On the error term of the mean square formula for the Riemann zeta-function in the critical strip $3/4 < \sigma < 1$

by

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1. Introduction. For $1/2 < \sigma < 1$, define

$$E_\sigma(t) = \int_0^t |\zeta(\sigma + iu)|^2 \, du - \left( \zeta(2\sigma)t + (2\pi)^{2\sigma-1}\frac{\zeta(2-2\sigma)}{2-2\sigma}t^{2-2\sigma}\right)$$

where $\zeta(s)$ is the Riemann zeta-function. This is an analogue of $E(t)$ for the case $\sigma = 1/2$, which was extensively studied. Comparatively, $E_\sigma(t)$ is new and the following mean square formula was obtained by Matsumoto and Meurman [8] and [9] within this decade:

$$\int_1^T E_\sigma(t)^2 \, dt = \begin{cases} A_1(\sigma)T^{5/2-2\sigma} + O(T) & \text{if } 1/2 < \sigma < 3/4, \\ A_0 T \log T + O(T) & \text{if } \sigma = 3/4, \\ O(T) & \text{if } 3/4 < \sigma < 1. \end{cases}$$

The $O$-term in the case $\sigma = 3/4$ in [8] was given with a slightly weaker estimate $O(T^{5/2})$ and Lam [5] improved it to $O(T)$. Then it is natural to wonder whether the $O$-terms are sharp. This question is meaningful, especially in the last case, because it provides the information how large $E_\sigma(t)$ ($3/4 < \sigma < 1$) can be. In fact, if we denote the error term $O(T)$ by $F_\sigma(T)$, it is conjectured that $F_\sigma(T) = A(\sigma)T + o(T)$ (see [4] and [7] for more details).

Concerning the case $3/4 < \sigma < 1$, in addition to the mean square estimate in (1.1), Ivić and Matsumoto [2] proved that

$$E_\sigma(t) \ll t^{4\kappa(1-\sigma)/(1+4\kappa(1-\sigma))}\log T$$

where $(\kappa, \lambda)$ is an exponent pair such that $\lambda = \kappa + 1/2$. It is the best upper bound to date. Furthermore, we found in [6] that

$$\int_1^T E_\sigma(t) \, dt = -2\pi\zeta(2\sigma - 1)T + O(\sqrt{T}).$$

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This immediately implies that $F_{\sigma}(T) = \int_1^T E_{\sigma}(t)^2 \, dt \gg T$ and partly answers the above-mentioned question (on whether the $O$-term in (1.1) is sharp). However, at the same time, it suggests that $E_{\sigma}(t)$ may consist of two parts: one is the constant $-2\pi\zeta(2\sigma - 1)$ and the other has a small mean value $O(\sqrt{T})$. This was also pointed out in Matsumoto [7, p. 256]. Thus we carry out this splitting, or in other words, we consider

$$E^*_\sigma(t) = \int_0^t |\zeta(\sigma + iu)|^2 \, du - \left( \frac{\zeta(2\sigma)t + (2\pi)^{2\sigma-1}\zeta(2-2\sigma)}{2-2\sigma} t^{2-2\sigma} - 2\pi\zeta(2\sigma-1) \right).$$

Then we have $E_{\sigma}(t) = -2\pi\zeta(2\sigma - 1) + E^*_\sigma(t)$, $\int_1^T E^*_\sigma(t) \, dt \ll \sqrt{T}$ and $\int_1^T E^*_\sigma(t)^2 \, dt \ll T$ by (1.1).

Our purpose here is to study the size of the last integral $\int_1^T E^*_\sigma(t)^2 \, dt$. It is interesting because if the integral is $o(T)$, then the conjecture (for the last case) is settled; otherwise, in view of its mean value, this shows that $E^*_\sigma(t)$ is highly oscillating and so is $E_{\sigma}(t)$. We find that the latter case occurs, as is anticipated (see the remark below).

