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On the values of Artin L-series at $s = 1$ and annihilation of class groups

by

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1. Introduction and statement of main results. Let L/K be a finite Galois extension of number fields of a group G. Fix a finite non-empty set of places S of K that contains the set S_∞ of archimedean places and also all non-archimedean places which ramify in L/K. Let $\mathrm{Ir}(G)$ denote the set of irreducible complex characters of G. For each ψ in $\mathrm{Ir}(G)$ write e_ψ for the primitive central idempotent $\psi(1)|G|^{-1}\sum_{g\in G}\psi(g^{-1})g$ of the complex group ring $\mathbb{C}[G]$, and $L_S(\psi, z)$ for the S-truncated Artin L-series of ψ. Then one obtains a $\mathbb{C}[G]$-valued meromorphic function of z by setting

$$\theta_{L/K,S}(z) := \sum_{\psi\in\mathrm{Ir}(G)} L_S(\check{\psi}, z)e_\psi, \tag{1}$$

where $\check{\psi}$ denotes the contragredient of ψ. We thereby obtain an invertible element of the centre of $\mathbb{C}[G]$ by setting

$$\theta^*_{L/K,S}(1) := \sum_{\psi\in\mathrm{Ir}(G)} L^*_S(\check{\psi}, 1)e_\psi,$$

where $L^*_S(\check{\psi}, 1)$ denotes the leading term at $z = 1$ of the series $L_S(\check{\psi}, z)$.

In this article we will always assume that L is a CM-field and write L^+ for its maximal real subfield. We will also assume that K is totally real (so $K \subseteq L^+$) and write τ for the (unique) generator of the subgroup $\mathrm{Gal}(L/L^+)$, which, we note, is central in G. We also fix an odd prime p and for any $\mathbb{Z}_p[G]$-module M we write M^- for the maximal $\mathbb{Z}_p[G]$-submodule $(1-\tau)M$ of M upon which τ acts as multiplication by -1. Finally, we fix an isomorphism of fields $j : \mathbb{C} \cong \mathbb{C}_p$.

For the moment we also assume that G is abelian. Then, motivated by the explicit integral refinement of the abelian Stark Conjecture that was

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formulated by Rubin in [29], in the articles [30, 31, 32] Solomon used the arithmetic of the various p-adic completions of L to construct a natural $\mathbb{C}_p[G]$-valued regulator and to then define a $\mathbb{Z}_p[G]$-submodule $\mathfrak{S}_{L/K}$ of $\mathbb{Q}_p[G]$ in terms of the product of this regulator and the element $\theta^*_{L/K,S}(1)$. Solomon then conjectured that $\mathfrak{S}_{L/K}$ is contained in $\mathbb{Z}_p[G]$ and further asked whether it annihilates the p-primary part $\mathrm{Cl}(\mathcal{O}_L)_p := \mathrm{Cl}(\mathcal{O}_L) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ of the ideal class group $\mathrm{Cl}(\mathcal{O}_L)$ of L. An affirmative answer to the latter question would constitute a natural p-adic analogue of a classical theorem of Stickelberger which asserts that if $K = \mathbb{Q}$ and G is abelian, then $\mathrm{Cl}(\mathcal{O}_L)$ is annihilated by an ideal of $\mathbb{Z}[G]$ that is defined using the value of $\theta_{L/K,S}(z)$ at $z = 0$.

In the unpublished 2007 PhD thesis [24] of the second-named author it is shown that (if G is abelian, then) the validity of a particular case of the Equivariant Tamagawa Number Conjecture of Burns and Flach implies that Solomon's ideal $\mathfrak{S}_{L/K}$ is contained in the Fitting ideal of the $\mathbb{Z}_p[G]$-module $\mathrm{Cl}(\mathcal{O}_L)_p$ and hence, in particular, belongs to $\mathbb{Z}_p[G]$ and annihilates $\mathrm{Cl}(\mathcal{O}_L)_p$, as had earlier been asked by Solomon.

In this article we shall prove a natural generalisation of the main results of [24] to the case that G is non-abelian. We therefore no longer assume that G is abelian.

We write $\mathbb{Q}^c$ for the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, and $\Sigma(L)$ for the set of embeddings of L into $\mathbb{Q}^c$. Then, with d_K denoting the degree of $K/\mathbb{Q}$, we also fix a set $\{\sigma_b : 1 \le b \le d_K\}$ of representatives for the orbits of the natural action of G on $\Sigma(L)$. We set

$$U^1_p(L) := \prod_{w|p} U^1(L_w),$$

where w runs over all p-adic places of L, and $U^1(L_w)$ denotes the subgroup of principal units in $L_w^\times$. For each element $\underline{u} = (u_a)_{1\le a\le d_K}$ in the direct sum $U^1_p(L)^{d_K}$ of d_K-copies of $U^1_p(L)$ we then define a matrix in $\mathrm{M}_{d_K}(\mathbb{C}_p[G])$ by setting

$$M^j(\underline{u}) := \left(\frac{1}{j(2\pi i)} \sum_{g\in G} \log_p(j \circ \sigma_b(g u_a)) g^{-1}\right)_{1\le a,b\le d_K} \tag{2}$$

where $\log_p$ denotes Iwasawa's p-adic logarithm. We also set

$$\mathcal{U}_{L/K,p} := \{\underline{u} \in U^1_p(L)^{d_K} : u_1, \dots, u_{d_K} \text{ are linearly independent over } \mathbb{Z}_p[G]\},$$

and

$$n_{p,S}(L/K) := \sum_{\chi\in \mathrm{Ir}(G)} e_\chi \Big(\prod_{v\in S_p\setminus S} \mathrm{N}v\Big)^{-\chi(1)},$$

where S_p denotes the union of S and the set of p-adic places of K.

We write $\zeta(R)$ for the centre of a ring R and we define the 'p-adic regulator lattice' $\mathcal{R}^j_{L/K,S}$ to be the $\zeta(\mathbb{Z}_p[G])$-submodule of $\zeta(\mathbb{C}_p[G])$ that is generated by the set

$$\{n_{p,S}(L/K)\cdot \mathrm{Nrd}_{\mathbb{C}_p[G]}(M^j(\underline{u})) : \underline{u}\in\mathcal{U}_{L/K}\}.$$

It is not difficult to show that the module $\mathcal{R}^j_{L/K,S}$ is both finitely generated over $\mathbb{Z}_p$ and independent of the precise choice of the embeddings σ_b.

We also write

$$j_* : \mathbb{C}[G]\to\mathbb{C}_p[G] \tag{3}$$

for the ring homomorphism that sends each element $\sum_{g\in G} z_g g$, with z_g in $\mathbb{C}$, to $\sum_{g\in G} j(z_g)g$, and

$$^{\#} : \mathbb{C}_p[G]\to\mathbb{C}_p[G]$$

for the $\mathbb{C}_p$-linear anti-involution which inverts elements of G.

We then define an element of $\zeta(\mathbb{C}_p[G])^-$ by setting

$$\mathcal{L}^j_{L/K,S} = j_*((1-\tau)\cdot\theta^*_{L/K,S}(1)^{\#}). \tag{4}$$

Finally, for each finitely generated G-module M we write $\mathrm{Fitt}_{\mathbb{Z}[G]}(M)$ for its generalised Fitting invariant, as introduced by Nickel in [26]. We recall, in particular, that $\mathrm{Fitt}_{\mathbb{Z}[G]}(M)$ is a certain equivalence class of $\zeta(\mathbb{Z}[G])$-lattices (see §3.1 for more details) and we shall write $(\mathrm{Fitt}_{\mathbb{Z}[G]}(M)\otimes_{\mathbb{Z}}\mathbb{Z}_p)^-$ for the corresponding equivalence class of $\zeta(\mathbb{Z}_p[G])$-lattices that contains $(X\otimes_{\mathbb{Z}}\mathbb{Z}_p)^-$ for any, and therefore every, lattice X in $\mathrm{Fitt}_{\mathbb{Z}[G]}(M)$.

We can now state our main results. They will be derived in §4 as a consequence of a result (Theorem 4.1) in which the role of the class group $\mathrm{Cl}(\mathcal{O}_L)$ is replaced by the Galois group over L of the maximal abelian tamely ramified pro-p extension of L that is unramified outside S. We note, in particular, that there are many cases in which the latter ray class group is strictly larger than $\mathrm{Cl}(\mathcal{O}_L)$ and so the result of Theorem 4.1 is in general much finer than the following consequence.

THEOREM 1.1. *If the Equivariant Tamagawa Number Conjecture of* [12, *Conjecture* 4] *is valid for the pair* $\big(h^0(\mathrm{Spec}(L))(1),\mathbb{Z}_p[G]^-\big)$, *then for each isomorphism of fields* $j:\mathbb{C}\cong\mathbb{C}_p$ *one has an inclusion*

$$\mathcal{R}^j_{L/K,S}\cdot\mathcal{L}^j_{L/K,S}\subseteq\big(\mathrm{Fitt}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_L))\otimes_{\mathbb{Z}}\mathbb{Z}_p\big)^-.$$

In the next result we write $\mathcal{H}(\mathbb{Z}_p[G])$ for the 'denominator ideal' in $\zeta(\mathbb{Z}_p[G])$ that is introduced by Nickel in [26]. We recall that this ideal has an elementary description that depends only upon G as an abstract group and can in many cases be computed explicitly: for example, it has recently been shown by Johnston and Nickel in [23] that $\mathcal{H}(\mathbb{Z}_p[G])=\zeta(\mathbb{Z}_p[G])$ if and only if p does not divide the order of the commutator subgroup of G. (For further details about $\mathcal{H}(\mathbb{Z}_p[G])$ see the discussion in §3.1.)

Corollary 1.2. *If the Equivariant Tamagawa Number Conjecture is valid for the pair* $\big(h^0(\mathrm{Spec}(L))(1), \mathbb{Z}_p[G]^-\big)$, *then one has an inclusion*

$$\mathcal{H}(\mathbb{Z}_p[G]) \cdot \mathcal{R}^j_{L/K,S} \cdot \mathcal{L}^j_{L/K,S} \subseteq \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_L)) \otimes_{\mathbb{Z}} \mathbb{Z}_p.$$

If G is abelian, then $\mathcal{H}(\mathbb{Z}_p[G]) = \mathbb{Z}_p[G]$ and the methods of [24] can be used to show that, when $K \neq \mathbb{Q}$, the $\mathbb{Z}_p[G]$-module that is generated by $\mathcal{R}^j_{L/K,S} \cdot \mathcal{L}^j_{L/K,S}$ (when j runs through all the possible isomorphisms $\mathbb{C} \cong \mathbb{C}_p$) coincides with the Solomon ideal $\mathfrak{S}_{L/K}$. Theorem 1.1 is therefore a natural generalisation (to non-abelian Galois extensions) of the result of [24, Theorem 4.1.1].

For the same reason, the following result extends [24, Corollary 4.1.7]. In this result we write L^{cl} for the Galois closure of L over $\mathbb{Q}$ (which, we note, is again a CM-field) and ζ_p for a choice of primitive pth root of unity in $\mathbb{Q}^c$.

We shall say that the 'Gross–Stark Conjecture is valid for L/K at p' if the conjectures that are formulated by Gross in [21, Conjectures 1.15 and 2.12] are valid for all irreducible $\mathbb{C}_p$-valued characters of G. In this regard, we also recall that important progress on these conjectures of Gross has recently been obtained by Darmon, Dasgupta and Pollack in [17].

