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Linear recurrences as sums of squares

by

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There is a beautiful theorem of Pourchet [4] stating that any rational polynomial in one variable that assumes only positive values can be written as a sum of five squares of rational polynomials. We provide here analogous theorems for linear recurrences of the second and third order with distinct eigenvalues. By *a rational recurrence sequence* we mean a linear recurrence sequence all of whose terms are rational and the coefficients of the recurrence relation are also rational. By analogy to the above result of Pourchet we have the following two theorems.

Theorem 1. *Consider the rational recurrence sequence*

$$x_n = ax_{n-1} + bx_{n-2} \quad \text{with } a, b, x_1, x_2 \in \mathbb{Q}$$

for $n \in \mathbb{Z}$. *Assume additionally that the roots of* $x^2 - ax - b$ *are distinct. If* $x_n > 0$ *for all* $n \in \mathbb{Z}$ *then there exist rational recurrence sequences* a_n, b_n, c_n, d_n, e_n *such that for all* $n \in \mathbb{Z}$,

$$x_n = a_n^2 + b_n^2 + c_n^2 + d_n^2 + e_n^2.$$

Theorem 2. *The sequence* $x_n = 4^n + 7$ *is not the sum of four squares of rational sequences of order less than* 3.

The last theorem concerns recurrence sequences of the third order.

Theorem 3. *Let* z_n *be a sequence defined by*

$$z_n = az_{n-1} + bz_{n-2} + cz_{n-3} \quad \text{with } a, b, c, z_1, z_2, z_3 \in \mathbb{Q}$$

for $n \in \mathbb{Z}$. *Assume that* z_n *is non-degenerate and that* $f(x) = x^3 - ax^2 - bx - c$ *is irreducible. Denote its roots by* α, β, γ *and consider the explicit representation*

$$z_n = \Delta\alpha^n + \Psi\beta^n + \Omega\gamma^n$$

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with $\Delta, \Psi, \Omega \in \mathbb{Q}(\alpha, \beta, \gamma)$. Then z_n is a sum of finitely many squares of real recurrence sequences if and only if all numbers $\alpha, \beta, \gamma, \Delta, \Psi, \Omega$ are positive. In this case six squares of rational recurrence sequences are sufficient.

It should be emphasized that there is a big difference between second order and third order recurrence sequences regarding their representability as sums of squares. The following example shows that in the latter case the condition $z_n > 0$ for all $n \in \mathbb{Z}$ is not sufficient for representability as the sum of finitely many squares.

Let the sequence (z_n) be given by the recurrence

$$z_{n+3} = 5z_{n+2} - 6z_{n+1} + z_n$$

with initial conditions

$$z_0 = 2, \quad z_1 = 1, \quad z_2 = 4.$$

The roots α, β, γ of the characteristic polynomial $f(x) = x^3 - 5x^2 + 6x - 1$ are

$$\alpha \approx 0.198062, \quad \beta \approx 1.55496, \quad \gamma \approx 3.24698$$

and the explicit representation is

$$z_n = [(2-\alpha)^2 - 1]\alpha^n + [(2-\beta)^2 - 1]\beta^n + [(2-\gamma)^2 - 1]\gamma^n.$$

Easy numerical calculations yield

$$\min_{n \in \mathbb{R}} z_n \approx 0.892786 \quad \text{at } n_{\min} \approx 0.747701.$$

Hence a fortiori

$$z_n > 0 \quad \text{for all } n \in \mathbb{Z}.$$

Because $(2-\beta)^2 - 1 \approx -0.80194 < 0$ we infer by Theorem 3 that z_n is not a sum of squares of finitely many real recurrence sequences! This resembles the counterexample to Hilbert's XVII problem given in 1967 by Motzkin [3]: the polynomial $g(x,y) = 1 - 3x^2y^2 + x^2y^4 + x^4y^2$ assumes only positive values for $x, y \in \mathbb{R}$ but is not the sum of finitely many squares of real polynomials. So in the case of exponential polynomials the counterexample does exist even in the one variable case!

For the proof of our theorems we will need five lemmas.