**Theorem.** Let $3/4 < \sigma < 1$ and $T_0$ be a sufficiently large constant. Then, for all $T \geq T_0$, we have $\int_1^T E^*_\sigma(t)^2 \, dt \gg_{\sigma} T$ where the implied constant depends on $\sigma$ only.

**Remark.** Let $\sigma_a(n) = \sum_{d|n} d^a$, and for $-1 < a < 0$ define

$$\Delta_a(x) = \sum_{n \leq x} \sigma_a(n) - \zeta(1-a)x - \frac{\zeta(1+a)}{1+a} x^{1+a} + \frac{1}{2} \zeta(-a).$$

This is a generalization of the classical error term $\Delta(x)$ in Dirichlet’s divisor problem. It is well known that there is an analogy between the error terms $\Delta(x)$ and $E(t)$. Such an analogy also appears between $\Delta_{1-2\sigma}(x)$ and $E_{\sigma}(t)$ (or more appropriately, $E^*_\sigma(t)$). (This was also discussed in [4] and [7].) Indeed, there is a mean square result for $\Delta_{1-2\sigma}(x)$, parallel to (1.1), obtained by Meurman [10] with a truncated Voronoi-type formula and delicate analysis. On the other hand, starting with another (finite) series representation, an asymptotic formula for $\int_1^T \Delta_{1-2\sigma}(x)^2 \, dx$ with $3/4 < \sigma < 1$ can be derived. This was worked out in [1] and [12] and, therefore, a similar result for $E^*_\sigma(t)$ is expected. The series representation used in [1] and [12], however, highly depends on the arithmetic nature of the divisor function and there is no counterpart of that for $E_{\sigma}(t)$ yet. (Our result is perhaps not insignificant in view of this difficulty.)

Finally, let us outline our approach. Define $G_{\sigma}(t) = \int_1^t E^*_\sigma(u) \, du$. Then $G_{\sigma}(t + h) - G_{\sigma}(t) = \int_t^{t+h} E^*_\sigma(u) \, du$ can be regarded as an approximation of $hE^*_\sigma(t)$. Thus, it leads to evaluate the mean square $\int_T^{2T} (G_{\sigma}(t+h) - G_{\sigma}(t))^2 \, dt$
which can be treated by Jutila’s argument in [3] and techniques in [9]. However, in order to make small $h$ admissible, we need a series representation of $G_{\sigma}(t)$ with a sufficiently small error term. To this end, we apply the argument in [8] to derive an averaged form of the series representation. However, we can only obtain a formula for $G_{\sigma}(2t) - G_{\sigma}(t)$ instead of $G_{\sigma}(t)$. This needs a little extra effort to handle (see Section 4).

In what follows, $3/4 < \sigma < 1$ and $c_i$ ($i = 1, 2, \ldots$) denote unspecified positive constants which depend on $\sigma$ only.

2. Averaged formula. We start with the formula

$$\int_2^T E_{\sigma}(t) \, dt = \Sigma_1(t, X) |_{\frac{T}{2}}^{|T|} + \pi^{-1/2} t^{5/2} I(t) |_{\frac{T}{2}}^{|T|} + O(1)$$

where $X \in [AT, T]$ is not an integer with a constant $0 < A < 1$, and

$$I(t) = \int_X^\infty \frac{\Delta_{1-2\sigma}(\xi)}{\xi^3 V^2 U^{1/2} (U - 1/2)^{\sigma} (U + 1/2)^{\sigma+2}} \, d\xi.$$ 

The function $\Sigma_1(t, X)$ is defined at the beginning of [6, Section 2] while $\Delta_{1-2\sigma}(\xi)$ is defined in [6, Lemma 3.2]. $U$ and $V$ are functions defined as in [6, Lemma 3.1] with $k$ replaced by $\xi$.

