{z/\lambda_j} - 1} \tilde{S}_j(z),$$

say. Note that $S_j(z)$ has poles at the points $z = 2\pi i m \lambda_j$ with $0 \neq m \in \mathbb{Z}$ such that $\tilde{S}_j(2\pi i m \lambda_j) \neq 0$. Denoting by $m_j$ the integer $m \neq 0$ with smallest absolute value for which $\tilde{S}_j(2\pi i m \lambda_j) \neq 0$, we have $|m_j| \leq N_j$. Indeed, if $\tilde{S}_j(2\pi i m \lambda_j) = 0$ for $m = 1,\ldots,N_j$ (or $m = -1,\ldots,-N_j$) then by (i) of Lemma 4.1 of [9] we deduce that $e^{2\pi i \mu_{j,k}} = 0$ for $k = 1,\ldots,N_j$, a contradiction. Note also that the poles of distinct $S_j(z)$ are all distinct since the $\lambda_j$ are not $\mathbb{Q}$-equivalent. Therefore, the left hand side of (2.2) is holomorphic in the disc

$$|z| < 2\pi \min_{1 \leq j \leq h_F} |m_j| \lambda_j$$

and in no larger disc, and hence

$$H_F^* = \limsup_{n \to \infty} \left( \frac{|H_F(n)|}{n!} \right)^{1/n} = \frac{1}{2\pi \min_{1 \leq j \leq h_F} |m_j| \lambda_j}.$$
But \( \min_{1 \leq j \leq h_F} |m_j| \lambda_j \leq \min_{1 \leq j \leq h_F} N_j \lambda_j \leq \sum_{j=1}^{h_F} N_j \lambda_j = d_F/2 \), and the result follows.

Proof of Theorem 2. We start with several preliminary lemmas. For \( \tau, \mu \in \mathbb{C} \) we write

\[
A(\tau) = \int_0^1 \left( e^{-\xi/\tau} - e^{-\xi} \right) \frac{d\xi}{\xi} + \int_1^\infty e^{-\xi/\tau} \frac{d\xi}{\xi}
\]

\[
B(\tau, \mu) = \int_0^1 \left( e^{-\xi} - \frac{\xi e^{-\xi(1+\mu/\tau)}}{\tau(1-e^{-\xi/\tau})} \right) \frac{d\xi}{\xi}
\]

\[
C(\tau, \mu) = \int_1^\infty \frac{\xi e^{-\xi(1+\mu/\tau)}}{1-e^{-\xi/\tau}} d\xi.
\]

Clearly, \( A(\tau) \) and \( B(\tau, \mu) \) are absolutely convergent and holomorphic for \( \Re \tau > 0 \), while \( C(\tau, \mu) \) is absolutely convergent and holomorphic for \( \Re \tau > 0 \) and \( \Re \frac{\mu}{\tau} > -1 \).

**Lemma 1.** For real \( \tau > 0 \) we have

\[ A(\tau) = \log \tau + O(1) \]

with an absolute constant in the \( O \)-symbol.

**Proof.** Suppose first that \( \tau \geq 1 \). Since \( e^{-\xi/\tau} - e^{-\xi} = O(\xi) \) for \( 0 \leq \xi \leq 1 \) we have

\[
\int_0^1 (e^{-\xi/\tau} - e^{-\xi}) \frac{d\xi}{\xi} \ll 1.
\]

Moreover,

\[
\int_1^\infty e^{-\xi/\tau} \frac{d\xi}{\xi} = \int_1^\tau \frac{d\xi}{\xi} + \int_1^\infty (e^{-\xi/\tau} - 1) \frac{d\xi}{\xi} + \int_{\tau}^\infty e^{-\xi/\tau} \frac{d\xi}{\xi}
\]

\[
= \log \tau + O\left( \int_1^\tau \frac{\xi}{\xi} d\xi \right) + O\left( \int_1^\infty \frac{e^{-\xi}}{\xi} d\xi \right) = \log \tau + O(1),
\]

as required. Let now \( 0 < \tau < 1 \). Then

\[
\int_1^\infty e^{-\xi/\tau} \frac{d\xi}{\xi} \leq \int_1^\infty e^{-\xi} \frac{d\xi}{\xi} \ll 1.
\]