LEMMA 1. *For any $a, b \in \mathbb{Q}^+$ there exist $c, d, f, g \in \mathbb{Q}$ satisfying*

$$a(b - c^2) = d^2 + f^2 + g^2.$$

Proof. Any positive definite quadratic form with rational coefficients and four variables assumes all positive rational values ([1, Chapter 1, §7.4]), hence there exist $d, f, g, c \in \mathbb{Q}$ such that

$$d^2 + f^2 + g^2 + ac^2 = ab.$$

LEMMA 2. *If $f(x) = ax^2 + bx + c \in \mathbb{Q}[x]$, $a > 0$, $\Delta = b^2 - 4ac < 0$ then there exist $a_i, b_i \in \mathbb{Q}$ for $i = 1, \ldots, 5$ such that*

$$f(x) = \sum_{i=1}^{5} (a_i x + b_i)^2.$$

Proof. By Lemma 1 there exist rational numbers g, α, β, γ satisfying

$$\frac{1}{4a}\left[\frac{-\Delta}{4a} - g^2\right] = \alpha^2 + \beta^2 + \gamma^2.$$

Hence

$$f(x) = 4a\left[\left(\frac{x}{2} + \frac{b}{4a}\right)^2 + \alpha^2 + \beta^2 + \gamma^2\right] + g^2$$

By Lagrange's theorem $4a$ is a sum of four rational squares and by Euler's identity we obtain the desired representation of $f(x)$.

LEMMA 3. *Consider k pairwise distinct, non-zero elements $\gamma_1, \ldots, \gamma_k$ of an infinite field K and k polynomials $f_1(n), \ldots, f_k(n) \in K[n]$. If*

$$\sum_{j=1}^{k} f_j(n)\gamma_j^n = 0 \quad \textit{for all } n \in \mathbb{Z}$$

then $f_j(n) \equiv 0$ for $j = 1, \ldots, k$.

Proof. Assuming that a counterexample exists, choose it in such a way that $k \geq 1$ is smallest possible and also $\min_{1 \leq j \leq k}\{\deg f_j\}$ is smallest possible. Assume without loss of generality that $\deg f_1$ is this smallest degree. From

$$\deg(\gamma_1 f_1(n+1) - \gamma_1 f_1(n)) < \deg f_1$$

and the identity

$$[\gamma_1 f_1(n+1) - \gamma_1 f_1(n)]\gamma_1^n + \sum_{j=2}^{k} [\gamma_j f_j(n+1) - \gamma_1 f_j(n)]\gamma_j^n \equiv 0$$

we infer that $\gamma_1 f_1(n+1) - \gamma_1 f_1(n) \equiv 0$. If k were greater than 1 then we would have

$$\gamma_2 f_2(n+1) - \gamma_1 f_2(n) \equiv 0,$$

which would lead to $\gamma_2 = \gamma_1$, a contradiction. But $k = 1$ is also impossible.

LEMMA 4 (Mordell [2]). *Let $F(x_1, x_2, x_3) = \sum_{i,j=1}^{3} d_{ij} x_i x_j$ be a positive definite quadratic form with rational coefficients. Then there exist $a_i, b_i, c_i \in \mathbb{Q}$ for $i = 1, \ldots, 6$ such that*

$$F(x_1, x_2, x_3) = \sum_{i=1}^{6} (a_i x_1 + b_i x_2 + c_i x_3)^2.$$

LEMMA 5. *Let H be a finitely generated subgroup of $\mathbb{R}^+$ containing multiplicatively independent numbers $\alpha_1, \ldots, \alpha_k$. For any $\beta_1, \ldots, \beta_k \in \mathbb{R}^+$ not necessarily distinct there exists a homomorphism $\phi : H \to \mathbb{R}^+$ such that $\phi(\alpha_j) = \beta_j$ for $j = 1, \ldots, k$.*

Proof of Theorem 1. Let us consider two cases regarding the eigenvalues of x_n.

First consider the case when both roots α and β of $x^2 - ax - b$ are rational. We have the explicit formula

$$x_n = A\alpha^n + B\beta^n \quad \text{with } A, B, \alpha, \beta \in \mathbb{Q}.$$

If $\alpha = -\beta$ then

$$x_{2n} = (A + B)(\alpha^n)^2 \quad \text{and} \quad x_{2n+1} = (A - B)\alpha(\alpha^n)^2$$

and again by Lagrange's theorem we infer that both subsequences x_{2n} and x_{2n+1} are sums of four squares of rational recurrence sequences. Hence the whole sequence x_n is also the sum of squares of four rational recurrence sequences.

