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# Metric properties for $p$-adic Oppenheim series expansions

by

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**1. Introduction.** Real numbers have several representations, such as continued fraction expansions, Lüroth series, Engel series, Sylvester series expansions and Cantor infinite products etc. (see [4] and [20]). Analogous to continued fraction expansions, certain types of $p$-adic continued fractions have been studied by many mathematicians; see for example, [15], [17], [13] and [14] etc. In [8]–[10], A. Knopfmacher and J. Knopfmacher introduced and studied some properties of various unique $p$-adic expansions as sums of reciprocals of $p$-adic numbers with $p$-adic valuations not less than 1. These expansions, including $p$-adic Lüroth series, Engel series, Sylvester series expansions and $p$-adic Cantor infinite products, were constructed to be analogous to the so-called Oppenheim series expansions of real numbers discussed in Galambos [4].

In the direction of metric and asymptotic results concerning digits, various results were established; in particular, for expansions of real numbers, by Jager and de Vroedt [6] and Salát [18] for real Lüroth series expansions, Erdős, Rényi and Szüsz [2] for real Engel and Sylvester series expansions, Rényi [16] for real Cantor infinite products and by Galambos [4] for more general situations, called Oppenheim series expansions of real numbers. Ruban [17] established $p$-adic metric theorems analogous to some of Khinchin [7] for real continued fractions. The corresponding results for $p$-adic Lüroth and Engel series expansions have been derived by A. Knopfmacher and J. Knopfmacher [11] and Grabner and A. Knopfmacher [5], respectively.

The main aim of this paper is to derive metric and asymptotic results for $p$-adic Oppenheim series expansions. We generalize the results obtained by A. Knopfmacher and J. Knopfmacher [11] and Grabner and A. Knopfmacher [5] for $p$-adic Lüroth and Engel series expansions. Also as special

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cases of our results, we give metric results for $p$-adic Sylvester series expansions and $p$-adic Cantor infinite products. The corresponding results for Oppenheim series expansions of Laurent series have been obtained by Fan and the author [3].

**2. The $p$-adic Oppenheim series expansions.** In order to explain the conclusions, we first fix some notations and describe the $p$-adic Oppenheim series expansions to be considered.

Let us give a brief account of $p$-adic numbers; more details can be found in the books by Koblitz [12] and Schikhof [19].

Let $p$ be a fixed prime number. Every non-zero rational number $A$ can be expressed uniquely in the form $A = p^a r/s$, where $(r,p) = (s,p) = 1$ and $a \in \mathbb{Z}$. The $p$-adic *valuation* $|\ |_p$ on $\mathbb{Q}$ is defined to be

$$|A|_p = p^{-a} \quad \text{if } A \neq 0, \qquad |0|_p = 0.$$

The completion of $\mathbb{Q}$ with respect to the $p$-adic metric $|\ |_p$ gives rise to the field $\mathbb{Q}_p$. Each element $A \in \mathbb{Q}_p$ has a unique series representation

$$A = \sum_{n=m}^{\infty} c_n p^n,$$

where $m \in \mathbb{Z}$ and the coefficients $c_n$ are rational integers satisfying $0 \leq c_n \leq p-1$ and $c_m \neq 0$. The integer $m$ is called the *order* of $A$ and denoted by $v(A)$, and $|A|_p = p^{-m}$. The valuation $|\ |_p$ defined on $\mathbb{Q}_p$ has the properties

$$|A|_p \geq 0, \qquad |A|_p = 0 \text{ if and only if } A = 0, \qquad |AB|_p = |A|_p |B|_p,$$
$$|A+B|_p \leq \max(|A|_p, |B|_p) \quad \text{with equality when } |A|_p \neq |B|_p.$$

For $v(A)$, we have

$$v(0) = \infty, \quad v(AB) = v(A) + v(B), \quad v(A/B) = v(A) - v(B) \quad \text{if } B \neq 0,$$
$$v(A+B) \geq \min(v(A), v(B)) \quad \text{with equality when } v(A) \neq v(B).$$

It is well known that the above non-Archimedean valuation leads to an ultrametric distance function $\varrho$, with $\varrho(A,B) = |A-B|_p$, making $\mathbb{Q}_p$ into a complete metric space with respect to $\varrho$.