Moreover, for \( 0 < \xi < \tau \) we have \( e^{-\xi/\tau} - e^{-\xi} = O(\xi/\tau) \), hence

\[
\int_0^\tau (e^{-\xi/\tau} - e^{-\xi}) \frac{d\xi}{\xi} \ll \frac{1}{\tau} \int_0^\tau d\xi \ll 1.
\]
and
\[
\int_{\tau}^{1} \frac{e^{-\xi/\tau} - e^{-\xi}}{\tau} \frac{d\xi}{\xi} = \int_{\tau}^{1} e^{-\xi/\tau} \frac{d\xi}{\xi} + \int_{\tau}^{1} (1 - e^{-\xi}) \frac{d\xi}{\xi} - \int_{\tau}^{1} \frac{d\xi}{\xi} = O(1) + O(1) + \log \tau,
\]
as required. ■

**Lemma 2.** For \( \Re \tau > 0, |\tau| > 1/2\pi \) and \( \mu \in \mathbb{C} \) we have
\[
B(\tau, \mu) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} B_n(\mu) c_n}{n!} \frac{1}{\tau^n} \quad \text{with} \quad c_n = \int_{0}^{1} e^{-\xi} \xi^{n-1} d\xi.
\]

**Proof.** By the substitution \( \xi \mapsto \tau \xi \) we get
(2.3) \[
B(\tau, \mu) = \int_{0}^{1/\tau} \left( \frac{-\xi e^{-\xi(\tau+\mu)}}{1 - e^{-\xi}} + e^{-\tau \xi} \right) \frac{d\xi}{\xi}.
\]
By (2) and (12) of Sect. 1.13 of Bateman’s Project [3], for \( |\xi| \leq 1/\tau \) \((< 2\pi)\) we have
\[
\xi e^{-\xi \mu} = \xi e^{\xi(1-\mu)} = \sum_{n=0}^{\infty} B_n(1-\mu) \xi^n = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n B_n(\mu) \xi^n}{n!}.
\]
Hence the integral in (2.3) becomes
\[
\int_{0}^{1/\tau} e^{-\tau \xi} \left( \sum_{n=1}^{\infty} \frac{(-1)^{n+1} B_n(\mu) \xi^{n-1}}{n!} \right) d\xi = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} B_n(\mu)^{1/\tau}}{n!} \int_{0}^{1} e^{-\xi} \xi^{n-1} d\xi,
\]
and the result follows by the substitution \( \xi \mapsto \xi/\tau \). ■

**Lemma 3.** For real \( \tau > 0 \) and \( \Re \mu \geq 0 \) we have
\[
C(\tau, \mu) = \frac{e^{-\mu/\tau}}{e(\tau + \mu)} + O(1)
\]
with an absolute constant in the \( O \)-symbol.

**Proof.** We have
\[
\frac{1}{\tau} \int_{1}^{\infty} \frac{e^{-\xi(1+\mu/\tau)}}{1 - e^{-\xi/\tau}} d\xi = \frac{1}{\tau} \sum_{k=0}^{\infty} \int_{1}^{\infty} e^{-\xi(1+(\mu+k)/\tau)} d\xi = \frac{1}{\tau} \sum_{k=0}^{\infty} \frac{e^{-(1+(\mu+k)/\tau)}}{1 + (\mu + k)/\tau}
\]
\[
= \frac{e^{-\mu/\tau}}{e(\tau + \mu)} + O \left( \sum_{k=1}^{\infty} \frac{e^{-k/\tau}}{\tau + k} \right)
\]
\[
= \frac{e^{-\mu/\tau}}{e(\tau + \mu)} + O \left( \frac{1}{\tau} \sum_{1 \leq k < \tau} 1 + \tau \sum_{k \geq \max(1, \tau)} \frac{1}{k^2} \right)
\]
\[
= \frac{e^{-\mu/\tau}}{e(\tau + \mu)} + O(1),
\]
as required. ■
Lemma 4. Let $c_2 > 0$ be the unique solution of the equation $\xi + \arctan \xi = \pi/2$. Then there exists an absolute constant $c_3 \geq 0$ such that for real $\tau > 0$ and $\mu \in \mathbb{C}$ satisfying $\Re \mu \geq 0$ and $|\Im \mu| \leq c_2 \tau$ we have
\[ \Re C(\tau, \mu) \geq -c_3. \]