Corollary 1.3. *Let L be a finite Galois CM-extension of a totally real number field K. Then the inclusions in Theorem* 1.1 *and Corollary* 1.2 *are valid unconditionally whenever all of the following conditions are satisfied:*

(i) *If either L^{cl} is contained in $(L^{\mathrm{cl}})^+(\zeta_p)$ or there exists a p-adic place w of L which is both wildly ramified in L/K and split in L/L^+, then the Gross–Stark Conjecture is valid for L/K at p.*
(ii) *If p divides the order of G, then the μ-invariant of the cyclotomic $\mathbb{Z}_p$-extension of L vanishes.*
(iii) *For any p-adic place w of L which is wildly ramified in L/K the extension $L_w/\mathbb{Q}_p$ is abelian.*

Proof. We must first discuss what is known concerning the validity of [12, Conjecture 4] for the pair $\big(h^0(\mathrm{Spec}(L))(0), \mathbb{Z}_p[G]^-\big)$.

In [27, Theorem 5.6] Nickel has shown that this conjecture is valid provided that all of the following conditions are satisfied: any p-adic place w of L which is wildly ramified in L/K does not split in L/L^+; the μ-invariant of the cyclotomic $\mathbb{Z}_p$-extension of L vanishes; one has $L^{\mathrm{cl}} \not\subseteq (L^{\mathrm{cl}})^+(\zeta_p)$. Further, the assumption on μ-invariants is needed to deduce the validity of a suitable main conjecture of non-commutative Iwasawa theory from the results of Ritter and Weiss in [28] and one knows that this condition is unnecessary if p does not divide the order of G (see, for example, the proof of [10, Corollary 2.8]).

In addition, in [11, Corollary 3.9] Burns has recently shown that the above case of [12, Conjecture 4] is also valid provided that the Gross–Stark Conjecture is valid for L/K at p and, in addition, if p divides the order of G then the μ-invariant of the cyclotomic $\mathbb{Z}_p$-extension of L vanishes.

At this stage we therefore know that [12, Conjecture 4] is valid for the pair $\big(h^0(\mathrm{Spec}(L))(0), \mathbb{Z}_p[G]^-\big)$ provided that the stated conditions (i) and (ii) are satisfied.

To now deduce the validity, under (i) and (ii), of [12, Conjecture 4] for the pair $\big(h^0(\mathrm{Spec}(L))(1), \mathbb{Z}_p[G]^-\big)$ one can argue just as in Nickel's proof of [27, Corollary 0.6] (which is given at end of §5 of loc. cit.). In fact, we can make a slight improvement of Nickel's argument. Indeed, aside from the observation that we make in Remark 1.4 below, the key point in Nickel's proof is that one knows the validity for L/K of the p-primary part of the central conjecture that is formulated by Bley and Burns in [3], and we recall that in [5, Proposition 4.6] Breuning has shown that this conjecture is valid whenever our stated condition (iii) holds (rather than requiring L/K to be tamely ramified at all p-adic places as is done in [27]).

Thus [12, Conjecture 4] is valid for $\big(h^0(\mathrm{Spec}(L))(1), \mathbb{Z}_p[G]^-\big)$ provided that (i)–(iii) hold and, given this fact, Corollary 1.3 follows directly from Theorem 1.1 and Corollary 1.2. ∎

Remark 1.4. In the proof of [27, Corollary 0.6] Nickel also assumes that Leopoldt's Conjecture is valid at p for the field L^{cl} that occurs in condition (i) of the statement of Corollary 1.3. This assumption is required because he uses the main result of Breuning and Burns in [7] which gives an explicit interpretation of [12, Conjecture 4] for the pair $\big(h^0(\mathrm{Spec}(L^{\mathrm{cl}}))(1), \mathbb{Z}_p[\mathrm{Gal}(L^{\mathrm{cl}}/\mathbb{Q})]\big)$ under the hypothesis that Leopoldt's Conjecture is valid at p for L^{cl}. However, the argument given in [7] shows that if one restricts to consider [12, Conjecture 4] for the pair $\big(h^0(\mathrm{Spec}(L^{\mathrm{cl}}))(1), \mathbb{Z}_p[\mathrm{Gal}(L^{\mathrm{cl}}/\mathbb{Q})]^-\big)$, then one can omit any assumption about Leopoldt's Conjecture. To explain this, note that for any number field F and each finite set of places Σ of F that contains all archimedean places, Leopoldt's Conjecture for F at p asserts the injectivity of the natural diagonal homomorphism

$$\Delta_{F,\Sigma} : \mathcal{O}_{F,\Sigma}^{\times} \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \prod_{w \in \Sigma_p} F_w^{\times} \hat{\otimes} \mathbb{Z}_p, \tag{5}$$

where each $F_w^{\times} \hat{\otimes}_{\mathbb{Z}} \mathbb{Z}_p$ denotes the pro-p completion of the multiplicative group $F_w^{\times}$. But if F is CM, with τ the generator of $\mathrm{Gal}(F/F^+)$, then the 'minus part' $(1-\tau)\Delta_{F,\Sigma}$ of this map is injective because the group $(1-\tau)(\mathcal{O}_F^{\times} \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ is finite (as is proved, for example, in [34, Theorem 4.12]) and it is this fact which is needed if one restricts the computations of [7] to the setting of $\big(h^0(\mathrm{Spec}(L^{\mathrm{cl}}))(1), \mathbb{Z}_p[\mathrm{Gal}(L^{\mathrm{cl}}/\mathbb{Q})]^-\big)$.

Corollary 1.5. *If L is an abelian extension of $\mathbb{Q}$ and K is any real subfield of L, then the inclusions in Theorem* 1.1 *and Corollary* 1.2 *are valid unconditionally.*

Proof. It suffices to show that all of the listed conditions in Corollary 1.3 are valid if L is abelian over $\mathbb{Q}$.

But in this case the validity of the Gross–Stark Conjecture for $L/\mathbb{Q}$ at p is proved by Gross in [21, Proposition 2.13 and §4]. In view of the well-known functorial properties of p-adic L-functions under addition and induction of characters, this result then implies the validity for any real subfield K of L of the Gross–Stark Conjecture for L/K at p so that condition (i) is satisfied.

In addition, the vanishing of the μ-invariant of the cyclotomic $\mathbb{Z}_p$-extension of L is in this case proved by Ferrero and Washington in [19], as required by condition (ii), and the assertion of condition (iii) is obviously valid. ∎

Remark 1.6. It is also possible to obtain the result of Corollary 1.5 as a direct consequence of Corollary 1.2 rather than by using Corollary 1.3. The point here is that if L is abelian over $\mathbb{Q}$, and K is any subfield of L, then the validity of [12, Conjecture 4] for the pair $\big(h^0(\mathrm{Spec}(L))(1), \mathbb{Z}[G]\big)$, and hence also for the associated pair $\big(h^0(\mathrm{Spec}(L))(1), \mathbb{Z}_p[G]^-\big)$, is proved by Burns and Flach in [13, Corollary 1.2].

The above results suggest several directions for further research.

Firstly, it would be interesting to know if there are any direct links between the modules $\mathcal{R}^j_{L/K,S} \cdot j_*(\theta^*_{L/K,S}(1)^{\#})$ that occur in Theorem 1.1 and Corollary 1.2 and the annihilators of ideal class groups that have recently been constructed from the values of p-adic Artin L-functions by Barrett and Burns in [2] and by Burns and Macias Castillo in [14] (see, in particular, Theorem 4.1 and Remark 4.2(iii) in loc. cit.).

It is also natural to ask if there are any direct links between our results and the important work of Greither in [20] which shows that the Equivariant Tamagawa Number Conjecture for the pair $\big(h^0(\mathrm{Spec}(L))(0), \mathbb{Z}[G]\big)$ has rather precise consequences for the Fitting invariant over $\mathbb{Z}[G]$ of the class group $\mathrm{Cl}(\mathcal{O}_L)$ in terms of the value of the series $\theta_{L/K,S}(z)$ at $z = 0$.

In another direction, it would seem natural to expect analogues of Theorem 1.1 and Corollary 1.2 concerning the structure of even-dimensional higher algebraic K-groups that extend the results proved for abelian extensions by Barrett in his PhD thesis [1] and, perhaps more optimistically, to hope that our approach might also usefully apply in the setting of the elliptic curve results that are proved by Burns, Macias Castillo and Wuthrich in [15].

The main contents of the present article is as follows. To prove Theorem 1.1 we must first rework the statement of the Equivariant Tamagawa Number Conjecture for $\big(h^0(\mathrm{Spec}(L))(1), \mathbb{Z}_p[G]^-\big)$ into a more explicit form that is better suited to our purposes (this is done in §2). We shall then prove a purely algebraic result which we feel may well be of some independent interest (this is Proposition 3.2) and finally derive Theorem 1.1 and Corollary 1.2 in §4 by combining this algebraic observation with our earlier computations.

2. The Equivariant Tamagawa Number Conjecture. In this section we quickly review the statement of the Equivariant Tamagawa Number Conjecture and then reinterpret the relevant special case in a more explicit and convenient form.

2.1. Relative K-theory. Let R be an integral domain of characteristic 0 and field of fractions F and let E be an extension field of F. Let $\mathfrak{A}$ be an R-order in a finite-dimensional semisimple F-algebra A and set $A_E := E \otimes_F A$.

We denote by $K_0(\mathfrak{A}, A_E)$ the algebraic K_0-group that is associated to the ring homomorphism $\mathfrak{A} \to A_E$. The constructions that we use depend both upon the description of the abelian group $K_0(\mathfrak{A}, A_E)$ in terms of explicit generators and relations that is given by Swan in [33, p. 215] and the fact, first observed by Burns and Flach [12, §2.8], that there is a canonical isomorphism of the form

$$\iota_{\mathfrak{A},A_E} : \pi_0(V(\mathfrak{A}, A_E)) \cong K_0(\mathfrak{A}, A_E), \tag{6}$$

where $V(\mathfrak{A}, A_E)$ is a suitable fibre product of categories of virtual objects in the sense of Deligne. In particular, we shall frequently construct elements of $K_0(\mathfrak{A}, A_E)$ by invoking the theory of 'refined Euler characteristics' that was introduced by Burns in [8, §1] and [9] (by using the description in terms of generators and relations) and later reworked and extended by Breuning and Burns in [6] (by invoking the theory of virtual objects).

We recall that $K_0(\mathfrak{A}, A_E)$ is functorial in the pair $(\mathfrak{A}, E)$ and sits inside a canonical long exact sequence of relative K-theory, and we make frequent use of the connecting homomorphism $\partial_{\mathfrak{A},A_E} : K_1(A_E) \to K_0(\mathfrak{A}, A_E)$ from this sequence. We also use the fact that the reduced norm of A_E induces an injective homomorphism

$$\mathrm{Nrd}_{A_E} : K_1(A_E) \to \zeta(A_E)^\times.$$

Let now G be a finite group. Then for every prime p and isomorphism of fields $j : \mathbb{C} \cong \mathbb{C}_p$ the ring homomorphism j_* defined in (3) restricts to give a group homomorphism

$$j_* : \zeta(\mathbb{R}[G])^\times \to \zeta(\mathbb{C}_p[G])^\times = \mathrm{im}(\mathrm{Nrd}_{\mathbb{C}_p[G]})$$

and also induces (by the functoriality of relative K-theory) a group homomorphism

$$j_* : K_0(\mathbb{Z}[G], \mathbb{R}[G]) \to K_0(\mathbb{Z}_p[G], \mathbb{C}_p[G]).$$

We recall that the 'extended boundary homomorphism'

$$\delta_G : \zeta(\mathbb{R}[G])^\times \to K_0(\mathbb{Z}[G], \mathbb{R}[G])$$

defined by Burns and Flach in [12, §2.8] is the unique homomorphism for which both $\delta_G \circ \mathrm{Nrd}_{\mathbb{R}[G]} = \partial_{\mathbb{Z}[G],\mathbb{R}[G]}$ and, in addition, one has $j_* \circ \delta_G = \delta_{G,p} \circ j_*$ (with the respective j_*), where we set

$$\delta_{G,p} := \partial_{\mathbb{Z}_p[G],\mathbb{C}_p[G]} \circ (\mathrm{Nrd}_{\mathbb{C}_p[G]})^{-1}$$

(which, we note, makes sense since the map $\mathrm{Nrd}_{\mathbb{C}_p[G]}$ is bijective).