If $\alpha \neq -\beta$ then all numbers A, B, α, β are positive, because $x_n > 0$ for all $n \in \mathbb{Z}$. If we apply Lemma 2 to $f(x) = Ax^2 + B$ we obtain

$$x_{2n} = \sum_{i=1}^{5} (a_i\alpha^n + b_i\beta^n)^2.$$

Similarly, applying Lemma 2 to $f(x) = A\alpha x^2 + B\beta$, we get

$$x_{2n+1} = \sum_{i=1}^{5} (c_i\alpha^n + d_i\beta^n)^2.$$

Now we consider the case when

$$x^2 - ax - b = (x - \gamma)(x - \bar{\gamma}) \quad \text{with } (\mathbb{Q}(\gamma) : \mathbb{Q}) = 2.$$

We will work with the explicit formula

$$x_n = \alpha\gamma^n + \bar{\alpha}\bar{\gamma}^n \quad \text{with } \alpha \in \mathbb{Q}(\gamma).$$

Let d be an integer such that $\mathbb{Q}(\gamma) = \mathbb{Q}(\sqrt{d})$ and write $\gamma = \varepsilon + \delta\sqrt{d}$ with $\varepsilon, \delta \in \mathbb{Q}$. Then

$$\gamma^n = A_n + B_n\sqrt{d}$$

where A_n, B_n are also rational recurrence sequences (with the same recurrence relation as x_n). The explicit formula for x_{2n} can now be rewritten as

$$x_{2n} = \alpha(A_n + B_n\sqrt{d})^2 + \bar{\alpha}(A_n - B_n\sqrt{d})^2.$$

Putting $\alpha = h + k\sqrt{d}$ with $h, k \in \mathbb{Q}$ we obtain finally

$$x_{2n} = 2h(A_n^2 + dB_n^2) + 4kdA_nB_n.$$

Because $x_n > 0$ for all $n \in \mathbb{Z}$ we infer first that $d > 0$ and further that $\alpha, \bar{\alpha}, \gamma, \bar{\gamma}$ are all positive as well. Now we verify easily that

$$f(x) = 2hx^2 + 4kdx + 2hd$$

satisfies the assumptions of Lemma 2, namely

$$2h = \alpha + \bar{\alpha} > 0 \quad \text{and} \quad \Delta = -16d\alpha\bar{\alpha} < 0.$$

Hence Lemma 2 gives the representation

$$x_{2n} = \sum_{i=1}^{5} (a_i A_n + b_i B_n)^2.$$

Using the explicit formula

$$x_{2n+1} = \alpha\gamma(A_n + B_n\sqrt{d})^2 + \bar{\alpha}\bar{\gamma}(A_n - B_n\sqrt{d})^2$$

and reasoning along the same lines we obtain a similar representation for x_{2n+1}.

Proof of Theorem 2. Let us assume that

$$a_n^2 + b_n^2 + c_n^2 + d_n^2 = 4^n + 7 \quad \text{for all } n \in \mathbb{Z} \tag{1}$$

with rational recurrence sequences a_n, b_n, c_n, d_n. A crucial observation is that all the eigenvalues γ appearing in a_n, b_n, c_n, d_n have modulus 1 or 2. This can be established as follows.

Each of the sequences a_n, b_n, c_n, d_n is of one of the forms:

(a) $\alpha\gamma^n + \bar{\alpha}\bar{\gamma}^n$ with $\gamma \notin \mathbb{R}, \alpha \neq 0$,
(b) $\alpha_1\gamma_1^n + \alpha_2\gamma_2^n$ where $\alpha_1, \alpha_2, \gamma_1, \gamma_2 \in \mathbb{R}^*$ and $|\gamma_1| > |\gamma_2|$,
(c) $\alpha_1\gamma_1^n$ with $\alpha_1, \gamma_1 \in \mathbb{R}^*$,
(d) $(an + b)\gamma^n$ where $a, b \in \mathbb{R}, a^2 + b^2 > 0, \gamma \in \mathbb{R}^*$,
(e) 0.

We could assume that $|\gamma_1| > |\gamma_2|$ in case (b), because in case $|\gamma_1| = |\gamma_2|$ we would restrict our analysis to even n and then we would be in case (c) or (e). Let us now list the squares of the above expressions:

- $\alpha^2(\gamma^2)^n + 2|\alpha|^2(|\gamma|^2)^n + \bar{\alpha}^2(\bar{\gamma}^2)^n$,
- $\alpha_1^2(\gamma_1^2)^n + 2\alpha_1\alpha_2(\gamma_1\gamma_2)^n + \alpha_2^2(\gamma_2^2)^n$,
- $\alpha_1^2\gamma_1^{2n}$,
- $(an + b)^2(\gamma^2)^n$,
- 0.

