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Sums of nine squares

by

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In this paper, the singular series $\varrho_9(n)$ for the number of representations of n as a sum of 9 squares is computed in two different ways. This gives a formula for the Dirichlet L-series $\sum_{m=1,\, m\,\mathrm{odd}}^{\infty} \left(\frac{n}{m}\right) m^{-4}$. Comparing with a result of Lomadze gives a strange identity.

Let $r_k(n)$ be the number of solutions $(x_1, \ldots, x_k) \in \mathbb{Z}^k$ of $x_1^2 + \ldots + x_k^2 = n$. For $k = 8$ we have Jacobi's formula

$$r_8(n) = 16\sigma_3^*(n), \tag{1}$$

where

$$\sigma_3^*(n) := \begin{cases} \sigma_3(n), & n \text{ odd}, \\ \sigma_3^{\mathrm{even}}(n) - \sigma_3^{\mathrm{odd}}(n), & n \text{ even}, \end{cases}$$

with

$$\sigma_3^{\mathrm{even}}(n) = \sum_{\substack{d \mid n \\ d\,\mathrm{even}}} d^3, \quad \text{etc.}$$

For $k = 9$ we have the recursion

$$r_9(n) = \sum_{|s| \le \sqrt{n}} r_8(n - s^2). \tag{2}$$

Combining (2) with (1) we get

$$r_9(n) = 16 \sum_{|s| \le \sqrt{n}} \sigma_3^*(n - s^2). \tag{3}$$

Now it is known (cf. [C]) that the genus of $I_9 = \langle 1, \ldots, 1 \rangle$ (sum of 9 squares) consists of 2 classes, the other one being $E_8 \oplus I_1 =: F_9$. Denote the number of representations of n by F_9 by $\alpha_9(n)$. Then

$$\alpha_9(n) = \sum_{|s| \le \sqrt{n}} r_{E_8}(n - s^2) = 240 \sum_{|s| \le \sqrt{n}} \sigma_3\left(\frac{n - s^2}{2}\right), \tag{4}$$

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where

$$\sigma_3\left(\frac{n-s^2}{2}\right) = 0 \quad \text{if } \frac{n-s^2}{2} \notin \mathbb{Z}.$$

The number of automorphisms of the two classes are:

$$A_1 := \#\,\mathrm{Aut}(I_9) = 2^9(9!), \qquad A_2 := \#\,\mathrm{Aut}(E_8 \oplus I_1) = 2^{14}\cdot 3^5 \cdot 5^2 \cdot 7 \cdot 2.$$

We have $A_2 = \frac{15}{2}A_1$. Thus the mass of I_9 is

$$M(I_9) = \frac{1}{A_1} + \frac{1}{A_2} = \frac{15+2}{15A_1} = \frac{17}{2786918400}$$

(cf. [C-S, p. 410]). The average value

$$\varrho_9(n) := \frac{r_9(n)/A_1 + \alpha_9(n)/A_2}{1/A_1 + 1/A_2} = \frac{15r_9(n) + 2\alpha_9(n)}{17} \tag{5}$$

can be evaluated by the Minkowski–Siegel theorem. For simplicity we only state the result for n odd and squarefree (see [G, p. 166]):

$$\varrho_9(n) = \frac{\pi^{9/2}}{\Gamma(9/2)} n^{7/2}\mathfrak{S}, \tag{6}$$

where

$$\mathfrak{S} = \mathfrak{S}_2 \frac{2^9(8!)}{(2^8-1)2^8\pi^8|B_8|} \sum_{\substack{m=1\\ m \text{ odd}}}^{\infty} \frac{\left(\frac{n}{m}\right)}{m^4},$$

$$\mathfrak{S}_2 = 1 + \frac{\cos\left(\pi\frac{2n-9}{4}\right)}{2^{7/2}} + \frac{\cos\left(\pi\frac{n-9}{4}\right)}{2^7}$$

(the factor 2 in $2n$ is missing in [G, p. 164, line 4 from bottom]). Here $|B_8| = 1/30$, so we get

$$\varrho_9(n) = \frac{24576 n^{7/2}}{17\pi^4}\mathfrak{S}_2 L\left(4, \left(\frac{n}{\cdot}\right)\right), \tag{7}$$

$$\mathfrak{S}_2 = \begin{cases} 15/16 & \text{if } n \equiv 3 \bmod 4, \\ 137/128 & \text{if } n \equiv 1 \bmod 8, \\ 135/128 & \text{if } n \equiv 5 \bmod 8. \end{cases} \tag{8}$$

Let us check this for $n = 1$:

$$\varrho_9(1) = \frac{24576}{17\pi^4} \cdot \frac{137}{128} \sum_{\substack{m=1\\ m \text{ odd}}}^{\infty} \frac{1}{m^4},$$

where the last sum is

$$\sum_{m=1}^{\infty} \frac{1}{m^4} - \frac{1}{2^4}\sum_{m=1}^{\infty}\frac{1}{m^4} = \frac{15}{16}\zeta(4) = \frac{15}{16}\cdot\frac{\pi^4}{90} = \frac{\pi^4}{96},$$

thus $\varrho_9(1) = 274/17$ (as in Siegel [S, I, p. 369]). On the other hand $\alpha_9(1) = 2, r_9(1) = 18$, thus (5) gives

$$\varrho_9(1) = \frac{15 \cdot 18 + 2 \cdot 2}{17} = \frac{274}{17}.$$

Comparing (5) with (7) we get a formula for the L-series $L\big(4, \big(\frac{n}{\cdot}\big)\big)$, namely