REMARK 2.1. Since the metric $\varrho$ is non-Archimedean, it follows that each point of a disc may be considered its center and thus if two discs intersect, then one contains the other.

For any $A \in \mathbb{Q}_p$, if $A = \sum_{n=v(A)}^{\infty} c_n p^n$, we call the finite series $\langle A \rangle = \sum_{v(A) \leq n \leq 0} c_n p^n$ the *fractional* part of $A$. Then $\langle A \rangle \in S_p$, where we define $S_p = \{\langle A \rangle : A \in \mathbb{Q}_p\} \subset \mathbb{Q}$. The set $S_p$ is multiplicatively but not additively closed. The function $\langle A \rangle$ and set $S_p$ have been used in the study of certain types of $p$-adic continued fractions by Mahler [15], Ruban [17] and Laohakosol [13] in particular.

For any $n \geq 1$, let $r_n, s_n$ be maps from $p^{-1}(S_p \backslash \{0\})$ to $\mathbb{Q} \backslash \{0\}$ satisfying, for any $a \in p^{-1}(S_p \backslash \{0\})$,

$$2v(a) - v(s_n(a)) + v(r_n(a)) \leq 0 \quad \text{for any } n \geq 1, \tag{1}$$

$$v(r_n(a)) = v(r_n(a')), \quad v(s_n(a)) = v(s_n(a')) \quad \text{if } v(a) = v(a'). \tag{2}$$

Given any $A \in \mathbb{Q}_p$, note that $\langle A \rangle = a_0 \in S_p$ if and only if $v(A - a_0) \geq 1$. Then define $A_1 = A - a_0$. As in [9], [10], if $A_n \neq 0$ with $v(A_n) \geq 1$ $(n \geq 1)$ is already defined, then define the "digit" $a_n = \langle 1/A_n \rangle$ and put

$$A_{n+1} = \left( A_n - \frac{1}{a_n} \right) \frac{s_n(a_n)}{r_n(a_n)}. \tag{3}$$

For any $m \geq 1$, if $A_m \neq 0$, by (1) and [10, (2.3)], we have $v(A_m) \geq 1$. If some $A_m = 0$, this recursive process stops. It was shown in [9], [10] that this algorithm leads to a finite or convergent series (relative to $\varrho$), called the *p-adic Oppenheim series expansion.*

THEOREM 2.2 ([9], [10]). *Every $x \in \mathbb{Q}_p$ has a finite or convergent (relative to $\varrho$) series expansion of the form*

$$x = a_0(x) + \frac{1}{a_1(x)} + \sum_{n=1}^{\infty} \frac{r_1(a_1(x)) \ldots r_n(a_n(x))}{s_1(a_1(x)) \ldots s_n(a_n(x))} \frac{1}{a_{n+1}(x)}, \tag{4}$$

*where $a_n(x) \in S_p$, $a_0(x) = \langle x \rangle$, and $v(a_1(x)) \leq 1$, for any $n \geq 1$,*

$$v(a_{n+1}(x)) \leq 2v(a_n(x)) - 1 + v(r_n(a_n(x))) - v(s_n(a_n(x))). \tag{5}$$

*The expansion is unique for $x$ subject to the above conditions on the "digits" $a_n(x)$.*

REMARK 2.3. The algorithm above is more restricted than the general algorithm described in [10] in order to obtain our metric results. (1) is used to guarantee that $v(a_n) \leq -1$ for any $n \geq 1$ if the process does not stop (see (5)).

Here are some special cases:

*p-adic Lüroth series expansion*: $s_n(a) = a(a-1)$, $r_n(a) = 1$;
*p-adic Engel expansion*: $s_n(a) = a$, $r_n(a) = 1$;
*p-adic Sylvester expansion*: $s_n(a) = 1$, $r_n(a) = 1$;
*p-adic Cantor infinite product*: $s_n(a) = a$, $r_n(a) = a + 1$.

Let $X_p = p\mathbb{Z}_p$ denote the maximal ideal in the ring $\mathbb{Z}_p$ of all $p$-adic integers, i.e. the set of $p$-adic numbers of order $\geq 0$. Then $X_p$ is compact. For any $A \in X_p$, $v(A) \geq 1$ and from Remark 2.3, $v(A_n) \geq 1$ for any $n \geq 1$ if the process does not stop. Let $\mathbf{P}$ be the probability measure with respect to

Haar measure on $\mathbb{Q}_p$ normalized by $\mathbf{P}(X_p)=1$. A convenient description of $\mathbf{P}$ on $X_p$ is given in Sprindžuk [21, pp. 67–70]. In particular, $\mathbf{P}(C)=p^{-m}$ for any disc

$$C=C(x,p^{-m-1}):=\{y\in\mathbb{Q}_p:|y-x|_p\le p^{-m-1}\}$$

of radius $p^{-m-1}$.