Assume now to the contrary that some eigenvalues ρ appearing in some of a_n, b_n, c_n, d_n satisfy $|\rho| \notin \{1, 2\}$. Let γ be one with the largest modulus. Then the resulting coefficients of $(|\gamma|^2)^n$ on both sides of equality (1) are equal (by Lemma 3), hence vanish. But this coefficient on the left hand side is a sum of some expressions of the form

- $2|\alpha|^2$,
- α_1^2 (assuming $|\gamma| = |\gamma_1|$),
- α_1^2,
- $(an+b)^2$,

all positive, and therefore the relevant sum must be empty. This contradicts the assumption that γ does appear among eigenvalues of some of a_n, b_n, c_n, d_n.

Now that we have proved the crucial observation, we turn to the proof proper and consider six non-trivial cases.

Case 1: All eigenvalues appearing in the relevant sequences are real. By the above, such an eigenvalue γ must satisfy $\gamma \in \{1, -1, 2, -2\}$. If we restict our attention to even n we can write (1) as

$$\sum_{j=1}^{4} (a_j 2^n + b_j)^2 = 4^n + 7 \quad \text{with } a_j, b_j \in \mathbb{Q}.$$

By Lemma 3 this is equivalent to the system of equations

$$\begin{cases} \sum_{j=1}^4 a_j^2 = 1, \\ \sum_{j=1}^4 b_j^2 = 7, \\ \sum_{j=1}^4 a_j b_j = 0, \end{cases}$$

in rational numbers a_j, b_j. By Euler's identity we obtain

$$7 = \Big(\sum_{j=1}^4 a_j^2\Big)\Big(\sum_{j=1}^4 b_j^2\Big) = \Big(\sum_{j=1}^4 a_j b_j\Big)^2 + t^2 + u^2 + v^2 = t^2 + u^2 + v^2$$

with rational t, u, v. But this contradicts the very well known fact that 7 is not the sum of three squares of rational numbers.

In the remaining cases we consider only n divisible by 4; in case $\gamma^2 = \bar{\gamma}^2$ we can then replace $(\alpha\gamma^n + \bar{\alpha}\bar{\gamma}^n)^2$ by $(a2^n + b)^2$ with $a, b \in \mathbb{Q}$ (and $ab = 0$):

$$(\alpha\gamma^{4k} + \bar{\alpha}\bar{\gamma}^{4k})^2 = (\alpha + \bar{\alpha})^2 (\gamma^4)^{2k} = ((\alpha + \bar{\alpha})\Lambda^{4k})^2 \quad \text{with } \Lambda \in \{1, 2\}.$$

Case 2: We have the representation

$$(\alpha_1\gamma^n + \bar{\alpha}_1\bar{\gamma}^n)^2 + (\alpha_2\gamma^n + \bar{\alpha}_2\bar{\gamma}^n)^2 + (a2^n + b)^2 + (c2^n + d)^2 = 4^n + 7$$

with $a, b, c, d \in \mathbb{Q}$ and $\gamma \notin \mathbb{R}$, $\alpha_1, \alpha_2 \in \mathbb{Q}(\gamma)$. If $\gamma^2 = \bar{\gamma}^2$ then we are in Case 1. If $\gamma^2 \neq \bar{\gamma}^2$ then by Lemma 3,

$$\alpha_1^2 + \alpha_2^2 = 0 \quad \text{and} \quad ab + cd = 0,$$

hence

$$(a^2 + c^2)4^n + (b^2 + d^2) + (2N(\alpha_1) + 2N(\alpha_2))(N\gamma)^n = 4^n + 7.$$

Now we distinguish two subcases.

- $N\gamma = 4$. Then $b^2 + d^2 = 7$, which is impossible.
- $N\gamma = 1$. Now

$$a^2 + c^2 = 1, \qquad 2N\alpha_1 + 2N\alpha_2 + b^2 + d^2 = 7.$$

$\alpha_1^2 = -\alpha_2^2$ implies that $\mathbb{Q}(\gamma) = \mathbb{Q}(i)$, $\alpha_1 = \epsilon + \delta i$ with $\epsilon, \delta \in \mathbb{Q}$ and $N\alpha_1 = N\alpha_2 = \epsilon^2 + \delta^2$. Hence

$$7 = (2\epsilon)^2 + (2\delta)^2 + b^2 + d^2.$$

Further

$$b^2 + d^2 = (b^2 + d^2)(a^2 + c^2) = (bc - ad)^2$$

and finally

$$7 = (2\epsilon)^2 + (2\delta)^2 + (bc - ad)^2,$$

which is the desired contradiction.