Proof. By Lemma 3 it suffices to show that $\Re \frac{e^{-\mu/\tau} - \mu/\tau}{\tau + \mu} \geq 0$ for $\tau$ and $\mu$ subject to the above conditions. But
\[
\Re \frac{e^{-\mu/\tau} - \mu/\tau}{\tau + \mu} = \frac{1}{\tau} \frac{e^{-|\Im \mu|/\tau}}{|1 + \mu/\tau|} \cos \left( \frac{|\Im \mu|}{\tau} + \arctan \frac{|\Im \mu/\tau|}{1 + \Re \mu/\tau} \right) \geq 0
\]
since
\[
0 \leq \frac{|\Im \mu|}{\tau} + \arctan \frac{|\Im \mu/\tau|}{1 + \Re \mu/\tau} \leq \frac{|\Im \mu|}{\tau} + \arctan \frac{|\Im \mu|}{\tau} \leq \frac{\pi}{2},
\]
and the result follows at once.

As usual we write
\[ \psi(s) = \frac{\Gamma'}{\Gamma}(s). \]

Thanks to equation (17) of Section 1.7.2 of the Bateman Project [3], for $\lambda > 0$, $\Re \mu \geq 0$, real $s > 0$ and $\Re \frac{\mu}{\lambda s} > -1$ we have
\begin{equation}
\psi(\lambda s + \mu) = A(\lambda s) + B(\lambda s, \mu) - C(\lambda s, \mu).
\end{equation}

Indeed, by a change of variable we see that
\[ A(\lambda s) + B(\lambda s, \mu) - C(\lambda s, \mu) = \int_0^\infty \left( e^{-\xi} - \frac{\xi e^{-(\lambda s + \mu)}}{1 - e^{-\xi}} \right) \frac{d\xi}{\xi}, \]
and the last integral equals $\psi(\lambda s + \mu)$ by the above-mentioned equation in [3].

Lemma 5. Let $F \in S^\#$ with $d_F > 0$ and $c_2$ be as in Lemma 4. Then for real $s \geq \max(2H^*_F, D_F/c_2)$ we have
\[
\Re \sum_{j=1}^r (\lambda_j \psi(\lambda_j s + \mu_j) - \lambda_j \log \lambda_j) \leq \frac{1}{2} d_F \log s + O(d_F)
\]
with an absolute constant in the $O$-symbol.

Proof. By (2.4) we rewrite the left hand side as
\[
\Re \sum_{j=1}^r (\lambda_j A(\lambda_j s) - \lambda_j \log \lambda_j) + \Re \sum_{j=1}^r \lambda_j B(\lambda_j s, \mu_j) - \Re \sum_{j=1}^r \lambda_j C(\lambda_j s, \mu_j)
\]
\[ = S_1 + S_2 - S_3, \]
say. Thanks to Lemma 1 we have
\[ S_1 = \frac{1}{2} d_F \log s + O(d_F). \]
Moreover, for $s > \frac{1}{2\pi} \max_{1 \leq j \leq r} \frac{1}{\lambda_j}$ from Lemma 2 and the definition of the $H_F(n)$’s we get
\[ \sum_{j=1}^{r} \lambda_j B(\lambda_j s, \mu_j) = -\sum_{j=1}^{r} \lambda_j \sum_{n=1}^{\infty} \frac{(-1)^n B_n(\mu_j) c_n}{n! \lambda_j^n} \frac{1}{s^n} \]
\[ = -\frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n H_F(n) c_n}{n!} \frac{1}{s^n}. \]
Since $|c_n| \leq 1$, the last series is certainly absolutely convergent for $s \geq 2H_F^*$ and we have
\[ |S_2| \leq \left| \sum_{j=1}^{r} \lambda_j B(\lambda_j s, \mu_j) \right| \ll \sum_{n=1}^{\infty} \left( \frac{H_F^*}{s} \right)^n \ll 1. \]
Finally, from Lemma 4, for $s \geq D_F/c_2$ we obtain
\[ S_3 \geq -\frac{1}{2} d_F c_3, \]
and the result follows. ■

Let $m_F$ denote the order of the pole of $F(s)$ at $s = 1$, with the convention that $-m_F$ is the order of zero if $F(1) = 0$, and write
\[ \xi(s) = s^{m_F} (1-s)^{m_F} Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s). \]
Then $\xi(s)$ is entire and non-vanishing at $s = 0$, $s = 1$ and satisfies $\xi(s) = \omega \overline{\xi}(1-s)$.