This formula comes from [6, (3.3)–(3.5), (3.8) and (3.14)], and a simple refinement of [6, (3.9)]

$$\int_2^T G_4^*(t) \, dt \ll T^{r-1/2} \log T \ll_{\sigma} 1.$$ 

The last upper bound relies on the fact that $r < 1/2$ when $3/4 < \sigma < 1$. (Note that $r = -(4\sigma^2 - 7\sigma + 2)/(4\sigma - 1)$, see [6, Lemma 3.2].)

To deal with $I(t)$, it can be observed that $I(t)$ is essentially the same as the integral $J$ in [8, p. 368] except that the exponent of $V$ is replaced by 2 and sin by cos in our case. We proceed with the argument in [8, Section 6] by replacing $\Delta_{1-2\sigma}(\xi)$ with its Voronoi-type series. The components corresponding to $J_2(n, b)$ and $O(T^{-\sigma-7/4})$ in [8, (6.1)] are treated in the same way. Then we carry out the smoothing process described in [8, Section 7], with $X = (L + \mu)^2$ ($0 \leq \mu \leq M$) and $L = M = \sqrt{T}/2$. It is easy to handle $\Sigma_1(t, X)$ but the remaining part in $I(t)$ needs a more complicated treatment, following the same lines of argument of the evaluation of $K_n$ in [8, p. 372]. Finally, we can obtain

$$\int_2^T E_{\sigma}(t) \, dt = \Sigma_1^*(t) |_{\frac{T}{2}}^{|T|} - \Sigma_2^*(t) |_{\frac{T}{2}}^{|T|} + O(1).$$

(2.1)
Here, corresponding to $\Sigma_{1,\sigma}^*(t, L, M)$ in [8], we have

$$\Sigma_1^*(t) = \sqrt{2} \left( \frac{t}{2\pi} \right)^{5/4-\sigma} \sum_{n \leq T} (-1)^n w_1(n) \sigma_1^2(n)^{1/4-\sigma} e_2(t, n) \sin f(t, n)$$

where $w_1(n) = 1$ if $n \leq T/4$ and $2(1 - \sqrt{n/T})$ if $T/4 < n \leq T$,

$$e_2(t, n) = \left( 1 + \frac{\pi n}{2t} \right)^{-1/4} \left( \sqrt{\frac{2t}{\pi n}} \arcsinh \sqrt{\frac{\pi n}{2t}} \right)^{-2},$$

$$f(t, n) = 2t \arcsinh \sqrt{\frac{\pi n}{2t}} + (2\pi nt + \pi^2 n^2)^{1/2} - \frac{\pi}{4}$$

with $\sigma_a(n) = \sum_{d|n} d^a$ and $\arcsinh x = \log(x + \sqrt{x^2 + 1})$. Moreover, following the proof of [8, (8.2)] (i.e. [8, p. 375]), $\Sigma_2^*(t)$ analogous to $\Sigma_{2,\sigma}^*(t, L, M)$ satisfies

$$\int_T^{2T} \Sigma_2^*(t)^2 dt \ll T$$

because $\Sigma_2^*(t)$ only differs from $\Sigma_{2,\sigma}^*(t, L, M)$ in the exponent of $\log(t/(2\pi n))$ (the exponent is $-2$ in $\Sigma_2^*(t)$) and the trigonometric function $\cos g(t, n)$ (sin $g(t, n)$ in $\Sigma_2^*(t)$).

Define

$$S(t) = \Sigma_1^*(t) - \Sigma_2^*(t) \quad \text{and} \quad G_\sigma(t) = \int_1^t E_{\sigma}^*(u) \, du.$$  

Then from (2.1) we get

$$G_\sigma(2T) - G_\sigma(T) = \int_T^{2T} E_{\sigma}^*(t) \, dt = S(2T) - S(T) + \mathcal{E}$$

where $|\mathcal{E}| \leq c_1$. Our next section is to study the mean square of the difference $S(t + h) - S(t)$.

\section{Mean square of $S(t + h) - S(t)$}

We prove the following.