Case 3: We have the representation

$$(a2^n + b)^2 + \sum_{j=1}^{3} (\alpha_j \gamma^n + \bar{\alpha_j}\bar{\gamma}^n)^2 = 4^n + 7$$

where $a, b \in \mathbb{Q}$ and $\gamma \notin \mathbb{R}$, $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Q}(\gamma)$. We will show that this case never holds. Assuming $\gamma^2 \neq \bar{\gamma}^2$ we get

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 0 \quad \text{and} \quad 2ab = 0 \tag{2}$$

and the representation takes the form

$$(2N\alpha_1 + 2N\alpha_2 + 2N\alpha_3)(N\gamma)^n + a^2 4^n + b^2 = 4^n + 7.$$

- $a = 0$ gives $N\gamma = 4$ and further $b^2 = 7$, an obvious nonsense.
- $b = 0$ gives $N\gamma = 1$, which leads to

$$2N\alpha_1 + 2N\alpha_2 + 2N\alpha_3 = 7.$$

But this is not possible by the following reasoning. Let $d \in \mathbb{N}$ be such that $\mathbb{Q}(\gamma) = \mathbb{Q}(\sqrt{-d})$. Then $\alpha_j = a_j + b_j\sqrt{-d}$ for $j = 1, 2, 3$ with rational a_j, b_j. Comparing the real parts of the two sides of the first equation of (2) gives

$$\sum_{j=1}^{3} (a_j^2 - db_j^2) = 0.$$

Taking into account that

$$\sum_{j=1}^{3} N(\alpha_j) = \sum_{j=1}^{3} (a_j^2 + db_j^2)$$

we obtain again an impossible equation

$$4a_1^2 + 4a_2^2 + 4a_3^2 = 7.$$

Case 4: We have the representation

$$(\alpha_1\gamma^n+\bar{\alpha_1}\bar{\gamma}^n)^2+(\alpha_2\gamma^n+\bar{\alpha_2}\bar{\gamma}^n)^2+(\beta_1\rho^n+\bar{\beta_1}\bar{\rho}^n)^2+(\beta_2\rho^n+\bar{\beta_2}\bar{\rho}^n)^2=4^n+7$$

with $\gamma,\rho\notin\mathbb{R}$, $N\gamma=1$, $N\rho=4$. If $\gamma^2=\bar{\gamma}^2$ then we are in Case 2. If $\gamma^2\neq\bar{\gamma}^2$ then

$$2N\alpha_1+2N\alpha_2=7 \quad \text{and} \quad \alpha_1^2+\alpha_2^2=0,$$

which results in $\mathbb{Q}(\alpha_1)=\mathbb{Q}(\alpha_2)=\mathbb{Q}(i)$, $N\alpha_1=N\alpha_2=\varepsilon^2+\eta^2$ with rational ε,η. Finally $(2\varepsilon)^2+(2\eta)^2=7$, which is impossible.

Case 5: We have

$$(\beta\rho^n+\bar{\beta}\bar{\rho}^n)^2+\sum_{j=1}^{3}(\alpha_j\gamma^n+\bar{\alpha_j}\bar{\gamma}^n)^2=4^n+7$$

with $\rho,\gamma\notin\mathbb{R}$ satisfying $\{N\rho,N\gamma\}=\{1,4\}$. If $\rho^2=\bar{\rho}^2$ then we are in Case 3. If $\rho^2\neq\bar{\rho}^2$ then the terms with ρ^{2n} on the left hand side will not cancel out.

Case 6: We have

$$\sum_{j=1}^{4}(\alpha_j\gamma^n+\bar{\alpha_j}\bar{\gamma}^n)^2=4^n+7$$

with $\gamma\notin\mathbb{R}$, $N\gamma\in\{1,4\}$. If $\gamma^2=\bar{\gamma}^2$ then we are in Case 1. If $\gamma^2\neq\bar{\gamma}^2$ then $\sum_{j=1}^{4}\alpha_j^2=0$ and

$$4^n+7\cdot 1^n=\Big(2\sum_{j=1}^{4}|\alpha_j|^2\Big)|\gamma|^{2n},$$

which is clearly impossible.