**Lemma 6.** For $F \in S$ with $d_F > 0$ and $\sigma > 1$ we have
\[ \Re \frac{\xi'}{\xi}(s) \geq 0. \]

**Proof.** By Hadamard’s theory we observe that
\[ \xi_F(s) = e^{As+B} \prod_{\rho} \left( 1 - \frac{s}{\rho} \right) e^{s/\rho}, \]
where $\rho = \beta + i\gamma$ runs over the zeros of $F(s)$, and hence
\[ \frac{\xi_F'}{\xi_F}(s) = A + \sum_{\rho} \left( \frac{1}{s-\rho} + \frac{1}{\rho} \right). \]
On the other hand the functional equation gives
\[ \frac{\xi_F'}{\xi_F}(s) = -\frac{\xi_F'}{\xi_F}(1-\bar{s}), \]
therefore
\[ A + \sum_{\rho} \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right) = -\bar{A} - \sum_{\rho} \left( \frac{1}{1 - s - \rho} + \frac{1}{\rho} \right). \]

Since \( \rho \) is a zero of \( F(s) \) if and only if \( 1 - \bar{\rho} \) is also a zero, the sums involving \( s - \rho \) and \( 1 - s - \bar{\rho} \) cancel, thus giving
\[ \Re A = - \sum_{\rho} \frac{1}{\rho}. \]

Consequently, for \( \sigma > 1 \),
\[ \Re \xi_F'(s) F(s) = \Re A + \sum_{\rho} \left( \frac{\sigma - \beta}{|s - \rho|^2} + \Re \frac{1}{\rho} \right) = \sum_{\rho} \frac{\sigma - \beta}{|s - \rho|^2} \geq 0, \]
and the result follows. \( \blacksquare \)

The proof of Theorem 2 is now easy. From Lemma 6, for real \( s > 1 \) we have
\[ \frac{m_F}{s} + \frac{m_F}{s - 1} + \log Q + \Re \sum_{j=1}^{r} \lambda_j \psi(\lambda_j s + \mu_j) + \Re \frac{F'(s)}{F(s)} \geq 0. \]

Moreover, since \( F \in S^* \) with \( d_F > 0 \), comparing with the Riemann zeta function we immediately see that \( m_F \leq d_F \) and \( (F'/F)(s) = O(d_F) \) for \( s \geq 2 \). Hence, recalling the definition of \( q_F \), for real \( s \geq 2 \) we get
\[ \frac{1}{2} \log q_F + \Re \sum_{j=1}^{r} (\lambda_j \psi(\lambda_j s + \mu_j) - \lambda_j \log \lambda_j) + O(d_F) \geq 0. \]

With the notation of Lemma 5, choosing \( s = \max(2H_F^*, D_F/c_2^2, 2) \) and applying Lemma 5 we obtain
\[ \frac{1}{2} \log q_F + \frac{1}{2} d_F \log s + O(d_F) \geq 0. \]

Hence for some constant \( c_4 > 0 \) we have
\[ q_F^{1/d_F} s \geq c_4, \]
and now Theorem 2 follows immediately upon recalling the fact that \( s = \max(2H_F^*, D_F/c_2^2, 2) \). \( \blacksquare \)

**Proof of Theorem 3.** Let \( F \in S^* \) with \( d_F > 0 \) and suppose that \( F(s) \) has a \( \gamma \)-factor of the form \([1.4]\); in particular, \( d_F \in \mathbb{N} \). We use again formula (2.3) of [9], which in this case reads
\[ \frac{z}{e^{2z} - 1} \sum_{j=1}^{d_F} e^{2z \mu_j} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{H_F(n)}{n!} z^n. \]
The right hand side of (2.5) converges absolutely for $|z| \leq 1/2H_F^*$, and for such $z$ we have

\begin{equation}
\sum_{n=0}^{\infty} \frac{|H_F(n)|}{n!} |z|^n \leq d_F + \sum_{n=1}^{\infty} \frac{1}{2^n} = d_F + 1.
\end{equation}

