ACTA ARITHMETICA
160.1 (2013)

A remark on Tate's algorithm and Kodaira types

by

TIM DOKCHITSER (Bristol) and VLADIMIR DOKCHITSER (Cambridge)

Let R be a complete discrete valuation ring with perfect residue field, fraction field K and valuation $v = v_K$. If E/K is an elliptic curve in Weierstrass form,

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \qquad (a_i \in K),$$

the celebrated algorithm of Tate ([3], [2, §IV.9]) determines the minimal model and the local invariants of E. In this paper we gently tweak the resulting models so that the Kodaira type can be simply read off from the valuations of the a_i:

THEOREM 1. *An elliptic curve E/K with additive reduction has a minimal Weierstrass model over R which depends on its Kodaira type as follows:*

	II	III	IV	I_0^*	$\mathrm{I}_{n>0}^*$	IV^*	III^*	II^*
$\min \frac{v(a_i)}{i}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{5}{6}$
extra condition			$v(b_6) = 2$	$v(d) = 6$	$v(d) > 6$, $v(a_2^2 - 3a_4) = 2$	$v(b_6) = 4$		

Here

$$b_6 = a_3^2 + 4a_6 = \mathrm{Disc}(y^2 + a_3y - a_6), \qquad d = \mathrm{Disc}(x^3 + a_2x^2 + a_4x + a_6).$$

Conversely, a Weierstrass model satisfying one of these conditions is minimal, with the corresponding Kodaira type.

There is a refinement for type I_n^* that recovers n as well:

PROPOSITION 2. *An elliptic curve E/K with Kodaira type I_n^*, $n > 0$, has a minimal model with*

$$v(a_2) = 1, \quad v(a_i) \geq \frac{i}{2} + \left\lfloor \frac{i-1}{2} \right\rfloor \frac{n}{2}, \quad \begin{cases} v(d) = n+6,\ v(b_6) \geq n+3 & \text{if } 2 \mid n, \\ v(d) \geq n+6,\ v(b_6) = n+3 & \text{if } 2 \nmid n. \end{cases}$$

2010 *Mathematics Subject Classification*: Primary 11G07.
Key words and phrases: Tate's algorithm, minimal model, elliptic curve, Kodaira type, minimal discriminant.

Conversely, a Weierstrass equation satisfying these conditions is minimal and defines an elliptic curve with Kodaira type I_n^*.

As an application, we deduce the behaviour of minimal discriminants and Kodaira types in tame extensions of local fields; our motivation came from Iwasawa theory of elliptic curves, where it is necessary to control local invariants of elliptic curves in towers of number fields (see [1]).

Theorem 3. *Let F/K be a tame extension of ramification degree e, and let E/K be an elliptic curve.*

(1) *If E/K has Kodaira type I_n, then E/F has type I_{en}.*
(2) *If E/K has Kodaira type I_n^*, then E/F has type I_{en}^* if e is odd and type I_{en} if e is even.*
(3) *In all other cases, the type of E/F is determined by*

$$\eth_{E/F} \equiv e\eth_{E/K} \bmod 12,$$

where $\eth = 0, 2, 3, 4, 6, 8, 9, 10$ *if E has Kodaira type* I_0, II, III, IV, I_0^*, IV^*, III^*, II^* *respectively.*

The valuations of minimal discriminants for E/K and E/F are related by

$$v_F(\Delta_{E/F}) = e\, v_K(\Delta_{E/K}) - 12\lfloor e\,\eth_{E/K}/12\rfloor,$$

where $\eth_{E/K} = 0$ *for* I_0 *and* I_n, 6 *for* I_n^* *and is as in* (3) *otherwise.*

Remark. If the residue characteristic is at least 5 and E/K has potentially good reduction, the fraction in the table in Theorem 1 is just $v(\Delta_{E/K})/12$, and $\eth_{E/K} = v(\Delta_{E/K})$ in Theorem 3. The conclusion of Theorem 3 is then equivalent to the standard fact that $v_F(\Delta_{E/F}) < 12$. The point is that $\eth$ gives the correct replacement for $v(\Delta)$ in residue characteristics 2 and 3. Note, however, that in residue characteristics 2 and 3 neither the Kodaira type nor the minimal discriminant behave as in Theorem 3 in wild extensions.