For any $x\in X_p$, let $\{\triangle_n(x):n\ge 0\}$ denote the sequence of random variables such that $\triangle_0(x)=v(a_1(x))$, $\triangle_n(x)=v(a_{n+1}(x))-2v(a_n(x))-v(r_n(a_n(x)))+v(s_n(a_n(x)))$ for $n\ge 1$.

Now we state our main results.

THEOREM 2.4. *For the p-adic Oppenheim series expansions described above*:

(i) $$\lim_{n\to\infty}\mathbf{P}\left\{x\in X_p:\ \frac{\sum_{j=0}^{n-1}\triangle_j(x)+\frac{p}{p-1}\,n}{\sqrt{np}/(p-1)}<t\right\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t}e^{-u^2/2}\,du.$$

(ii) *For* $\mathbf{P}$*-almost all* $x\in X_p$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\triangle_j(x)=-\frac{p}{p-1}.$$

(iii) *For* $\mathbf{P}$*-almost all* $x\in X_p$,

$$\limsup_{n\to\infty}\frac{\sum_{j=0}^{n-1}\triangle_j(x)+\frac{p}{p-1}\,n}{\sqrt{2n\log\log n}}=\frac{\sqrt{p}}{p-1},$$

$$\liminf_{n\to\infty}\frac{\sum_{j=0}^{n-1}\triangle_j(x)+\frac{p}{p-1}\,n}{\sqrt{2n\log\log n}}=-\frac{\sqrt{p}}{p-1}.$$

Furthermore, we consider the random variables

$$\left|\frac{a_{n+1}(x)s_n(a_n(x))}{a_n(x)^2r_n(a_n(x))}\right|_p=p^{-\triangle_n(x)},\qquad n=1,2,\ldots$$

In Proposition 3.5, we will show that these are independent and identically distributed with infinite expectation. However, we have the following result:

THEOREM 2.5. *For any fixed* $\varepsilon>0$,

$$\lim_{n\to\infty}\mathbf{P}\left\{x\in X_p:\left|\frac{1}{n\log_p n}\sum_{j=1}^{n}\left|\frac{a_{j+1}(x)s_j(a_j(x))}{a_j(x)^2r_j(a_j(x))}\right|_p-(p-1)\right|>\varepsilon\right\}=0,$$

*i.e.*

$$\frac{1}{n\log_p n}\sum_{j=1}^{n}\left|\frac{a_{j+1}(x)s_j(a_j(x))}{a_j(x)^2r_j(a_j(x))}\right|_p\to p-1\quad\textit{in probability.}$$

This paper is organized as follows. In Section 3, we give the proof of Theorem 2.4. Section 4 is devoted to the proof of Theorem 2.5.

**3. Proof of Theorem 2.4.** In order to prove Theorem 2.4, we need some preliminary results.

LEMMA 3.1. *For any* $k_1, \dots, k_n \in S_p$ *satisfying*

$$v(k_1) \le -1, \quad v(k_{j+1}) \le 2v(k_j) - 1 + v(r_j(k_j)) - v(s_j(k_j)), \quad 1 \le j \le n-1,$$

*we have*

$$\mathbf{P}\{x \in X_p : a_1(x) = k_1, \dots, a_n(x) = k_n\} = p^{-\sum_{j=1}^{n-1}(v(r_j(k_j))-v(s_j(k_j)))+2v(k_n)}.$$

*Proof.* From (5), we have

$$v\left(\frac{r_1(k_1)\dots r_n(k_n)}{s_1(k_1)\dots s_n(k_n)}\,\frac{1}{a_{n+1}(x)}\right) \ge v\left(\frac{r_1(k_1)\dots r_{n-1}(k_{n-1})}{s_1(k_1)\dots s_{n-1}(k_{n-1})}\,\frac{1}{a_n(x)}\right) + 1.$$