Choosing $z = z_l^\pm = \pm il/4H_F^*d_F$ with $d_F \leq l \leq 2d_F$ we have $|z_l^\pm| \leq 1/2H_F^*$ and hence from (2.5) and (2.6) we obtain

\begin{equation}
\left| \sum_{j=1}^{d_F} (e^{\pm i \mu_j/2H_F^*d_F})^l \right| \leq \left| e^{2z_l^\pm} - 1 \right| \frac{1}{2} (d_F + 1) \ll d_F + 1
\end{equation}

for every $d_F \leq l \leq 2d_F$. By Turán’s Second Main Theorem (see Theorem 8.1 of Turán [16] with $b_j = 1$, $m = 0$ and $n = d_F$), there exists $l_0$ in the above range such that the left hand side of (2.7) is

\begin{equation}
\geq \max_{1 \leq j \leq r} |e^{\pm i \mu_j/2H_F^*d_F}| l_0 c_5^{-d_F}
\end{equation}

with a suitable absolute constant $c_5 > 0$. Note that the max in (2.8) is due to a normalization in the above-cited Theorem 8.1. Recalling the definition of $D_F$, since $\lambda_j = 1/2$ for every $j$ we can choose the signs $\pm$ in such a way that

\begin{equation}
\max_{1 \leq j \leq r} |e^{\pm i \mu_j/2H_F^*d_F}| = e^{D_F/4H_F^*d_F}.
\end{equation}

Therefore, from (2.7)–(2.9) we deduce that

\[ e^{D_F l_0/4H_F^*d_F} \ll c_5^{d_F} (d_F + 1) \leq c_6^{d_F} \]

with some absolute constant $c_6 > 0$, and hence

\[ D_F \ll H_F^*d_F.\]

The result now follows from Theorems 1 and 2.

3. Special cases. In this section we collect further results and problems, and consider several special cases. Suppose first that the $\gamma$-factor of $F \in S^z$ has the form

\[ \gamma(s) = Q^s \Gamma(\lambda s + \mu)^m, \quad m \in \mathbb{N}. \]

Specializing formula (2.3) of [9] and arguing similarly to (2.5)–(2.7) above, choosing $z = \varepsilon i/2H_F^*$ with $\varepsilon = -\text{sgn} \Im \mu$ we obtain

\[ e^{D_F/2H_F^*} \leq \frac{H_F^*}{m} (d_F + 1) \leq 2H_F^*(d_F + 1), \]

and hence

\begin{equation}
(1 + d_F)H_F^* \geq \frac{1}{2} \quad \text{and} \quad D_F \leq 2H_F^* \log(2H_F^*(d_F + 1)).
\end{equation}
If, in addition, we have $\lambda \leq 1$ then $(d_F + 1)/m \leq (2m + 1)/m \leq 3$, thus $e^{D_F/2H_F^*} \leq 6H_F^*$ and therefore

$$H_F^* \geq \frac{1}{6} \quad \text{and} \quad D_F \leq 2H_F^* \log(6H_F^*).$$

Suppose now that the $\gamma$-factor of $F(s)$ has the form

$$\gamma(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda s + \mu_j), \quad \mu_j = i\kappa_j \text{ with } \kappa_j \in \mathbb{R}.$$ 

By an analogous argument, with $\varepsilon = \pm 1$, we obtain

$$\left| \frac{1}{2H_F^*} \sum_{j=1}^{r} e^{-\varepsilon \kappa_j/2H_F^*} \right| \leq d_F + 1.$$ 

Choosing $\varepsilon = -\text{sgn} \max_j \kappa_j$ we obtain

$$e^{D_F/2H_F^*} \leq 2(d_F + 1)H_F^*,$$

and inequalities (3.1) follow in this case as well.

A subset $F$ of $S^*$ is called an $H$-family if for every $F, G \in F$ we have $H_F(n) = H_G(n)$ for all $n \geq 0$. For example, the set of the Dedekind zeta functions associated with all fields with given signature $(r_1, r_2)$ is an $H$-family. We have

**Corollary.** Given an $H$-family $F$ there exists a constant $c(F) > 0$ such that for every $F \in F$,

$$q_F \geq c(F).$$

**Proof.** Clearly, $H_F^*$ and $d_F$ are constant for $F \in F$. Let $F, G \in F$ and $\gamma(s)$, $\gamma'(s)$ be $\gamma$-factors of $F(s)$ and $G(s)$, respectively. From p. 99 of [9] we know that $\gamma(s)$ and $\gamma'(s)$ have the same poles. But the poles $\rho$ of $\gamma(s)$ (resp. $\gamma'(s)$) coincide, apart possibly from $\rho = 0$, with the trivial zeros of $F(s)$ (resp. $G(s)$), therefore

$$D_F = \max_{\rho} |\Im \rho| = D_G.$$ 

Hence $D_F$ is also constant, and the result follows from Theorem 2. 