**Lemma 3.1.** Let $B$ and $B'$ be sufficiently large but fixed constants. Then

$$K_1 Th^{5-4\sigma} \leq \int_T^{2T} (S(t + h) - S(t))^2 \, dt \leq K_2 Th^{5-4\sigma}$$

uniformly for $B \leq h \leq \sqrt{T}/B'$, where $K_1$ and $K_2$ are positive constants depending on $\sigma$ only.
Proof. Using \((t + h)^{5/4 - \sigma} - t^{5/4 - \sigma} \ll t^{1/4 - \sigma} h\) and
\[
e_2(t + h, n) - e_2(t, n) = h \frac{\partial}{\partial t} \bigg|_{t=\xi} e_2(t, n) \ll nht^{-2}
\]
with \(\xi \in [t, t + h]\), we have, for \(t \in [T, 2T]\),
\[
\Sigma^*_1(t + h) - \Sigma^*_1(t) = \sqrt{2} \left( \frac{t}{2\pi} \right)^{5/4 - \sigma} \text{Im} \sum_{n \leq T} a_n k_n(t) e^{if(t,n)} + O(T^{-1/2} h)
\]
where
\[
a_n = (-1)^n w_1(n) \sigma_{1-2\sigma}(n)n^{\sigma-7/4},
\]
\[
k_n(t) = e_2(t, n)(\exp(i(f(t + h, n) - f(t, n))) - 1).
\]
This yields
\[
(3.1) \quad S(t + h) - S(t) = \sqrt{2} \left( \frac{t}{2\pi} \right)^{5/4 - \sigma} \text{Im} \sum_{n \leq T} a_n k_n(t) e^{if(t,n)} + F(t, h)
\]
where \(F(t, h) = \Sigma^*_2(t + h) - \Sigma^*_2(t) + O(1)\). In view of (2.2), it suffices to evaluate
\[
J = 2 \int_T^{2T} \left( \frac{t}{2\pi} \right)^{5/2 - 2\sigma} \left( \text{Im} \sum_{n \leq T} a_n k_n(t) e^{if(t,n)} \right)^2 dt.
\]
Plainly,
\[
(3.2) \quad J = \int_T^{2T} \left( \frac{t}{2\pi} \right)^{5/2 - 2\sigma} \left[ \sum_{n \leq T} a_n k_n(t) e^{if(t,n)} \right]^2 dt
\]
\[
= (2\pi)^{2\sigma-5/2} \sum_{n \leq T} a_n^2 \int_T^{2T} t^{5/2 - 2\sigma} |k_n(t)|^2 dt + O(T^{5/2 - 2\sigma}(|J_1| + |J_2|))
\]
by the mean value theorem for integrals, where for some \(T_i \in [T, 2T]\) \((i = 1, 2, 3, 4)\),
\[
J_1 = \int_{T_1}^{T_2} \sum_{m \neq n \leq T} a_m a_n k_m(t) k_n(t) e^{i(f(m) - f(n))} dt,
\]
\[
J_2 = \int_{T_3}^{T_4} \sum_{m, n \leq T} a_m a_n k_m(t) k_n(t) e^{i(f(m) + f(n))} dt.
\]
To bound \(J_1\), we write \(k(t) = k_m(t) k_n(t)\) and \(\phi(t) = f(t, m) - f(t, n)\). Then
\[
J_1 = \sum_{m \neq n \leq T} a_m a_n \left\{ k(t) \frac{e^{i\phi(t)}}{\phi'(t)} \right|_{T_1}^{T_2} - \int_{T_1}^{T_2} \left( \frac{k'(t)}{\phi'(t)} - k(t) \frac{\phi''(t)}{\phi'(t)^2} \right) e^{i\phi(t)} dt \right\}.
\]
Noting that
\[(3.3) \quad f(t + h, n) - f(t, n) = 2h \ar sinh \sqrt{\frac{\pi n}{2\xi}} \quad \text{where } \xi \in [t, t + h],\]
we have \(k_n(t) \ll \min(1, h\sqrt{n/T})\) as \(\ar sinh x \ll x\) for all \(x \geq 0\). Applying Hilbert’s inequality (see [11]) and \(\min_{m \neq n} |\phi'(t)| \gg \sqrt{nt}\), we get
\[