Proof of Theorem 3. First consider the case when $\alpha,\beta,\gamma,\Delta,\Psi,\Omega$ are all positive. We consider only the case of even indices. Let A_n,B_n,C_n be rational recurrence sequences defined implicitly by

$$\begin{aligned}\alpha^n&=A_n+B_n\alpha+C_n\alpha^2,\\ \beta^n&=A_n+B_n\beta+C_n\beta^2,\\ \gamma^n&=A_n+B_n\gamma+C_n\gamma^2.\end{aligned}$$

We can write

$$\begin{aligned}z_{2n}&=\Delta(A_n+B_n\alpha+C_n\alpha^2)^2+\Psi(A_n+B_n\beta+C_n\beta^2)^2\\ &\quad+\Omega(A_n+B_n\gamma+C_n\gamma^2)^2\\ &=G(A_n,B_n,C_n)\end{aligned}$$

where G is a real quadratic form in A_n,B_n,C_n. Obviously G is rational and positive definite. Therefore we can apply Lemma 4 to obtain the desired representation of z_{2n} as a sum of six squares. The proof for z_{2n+1} is very similar.

We proceed with the case of positive eigenvalues α, β, γ but now at least one of Δ, Ψ, Ω is negative. Without loss of generality let $\alpha < \beta < \gamma$. If $\Omega < 0$ or $\Delta < 0$ then there exists $n \in \mathbb{Z}$ (positive or negative) such that $z_n < 0$, hence the representation of z_n as a sum of squares is trivially impossible.

Let us now consider the most interesting case when $\Delta > 0, \Psi < 0, \Omega > 0$. We emphasize that in this case it can happen that $z_n > 0$ for all *real* n but, as we will prove, z_n is not a sum of squares of finitely many real recurrence sequences.

Assume to the contrary that there exist real recurrence sequences $t_n^{(1)}, \dots, t_n^{(K)}$ satisfying the identity

$$\sum_{j=1}^{K} t_n^{(j)2} = z_n. \tag{3}$$

Let E be the set of the moduli of all eigenvalues appearing in $t_n^{(1)}, \dots, t_n^{(K)}$ and put $H = \langle \alpha, \beta, \gamma, E \rangle$. By elementary Galois theory we infer that

$$\alpha^k \beta^l \gamma^m = 1 \quad \text{with } k, l, m \in \mathbb{Z}$$

implies $k = l = m$ and $\alpha\beta\gamma = 1$ or $k + l + m = 0$. The latter is possible only for $k = l = m = 0$ because z_n is non-degenerate. So, in any case we can apply Lemma 5 for $\alpha_1 = \alpha$, $\alpha_2 = \gamma$ to obtain a homomorphism $\phi : H \to \mathbb{R}^+$ such that

$$\phi(\alpha) = \varepsilon, \quad \phi(\gamma) = \varepsilon \quad \text{and} \quad \phi(\beta) = 1/\varepsilon^2.$$

If $t_n = \sum_{j=1}^{k} f_j(n)\gamma_j^n$ is one of the sequences $t_n^{(1)}, \dots, t_n^{(K)}$ then we define

$$\Phi(t_n) = \sum_{j=1}^{k} f_j(n) \left(\phi(|\gamma_j|) \frac{\gamma_j}{|\gamma_j|} \right)^n.$$

The crucial observation is that $\Phi(t_n^{(j)})$ are also real and that the identity (3) goes into

$$\sum_{j=1}^{K} \Phi(t_n^{(j)})^2 = \Delta\varepsilon^n + \Omega\varepsilon^n + \Psi \left(\frac{1}{\varepsilon^2} \right)^n.$$

But the above cannot be true for $\varepsilon > 0$ sufficiently small and $n = 1$. So we have obtained the desired contradiction.

Now consider the case when all eigenvalues α, β, γ are real but not all are positive. If not all Δ, Ψ, Ω are positive then we consider the sequence $u_n = z_{2n}$ and the previous case applies. If all Δ, Ψ, Ω are positive we put $v_n = z_{2n+1} = \Delta\alpha(\alpha^2)^n + \Psi\beta(\beta^2)^n + \Omega\gamma(\gamma^2)^n$ and the previous case applies again.

Finally let $\alpha \in \mathbb{R}$, $\gamma = \overline{\beta} \notin \mathbb{R}$. If $|\alpha| \neq |\beta|$ then by an easy reasoning $z_n < 0$ for infinitely many $n \in \mathbb{Z}$. The case $|\alpha| = |\beta| = |\gamma|$ is not possible because it would lead to $\alpha^3 = \alpha\beta\gamma \in \mathbb{Q}$, while z_n is assumed to be non-degenerate.

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