For any noetherian ring R we shall write $D^{\mathrm{perf}}(R)$ for the derived category of complexes of R-modules.

2.2. Statement of the conjecture. Let L/K be a finite Galois extension of number fields of a group G. For any motive M defined over K we regard the motive $M_L := h^0(\mathrm{Spec}(L)) \otimes_{h^0(\mathrm{Spec}(K))} M$ as defined over K and endowed with a natural left action of the group ring $\mathbb{Q}[G]$ (via the first factor). We write $L^*(M_L)$ for the leading term in the Taylor expansion at $z = 0$ of the natural $\zeta(\mathbb{C}[G])$-valued L-function of M_L.

Then the 'Equivariant Tamagawa Number Conjecture' of [12, Conjecture 4] predicts for the pair $(M_L, \mathbb{Z}[G])$ an equality in $K_0(\mathbb{Z}[G], \mathbb{R}[G])$ of the form

$$\delta_G(L^*(M_L)) = \chi(M_L), \tag{7}$$

where $\chi(M_K)$ is an Euler characteristic that is defined by using virtual objects arising from the various motivic cohomology groups, realisations, comparison isomorphisms and regulators associated to both M_K and its Kummer dual. This equality refines the seminal 'Tamagawa Number Conjecture' that was originally formulated by Bloch and Kato in [4] and later extended and refined by Fontaine and Perrin-Riou [18].

The nature of the above conjectural equality (7) is in general rather involved and so in this section we begin the task of reworking the appropriate special case into a form that is better suited to our purposes. Our starting point for this are the explicit computations of Breuning and Burns in [7].

We thus fix a finite set S of places of K which contains all the archimedean places and all places which ramify in L/K, and for each prime p we write S_p for the union of S and the set of p-adic places of K. We also write $\mathcal{O}_{L,S_p}$ for the subring of L comprising elements that are integral at all places which do not lie above S_p. We write $\mathbb{Z}_p(1)$ for the inverse limit $\varprojlim_n \mu_{p^n}(\mathbb{Q}^c)$, where $\mu_{p^n}(\mathbb{Q}^c)$ is the subgroup of p^nth roots of unity in $\mathbb{Q}^c$, and the limit is taken with respect to the pth power maps. We regard $\mathbb{Z}_p(1)$ as an étale sheaf on $\mathrm{Spec}(\mathcal{O}_{L,S_p})$ in the natural way and write $R\Gamma_c(\mathcal{O}_{L,S_p}, \mathbb{Z}_p(1))$ for the compactly supported étale cohomology of $\mathbb{Z}_p(1)$, as defined, for example, in [12, §3.2].

Then, following the description given in [7, §5.3], one knows that the conjectural equality (7) is valid for the motive $M = h^0(\mathrm{Spec}(K))(1)$ if and only if for each p, and each isomorphism of fields $j : \mathbb{C} \cong \mathbb{C}_p$, one has in $K_0(\mathbb{Z}_p[G], \mathbb{C}_p[G])$ an equality

$$\delta_{G,p}\big(j_*(\theta^*_{L/K,S_p}(1)^{\#})\big) = -\iota_{\mathbb{Z}_p[G],\mathbb{C}_p[G]}\big([R\Gamma_c(\mathcal{O}_{L,S_p}, \mathbb{Z}_p(1))]_{\mathbb{Z}_p[G]}, \omega^j\big). \tag{8}$$

Here $[R\Gamma_c(\mathcal{O}_{L,S_p}, \mathbb{Z}_p(1))]_{\mathbb{Z}_p[G]}$ is the virtual object over $\mathbb{Z}_p[G]$ associated to the complex $R\Gamma_c(\mathcal{O}_{L,S_p}, \mathbb{Z}_p(1))$ in $D^{\mathrm{perf}}(\mathbb{Z}_p[G])$, and ω^j is a canonical morphism of virtual objects over $\mathbb{C}_p[G]$,

$$\begin{aligned}[][R\Gamma_c(\mathcal{O}_{L,S_p}, \mathbb{Z}_p(1))]_{\mathbb{Z}_p[G]} &\otimes_{\mathbb{Z}_p[G]} \mathbb{C}_p[G] \\ &= [R\Gamma_c(\mathcal{O}_{L,S_p}, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \mathbb{C}_p]_{\mathbb{C}_p[G]} \to \mathbf{1}_{\mathbb{C}_p[G]},\end{aligned}$$

that we shall make more precise in §2.4. (Note that one must use S_p rather than S on the left hand side of the equality (8) because the explicit computations made in [7, §5] always assume that S contains all p-adic places of K.)

In the following we shall always assume, as in §1, that L is a CM-field, K is totally real and p is an odd prime. We define a central idempotent of $\mathbb{Z}_p[G]$ by setting

$$e^- := (1-\tau)/2 \in \zeta(\mathbb{Z}_p[G])$$

and then for any element x of a $\mathbb{Z}_p[G]$-module M we write x^- and M^- in place of e^-x and e^-M respectively. We will also use a similar convention for morphisms and complexes of $\mathbb{Z}_p[G]$-modules.

2.3. Compactly supported étale cohomology. Before proceeding it is convenient to describe the complex $R\Gamma_c(\mathcal{O}_{L,S_p}, \mathbb{Z}_p(1))$ more explicitly and so this is what we do now.

We write $M_{S_p}(L)$ for the maximal abelian pro-p extension of L that is unramified outside all places of L that lie above those in S_p, and note that, since $M_{S_p}(L)$ is Galois over K, the group $\mathrm{Gal}(M_{S_p}(L)/L)$ has a natural conjugation action of the algebra $\mathbb{Z}_p[G]$.

We also consider the direct sum

$$H_{B,p} := \bigoplus_{\Sigma(L)} \mathbb{Z}_p(1)$$

as a $\mathbb{Z}_p[G \times \mathrm{Gal}(\mathbb{C}/\mathbb{R})]$-module, where G acts via L and $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts diagonally. We then let $H^+_{B,p}$ denote the $\mathbb{Z}_p[G]$-submodule of $H_{B,p}$ comprising elements that are invariant under the given action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ and note that $H^+_{B,p}$ is a free $\mathbb{Z}_p[G]^-$-module with basis $\{\{\exp(2\pi i/p^n)\}_{n\geq 0} \cdot$

$(\sigma_b - c \circ \sigma_b)\}_{1\le b\le d_K}$, where c is the generator of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ and we regard $\{\exp(2\pi i/p^n)\}_{n\ge 0}$ as a generator of the $\mathbb{Z}_p$-module $\mathbb{Z}_p(1)$.

We recall that, for any noetherian ring R, a complex of R-modules is said to be 'cohomologically perfect' if it can be represented by a bounded complex of finitely generated R-modules and is also such that each of its cohomology modules has finite projective dimension over R. We further note that a cohomologically perfect complex of R-modules is automatically a perfect complex of R-modules.

We write $S_p(L)$ for the set of places of L above those in S_p and use the diagonal homomorphism $\Delta_{L,S_p(L)}$ from (5).

PROPOSITION 2.1. *We fix L/K and S as above and we set $C^\bullet := R\Gamma_c(\mathcal{O}_{L,S_p}, \mathbb{Z}_p(1))$. Moreover, we set $\mathfrak{A} := \mathbb{Z}_p[G]^-$. Then $C^{\bullet,-}$ is a cohomologically perfect complex of $\mathfrak{A}$-modules that is acyclic outside degrees one and two. In addition, there is a canonical identification of $H^1(C^{\bullet,-})$ with the free $\mathfrak{A}$-module $H^+_{B,p}$ and of $H^2(C^{\bullet,-})$ with the Galois group* $\mathrm{Gal}(M_{S_p}(L)/L)^-$.

Proof. It is well known that the complex $C^\bullet$ has all of the following properties (see, for example, the proof of [7, Lemma 4.1]):

(P1) $C^\bullet$ is a perfect complex of $\mathbb{Z}_p[G]$-modules that is acyclic outside degrees one, two and three.

(P2) There are canonical short exact sequences of $\mathbb{Z}_p[G]$-modules

$$0 \to H^+_{B,p} \to H^1(C^\bullet) \to \ker(\Delta_{L,S_p(L)}) \to 0$$

and

$$0 \to \mathrm{cok}(\Delta_{L,S_p(L)}) \to H^2(C^\bullet) \to \mathrm{Cl}(\mathcal{O}_{L,S_p}) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to 0.$$

(P3) The $\mathbb{Z}_p[G]$-module $H^3(C^\bullet)$ is canonically isomorphic to $\mathbb{Z}_p$ (with trivial G-action).

Clearly, $\mathbb{Z}_p^-$ vanishes and hence (P1) and (P3) combine to imply that $C^{\bullet,-}$ is a perfect complex of $\mathfrak{A}$-modules that is acyclic outside degrees one and two. A standard construction of homological algebra then shows that $C^{\bullet,-}$ is quasi-isomorphic to a complex of $\mathfrak{A}$-modules $\Psi^\bullet$ of the form $\Psi^1 \to \Psi^2$ where Ψ^2 is a finitely generated free $\mathfrak{A}$-module and Ψ^1 is a finitely generated $\mathfrak{A}$-module that has finite projective dimension (and is placed in degree one). Using such a quasi-isomorphism we can identify the cohomology groups $H^1(\Psi^\bullet)$ and $H^2(\Psi^\bullet)$ with $H^1(C^{\bullet,-})$ and $H^1(C^{\bullet,-})$ and hence obtain an exact sequence of $\mathfrak{A}$-modules of the form

$$0 \to H^1(C^{\bullet,-}) \to \Psi^1 \to \Psi^2 \to H^2(C^{\bullet,-}) \to 0.$$

But, as already observed in Remark 1.4, the group $\ker(\Delta_{L,S_p(L)})^-$ vanishes and so (P2) induces a canonical identification of $H^1(C^{\bullet,-}) = H^1(C^\bullet)^-$ with $(H^+_{B,p})^- = H^+_{B,p}$. In particular, since $H^1(C^{\bullet,-})$ is a free $\mathfrak{A}$-module, the last displayed exact sequence implies that $H^2(C^{\bullet,-})$ has finite projective dimension as an $\mathfrak{A}$-module and hence that $C^{\bullet,-}$ is a cohomologically perfect complex of $\mathfrak{A}$-modules, as claimed.

At this stage to complete the proof it suffices to note that there are canonical isomorphisms of $\mathbb{Z}_p[G]$-modules of the form

$$\begin{aligned} H^2(C^\bullet) &\cong \mathrm{Hom}_{\mathrm{cont}}(H^1(\mathcal{O}_{L,S_p}, \mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Q}_p/\mathbb{Z}_p) \\ &\cong \mathrm{Hom}_{\mathrm{cont}}\Big(\varinjlim_n H^1(\mathcal{O}_{L,S_p}, \mathbb{Z}/p^n), \mathbb{Q}_p/\mathbb{Z}_p\Big) \\ &\cong \varprojlim_n \mathrm{Hom}_{\mathrm{cont}}(H^1(\mathcal{O}_{L,S_p}, \mathbb{Z}/p^n), \mathbb{Q}_p/\mathbb{Z}_p) \\ &\cong \varprojlim_n \mathrm{Hom}_{\mathrm{cont}}\big(\mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(M_{S_p}(L)/L), \mathbb{Z}/p^n), \mathbb{Q}_p/\mathbb{Z}_p\big) \\ &\cong \varprojlim_n \mathrm{Gal}(M_{S_p}(L)/L)/p^n \cong \mathrm{Gal}(M_{S_p}(L)/L). \end{aligned}$$

All isomorphisms here are clear except for the first, which is induced by the Artin–Verdier Duality Theorem, and the fourth, which is induced by the canonical identification of $H^1(\mathcal{O}_{L,S_p}, \mathbb{Z}/p^n)$ with

$$\mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(N_{S_p}(L)/L), \mathbb{Z}/p^n) \cong \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(M_{S_p}(L)/L), \mathbb{Z}/p^n).$$

Here $N_{S_p}(L)$ denotes the maximal pro-p extension of L that is unramified outside of $S_p(L)$ and so the displayed isomorphism follows from the fact that $\mathrm{Gal}(M_{S_p}(L)/L)$ identifies with the abelianisation of $\mathrm{Gal}(N_{S_p}(L)/L)$. ∎

2.4. A useful reinterpretation. In this subsection we shall give a more explicit description of the 'minus part' of the conjectural equality (8) in terms of the refined Euler characteristics $\chi(-,-)$ that are defined in [6].