\sum_{m \neq n \leq T} a_m a_n k(t) \frac{e^{i\phi'(t)}}{\phi'(t)} |T_2| \leq \sum_{n \leq T} a_n^2 |k_n(t)|^2 \sqrt{nT} \leq T^{1/2} \sum_{n \leq T} \sigma_{1-2\sigma} (n)^2 n^{2\sigma - 3} \min(1, h\sqrt{n/T})^2 \leq T^{2\sigma - 3/2} h^{4 - 4\sigma}.
\]
Using the fact that \(k_n'(t) \ll \sqrt{n} h t^{-3/2}\), we have
\[
\int_{T_1} \sum_{m \neq n \leq T} a_m a_n k'(t) \frac{e^{i\phi(t)}}{\phi'(t)} dt \leq T \left( \sum_{n \leq T} a_n^2 |k_n'(t)|^2 \sqrt{nT} \right)^{1/2} \left( \sum_{n \leq T} a_n^2 |k_n(t)|^2 \sqrt{nT} \right)^{1/2} \leq T^{2\sigma - 3/2} h^{3 - 2\sigma}.
\]
Finally, from [9, p. 379], we have
\[
\frac{\phi''(t)}{\phi'(t)^2} = -\frac{1}{2t\phi'(t)} \left( \cosh \left( \frac{1}{2} f'(t, m) \right) \right)^{-2} + O(t^{-1}).
\]
Then we see that as \(\cosh x \geq 1\),
\[
\int_{T_1} \sum_{m \neq n \leq T} a_m a_n k(t) \frac{\phi''(t)}{\phi'(t)^2} e^{i\phi(t)} dt \leq \sum_{n \leq T} a_n^2 |k_n(t)|^2 \sqrt{nT} + \left( \sum_{n \leq T} |a_n k_n(t)| \right)^2 \leq T^{2\sigma - 3/2} h^{4 - 4\sigma}.
\]
To sum up, \(J_1 \ll T^{2\sigma - 3/2} h^{3 - 2\sigma}\). The estimation of \(J_2\) is easier. Taking \(k(t) = k_m(t) k_n(t)\) and \(\phi(t) = f'(t, m) + f'(t, n)\), we then have \(k(t) \ll \min(1, h\sqrt{n/T}), k'(t) \ll \sqrt{n} h t^{-3/2}, \phi'(t) \gg (\sqrt{m} + \sqrt{n})/\sqrt{T}\) and \(\phi''(t) \ll (\sqrt{m} + \sqrt{n}) T^{-3/2}\). The same treatment yields \(J_2 \ll T^{2\sigma - 3/2} h\) and so (3.2) becomes
\[
J = (2\pi)^{2\sigma - 5/2} \sum_{n \leq T} a_n^2 \int_{T}^{2T} t^{5/2 - 2\sigma} |k_n(t)|^2 dt + O(T h^{3 - 2\sigma}).
\]
From this, we can deduce that
\[ J \leq c_2 T^{7/2-2\sigma} \sum_{n \leq T} \sigma_1(n) n^{2\sigma-7/2} \min(1, \sqrt{n/T})^2 + c_3 T h^{3-2\sigma} \]
\[ \leq c_4 T h^{5-4\sigma}. \]
Since \( x/2 \leq \text{arsinh} x \leq 2x \) when \( 0 \leq x \leq 1 \), we see from (3.3) that \( c_5 h \sqrt{n/T} \leq f(t+h,n) - f(t,n) \leq \pi/2 \) when \( n \leq \delta T/h^2 \) for some small constant \( \delta > 0 \). Therefore, provided \( \delta B' \geq c_6 (>1) \), we get
\[ J \geq c_7 T^{5/2-2\sigma} h^2 \sum_{n \leq \delta T/h^2} \sigma_1(n) n^{2\sigma-5/2} - c_8 T h^{3-2\sigma} \]
\[ \geq \left( c_9 - \frac{c_8}{B^2-2a} \right) T h^{5-4\sigma} \geq c_{10} T h^{5-4\sigma} \]
provided \( B > c_{11} \). From (2.2), we see that \( \int_T^{2T} F(t,h)^2 \, dt \ll T \), and thus from (3.1) and (3.4),
\[ \int_T^{2T} (S(t+h) - S(t))^2 \, dt = J + O(T + \sqrt{JT}) = J + O(T h^{5/2-2\sigma}). \]
This completes the proof of Lemma 3.1, with (3.4), (3.5) and a sufficiently large \( B \).