Example 4. The curve $E : y^2 = x^3 - 2x$ over $K = \mathbb{Q}_2$ has Kodaira type III (as $\min v(a_i)/i = 1/4$) and $v(\Delta) = 9$. By Theorem 3, over the tame extensions $F_n = \mathbb{Q}_2(\sqrt[5^n]{2})$ the reduction remains of Type III, and the valuations are

$$v_{F_n}(\Delta_{E/F_n}) 9 \cdot 5^n - 12\left\lfloor \frac{3 \cdot 5^n}{12} \right\rfloor = 6 \cdot 5^n + 3 = 33{,}153{,}753, \ldots.$$

In particular, they are not bounded by 12 (or by anything) as they would be in residue characteristics ≥ 5. Over the wild quartic extensions $\mathbb{Q}_2(\sqrt[4]{2})$, $\mathbb{Q}_2(\sqrt[4]{-2})$, $\mathbb{Q}_2(\zeta_8)$, the Kodaira types of E are III^*, I_3^*, I_4^*, and the valuations of the minimal discriminants are 12, 12, 24, respectively. So these cannot be recovered just from $E/\mathbb{Q}_2$ and the ramification degree.

In the proofs below we follow the steps of Tate's algorithm, numbered as in [3] and [2, §IV.9].

1. Proof of Theorem 1. Let π be a uniformiser of K.

By Steps 1–2 of Tate's algorithm, an elliptic curve with additive reduction over K has an integral model with $\pi \mid a_3, a_4, a_6$, $\pi \mid b_2 = a_1^2 + 4a_2$. If K has residue characteristic 2, this means $\pi \mid a_1$, and, shifting $y \mapsto y - \alpha x$ for any $\alpha \in R$ with $\alpha^2 \equiv a_2 \bmod \pi$, we can get $\pi \mid a_2$ as well. Similarly, if K has odd residue characteristic, the substitution $y \mapsto y - (a_1/2)x$ makes both a_1 and a_2 divisible by π. Now we run Tate's algorithm through this equation, and inspect the model that comes out of it:

Type II (Step 3): Here $\pi^2 \nmid a_6$ and the valuations of the a_i are ≥ 1, ≥ 1, ≥ 1, ≥ 1, $= 1$, so $\min v(a_i)/i = 1/6$.

Type III (Step 4): Here $\pi^2 \mid a_6$ and

$$\pi^3 \nmid b_8 = a_1^2 a_6 + 4a_2 a_6 - a_1 a_3 a_4 + a_2 a_3^2 - a_4^2 \equiv -a_4^2 \bmod \pi^3.$$

So $\pi^2 \nmid a_4$, the valuations of the a_i are ≥ 1, ≥ 1, ≥ 1, $= 1$, ≥ 2 and $\min v(a_i)/i = 1/4$.

Type IV (Step 5): Here $\pi^2 \mid a_6$, $\pi^3 \mid b_8 \Rightarrow \pi^2 \mid a_4$, and $\pi^3 \nmid b_6 = a_3^2 + 4a_6$. The valuations of the a_i are $\geq 1, \geq 1, \geq 1, \geq 2, \geq 2$, and either $v(a_3) = 1$ or $v(a_6) = 2$ since $\pi^3 \nmid a_3^2 + 4a_6$. So $\min v(a_i)/i = 1/3$.