Now we compute $H_F^*$ and $H_F^*$ for $F(s) = \zeta(s)^k$ ($k \geq 1$ integer) and $\zeta_K(s)$. Recalling the definition of $H_F(n)$, for $n \geq 0$ we have

$$H_\zeta(n) = 2^n B_n(0) = 2^n B_n = \begin{cases} (-1)^{n/2-1}2^{n+1}n!(2\pi)^{-n}\zeta(n), & 2 \mid n, \\ 0, & 2 \nmid n, n > 1, \\ 1, & n = 1. \end{cases}$$

Writing the even $n$ as $n = 2m \geq 2$ we get

$$\left( \frac{|H_\zeta(2m)|}{(2m)!} \right)^{1/2m} = \left( \frac{2\zeta(2m)}{\pi} \right)^{1/2m}.$$


hence
\[ H_\zeta^\bullet = \limsup_{m \to \infty} \frac{(2\zeta(2m))^{1/2m}}{\pi} = \frac{1}{\pi}. \]

By Theorem 1 we have \( d_F H_F^\bullet \geq 1/\pi \), therefore the infimum of \( d_F H_F^\bullet \) for \( F \in S^\sharp \) is \( H_\zeta^\bullet \). Moreover,
\[ (2\zeta(2m))^{1/2m} = \exp\left(\frac{1}{2m} \sum_p \sum_{k=1}^\infty \frac{1}{kp^{2mk}} + \frac{\log 2}{2m}\right) \leq (2\zeta(2))^{1/2} = \frac{\pi}{\sqrt{3}}, \]
since the argument of the exponential is decreasing in \( m \), and \( |H_\zeta(1)| = |2B_1| = 1 \). Hence
\[ H_\zeta^\bullet = 1. \]

Since for integers \( n, k \geq 1 \) we have
\[ H_\zeta^k(n) = kH_\zeta(n), \]
the above results also give
\[ H_\zeta^k = \frac{1}{\pi}, \quad H_\zeta^k = k. \]

Consider now \( \zeta_K(s) \) with \( K \) of signature \((r_1, r_2)\). We have
\[ H_{\zeta_K}(n) = (r_1 2^n + 2r_2)B_n, \]
and writing \( n = 2m \) we deduce as before that
\[ \left( \frac{|H_{\zeta_K}(2m)|}{(2m)!} \right)^{1/2m} = (r_1 2^{2m} + 2r_2)^{1/2m} \frac{(2\zeta(2m))^{1/2m}}{2\pi}, \]
hence
\[ H_{\zeta_K}^\bullet = \frac{1}{\pi}. \]

Moreover,
\[ \left( \frac{|H_{\zeta_K}(2m)|}{(2m)!} \right)^{1/2m} \leq \frac{(4r_1 + 2r_2)^{1/2}}{2\pi} (2\zeta(2))^{1/2} = \sqrt{\frac{2r_1 + r_2}{6}}, \]
since \( (a2^\xi + b)^{1/\xi} \) is decreasing. Finally, \( H_{\zeta_K}(1) = r_1 + r_2 \) and therefore
\[ H_{\zeta_K}^\bullet = r_1 + r_2. \]

We conclude with two problems.

**Problem 1.** Does there exist a function \( \Phi \) such that
\[ d_F \leq \Phi(H_F^\bullet) \]
(i.e. the degree is controlled by the invariant \( H_F^\bullet \))? We believe that there exists an absolute constant \( c > 0 \) such that \( d_F \ll (H_F^\bullet)^c \), or even \( d_F \ll (H_F^\bullet)^{1+\varepsilon} \).
Problem 2. Is it true that the infimum of $d_F H^*_F$ is $H^*_F$? Here the infimum is over $S^\sharp$, or $S$, or $S^*$. We know that this is true with $H^*_F$ in place of $H^*_F$.

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