Below we often use the following convention: if E is an extension field of $\mathbb{Q}_p$, then in each degree i we set

$$H^i_c(E(1)) := E \otimes_{\mathbb{Z}_p} H^i(R\Gamma_c(\mathcal{O}_{L,S_p}, \mathbb{Z}_p(1))),$$

which is to be regarded as an $E[G]$-module in the natural way.

PROPOSITION 2.2. *There exists a canonical isomorphism of $\mathbb{C}_p[G]$-modules* (*that is described explicitly in the course of the proof below*)

$$\Phi^j_{L/K} : H^2_c(\mathbb{C}_p(1))^- \to H^1_c(\mathbb{C}_p(1))^-$$

for which one has

$$\iota_{\mathbb{Z}_p[G],\mathbb{C}_p[G]}\big([R\Gamma_c(\mathcal{O}_{L,S_p}, \mathbb{Z}_p(1))]_{\mathbb{Z}_p[G]}, \omega^j\big)^- = \chi\big(R\Gamma_c(\mathcal{O}_{L,S_p}, \mathbb{Z}_p(1))^-, \Phi^j_{L/K}\big)$$

in $K_0(\mathbb{Z}_p[G]^-, \mathbb{C}_p[G]^-)$.

Proof. Recalling that $\Sigma(L)$ denotes the set of all complex embeddings $L \to \mathbb{C}$ we consider the direct sum $\bigoplus_{\Sigma(L)} \mathbb{C}$ as a $G \times \mathrm{Gal}(\mathbb{C}/\mathbb{R})$-module, where G acts via L and $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts diagonally. We write H_B for the $G \times$

$\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-submodule $\bigoplus_{\Sigma(L)} 2\pi i\cdot\mathbb{Z}$ of $\bigoplus_{\Sigma(L)}\mathbb{C}$ and let H_B^+ and $(\bigoplus_{\Sigma(L)}\mathbb{C})^+$ denote the G-submodules comprising elements invariant under the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

Then, by taking the 'minus part' of (that is, by applying the exact functor $\mathbb{C}_p[G]^-\otimes_{\mathbb{C}_p[G]} -$ to) the explicit description of the morphism ω^j that is given by Breuning and Burns in [7, (26)] one finds that $\omega^{j,-}$ is equal to the following composite morphism:

$$\begin{aligned}(9)\qquad [(R\Gamma_c(\mathcal{O}_{L,S_p},\mathbb{Z}_p(1))\otimes_{\mathbb{Z}_p}\mathbb{C}_p)^-]_{\mathbb{C}_p[G]^-} &\\ \cong [H_c^1(\mathbb{C}_p(1))^-]^{-1}_{\mathbb{C}_p[G]^-}\otimes[H_c^2(\mathbb{C}_p(1))^-]_{\mathbb{C}_p[G]^-}&\\ \cong [H_B^+\otimes_{\mathbb{Z}}\mathbb{C}_p]^{-1}_{\mathbb{C}_p[G]^-}\otimes[H_B^+\otimes_{\mathbb{Z}}\mathbb{C}_p]_{\mathbb{C}_p[G]^-}\cong 1_{\mathcal{V}(\mathbb{C}_p[G]^-)}.&\end{aligned}$$

The first morphism here is the canonical 'passage to cohomology' morphism, the third is induced by the very definition of inverse virtual object, and the second is equal to $[\psi_1]^{-1}_{\mathbb{C}_p[G]^-}\otimes[\psi_2]_{\mathbb{C}_p[G]^-}$, where we use the composite isomorphisms of $\mathbb{C}_p[G]^-$-modules

$$(10)\qquad \psi_1: H_c^1(\mathbb{C}_p(1))^-\cong H_{B,p}^+\otimes_{\mathbb{Z}_p}\mathbb{C}_p\cong H_B^+\otimes_{\mathbb{Z}}\mathbb{C}_p$$

and

$$\begin{aligned}(11)\qquad \psi_2: H_c^2(\mathbb{C}_p(1))^-&\cong U_p^1(L)^-\otimes_{\mathbb{Z}_p}\mathbb{C}_p\\ &\cong\Big(\prod_{w\mid p}L_w\Big)^-\otimes_{\mathbb{Q}_p}\mathbb{C}_p\cong(L\otimes_{\mathbb{Q}}\mathbb{C}_p)^-\cong H_B^+\otimes_{\mathbb{Z}}\mathbb{C}_p.\end{aligned}$$

Here the first isomorphism in (10) comes from Proposition 2.1 and the second is the scalar extension of the isomorphism of $\mathbb{Z}_p[G]^-$-modules

$$(12)\qquad H_{B,p}^+\to H_B^+\otimes_{\mathbb{Z}}\mathbb{Z}_p$$

that is the restriction of the map $H_{B,p}\to H_B\otimes_{\mathbb{Z}}\mathbb{Z}_p$ which sends each element $(n_\sigma\{\exp(2\pi i/p^n)\}_{n\geq 0})_{\sigma\in\Sigma(L)}$ to $(n_\sigma(2\pi i))_{\sigma\in\Sigma(L)}$; the first isomorphism in (11) is induced by the isomorphism $H_c^2(\mathbb{C}_p(1))^-\cong\mathrm{cok}(\Delta_{L,S_p})^-\otimes_{\mathbb{Z}_p}\mathbb{C}_p$ coming from Proposition 2.1 together with the obvious equalities

$$\mathrm{cok}(\Delta_{L,S_p})^-\otimes_{\mathbb{Z}_p}\mathbb{C}_p=\mathrm{cok}(\Delta_{L,S'})^-\otimes_{\mathbb{Z}_p}\mathbb{C}_p=U_p^1(L)^-\otimes_{\mathbb{Z}_p}\mathbb{C}_p,$$

where S' denotes the set of places of K which are either p-adic or archimedean; the second isomorphism in (11) is induced by the canonical isomorphism

$$(13)\qquad U_p^1(L)\to\prod_{w\mid p}L_w,\qquad (u_w)_w\mapsto(\log_p u_w)_w;$$

the third isomorphism in (11) is induced by the isomorphism

$$(14)\qquad L\otimes_{\mathbb{Q}}\mathbb{Q}_p\to\prod_{w\mid p}L_w$$

coming from the natural diagonal map $L \to \prod_{w|p} L_w$; the last isomorphism in (11) is induced by the image under $- \otimes_{\mathbb{R},j} \mathbb{C}_p$ of the isomorphism of $\mathbb{R}[G]$-modules

$$L^- \otimes_{\mathbb{Q}} \mathbb{R} \cong H_B^+ \otimes_{\mathbb{Z}} \mathbb{R} \tag{15}$$

that is obtained by restricting the isomorphism $L \otimes_{\mathbb{Q}} \mathbb{R} \cong (\bigoplus_{\Sigma(L)} \mathbb{C})^+$ which sends each element $a \otimes z$ to $(\sigma(a)z)_{\sigma \in \Sigma(L)}$.

We now define an isomorphism of $\mathbb{C}_p[G]^-$-modules by setting

$$\Phi^j_{L/K} := \psi_1^{-1} \circ \psi_2 : H^2_c(\mathbb{C}_p(1))^- \to H^1_c(\mathbb{C}_p(1))^-.$$

We recall that an explicit description of the isomorphism $\iota_{\mathbb{Z}_p[G]^-,\mathbb{C}_p[G]^-}$ that occurs in (6) is given in the proof of [12, Proposition 2.5] (and is also derived, with more details, in [7, Lemma 5.1, Theorem 6.2 and Lemma 6.3]). By combining this description with the explicit description of the morphism $\omega^{j,-}$ given in (9) and the very definition of the isomorphism $\Phi^j_{L/K}$ it is clear that

$$\begin{aligned}
&\iota_{\mathbb{Z}_p[G],\mathbb{C}_p[G]}([R\Gamma_c(\mathcal{O}_{L,S_p}, \mathbb{Z}_p(1))]_{\mathbb{Z}_p[G]}, \omega^j)^- \\
&\qquad\qquad = \iota_{\mathbb{Z}_p[G]^-,\mathbb{C}_p[G]^-}\big([R\Gamma_c(\mathcal{O}_{L,S_p}, \mathbb{Z}_p(1))^-]_{\mathbb{Z}_p[G]^-}, \omega^{j,-}\big) \\
&\qquad\qquad = \chi\big(R\Gamma_c(\mathcal{O}_{L,S_p}, \mathbb{Z}_p(1))^-, \Phi^j_{L/K}\big)
\end{aligned}$$

in $K_0(\mathbb{Z}_p[G]^-, \mathbb{C}_p[G]^-)$, as we wanted to prove. ∎

2.5. A comparison of p-adic regulators. In this section we prove the following useful comparison result.

PROPOSITION 2.3. *For each element $\underline{u}$ of $U^1_p(L)^{d_K}$ there exists a homomorphism of $\mathbb{Z}_p[G]^-$-modules*

$$\phi_{\underline{u}} : H^1_c(\mathcal{O}_{L,S_p}, \mathbb{Z}_p(1))^- \to H^2_c(\mathcal{O}_{L,S_p}, \mathbb{Z}_p(1))^-$$

for which one has

$$\mathrm{Nrd}_{\mathbb{C}_p[G]}(\Phi^j_{L/K} \circ (\mathbb{C}_p \otimes_{\mathbb{Z}_p} \phi_{\underline{u}})) = \mathrm{Nrd}_{\mathbb{C}_p[G]}(M^j(\underline{u}))$$

where $M^j(\underline{u})$ is the matrix in $\mathrm{M}_{d_K}(\mathbb{C}_p[G])$ that is defined in (2). *Furthermore, the homomorphism $\phi_{\underline{u}}$ is injective whenever $\underline{u}$ belongs to the set $\mathcal{U}_{L/K,p}$.*

Proof. First we fix, just as in §1, a set

$$Z := \{\sigma_b : 1 \le b \le d_K\}$$

of representatives for the orbits of the natural action of G on $\Sigma(L)$.

Then, since each element of $\Sigma(L)$ can be written uniquely in the form $\sigma_b \circ g$ for some index b and element g of G, this choice gives rise to an isomorphism of $\mathbb{Z}_p[G]$-modules

$$h'_Z : H_B \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \bigoplus_{b=1}^{d_K} \mathbb{Z}_p[G]$$

which sends each element $(2\pi i \cdot n_\rho)_{\rho \in \Sigma(L)} \otimes_{\mathbb{Z}} 1$ of $H_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$ to the element $(\sum_{g\in G} n_{\sigma_b g} g^{-1})_{1\le b\le d_K}$ of $\bigoplus_{b=1}^{d_K} \mathbb{Z}_p[G]$. In the following we write

$$h_Z : H_B^+ \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \bigoplus_{b=1}^{d_K} \mathbb{Z}_p[G]^-$$

for the isomorphism of $\mathbb{Z}_p[G]^-$-modules that is obtained by restricting h'_Z.