Type I_0^* (Step 6): Here $\pi^2 \mid a_3$, $\pi^2 \mid a_4$, $\pi^3 \mid a_6$, $v(d) = 6$, the valuations of the a_i are $\geq 1, \geq 1, \geq 2, \geq 2, \geq 3$, so $\min v(a_i)/i \geq 1/2$. Because

$$6 = v(d) = \pi^6 \operatorname{Disc}\left(x^3 + \frac{a_2}{\pi} x^2 + \frac{a_4}{\pi^2} x + \frac{a_6}{\pi^3} \right),$$

at least one of a_2/π, a_4/π^2 and a_6/π^3 is a unit, so the minimum is exactly 1/2.

Type I_n^*, $n \geq 1$ (Step 7): Here $\pi^2 \mid a_3$, $\pi^2 \mid a_4$, $\pi^3 \mid a_6$, $v(d) > 6$ and $\pi^2 \nmid a_2$, so $\min v(a_i)/i = 1/2$, attained for $i = 2$. Moreover, the cubic $x^3 + (a_2/\pi)x^2 + (a_4/\pi^2)x + (a_6/\pi^3)$ has a double root which is not a triple root. A cubic polynomial $x^3 + ax^2 + bx + c$ has a triple root if and only if its discriminant is 0 and $a^2 - 3b = 0$, and this gives the two extra stated conditions ([1]).

Type IV^* (Step 8): Here $\pi^2 \mid a_2$, $\pi^2 \mid a_3$, $\pi^3 \mid a_4$, $\pi^4 \mid a_6$ and $\pi^5 \nmid b_6 = a_3^2 + 4a_6$. The valuations of the a_i are $\geq 1, \geq 2, \geq 2, \geq 3, \geq 4$, and either $v(a_3) = 2$ or $v(a_6) = 4$ since $\pi^5 \nmid a_3^2 + 4a_6$. So $\min v(a_i)/i = 2/3$.

([1]) If the roots of $x^3 + ax^2 + bx + c$ are α, β, γ, then the discriminant condition is equivalent to two of them being equal, say $\alpha = \beta$, in which case $a^2 - 3b = (\alpha - \gamma)^2$ measures whether it is a triple root.

Type III* (Step 9): Here $\pi^2 \mid a_2$, $\pi^3 \mid a_3$, $\pi^3 \mid a_4$, $\pi^5|a_6$ and $\pi^4 \nmid a_4$. The valuations of the a_i are $\geq 1, \geq 2, \geq 3, = 3, \geq 5$, so $\min v(a_i)/i = 3/4$.

Type II* (Step 10): Here $\pi^2 \mid a_2$, $\pi^3 \mid a_3$, $\pi^4 \mid a_4$, $\pi^5|a_6$ and $\pi^6 \nmid a_6$. The valuations of the a_i are $\geq 1, \geq 2, \geq 3, \geq 4, = 5$, so $\min v(a_i)/i = 5/6$.

Conversely, any model satisfying one of the conditions in the table is minimal with the right Kodaira type, which is immediate from the corresponding step of Tate's algorithm. (The steps do not change such a model.)

2. Proof of Proposition 2. From Step 7 of Tate's algorithm it follows readily that a curve E/K of type I_n^* has a minimal model with

$$v(a_2) = 1, \quad v(a_i) \geq \frac{i}{2} + \left\lfloor \frac{i-1}{2} \right\rfloor \frac{n}{2}, \quad \begin{cases} v(D) = n+4,\ v(b_6) \geq n+3 & \text{if } 2 \mid n, \\ v(D) \geq n+4,\ v(b_6) = n+3 & \text{if } 2 \nmid n, \end{cases}$$

where $D = \mathrm{Disc}(a_2x^2 + a_4x + a_6)$. Because $n \geq 1$,

$$d = -4a_2^3a_6 + a_2^2a_4^2 - 4a_4^3 - 27a_6^2 + 18a_2a_4a_6 \equiv a_2^2 D \bmod \pi^{n+7},$$

the conditions on D are equivalent to those on d in the proposition.

