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On a multiple trigonometric series

by

M. Z. Garaev (Taipei)

1. Introduction. Throughout the text $c, c_1, \ldots, c_5$ denote absolute positive constants not necessarily the same in different cases. The constants implicit in the O symbols are also absolute.

G. I. Arkhipov and K. I. Oskolkov [1] proved that for any polynomial $P(x) = \alpha_1 x^n + \ldots + \alpha_n x + \alpha_{n+1}$ with real coefficients $\alpha_1, \ldots, \alpha_{n+1}$ the sequence

$$H_N = \sum_{1 \le |x| \le N} \frac{e^{2\pi i P(x)}}{x}$$

converges as $N \to \infty$. They applied Vinogradov's method of trigonometric sums. Afterwards K. I. Oskolkov [3, 4] discovered a nontrivial application of this result and Vinogradov's method to the investigation of the properties of solutions of Schrödinger type PDEs. To PDEs with mixed derivatives correspond multiple trigonometric sums. The simplest type of such sums is $h_N = h_N(\alpha)$ defined as follows:

$$h_N = \sum_{x=1}^{N} \sum_{y=1}^{N} \frac{\sin(\alpha xy)}{xy}. \tag{1}$$

The question of convergence of $\{h_N\}_{N=1}^{\infty}$ was discussed several times by G. I. Arkhipov, V. N. Chubarikov and the author. But still it has not been known to us whether this sequence is convergent for all real α or not.

In this note we give an answer to this question.

2. Theorem. *There exists a real number α such that the sequence* (1) *diverges as $N \to \infty$.*

We will use the following well known statement (see e.g. [2, p. 473]):

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Lemma 1. *For any real number t we have*

$$\varrho(t) = \sum_{1\le y\le Y} \frac{\sin(2\pi yt)}{\pi y} + O\left(\frac{1}{1+Y|\sin\pi t|}\right)$$

where $\varrho(t) = 1/2 - \{t\}$.

Lemma 2. *For any positive integers x, q, N we have*

$$\sum_{1\le y\le qN} \frac{\sin\left(2\pi\frac{xy}{N}\right)}{y} = \sum_{y=1}^{qN-1} \frac{T_y}{y(y+1)}$$

where $T_y = \sum_{k=1}^{y} \sin\left(2\pi\frac{xk}{N}\right)$.

Lemma 2 follows from Abel transformation and $T_{qN} = 0$.

Lemma 3. *For any positive integers q, N we have*

$$S = \sum_{x=1}^{N}\sum_{y=1}^{qN} \frac{\sin\left(2\pi\frac{xy}{N}\right)}{xy} > c_1 \log N - c_2.$$

Proof. As $\sum_{1\le y\le Y} \sin(\alpha y)/y = O(1)$ uniformly in α and Y, by Lemma 2 we have

$$S = \sum_{1\le x\le N/8} x^{-1} \sum_{y=1}^{qN} \frac{T_y}{y(y+1)} + O\Big(\sum_{N/8<x\le N} x^{-1}\Big)$$

where

$$T_y = \sum_{k=1}^{y} \sin\left(2\pi\frac{xk}{N}\right) = \frac{\sin^2\left(\pi\frac{x}{N}y\right)\cos\left(\pi\frac{x}{N}\right)}{\sin\left(\pi\frac{x}{N}\right)} + \frac{1}{2}\sin\left(2\pi\frac{xy}{N}\right).$$

Using again $\sum_{1\le x\le N/8} \sin(\alpha x)/x = O(1)$ we have

$$S > \sum_{1\le x\le N/8} x^{-1} \sum_{1\le y\le N/(2x)} \frac{\sin^2\left(\pi\frac{x}{N}y\right)\cos\left(\pi\frac{x}{N}\right)}{y(y+1)\sin\left(\pi\frac{x}{N}\right)} - O(1).$$

The ranges of variables are such that

$$\sin^2\left(\pi\frac{x}{N}y\right) \ge \frac{4x^2y^2}{N^2}, \quad \cos\left(\pi\frac{x}{N}\right) \ge \frac{1}{2}, \quad 0 < \sin\left(\pi\frac{x}{N}\right) < \pi\frac{x}{N}.$$

Therefore

$$S > \sum_{1\le x\le N/8} x^{-1} \sum_{1\le y\le N/(2x)} \frac{x}{\pi N} - O(1) > c_1 \log N - c_2.$$

Lemma 3 is proved.

3. Proof of the Theorem. We put

$$\alpha = 2\pi \sum_{n=1}^{\infty} \frac{1}{q_n}$$

where the sequence $q_1, q_2, \ldots$ is defined as follows: $q_1 = 2$, $q_{n+1} = q_n^{nq_n+1}$ for all positive integers n. In order to prove the Theorem it is enough to establish that for this α the subsequence h_{q_n} of (1) diverges as $n \to \infty$.

For a given n set $q = q_n, N = q_n^{nq_n}$. We will prove that

$$h_{q_{n+1}} = h_{qN} > c_1 q^{-1} \log N - c_2 \log q > cn$$

for all large enough n.