We then consider the composite isomorphism of $\mathbb{Z}_p[G]^-$-modules

$$\psi_Z : H^1_c(\mathbb{Z}_p(1))^- \cong H^+_{B,p} \cong H^+_B \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \bigoplus_{b=1}^{d_K} \mathbb{Z}_p[G]^-,$$

where the first isomorphism comes from Proposition 2.2, the second one is (12) and the last one is h_Z.

Recalling that $\underline{u} = (u_b)_{1\le b\le d_K}$, we next write

$$\Phi_{\underline{u}} : \bigoplus_{b=1}^{d_K} \mathbb{Z}_p[G]^- \to U^1_p(L)^- \tag{16}$$

for the natural homomorphism of $\mathbb{Z}_p[G]^-$-modules which sends each element $(z_b)_{1\le b\le d_K}$ to $\prod_{b=1}^{d_K} u_b^{z_b}$ (where we write the natural action of $\mathbb{Z}_p[G]$ on unit groups exponentially) and also

$$a_{L,S,p} : U^1_p(L) = \prod_{w|p} U^1(L_w) \subset \prod_{w|p} L_w^\times \hat{\otimes}_{\mathbb{Z}} \mathbb{Z}_p \xrightarrow{A_{L,S,p}} \mathrm{Gal}(M_{S_p}(L)/L) \tag{17}$$

for the composite homomorphism of $\mathbb{Z}_p[G]$-modules in which $A_{L,S,p}$ denotes the homomorphism that is induced by (restriction of) the global Artin reciprocity map.

We then define

$$\phi_{\underline{u},Z} : H^1_c(\mathcal{O}_{L,S_p}, \mathbb{Z}_p(1))^- \to H^2_c(\mathcal{O}_{L,S_p}, \mathbb{Z}_p(1))^- \cong \mathrm{Gal}(M_{S_p}(L)/L)^-$$

to be the unique homomorphism of $\mathbb{Z}_p[G]^-$-modules which makes the following diagram commute:

$$\begin{tikzcd} H^1_c(\mathcal{O}_{L,S_p}, \mathbb{Z}_p(1))^- \arrow[r, "\phi_{\underline{u},Z}"] \arrow[d, "\psi_Z"'] & H^2_c(\mathcal{O}_{L,S_p}, \mathbb{Z}_p(1))^- \\ \bigoplus_{b=1}^{d_K} \mathbb{Z}_p[G]^- \arrow[r, "\Phi_{\underline{u}}"] & U^1_p(L)^- \arrow[u, "a^-_{L,S,p}"'] \end{tikzcd} \tag{18}$$

Now for each homomorphism of $\mathbb{Z}_p[G]^-$-modules $\theta : M \to N$ we write $\theta_{\mathbb{C}_p}$ for the induced homomorphism of $\mathbb{C}_p[G]^-$-modules $\mathbb{C}_p \otimes_{\mathbb{Z}_p} M \to \mathbb{C}_p \otimes_{\mathbb{Z}_p} N$.

We then define Ξ^j_Z to be the unique isomorphism of $\mathbb{C}_p[G]^-$-modules which makes the (second square of the) following diagram commute:

$$
\text{(19)} \qquad
\begin{tikzcd}
H^1_c(\mathbb{C}_p(1))^- \arrow[r, "(\phi_{\underline{u},Z})_{\mathbb{C}_p}"] \arrow[d, "(\psi_Z)_{\mathbb{C}_p}"'] & H^2_c(\mathbb{C}_p(1))^- \arrow[r, "\Phi^j_{L/K}"] & H^1_c(\mathbb{C}_p(1))^- \arrow[d, "(\psi_Z)_{\mathbb{C}_p}"'] \\
\bigoplus_{b=1}^{d_K} \mathbb{C}_p[G]^- \arrow[r, "(\Phi_{\underline{u}})_{\mathbb{C}_p}"] & \mathbb{C}_p \otimes_{\mathbb{Z}_p} U^1_p(L)^- \arrow[r, "\Xi^j_Z"] \arrow[u, "(a^-_{L,S,p})_{\mathbb{C}_p}"'] & \bigoplus_{b=1}^{d_K} \mathbb{C}_p[G]^-
\end{tikzcd}
$$

In particular, since the vertical maps in this diagram are bijective we find that

$$\mathrm{Nrd}_{\mathbb{C}_p[G]}(\Phi^j_{L/K} \circ (\phi_{\underline{u},Z})_{\mathbb{C}_p}) = \mathrm{Nrd}_{\mathbb{C}_p[G]}(\Xi^j_Z \circ (\Phi_{\underline{u}})_{\mathbb{C}_p}).$$

This equality shows that the claimed result will follow (with $\phi_{\underline{u}} = \phi_{\underline{u},Z}$) if we can show that $M^j(\underline{u})$ is equal to the matrix of the composite map $\Xi^j_Z \circ (\Phi_{\underline{u}})_{\mathbb{C}_p}$ with respect to the obvious $\mathbb{C}_p[G]^-$-basis of the (free) module $\bigoplus_{b=1}^{d_K} \mathbb{C}_p[G]^-$.

To check this we recall that $\Phi^j_{L/K}$ is defined to be the composite $\psi_1^{-1} \circ \psi_2$ whilst the explicit definitions of the homomorphisms ψ_Z, h_Z and ψ_1 ensures that $(\psi_Z)_{\mathbb{C}_p}$ is equal to $(h_Z)_{\mathbb{C}_p} \circ \psi_1$. From the commutativity of the second square in (19) we therefore find that

$$\text{(20)} \qquad \Xi^j_Z = (\psi_Z)_{\mathbb{C}_p} \circ \Phi^j_{L/K} \circ (a^-_{L,S,p})_{\mathbb{C}_p} = (h_Z)_{\mathbb{C}_p} \circ \psi_2 \circ (a^-_{L,S,p})_{\mathbb{C}_p}.$$

Now $(a^-_{L,S,p})_{\mathbb{C}_p}$ is the inverse of the first map that occurs in the definition (11) of the isomorphism ψ_2 and so $\psi_2 \circ (a^-_{L,S,p})_{\mathbb{C}_p}$ is equal to the composite homomorphism

$$U^1_p(L)^- \otimes_{\mathbb{Z}_p} \mathbb{C}_p \cong \Big(\prod_{w|p} L_w\Big)^- \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong (L \otimes_{\mathbb{Q}} \mathbb{C}_p)^- \cong H^+_B \otimes_{\mathbb{Z}} \mathbb{C}_p \subset \prod_{\Sigma(L)} \mathbb{C}_p$$

which sends each element $u \otimes_{\mathbb{Z}_p} 1$ to $\big(\log_p\big(j(\rho(u)) \otimes_{\mathbb{Z}_p} 1\big)\big)_{\rho\in\Sigma(L)}$, where $\log_p$ is the $\mathbb{C}_p$-linear extension of Iwasawa's p-adic logarithm to $U^1_p(L)^- \otimes_{\mathbb{Z}_p} \mathbb{C}_p$.

By applying $(h_Z)_{\mathbb{C}_p}$ to this map it then follows from (20) that for every element u in $U_p(L)^-$ one has an equality

$$\begin{aligned}
\Xi^j_Z(u \otimes_{\mathbb{Z}_p} 1) &= (h_Z)_{\mathbb{C}_p}\big(\big(\log_p\big(j(\rho(u)) \otimes_{\mathbb{Z}_p} 1\big)\big)_{\rho\in\Sigma(L)}\big) \\
&= \bigg(\frac{1}{j(2\pi i)} \sum_{g\in G} \log_p\big(j \circ \sigma_b(g(u))\big) g^{-1}\bigg)_{1\le b\le d_K}.
\end{aligned}$$

After recalling the explicit definition (16) of the homomorphism $\Phi_{\underline{u}}$ it is then clear that, with respect to the obvious $\mathbb{C}_p[G]^-$-basis of $\bigoplus_{b=1}^{d_K} \mathbb{C}_p[G]^-$, the matrix of the composite $\Xi^j_Z \circ (\Phi_{\underline{u}})_{\mathbb{C}_p}$ is equal to

$$M^j(\underline{u}) := \left(\frac{1}{j(2\pi i)}\sum_{g\in G}\log_p\big(j\circ\sigma_b(g(u_a))\big)g^{-1}\right)_{1\le a,b\le d_K},$$

as required.

It now only remains to check that the homomorphism $\phi_{\underline{u},Z}$ is injective provided that $\underline{u}$ belongs to the set $\mathcal{U}_{L/K,p}$ defined just after (2). In this case the map $\Phi_{\underline{u}}$ is obviously injective and so $\mathrm{im}(\Phi_{\underline{u}})$ is a free $\mathbb{Z}_p[G]^-$-module. On the other hand, from the lower row of the exact commutative diagram in [25, Lemma (10.3.12)] (with S equal to $S_p(L)$ in our notation) one finds that the kernel of the homomorphism $A_{L,S,p}$ that occurs in (17) is equal to the intersection

$$\mathrm{im}(\Delta_{L,S(L)})\cap\Big(\prod_{w|p}L^\times_w\,\hat{\otimes}_{\mathbb{Z}}\,\mathbb{Z}_p\Big)\subset\prod_{w\in S_p(L)}L^\times_w\,\hat{\otimes}_{\mathbb{Z}}\,\mathbb{Z}_p,$$

where the diagonal homomorphism $\Delta_{L,S(L)}$ is as in (5).

Given the commutativity of the diagram (18) this implies that the composite map $\Phi_{\underline{u}}\circ\psi_Z$ induces an isomorphism

$$\ker(\phi_{\underline{u},Z})\cong\mathrm{im}(\Phi_{\underline{u}})\cap\mathrm{im}(\Delta_{L,S(L)}). \tag{21}$$

Now this intersection is equal to

$$\mathrm{im}(\Phi_{\underline{u}})\cap\Delta_{L,S(L)}(\mathcal{O}^\times_L\otimes_{\mathbb{Z}}\mathbb{Z}_p)^- = \mathrm{im}(\Phi_{\underline{u}})\cap\Delta_{L,S(L)}((\mathcal{O}^\times_L\otimes_{\mathbb{Z}}\mathbb{Z}_p)^-)$$

and this module is trivial since $\mathrm{im}(\Phi_{\underline{u}})$ is $\mathbb{Z}_p$-free whilst $(\mathcal{O}^\times_L\otimes_{\mathbb{Z}}\mathbb{Z}_p)^-$ is finite (as already noted in Remark 1.4). The isomorphism (21) therefore implies that $\phi_{\underline{u},Z}$ is injective, as claimed.

This completes the proof of Proposition 2.3. ∎

3. A useful algebraic observation. In this section we prove a purely algebraic result that plays an important role in our proof of Theorem 1.1.

To do this we fix a Dedekind domain R with field of fractions F and an extension field E of F. We also fix an R-order $\mathfrak{A}$ in a finitely generated semisimple F-algebra A and we set $A_E := E\otimes_F A$. (The standard example we have in mind for this is the case $R=\mathbb{Z}_p$, $E=\mathbb{C}_p$ and $\mathfrak{A}=\mathbb{Z}_p[G]^-$ and $A=\mathbb{Q}_p[G]^-$ for some finite group G.)

3.1. Fitting invariants and the denominator ideal. In this subsection we quickly review the invariants that are introduced by Nickel in [26].