4. Proof of Theorem. We choose an integer \( R \) such that \( 2^{R(5-4\sigma)} \geq 4K_2/K_1 \) where \( K_1 \) and \( K_2 \) are defined as in Lemma 3.1. Let
\[ h = \max(B, (12(c_1 R)^2/K_2)^{1/(5-4\sigma)}) \]
and \( T \) be any sufficiently large number (in particular, \( T \geq 2^R (B'h)^2 \)). From (2.3), we have
\[ (G\sigma(2^r(t+h)) - G\sigma(2^r t)) - (G\sigma(2^{r-1}(t+h)) - G\sigma(2^{r-1} t)) \]
\[ = (G\sigma(2^r(t+h)) - G\sigma(2^{r-1}(t+h))) - (G\sigma(2^r t) - G\sigma(2^{r-1} t)) \]
\[ = (S(2^r(t+h)) - S(2^{r-1}(t+h))) - (S(2^r t) - S(2^{r-1} t)) + \mathcal{E}'_r \]
\[ = (S(2^r(t+h)) - S(2^r t)) - (S(2^{r-1}(t+h)) - S(2^{r-1} t)) + \mathcal{E}'_r \]
where \( |\mathcal{E}'_r| \leq 2c_1 \). Summing over \( r = 1, \ldots, R \), yields
\[ (G\sigma(2^R(t+h)) - G\sigma(2^R t)) - (G\sigma(t+h) - G\sigma(t)) \]
\[ = (S(2^R(t+h)) - S(2^R t)) - (S(t+h) - S(t)) + \sum_{r=1}^{R} \mathcal{E}'_r. \]
Now, we square both sides and integrate over \([T, 2T]\). Using the inequality \( 2(a^2 + b^2) \geq (a - b)^2 \geq a^2/2 - b^2 \), we infer that
\[
\frac{2T}{T} \int \left( G_\sigma(2^R(t + h)) - G_\sigma(2^Rt) \right)^2 dt + \frac{2T}{T} \int \left( G_\sigma(t + h) - G_\sigma(t) \right)^2 dt
\]

\[
\geq 4^{-1} \int \left( (S(2^R(t + h)) - S(2^Rt)) - (S(t + h) - S(t)) \right)^2 dt - \frac{T}{2} \sum_{r=1}^{R} E_r^2
\]

\[
\geq 4^{-1} \left( \frac{1}{2^{R+1}T} \int (S(t + 2^Rh) - S(t))^2 dt - \int (S(t + h) - S(t))^2 dt \right)
\]

\[
- 2(c_1R)^2T
\]

\[
\geq \frac{K^2}{4} Th^{5-4\sigma} - 2(c_1R)^2T \geq (c_1R)^2T
\]

by Lemma 3.1 with our choices of \( R \) and \( h \). Using the Cauchy–Schwarz inequality, we see that

\[
\frac{2T}{T} \int \left( G_\sigma(t + h) - G_\sigma(t) \right)^2 dt = \int \left( \int_t^{t+h} E_\sigma^*(u) du \right)^2 dt \leq h^2 \int T \left( E_\sigma^*(u) \right)^2 du.
\]

As \( R \) and \( h \) are fixed constants, we conclude that \( \int \left( E_\sigma^*(u) \right)^2 du \gg T \) and hence the result.

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