Conversely, such a model has $v(a_2^2 - 3a_4) = 2$, and so the polynomial $x^3 + (a_2/\pi)x^2 + (a_4/\pi^2)x + a_6/\pi^3$ has a double root, but not a triple root. Step 7 of Tate's algorithm shows the model to be minimal of type I_n^*.

3. Proof of Theorem 3. (1) If E has good or multiplicative reduction (types I_0, $\mathrm{I}_{n>0}$), the minimal model stays minimal in all extensions, and the reduction stays good, respectively multiplicative. In the multiplicative case, $-n$ is the valuation of the j-invariant of E, so it gets scaled by e in F/K; cf. [2, §IV.9, Table 4.1].

(2), (3) Fix a uniformiser π of F, and write l for the residue characteristic. As F/K is tame, $l \nmid e$.

Assume that E/K has additive reduction, and is in Weierstrass form as in Theorem 1 (and as in Proposition 2 for type I_n^*). Then $\min v_K(a_i)/i = \eth_{E/K}/12$ and $\min v_F(a_i)/i = e\eth_{E/K}/12$. Over F this model can be rescaled $\lfloor e\eth_{E/K}/12 \rfloor$ times with the standard substitution $y \mapsto \pi^3 y$, $x \mapsto \pi^2 x$; call the new Weierstrass coefficients A_1, A_2, A_3, A_4, A_6. So now

$$\min_i \frac{v_F(A_i)}{i} \in \left\{0, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}\right\}.$$

We proceed to show that the resulting equation satisfies the 'extra conditons' of Theorem 1, and, if $\min v_F(A_i)/i = 0$, that E/F has good reduction (Type I_0). This implies all the claims in Theorem 3.

It is a simple consequence of the fact that the tame inertia is cyclic that tame extensions can be built up from unramified ones and ramified extensions of prime degree. If F/K is unramified ($e = 1$), there is nothing to prove. So for simplicity we may and will assume that $[F:K]=e=p$ is prime, $p \neq l$.

We first deal with the cases when E acquires good reduction:

Type IV, IV*, $p=3$, $l \neq 3$: the valuations of the a_i are $>1/3, >2/3, \geq 1, >4/3, \geq 2$ for Type IV and $>2/3, >4/3, \geq 2, >8/3, \geq 4$ for Type IV*. The valuations of the A_i are therefore $>0, >0, \geq 0, >0, \geq 0$, so the model reduces to $y^2+\alpha y = x^3+\beta$ over the residue field of F. It has discriminant $-27(\alpha^2+4\beta)^2$, which is non-zero, since $l \neq 3$ and $\alpha^2+4\beta \neq 0$ from the b_6 condition for E/K. So E/F has good reduction.

Type I$_0^*$, $p=2$, $l \neq 2$: In the same manner, the valuations of the A_i are $>0, \geq 0, >0, \geq 0, \geq 0$, the model reduces to $y^2=x^3+\alpha x^2+\beta x+\gamma$ and this has non-zero discriminant since $l \neq 2$ and $v_K(d)=6$ for E/K.

Now we look at the remaining cases, all entirely similar.

Type IV, IV*, $p \neq 3$: The extra condition in the table for E/K automatically rescales to give the one for E/F.

Type I$_0^*$, $p \neq 2$: The condition for E/K rescales to give the one for E/F.

Type II, $p=2$, $l \neq 2$: $v_K(a_3) \geq 1$, $v_K(a_6)=1$ gives $v_F(A_3) \geq 2$, $v_F(A_6)=2$. So $\pi^3 \nmid A_3^2+4A_6=B_6$, which is the condition for type IV.

Type II*, $p=2$, $l \neq 2$: $v_K(a_3) \geq 3, v_K(a_6)=5$ gives (after one rescaling) $v_F(A_3) \geq 3$, $v_F(A_6)=4$. So $\pi^5 \nmid A_3^2+4A_6=B_6$, which is the conditon for IV*.