For $\zeta(\mathfrak{A})$-submodules of an R-torsionfree $\zeta(\mathfrak{A})$-module X Nickel introduces an equivalence relation of ‘$\mathrm{Nrd}(\mathfrak{A})$-equivalence’ and a notion of inclusion among its classes. More precisely, two such submodules M and N of X are said to be $\mathrm{Nrd}(\mathfrak{A})$-equivalent if there are a natural number n and a matrix $U\in\mathrm{GL}_n(\mathfrak{A})$ such that $M=\mathrm{Nrd}(U)N$ and the corresponding equivalence class is denoted by $[M]_{\mathrm{Nrd}(\mathfrak{A})}$. There are also natural notions of inclusion and containment for these classes: one writes $[M]_{\mathrm{Nrd}(\mathfrak{A})} \subset [N]_{\mathrm{Nrd}(\mathfrak{A})}$ if for each $M' \in [M]_{\mathrm{Nrd}(\mathfrak{A})}$ one has $M' \subseteq N'$ for some $N' \in [N]_{\mathrm{Nrd}(\mathfrak{A})}$, and for x in X one writes $x \in [M]_{\mathrm{Nrd}(\mathfrak{A})}$ if $x \in M'$ for some M' in $[M]_{\mathrm{Nrd}(\mathfrak{A})}$.

Let M be a finitely presented $\mathfrak{A}$-module M and

$$\mathfrak{A}^m \xrightarrow{h} \mathfrak{A}^n \to M \to 0 \tag{22}$$

a finite presentation of M. Let $S(h)$ denote the set of all $n \times n$ submatrices of the associated matrix of h (in the canonical basis) if $m \geq n$. The (non-commutative) *Fitting invariant* of h over $\mathfrak{A}$ is then defined to be the equivalence class

$$\mathrm{Fitt}_{\mathfrak{A}}(h) = \begin{cases} [\langle \mathrm{Nrd}(H) : H \in S(h)\rangle]_{\mathrm{Nrd}(\mathfrak{A})} & \text{if } m \geq n, \\ [\{0\}]_{\mathrm{Nrd}(\mathfrak{A})} & \text{if } m < n. \end{cases}$$

One also defines $\mathrm{Fitt}^{\max}_{\mathfrak{A}}(M)$ to be the unique Fitting invariant of M over $\mathfrak{A}$ which is maximal among all Fitting invariants of M with respect to the partial order '$\subset$'.

In addition, if M admits a quadratic presentation h (that is, there exists an exact sequence (22) with $m = n$), then one sets

$$\mathrm{Fitt}_{\mathfrak{A}}(M) := \mathrm{Fitt}_{\mathfrak{A}}(h).$$

This equivalence class can be shown to be independent of the chosen quadratic presentation h, and therefore coincides with $\mathrm{Fitt}^{\max}_{\mathfrak{A}}(M)$.

We now recall another important construction of Nickel, the so-called 'denominator ideal'. To do this note that for every matrix H in $\mathrm{M}_d(\mathfrak{A})$ there is a unique matrix H^* in $\mathrm{M}_d(A)$ with $HH^* = H^*H = \mathrm{nr}_A(H)I_d$ and such that for every primitive central idempotent e of $F[G]$ the matrix H^*e is invertible if and only if the reduced norm $\mathrm{Nrd}_A(H)e$ is non-zero. The *denominator ideal* of $\mathfrak{A}$ is then defined to be the ideal of $\zeta(\mathfrak{A})$ that is obtained by setting

$$\mathcal{H}(\mathfrak{A}) := \{x \in \zeta(\mathfrak{A})\colon \text{ if } d > 0 \text{ and } H \in \mathrm{M}_d(\mathfrak{A}) \text{ then } xH^* \in \mathrm{M}_d(\mathfrak{A})\}.$$

Note that for each $H \in \mathrm{M}_d(\mathfrak{A})$ the matrix H^* belongs to $\mathrm{M}_d(\mathfrak{M})$ for any maximal order $\mathfrak{M}$ in A that contains $\mathfrak{A}$ [27, Lemma 4.1]. In particular therefore, if $\mathfrak{A} = R[G]$ for a finite group G, then Jacobinski's description in [22] of the central conductor of $\mathfrak{M}$ in $R[G]$ implies that for any F-valued character ψ of G the element $\psi(1)^{-1}|G|e_\psi$ belongs to $\mathcal{H}(R[G])$, and so one has $|G|\zeta(\mathfrak{M}) \subseteq \mathcal{H}(R[G]) \subseteq \zeta(R[G])$.

In [23] Johnston and Nickel have recently given some more precise information on the ideal $\mathcal{H}(\mathfrak{A})$ and have also explicitly computed it in several important cases: for example, they have shown that $\mathcal{H}(\mathbb{Z}_p[G]) = \zeta(\mathbb{Z}_p[G])$ if and only if p does not divide the order of the commutator subgroup of G

(as is obviously the case if G is abelian and also, for example, if G is the alternating group A_4 on four letters and p is odd).

We shall make much use of the following result of Nickel (which coincides with [26, Theorem 4.2]).

PROPOSITION 3.1. *If R is an integrally closed complete commutative noetherian local ring and M is a finitely presented $\mathfrak{A}$-module, then one has an inclusion*

$$\mathcal{H}(\mathfrak{A}) \cdot \mathrm{Fitt}_{\mathfrak{A}}^{\max}(M) \subset \mathrm{Ann}_{\mathfrak{A}}(M).$$

3.2. The annihilation result. Motivated by Proposition 2.1, in this subsection we assume to be given data of the following form:

(H1) a cohomologically perfect complex $C^\bullet$ of $\mathfrak{A}$-modules that is acyclic outside degrees one and two and is such that the $\mathfrak{A}$-module $H^1(C^\bullet)$ is projective;
(H2) an isomorphism of A_E-modules $\lambda : E \otimes_R H^2(C^\bullet) \cong E \otimes_R H^1(C^\bullet)$;
(H3) an element $\mathcal{L}$ of $\zeta(A_E)^\times$ with $\delta_{\mathfrak{A},A_E}(\mathcal{L}) = -\chi(C^\bullet, \lambda)$;
(H4) an injective homomorphism of $\mathfrak{A}$-modules $\phi : H^1(C^\bullet) \to H^2(C^\bullet)$.

Here we write $\delta_{\mathfrak{A},A_E}$ for the composite of the inverse of the (bijective) reduced norm homomorphism $K_1(A_E) \to \zeta(A_E)^\times$ and the standard connecting homomorphism of relative K-theory $\partial_{\mathfrak{A},A_E} : K_1(A_E) \to K_0(\mathfrak{A}, A_E)$. We have then a commutative diagram of the form

$$\begin{tikzcd}
\zeta(\mathfrak{A})^\times \arrow[r, hook] & \zeta(A_E)^\times \arrow[dr, "\delta_{\mathfrak{A},A_E}"] & \\
K_1(\mathfrak{A}) \arrow[r] & K_1(A_E) \arrow[u, "\mathrm{Nrd}_{A_E}"] \arrow[r, "\partial_{\mathfrak{A},A_E}"] & K_0(\mathfrak{A}, A_E)
\end{tikzcd}$$

in which the lower row is exact.

We shall also assume that the order $\mathfrak{A}$ has the following property:

($*$) any finitely generated projective (left) $\mathfrak{A}$-module P is free if and only if the associated A-module P_F is free.

We recall in particular that, by a well-known result of Swan (see, for example, [16, Theorem (32.1)]), for any finite group G any direct factor $\mathfrak{A}$ of $\mathbb{Z}_p[G]$ has the property ($*$).

We shall now prove the following result.

PROPOSITION 3.2. *Let $C^\bullet$, λ, $\mathcal{L}$, and ϕ be data satisfying the hypotheses* (H1)–(H4). *If $\mathfrak{A}$ has the property* ($*$) *then*

$$\mathrm{Nrd}_{A_E}(\lambda \circ (E \otimes_R \phi)) \cdot \mathcal{L} \in \mathrm{Fitt}_{\mathfrak{A}}(\mathrm{cok}(\phi)).$$

Proof. To prove this result we fix a complex $\Psi^1 \xrightarrow{d} \Psi^2$ that is isomorphic to $C^\bullet$ in $D^{\mathrm{perf}}(\mathfrak{A})$, where the $\mathfrak{A}$-modules Ψ^1 and Ψ^2 are both finitely generated and of finite projective dimension and Ψ^1 is placed in degree one. After fixing such an isomorphism we obtain an exact sequence of $\mathfrak{A}$-modules of the form

$$0 \to H^1 \xrightarrow{\iota} \Psi^1 \xrightarrow{d} \Psi^2 \xrightarrow{\pi} H^2 \to 0,$$

where we have set $H^1 := H^1(C^\bullet)$ and $H^2 := H^2(C^\bullet)$. Since H^1 is projective we can also fix a lift $\widetilde{\phi}$ of ϕ through the surjective homomorphism π and so obtain an exact commutative diagram of $\mathfrak{A}$-modules

$$\begin{tikzcd}
H^1 \arrow[r, "="] \arrow[d, "="] & H^1 \arrow[r, "0"] \arrow[d, hook, "\iota"] & H^1 \arrow[r, "="] \arrow[d, hook, "\widetilde{\phi}"] & H^1 \arrow[d, hook, "\phi"] \\
H^1 \arrow[r, hook, "\iota"] & \Psi^1 \arrow[r, "d"] \arrow[d] & \Psi^2 \arrow[r, two heads, "\pi"] \arrow[d] & H^2 \arrow[d, "\kappa"] \\
 & \mathrm{cok}(\iota) \arrow[r, hook, "d'"] & \mathrm{cok}(\widetilde{\phi}) \arrow[r, two heads, "\pi'"] & \mathrm{cok}(\phi)
\end{tikzcd} \tag{23}$$

In this diagram the lower vertical arrows are the natural projection maps and the homomorphisms d' and π' are those that are induced by d and π respectively; exactness of the diagram is therefore an easy consequence of the fact that ϕ is assumed to be injective (by condition (H4)).

In particular, the exactness of the second and third columns in (23) combines with the projectivity of H^1 (by condition (H1)) and our choice of modules Ψ^i to imply that the $\mathfrak{A}$-modules $\mathrm{cok}(\iota)$ and $\mathrm{cok}(\widetilde{\phi})$ are of finite projective dimension, and then the exactness of the lower row implies that the $\mathfrak{A}$-module $\mathrm{cok}(\phi)$ is also of finite projective dimension.

We write $\Psi_1^\bullet$ and $\Psi_2^\bullet$ for the complexes $H^1 \xrightarrow{0} H^1$ and $\mathrm{cok}(\iota) \xrightarrow{d'} \mathrm{cok}(\widetilde{\phi})$, where in both cases the first term is placed in degree one. Then $\Psi_2^\bullet$ is acyclic outside degree two, π' induces an identification of $H^2(\Psi_2^\bullet)$ with $\mathrm{cok}(\phi)$ and the above diagram gives rise to a short exact sequence of perfect complexes of $\mathfrak{A}$-modules

$$0 \to \Psi_1^\bullet \xrightarrow{\alpha} \Psi^\bullet \xrightarrow{\beta} \Psi_2^\bullet \to 0 \tag{24}$$

in which $H^1(\alpha)$ is the identity map on H^1, $H^2(\alpha) = \phi$, $H^1(\beta)$ is the zero map and $H^2(\beta) = \kappa$.