$$(9)\quad L\left(4, \left(\frac{n}{*}\right)\right) = \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} \frac{\left(\frac{n}{m}\right)}{m^4} = \frac{17\pi^4}{24576 n^{7/2} \mathfrak{S}_2} \cdot \frac{15 r_9(n) + 2\alpha_9(n)}{17}$$
$$= \frac{\pi^4}{24576 n^{7/2} \mathfrak{S}_2} (15 r_9(n) + 2\alpha_9(n))$$
$$= \frac{\pi^4}{24576 n^{7/2} \mathfrak{S}_2} \left(15 \cdot 16 \sum_{|s| \le \sqrt{n}} \sigma_3^*(n - s^2) + 2 \cdot 240 \sum_{|s| \le \sqrt{n}} \sigma_3\left(\frac{n - s^2}{2}\right)\right)$$
$$= \frac{\pi^4 \cdot 5}{512 n^{7/2} \mathfrak{S}_2} \sum_{|s| \le \sqrt{n}} \left(\sigma_3^*(n - s^2) + 2\sigma_3\left(\frac{n - s^2}{2}\right)\right),$$

which is an explicit evaluation after insertion of $\mathfrak{S}_2$ from (8).

If n is not odd and squarefree the value for $\varrho_9(n)$ is explicitly given by Walfisz ([V, p. 195]):

$$(10)\qquad \varrho_9(n) = \frac{24576}{17\pi^4} \mathfrak{S}_2 L\left(4, \left(\frac{n}{*}\right)\right) n^{7/2} \tau_4(n)$$

with

$$\tau_4(n) = \prod_{p|n} \frac{1}{1 - 1/p^7} \prod_{\substack{p|n \\ 2\nmid a}} \left(1 - \frac{1}{p^{(7/2)(a+1)}}\right) \cdot \prod_{\substack{p|n \\ 2|a,\, a>0}} \left(1 + \frac{\left(\frac{P}{p}\right) - \frac{1}{p^3}}{1 - \frac{\left(\frac{P}{p}\right)}{p^4}} \cdot \frac{1}{p^{7a/2+4}}\right),$$

where $p^a \,\|\, n$, $P = p^{-a} n$ and

$$\mathfrak{S}_2 = \begin{cases} \dfrac{135}{127} - \dfrac{255}{127 \cdot 16} \cdot \dfrac{1}{2^{7h}} & \text{if } c = 2h+1 \text{ or } N \equiv 3 \bmod 4,\ c = 2h, \\ \dfrac{135}{127} + \dfrac{119}{127 \cdot 128} \cdot \dfrac{1}{2^{7h}} & \text{if } N \equiv 1 \bmod 8,\ c = 2h, \\ \dfrac{135}{127} - \dfrac{135}{127 \cdot 128} \cdot \dfrac{1}{2^{7h}} & \text{if } N \equiv 5 \bmod 8,\ c = 2h, \end{cases}$$

where $n = 2^c N$ (formula (4.59) in [V, p. 195] has a sign error in the middle term).

For n odd and squarefree we get $\tau_4(n) = 1$, so formula (10) coincides with (7). As before we can evaluate $L\bigl(4, \bigl(\frac{n}{*}\bigr)\bigr)$ by inserting in formula (10).

Recently Lomadze has shown the following formula (see [L, (1.1)] and [I, p. 187]):

$$r_9(n) = \varrho_9(n) + \frac{4}{17} \sum_{x_1^2+x_2^2+x_3^2=3n} \left(\frac{x_1x_2x_3}{3}\right) x_1x_2x_3.$$

Comparing this with formula (10) gives

$$\frac{24576 n^{7/2} \tau_4(n)}{17\pi^4} \mathfrak{S}_2 L\left(4, \left(\frac{n}{*}\right)\right) = r_9(n) - \frac{4}{17} \sum_{x_1^2+x_2^2+x_3^2=3n} \left(\frac{x_1x_2x_3}{3}\right) x_1x_2x_3.$$

Inserting (3) we get

$$L\left(4, \left(\frac{n}{*}\right)\right) = \frac{17\pi^4}{48 \cdot 512 n^{7/2} \tau_4(n) \mathfrak{S}_2} \left(16 \sum_{|s| \le \sqrt{n}} \sigma_3^*(n - s^2) - \frac{4}{17} \sum \left(\frac{x_1x_2x_3}{3}\right) x_1x_2x_3\right).$$

Comparing with (9) gives

$$\frac{17}{48\tau_4(n)} \left(16 \sum_{|s| \le \sqrt{n}} \sigma_3^*(n - s^2) - \frac{4}{17} \sum \left(\frac{x_1x_2x_3}{3}\right) x_1x_2x_3\right) = 5 \sum_{|s| \le \sqrt{n}} \left(\sigma_3^*(n - s^2) + 2\sigma_3\left(\frac{n - s^2}{2}\right)\right).$$

Restricting to the case of n odd and squarefree, where $\tau_4(n) = 1$, we get

$$\frac{2}{3} \sum_{|s| \le \sqrt{n}} \sigma_3^*(n - s^2) - \frac{1}{12} \sum \left(\frac{x_1x_2x_3}{3}\right) x_1x_2x_3 = 10 \sum_{|s| \le \sqrt{n}} \sigma_3\left(\frac{n - s^2}{2}\right)$$

or

$$(11) \qquad \sum_{x_1^2+x_2^2+x_3^2=3n} \left(\frac{x_1x_2x_3}{3}\right) x_1x_2x_3 = \sum_{|s| \le \sqrt{n}} \left(8\sigma_3^*(n - s^2) - 120\sigma_3\left(\frac{n - s^2}{2}\right)\right)$$

for n odd and squarefree.

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