Type II, $p=3$, $l \neq 3$: $v_K(a_2) \geq 1$, $v_K(a_4) \geq 1$, $v_K(a_6)=1$ gives $v_F(A_2) \geq 3$, $v_F(A_4) \geq 3$, $v_F(A_6)=3$, so

$$x^3+\frac{A_2}{\pi}x^2+\frac{A_4}{\pi^2}x+\frac{A_6}{\pi^3} \equiv x^3+\text{unit} \mod \pi,$$

which has non-zero discriminant as $l \neq 3$. So $v_F(\mathrm{Disc}(x^3+A_2x^2+A_4x+A_6))=6$ as required for type I$_0^*$. Type II*, $p=3$ is similar.

Type III, $p=2$, $l \neq 2$: $v_K(a_2) \geq 1$, $v_K(a_4)=1$, $v_K(a_6) \geq 2$ gives $v_F(A_2) \geq 2$, $v_F(A_4)=2$, $v_F(A_6) \geq 4$, so

$$x^3+\frac{A_2}{\pi}x^2+\frac{A_4}{\pi^2}x+\frac{A_6}{\pi^3} \equiv x^3+\text{unit}\cdot x \mod \pi,$$

which has non-zero discriminant as $l \neq 2$. So $v_F(\mathrm{Disc}(x^3+A_2x^2+A_4x+A_6))=6$ as required for type I$_0^*$. Type III*, $p=2$ is similar.

Type I$_n^*$, $p=2$, $l \neq 2$: E has non-integral j-invariant ([2, §IV.9, Table 4.1]), and so acquires multiplicative reduction over F ([2, Thm. V.5.3]).

Comparing the valuations of the j-invariants and the discriminants, we find that E/F has type I_{2n}, and the A_i define a minimal equation.

Type I_n^*, $p \neq 2$: The valuations of $a_1, \ldots, a_6, b_6, d$ are ≥ 1, $= 1$, $\geq (n+3)/2$, $\geq (n+4)/2$, $\geq n+3$, $\geq n+3$, $\geq n+6$ with equality for one of the last two (depending on whether n is even or odd). Over F they become $\geq p$, $= p$, $\geq p\frac{n+3}{2}$, $\geq p\frac{n+4}{2}$, $\geq p(n+3)$, $\geq p(n+3)$, $\geq p(n+6)$. After rescaling the model $\lfloor 6e/12 \rfloor = (p-1)/2$ times, we see that the valuations of $A_1, \ldots, A_6, B_6, D$ for the new model are $\geq (p+1)/2$, $= 1$, $\geq (pn+3)/2$, $\geq (pn+4)/2$, $\geq pn+3$, $\geq pn+3$, $\geq pn+6$, again with equality for one of the last two. In other words, the model satisfies the conditions of Proposition 2 for Type I_{pn}^*.

Acknowledgements. The first author is supported by a Royal Society University Research Fellowship. We would like to thank Matthias Schuett and the referee for carefully reading the manuscript.

References

[1] T. Dokchitser and V. Dokchitser, *Growth of Ш in towers for isogenous curves*, arXiv:1301.4257, 2013.

[2] J. H. Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*, Grad. Texts in Math. 151, Springer, New York, 1994.

[3] J. Tate, *Algorithm for determining the type of a singular fiber in an elliptic pencil*, in: Modular Functions of One Variable IV, B. J. Birch and W. Kuyk (eds.), Lecture Notes in Math. 476, Springer, Berlin, 1975, 33–52.

Tim Dokchitser
Department of Mathematics
University of Bristol
University Walk
Bristol BS8 1TW, United Kingdom
E-mail: tim.dokchitser@bristol.ac.uk

Vladimir Dokchitser
Emmanuel College
Cambridge CB2 3AP, United Kingdom
E-mail: v.dokchitser@dpmms.cam.ac.uk

Received on 22.1.2013
and in revised form on 22.2.2013 (7323)