Now the exactness of the last column in (23) combines with the fact that the A-modules H_F^1 and H_F^2 are isomorphic (as follows from condition (H2)) to imply that $H^2(\Psi_2^\bullet) = \mathrm{cok}(\phi)$ is a torsion R-module. Thus, after applying the functor $E \otimes_R -$ to the long exact sequence of cohomology of the exact sequence (24) we obtain an exact commutative diagram of A_E-modules

$$\begin{tikzcd} 0 \arrow[r] & H^2(\Psi_1^\bullet)_E \arrow[r, "H^2(\alpha)_E"] \arrow[d, "\lambda\circ\phi_E"'] & H^2(\Psi^\bullet)_E \arrow[r, "H^2(\beta)_E"] \arrow[d, "\lambda"] & H^2(\Psi_2^\bullet)_E \arrow[r] \arrow[d, "0"] & 0 \\ 0 \arrow[r] & H^1(\Psi_1^\bullet)_E \arrow[r, "H^1(\alpha)_E"] & H^1(\Psi^\bullet)_E \arrow[r, "H^1(\beta)_E"] & H^1(\Psi_2^\bullet)_E \arrow[r] & 0 \end{tikzcd}$$

By applying the additivity criterion for refined Euler characteristics of Breuning and Burns (in the form of [6, Lemma 5.7]) to both this diagram and the exact sequence (24) we obtain the following equality in $K_0(\mathfrak{A}, A_E)$:

$$\chi(\Psi_1^\bullet, \lambda \circ \phi_E) + \chi(\Psi_2^\bullet, 0) = \chi(\Psi^\bullet, \lambda). \tag{25}$$

Now since $\mathfrak{A}$ has Krull dimension one and $\operatorname{cok}(\phi)$ has finite projective dimension, there exists a natural number n and an exact sequence of $\mathfrak{A}$-modules of the form $0 \to P \to \mathfrak{A}^n \to \operatorname{cok}(\phi) \to 0$ in which P is projective. Since $\operatorname{cok}(\phi)_F$ vanishes, this sequence implies that the A-modules P_F and $\mathfrak{A}^n_F = A^n$ are isomorphic and so condition $(*)$ implies that P is isomorphic to $\mathfrak{A}^n$. Hence there exists an exact sequence of $\mathfrak{A}$-modules of the form

$$0 \to \mathfrak{A}^n \xrightarrow{\psi} \mathfrak{A}^n \to \operatorname{cok}(\phi) \to 0.$$

We can interpret this exact sequence as an isomorphism in $D^{\mathrm{perf}}(\mathfrak{A})$ between $\Psi_2^\bullet \cong \operatorname{cok}(\phi)[-2]$ and the complex $\Psi_3^\bullet$ that is equal to $\mathfrak{A}^n \xrightarrow{\psi} \mathfrak{A}^n$, where the first term occurs in degree one. By Lemma 3.4 below, one then has equalities

$$\begin{aligned}\chi(\Psi_1^\bullet, \lambda \circ \phi_E) &= [H^1, \lambda \circ \phi_E, H^1] = \partial_{\mathfrak{A},A_E}([H^1_E, \lambda \circ \phi_E]) \\ &= \delta_{\mathfrak{A},A_E}(\mathrm{Nrd}_{A_E}(\lambda \circ \phi_E))\end{aligned}$$

and

$$\begin{aligned}\chi(\Psi_2^\bullet, 0) = \chi(\Psi_3^\bullet, 0) &= [\mathfrak{A}^n, \psi_E^{-1}, \mathfrak{A}^n] = \partial_{\mathfrak{A},A_E}([A^n_E, \psi_E^{-1}]) \\ &= \delta_{\mathfrak{A},A_E}(\mathrm{Nrd}_{A_E}(\psi_E^{-1})).\end{aligned}$$

Finally, we combine the fact that the complexes $\Psi^\bullet$ and $C^\bullet$ are isomorphic in $D^{\mathrm{perf}}(\mathfrak{A})$ with the definition of the element $\mathcal{L}$ that occurs in condition (H3) to deduce that

$$\chi(\Psi^\bullet, \lambda) = \chi(C^\bullet, \lambda) = -\delta_{\mathfrak{A},A_E}(\mathcal{L}).$$

By substituting the last three displayed formulas into (25) we then obtain an equality

$$\delta_{\mathfrak{A},A_E}(\mathrm{Nrd}_{A_E}(\lambda \circ \phi_E)\mathrm{Nrd}_{A_E}(\psi_E^{-1})\mathcal{L}) = 0.$$

Now the kernel of $\delta_{\mathfrak{A},A_E}$ is equal to $\mathrm{Nrd}_{A_E}(\ker(\partial_{\mathfrak{A},A_E})) = \mathrm{Nrd}_{A_E}(K_1(\mathfrak{A}))$ and, since $\mathfrak{A}$ is semilocal, the natural homomorphism $\mathrm{GL}_n(\mathfrak{A}) \to K_1(\mathfrak{A})$ is surjective (by, for example, [16, (40.41), (40.42)]).

The last displayed equality therefore implies the existence of a matrix ε in $\mathrm{GL}_n(\mathfrak{A}) = \mathrm{Aut}_{\mathfrak{A}}(\mathfrak{A}^n)$ with

$$\mathrm{Nrd}_{A_E}(\lambda \circ \phi_E)\mathcal{L} = \mathrm{Nrd}_{A_E}(\psi_E)\mathrm{Nrd}_{A_E}(\varepsilon_E) = \mathrm{Nrd}_{A_E}((\psi \circ \varepsilon)_E). \tag{26}$$

But

$$\mathfrak{A}^n \xrightarrow{\psi\circ\varepsilon} \mathfrak{A}^n \to \mathrm{cok}(\phi) \to 0$$

is a quadratic presentation of the $\mathfrak{A}$-module $\mathrm{cok}(\phi)$, so $\mathrm{Nrd}_{A_E}((\psi \circ \varepsilon)_E)\zeta(\mathfrak{A})$ is a $\mathrm{Nrd}(\mathfrak{A})$-representative of the invariant $\mathrm{Fitt}_{\mathfrak{A}}(\mathrm{cok}(\phi))$. In particular therefore, the equality (26) implies that $\mathrm{Nrd}_{A_E}(\lambda\circ\phi_E)\mathcal{L}$ belongs to $\mathrm{Fitt}_{\mathfrak{A}}(\mathrm{cok}(\phi))$, as required to complete the proof of Proposition 3.2. ∎

Corollary 3.3. *Let $C^\bullet$, λ and $\mathcal{L}$ be any data satisfying the hypotheses* (H1)–(H3) *and assume that $\mathfrak{A}$ has the property* $(*)$. *Then for any injective homomorphism of $\mathfrak{A}$-modules*

$$\phi : H^1(C^\bullet) \to H^2(C^\bullet)$$

one has an inclusion

$$\mathcal{H}(\mathfrak{A}) \cdot \mathrm{Nrd}_{A_E}(\lambda \circ (E \otimes_R \phi)) \cdot \mathcal{L} \subset \mathrm{Ann}_{\mathfrak{A}}(\mathrm{cok}(\phi)).$$

Proof. This result follows as a direct consequence of combining Propositions 3.2 and 3.1. ∎

We conclude this subsection by making an explicit calculation of the refined Euler characteristics that occur in the proof of Proposition 3.2.

Lemma 3.4. *Let $P^\bullet = [P^1 \xrightarrow{d} P^2]$ be a complex of finitely generated projective $\mathfrak{A}$-modules with P^1 placed in degree one and set $H^k := H^k(P^\bullet)$ for both $k = 1, 2$. Then for each isomorphism of A_E-modules $\tau : H^2_E \cong H^1_E$ the following assertions are valid:*

(i) *If $H^1_E = H^2_E = 0$ then $\chi(P^\bullet, \tau) = [P^2, (d_E)^{-1}, P^1]$.*
(ii) *If $d_E = 0$ then $\chi(P^\bullet, \tau) = [P^2, \tau, P^1]$.*

Proof. Let $\pi : P^2 \to H^2$ be the canonical projection and, since A_E is semsimple, we fix splitting maps μ and ν for the following exact sequences of A_E-modules:

$$0 \longrightarrow H^1_E \longrightarrow P^1_E \underset{\mu}{\overset{d_E}{\rightleftarrows}} \ker(\pi_E) \longrightarrow 0$$

and

$$0 \longrightarrow \ker(\pi_E) \longrightarrow P^2_E \underset{\nu}{\overset{\pi_E}{\rightleftarrows}} H^2_E \longrightarrow 0.$$

We write $\hat{\tau}$ for the unique map which makes the following diagram commute:

$$\begin{tikzcd} P_E^2 \arrow[r, "\hat{\tau}"] \arrow[d, "\cong"'] & P_E^1 \arrow[d, "\cong"'] \\ H_E^2 \oplus \ker(\pi_E) \arrow[r, "\tau\oplus\mathrm{Id}"] & H_E^1 \oplus \ker(\pi_E) \end{tikzcd}$$

Then the Euler characteristic $\chi(P^\bullet, \tau)$ is defined to be equal to the element $[P^2, \hat{\tau}, P^1]$ of $K_0(\mathfrak{A}, A_E)$.

In addition, a simple computation shows that for each $c \in P_E^2$ one has

$$\hat{\tau}(c) = \tau(\pi_E(c)) + \mu\big(c - \nu(\pi_E(c))\big)$$

and from here it is easy to see that

- if $H_E^1 = H_E^2 = 0$ then $\pi_E = 0$, d_E is an isomorphism and $\hat{\tau} = \mu = (d_E)^{-1}$, and
- if $d_E = 0$ then $\mu = 0$, π_E is the identity map and $\hat{\tau} = \tau$.

The claims (i) and (ii) are now clear. ∎

4. The proofs of Theorem 1.1 and Corollary 1.2. We shall first combine the results of Propositions 2.1, 2.2 and 3.2 to obtain the theorem below.

In this result we use both the equivariant leading term $\mathcal{L}^j_{L/K,S}$ and the p-adic regulator lattice $\mathcal{R}^j_{L/K,S}$ that occur in Theorem 1.1.

We also write $M_S^t(L)$ for the maximal abelian pro-p extension of L that is unramified outside the set of places that lie above those in S but are not p-adic, and we note that $\mathrm{Gal}(M_S^t(L)/L)$ is endowed with a natural conjugation action of $\mathbb{Z}_p[G]$.

THEOREM 4.1. *If the conjectural equality* (8) *is valid, then*

$$\mathcal{R}^j_{L/K,S} \cdot \mathcal{L}^j_{L/K,S} \subset \mathrm{Fitt}_{\mathbb{Z}_p[G]^-}\big(\mathrm{Gal}(M_S^t(L)/L)^-\big),$$

and hence also

$$\begin{aligned} \mathcal{H}(\mathbb{Z}_p[G]^-) \cdot \mathcal{R}^j_{L/K,S} \cdot \mathcal{L}^j_{L/K,S} &\subset \mathrm{Ann}_{\mathbb{Z}_p[G]^-}\big(\mathrm{Gal}(M_S^t(L)/L)^-\big) \\ &\subset \mathrm{Ann}_{\mathbb{Z}_p[G]}\big(\mathrm{Gal}(M_S^t(L)/L)\big) \\ &= \mathrm{Ann}_{\mathbb{Z}[G]}\big(\mathrm{Gal}(M_S^t(L)/L)\big) \otimes_{\mathbb{Z}} \mathbb{Z}_p. \end{aligned}$$

Before proving this theorem we note that it leads directly to the proof of our main results.