Thus by Theorem 2.2, $\{x \in X_p : a_1(x) = k_1, \dots, a_n(x) = k_n\}$ is a disc with center at

$$\frac{1}{k_1} + \sum_{j=2}^{n} \frac{r_1(k_1)\dots r_{j-1}(k_{j-1})}{s_1(k_1)\dots s_{j-1}(k_{j-1})}\,\frac{1}{k_j}$$

and diameter

$$p^{-\sum_{j=1}^{n-1}(v(r_j(k_j))-v(s_j(k_j)))+2v(k_n)-1}.$$

Thus

$$\mathbf{P}\{x \in X_p : a_1(x) = k_1, \dots, a_n(x) = k_n\} = p^{-\sum_{j=1}^{n-1}(v(r_j(k_j))-v(s_j(k_j)))+2v(k_n)}. \qquad \blacksquare$$

PROPOSITION 3.2. *For any* $k_1, \dots, k_{n+1} \in S_p$ *satisfying*

$$v(k_1) \le -1, \quad v(k_{j+1}) \le 2v(k_j) - 1 + v(r_j(k_j)) - v(s_j(k_j)), \quad 1 \le j \le n,$$

*we have*

$$\begin{aligned}\mathbf{P}\{a_{n+1}(x) = k_{n+1} \mid a_n(x) = k_n\} &\\ &= \mathbf{P}\{a_{n+1}(x) = k_{n+1} \mid a_1(x) = k_1, \dots, a_n(x) = k_n\} \\ &= \left|\frac{r_n(k_n)}{s_n(k_n)}\right|_p \frac{|k_n|_p^2}{|k_{n+1}|_p^2},\end{aligned}$$

*i.e.* $\{a_n(x) : n \ge 1\}$ *forms a Markov chain with transition probabilities,*

$$\mathbf{P}\{a_{n+1}(x) = l_{n+1} \mid a_n(x) = l_n\} = \left|\frac{r_n(l_n)}{s_n(l_n)}\right|_p \frac{|l_n|_p^2}{|l_{n+1}|_p^2}$$

*if* $v(l_{n+1}) \le 2v(l_n) - 1 + v(r_n(l_n)) - v(s_n(l_n))$, *and* $0$ *otherwise.*

*Proof.* By Lemma 3.1,

$$\begin{aligned}\mathbf{P}\{a_{n+1}(x) = k_{n+1} \mid a_n(x) = k_n, \dots, a_1(x) = k_1\} &\\ &= \frac{\mathbf{P}\{a_1(x) = k_1, \dots, a_n(x) = k_n,\, a_{n+1}(x) = k_{n+1}\}}{\mathbf{P}\{a_1(x) = k_1, \dots, a_{n-1}(x) = k_{n-1},\, a_n(x) = k_n\}} \\ &= \frac{p^{-\sum_{j=1}^{n}(v(r_j(k_j))-v(s_j(k_j)))+2v(k_{n+1})}}{p^{-\sum_{j=1}^{n-1}(v(r_j(k_j))-v(s_j(k_j)))+2v(k_n)}} = \left|\frac{r_n(k_n)}{s_n(k_n)}\right|_p \frac{|k_n|_p^2}{|k_{n+1}|_p^2}.\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\mathbf{P}\{a_{n+1}(x) = k_{n+1} \mid a_n(x) = k_n\} \\
&\quad = \frac{\mathbf{P}\{a_n(x) = k_n,\, a_{n+1}(x) = k_{n+1}\}}{\mathbf{P}\{a_n(x) = k_n\}} \\
&\quad = \frac{\sum \mathbf{P}\{a_j(x) = l_j,\, 1 \le j \le n-1,\, a_n(x) = k_n,\, a_{n+1}(x) = k_{n+1}\}}{\sum \mathbf{P}\{a_j(x) = m_j,\, 1 \le j \le n-1,\, a_n(x) = k_n\}} \qquad (6) \\
&\quad = \frac{\sum p^{-\sum_{j=1}^{n-1}(v(r_j(l_j)) - v(s_j(l_j))) - (v(r_n(k_n)) - v(s_n(k_n))) + 2v(k_{n+1})}}{\sum p^{-\sum_{j=1}^{n-1}(v(r_j(m_j)) - v(s_j(m_j))) + 2v(k_n)}} \qquad (7) \\
&\quad = \left|\frac{r_n(k_n)}{s_n(k_n)}\right|_p \frac{|k_n|_p^2}{|k_{n+1}|_p^2},
\end{aligned}$$