Since

$$\alpha = 2\pi\left(\frac{a}{q} + \frac{1}{qN}\right) + O\left(\frac{1}{q^3N^3}\right),$$

we have

$$h_{q_{n+1}} = \sum_{x=1}^{qN}\sum_{y=1}^{qN} \frac{\sin\left(2\pi\left(\frac{a}{q} + \frac{1}{qN}\right)xy\right)}{xy} + O(1).$$

Put

$$S_1 = {\sum_{x\le qN}}' \sum_{y=1}^{qN} \frac{\sin\left(2\pi\left(\frac{a}{q} + \frac{1}{qN}\right)xy\right)}{xy}$$

and

$$S_2 = {\sum_{x\le qN}}'' \sum_{y=1}^{qN} \frac{\sin\left(2\pi\left(\frac{a}{q} + \frac{1}{qN}\right)xy\right)}{xy}$$

where the prime indicates the additional condition $x \equiv 0 \pmod q$ while two primes mean $x \not\equiv 0 \pmod q$.

We have

$$h_{q_{n+1}} = S_1 + S_2 + O(1). \tag{2}$$

From Lemma 3 it follows that $S_1 > c_1 q^{-1} \log N - c_2 = c_1 n \log q - c_2$, i.e.

$$S_1 > c_1 n \log q - c_2. \tag{3}$$

Now we prove that $S_2 > -c_3 \log q$.

The subsum of S_2 over $0 < x < q$ is $O(\log q)$. Therefore according to Lemma 1 we have

$$S_2 = \pi S_3 + \pi S_4 + O(\log q) \tag{4}$$

where

$$S_3 = \sum_{q<x<qN}'' x^{-1} \varrho\Big(\frac{ax}{q} + \frac{x}{qN}\Big),$$

$$S_4 = \sum_{q<x<qN}'' x^{-1} O\bigg(\frac{1}{1 + qN\big|\sin\pi\big(\frac{ax}{q} + \frac{x}{qN}\big)\big|}\bigg).$$

Further, since $x^{-1}O(f) = O(x^{-2}) + O(f^2)$, we have

$$S_4 = O(1) + O\bigg(\sum_{x=1}^{qN} \frac{1}{1 + q^2N^2\sin^2\pi\big(\frac{ax}{q} + \frac{x}{qN}\big)}\bigg).$$

Taking into account that q and N are powers of 2 we see that when x runs through a complete system (mod qN), then so does $(aN+1)x$. Hence

$$\begin{aligned} S_4 &= O(1) + O\bigg(\sum_{x=1}^{qN} \frac{1}{1 + q^2N^2\sin^2\frac{\pi x}{qN}}\bigg) \\ &= O(1) + O\bigg(\sum_{x=1}^{qN/2} \frac{1}{1 + q^2N^2\sin^2\frac{\pi x}{qN}}\bigg). \end{aligned}$$

Using the fact that

$$\sin\frac{\pi x}{qN} > \frac{x}{qN} \qquad \text{for } 1 \le x \le qN/2$$

we obtain

$$S_4 = O(1). \tag{5}$$

In order to estimate S_3 we note that

$$\Big\{\frac{ax}{q} + \frac{x}{qN}\Big\} \le \Big\{\frac{ax}{q}\Big\} + \Big\{\frac{x}{qN}\Big\} = \Big\{\frac{ax}{q}\Big\} + \frac{x}{qN}.$$

Therefore

$$S_3 \ge \sum_{q<x<qN}'' x^{-1}\Big(\varrho(axq^{-1}) - \frac{x}{qN}\Big) > -1 + \sum_{q<x<qN}'' x^{-1}\varrho(axq^{-1}).$$

Substitution $x = qu + l$ where $1 \le u \le N-1$, $1 \le l \le q-1$ gives us

$$\begin{aligned} 1 + S_3 &> \sum_{u=1}^{N-1}\sum_{l=1}^{q/2-1} (qu+l)^{-1}\varrho(alq^{-1}) + \sum_{u=1}^{N-1}\sum_{l=q/2+1}^{q-1} (qu+l)^{-1}\varrho(alq^{-1}) \\ &= \sum_{u=1}^{N-1}\sum_{l=1}^{q/2-1} (qu+l)^{-1}\varrho(alq^{-1}) + \sum_{u=1}^{N-1}\sum_{l=1}^{q/2-1} (qu+q-l)^{-1}\varrho(-alq^{-1}) \end{aligned}$$

where we have used $\varrho(1/2) = 0$.

Now note that $\varrho(-alq^{-1}) = -\varrho(alq^{-1})$. Therefore

$$1 + S_3 > -\sum_{u=1}^{N-1} \sum_{l=1}^{q/2-1} \left(\frac{1}{qu+l} - \frac{1}{qu+q-l} \right) > -\sum_{u=1}^{N-1} \sum_{l=1}^{q/2-1} \frac{q}{q^2u^2} > -2,$$

whence $S_3 > -3$. Together with (5), (4), (3), (2) and $q = q_n$ we obtain

$$h_{q_{n+1}} > c_1 n \log q_n - c_4 \log q_n - c_5.$$

The Theorem is proved.

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References

[1] G. I. Arkhipov and K. I. Oskolkov, *On special trigonometric series and its application*, Math. USSR-Sb. 62 (1989), 145–155.

[2] G. I. Arkhipov, V. A. Sadovnichii and V. N. Chubarikov, *Lectures in Mathematical Analysis*, Vysshaya Shkola, Moscow, 1999 (in Russian).

[3] K. I. Oskolkov, *Vinogradov series and integrals and their applications*, Proc. Steklov Inst. Math. 190 (1992), 193–229.

[4] —, *Vinogradov series in the Cauchy problem for equations of Schrödinger type*, ibid. 200 (1993), 291–315.

Institute of Mathematics
Academia Sinica
Taipei 11529, Taiwan
E-mail: garaev@math.sinica.edu.tw

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