COROLLARY 4.2. *Theorem* 1.1 *and Corollary* 1.2 *are valid.*

Proof. The inclusion of Theorem 1.1, respectively Corollary 1.2, follows immediately from Theorem 4.1 and the inclusion

$$\begin{aligned}\mathrm{Fitt}_{\mathbb{Z}_p[G]^-}\big(\mathrm{Gal}(M^t_{S_p}(L)/L)^-\big) &\subseteq \mathrm{Fitt}_{\mathbb{Z}_p[G]^-}\big((\mathrm{Cl}(\mathcal{O}_L)\otimes_{\mathbb{Z}}\mathbb{Z}_p)^-\big)\\ &= \big(\mathrm{Fitt}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_L))\otimes_{\mathbb{Z}}\mathbb{Z}_p\big)^-,\end{aligned}$$

respectively

$$\mathrm{Ann}_{\mathbb{Z}_p[G]}\big(\mathrm{Gal}(M^t_{S_p}(L)/L)\big) \subseteq \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_L))\otimes_{\mathbb{Z}}\mathbb{Z}_p,$$

proved in Lemma 4.3 below. ∎

LEMMA 4.3. *Fix an element $\underline{u}$ of $U^1_p(L)^{d_K}$ and consider the homomorphism $\phi_{\underline{u}}$ that is constructed in Proposition* 2.3. *Then there are natural surjective homomorphisms of $\mathbb{Z}_p[G]$-modules*

$$\mathrm{cok}(\phi_{\underline{u}}) \twoheadrightarrow \mathrm{Gal}(M^t_{S_p}(L)/L)^- \quad \textit{and} \quad \mathrm{Gal}(M^t_{S_p}(L)/L) \twoheadrightarrow \mathrm{Cl}(\mathcal{O}_L)\otimes_{\mathbb{Z}}\mathbb{Z}_p \tag{27}$$

and hence also inclusions of non-commutative Fitting invariants

$$\begin{aligned}\mathrm{Fitt}_{\mathbb{Z}_p[G]^-}(\mathrm{cok}(\phi_{\underline{u}})) &\subset \mathrm{Fitt}_{\mathbb{Z}_p[G]^-}\big(\mathrm{Gal}(M^t_{S_p}(L)/L)^-\big)\\ &\subset \mathrm{Fitt}_{\mathbb{Z}_p[G]^-}\big((\mathrm{Cl}(\mathcal{O}_L)\otimes_{\mathbb{Z}}\mathbb{Z}_p)^-\big)\end{aligned}$$

and of annihilator ideals

$$\begin{aligned}\mathrm{Ann}_{\mathbb{Z}_p[G]}\big(\mathrm{Gal}(M^t_{S_p}(L)/L)\big) &\subseteq \mathrm{Ann}_{\mathbb{Z}_p[G]}(\mathrm{Cl}(\mathcal{O}_L)\otimes_{\mathbb{Z}}\mathbb{Z}_p)\\ &= \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_L))\otimes_{\mathbb{Z}}\mathbb{Z}_p.\end{aligned}$$

Proof. It is enough for us to prove that there exist surjective homomorphisms as in (27), since then the inclusions between the annihilator ideals are obvious and those between the respective Fitting invariants follow immediately from the result of Nickel in [26, Proposition 3.5]. Further, the existence of the second-occurring surjective homomorphism in (27) is an obvious consequence of the fact that class field theory identifies $\mathrm{Cl}(\mathcal{O}_L)\otimes_{\mathbb{Z}}\mathbb{Z}_p$ with the Galois group of $H_{L,p}/L$ where $H_{L,p}$ is the Hilbert p-class field of L and so is contained in $M^t_{S_p}(L)$.

To prove the existence of the first-occurring surjective homomorphism in (27) we note that the explicit definition of the homomorphism $\phi_{\underline{u}}$ $(= \phi_{\underline{u},Z})$ via the commutative diagram (18) ensures that its image is contained in the image of $a^-_{L,S,p}$, where the homomorphism $a_{L,S,p}$ is defined in (17) using the Artin reciprocity map. Now class field theory implies that the image of $a_{L,S,p}$ is equal to the subgroup of $\mathrm{Gal}(M_{S_p}(L)/L)$ that is generated by the inertia subgroups of each place of L above p. The definition of the field $M^t_{S_p}(L)$ therefore implies that this subgroup is equal to $\mathrm{Gal}(M_{S_p}(L)/M^t_{S_p}(L))$ and so there are natural surjective homomorphisms of the form

$$\begin{aligned}\mathrm{cok}(\phi_{\underline{u}}) \twoheadrightarrow \mathrm{cok}(a^-_{L,S,p}) &= \mathrm{cok}(a_{L,S,p})^- \\ &\cong \mathrm{Gal}(M_{S_p}(L)/L)^-/\mathrm{Gal}(M_{S_p}(L)/M^t_{S_p}(L))^- \\ &\cong \mathrm{Gal}(M^t_{S_p}(L)/L)^-,\end{aligned}$$

as required. ∎

Proof of Theorem 4.1. Set $C^\bullet := R\Gamma_c(\mathcal{O}_{L,S_p}, \mathbb{Z}_p(1))$. Then Proposition 2.1 implies that the complex $C^{\bullet,-}$ and isomorphism $\Phi^j_{L/K}$ that occurs in Proposition 2.2 together satisfy the hypotheses (H1) and (H2) of §3.2.

In addition, by combining the conjectural equality (8) with the result of Proposition 2.2 one has an equality

$$\delta_{G,p}(j_*(\theta^*_{L/K,S_p}(1)^{\#,-}) = -\chi\big(R\Gamma_c(\mathcal{O}_{L,S_p}, \mathbb{Z}_p(1))^-, \Phi^j_{L/K}\big).$$

This shows that the element $\mathcal{L} := j_*(\theta^*_{L/K,S_p}(1)^{\#,-})$ satisfies the hypothesis (H3) of §3.2 with respect to the pair $C^{\bullet,-}$ and $\Phi^j_{L/K}$.

In addition, for any element $\underline{u}$ of $\mathcal{U}^-_{L/K,p}$, the homomorphism $\phi_{\underline{u}}$ of $\mathbb{Z}_p[G]^-$-modules that is constructed in Proposition 2.3 satisfies (H4).

Thus, since $\mathbb{Z}_p[G]^-$ has the property $(*)$ (by Swan's Theorem), Propositions 3.2 and 2.3 can be applied to this data to obtain the following inclusion (in the sense discussed in §3.1):

$$\mathrm{Nrd}_{\mathbb{C}_p[G]}(M^j(\underline{u})) \cdot j_*(\theta^*_{L/K,S_p}(1)^{\#,-}) \subset \mathrm{Fitt}_{\mathbb{Z}_p[G]^-}(\mathrm{cok}(\phi_{\underline{u}})),$$

and hence, by Lemma 4.3, also an inclusion

$$\mathrm{Nrd}_{\mathbb{C}_p[G]}(M^j(\underline{u})) \cdot j_*(\theta^*_{L/K,S_p}(1)^{\#,-}) \subset \mathrm{Fitt}_{\mathbb{Z}_p[G]^-}\big(\mathrm{Gal}(M^t_S(L)/L)^-\big). \tag{28}$$

Now, by taking the minus part of the formula of Lemma 4.4 below, one has an equality

$$n_{p,S}(L/K) \cdot \mathcal{L}^j_{L/K,S} \equiv j_*(\theta^*_{L/K,S_p}(1)^{\#,-}) \bmod \mathrm{Nrd}_{\mathbb{Q}_p[G]^-}(K_1(\mathbb{Z}_p[G]^-)).$$

Thus, after taking account of the fact that the non-commutative Fitting invariants of $\mathbb{Z}_p[G]^-$-modules are preserved under multiplication by elements of the group $\mathrm{Nrd}_{\mathbb{Q}_p[G]^-}(K_1(\mathbb{Z}_p[G]^-))$, in the inclusion (28) one can replace the element $j_*(\theta^*_{L/K,S_p}(1)^{\#,-})$ by $n_{p,S}(L/K) \cdot \mathcal{L}^j_{L/K,S}$ and so obtain an inclusion

$$\mathrm{Nrd}_{\mathbb{C}_p[G]}(M^j(\underline{u})) \cdot n_{p,S}(L/K) \cdot \mathcal{L}^j_{L/K,S} \subset \mathrm{Fitt}_{\mathbb{Z}_p[G]^-}\big(\mathrm{Gal}(M^t_S(L)/L)^-\big).$$

This proves the first inclusion in the statement of Theorem 4.1 and the second inclusion then follows directly from the algebraic result of Proposition 3.1. This completes the proof of Theorem 4.1. ∎

We end this section by proving the technical result which was used in the argument given above. We use the element $n_{p,S}(L/K)$ of $\zeta(\mathbb{Q}[G])$ that is defined in §1.

LEMMA 4.4. *In $\zeta(\mathbb{C}_p[G])^\times$ one has a congruence*

$$n_{p,S}(L/K)\cdot\theta^*_{L/K,S}(1) \equiv \theta^*_{L/K,S_p}(1) \bmod \mathrm{Nrd}_{\mathbb{Q}_p[G]}(K_1(\mathbb{Z}_p[G])).$$

Proof. For any element x of $\mathbb{Q}_p[G]^\times$ we shall write $[x]_{\mathrm{r}}$ for the automorphism of $\mathbb{Q}_p[G]$, considered as a left $\mathbb{Q}_p[G]$-module, that is given by right multiplication by the element x. For each place v of K that is unramified in L we shall also fix a place w of L above v and write Fr_w for the corresponding Frobenius element in G.

Then, since S contains all places of K which ramify in L, the definition of (truncated) Stickelberger functions implies that

$$\begin{aligned}\theta^*_{L/K,S_p}(1) &= \theta^*_{L/K,S}(1)\prod_{v\in S_p\setminus S}\mathrm{Nrd}_{\mathbb{Q}_p[G]}([\mathrm{id}-\mathrm{Fr}_w\mathrm{N}v^{-1}]_{\mathrm{r}})\\ &= \theta^*_{L/K,S}(1)\prod_{v\in S_p\setminus S}\mathrm{Nrd}_{\mathbb{Q}_p[G]}([\mathrm{N}v]_{\mathrm{r}})^{-1}\mathrm{Nrd}_{\mathbb{Q}_p[G]}([\mathrm{N}v-\mathrm{Fr}_w]_{\mathrm{r}})\\ &\equiv \theta^*_{L/K,S}(1)\prod_{v\in S_p\setminus S}\mathrm{Nrd}_{\mathbb{Q}_p[G]}([\mathrm{N}v]_{\mathrm{r}})^{-1} \bmod \mathrm{Nrd}_{\mathbb{Q}_p[G]}(K_1(\mathbb{Z}_p[G])).\end{aligned}$$

The congruence here is valid because each place v in $S_p\setminus S$ is p-adic. Indeed, for this reason the equality

$$(\mathrm{N}v-\mathrm{Fr}_w)\sum_{i=0}^{f-1}(\mathrm{N}v)^{f-1-i}(\mathrm{Fr}_w)^i = (\mathrm{N}v)^f-(\mathrm{Fr}_w)^f = (\mathrm{N}v)^f-1,$$

where f denotes the order of Fr_w in G, implies that $\mathrm{N}v-\mathrm{Fr}_w$ is a unit in $\mathbb{Z}_p[G]$ and hence that the element $\mathrm{Nrd}_{\mathbb{Q}_p[G]}([\mathrm{N}v-\mathrm{Fr}_w]_{\mathrm{r}})$ belongs to $\mathrm{Nrd}_{\mathbb{Q}_p[G]}(K_1(\mathbb{Z}_p[G]))$.

The claimed result now follows from the last displayed congruence because the definition of $n_{p,S}(L/K)$ ensures that for each character χ in $\mathrm{Ir}(G)$ one has equalities

$$\begin{aligned}e_\chi\prod_{v\in S_p\setminus S}\mathrm{Nrd}_{\mathbb{Q}_p[G]}([\mathrm{N}v]_{\mathrm{r}}) &= \prod_{v\in S_p\setminus S}e_\chi\mathrm{Nrd}_{\mathbb{Q}_p[G]}([\mathrm{N}v]_{\mathrm{r}})\\ &= e_\chi\prod_{v\in S_p\setminus S}\det_{\mathbb{Q}_p^c}(\mathrm{N}v\mid V_\chi)\\ &= e_\chi\prod_{v\in S_p\setminus S}(\mathrm{N}v)^{\chi(1)} = e_\chi\cdot n_{p,S}(L/K)^{-1}\end{aligned}$$

and hence that $\prod_{v\in S_p\setminus S}\mathrm{Nrd}_{\mathbb{Q}_p[G]}([\mathrm{N}v]_{\mathrm{r}})^{-1} = n_{p,S}(L/K)$. ∎

This completes the proof of all of the claims that we made above.

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