where the summations in the numerators of (6) and (7) are over all $l_1, \ldots, l_{n-1} \in S_p$ satisfying $v(l_1) \le -1$, $v(l_{j+1}) \le 2v(l_j) - 1 + v(r_j(l_j)) - v(s_j(l_j))$ for $1 \le j \le n-2$ and $v(k_n) \le 2v(l_{n-1}) - 1 + v(r_{n-1}(l_{n-1})) - v(s_{n-1}(l_{n-1}))$, and the summations in the denominators of (6) and (7) are over all $m_1, \ldots, m_{n-1} \in S_p$ satisfying $v(m_1) \le -1$, $v(m_{j+1}) \le 2v(m_j) - 1 + v(r_j(m_j)) - v(s_j(m_j))$ for $1 \le j \le n-2$ and $v(k_n) \le 2v(m_{n-1}) - 1 + v(r_{n-1}(m_{n-1})) - v(s_{n-1}(m_{n-1}))$. ∎

Next we show that $\{v(a_n(x)) : n \ge 1\}$ forms a Markov chain.

LEMMA 3.3. *For any $k_1, \ldots, k_n \in S_p$ as in Lemma 3.1, we have*

$$\begin{aligned}
&\mathbf{P}\{x \in X_p : v(a_1(x)) = v(k_1), \ldots, v(a_n(x)) = v(k_n)\} \\
&\qquad = (p-1)^n p^{-\sum_{j=1}^{n-1} v(k_j) + v(k_n)} p^{-\sum_{j=1}^{n-1}(v(r_j(k_j)) - v(s_j(k_j)))}.
\end{aligned}$$

*Proof.* By Lemma 3.1 and (2),

$$\begin{aligned}
&\mathbf{P}\{x \in X_p : v(a_1(x)) = v(k_1), \ldots, v(a_n(x)) = v(k_n)\} \\
&\qquad = \sum \mathbf{P}\{a_1(x) = l_1, \ldots, a_n(x) = l_n\} \qquad (8) \\
&\qquad = \sum p^{-\sum_{j=1}^{n-1}(v(r_j(l_j)) - v(s_j(l_j))) + 2v(l_n)} \qquad (9) \\
&\qquad = \sum p^{-\sum_{j=1}^{n-1}(v(r_j(k_j)) - v(s_j(k_j))) + 2v(k_n)} \qquad (10) \\
&\qquad = (p-1)^n p^{-\sum_{j=1}^{n} v(k_j) + 2v(k_n)} p^{-\sum_{j=1}^{n-1}(v(r_j(k_j)) - v(s_j(k_j)))} \\
&\qquad = (p-1)^n p^{-\sum_{j=1}^{n-1} v(k_j) + v(k_n)} p^{-\sum_{j=1}^{n-1}(v(r_j(k_j)) - v(s_j(k_j)))},
\end{aligned}$$

where the summations in (8), (9) and (10) are over all $l_1, \ldots, l_n \in S_p$ such that $v(l_j) = v(k_j)$, $1 \le j \le n$. ∎

PROPOSITION 3.4. *For any $k_1,\ldots,k_{n+1}\in S_p$ as in Proposition 3.2, we have*

$$\begin{aligned}
&\mathbf{P}\{v(a_{n+1}(x))=v(k_{n+1})\,|\,v(a_n(x))=v(k_n)\}\\
&\qquad=\mathbf{P}\{v(a_{n+1}(x))=v(k_{n+1})\,|\,v(a_1(x))=v(k_1),\ldots,v(a_n(x))=v(k_n)\}\\
&\qquad=(p-1)p^{v(k_{n+1})-2v(k_n)-v(r_n(k_n))+v(s_n(k_n))}.
\end{aligned}$$

*Proof.* By Lemma 3.3,

$$\begin{aligned}
\mathbf{P}\{v(a_{n+1}(x))&=v(k_{n+1})\,|\,v(a_n(x))=v(k_n),\ldots,v(a_1(x))=v(k_1)\}\\
&=\frac{\mathbf{P}\{v(a_1(x))=v(k_1),\ldots,v(a_{n+1}(x))=v(k_{n+1})\}}{\mathbf{P}\{v(a_1(x))=v(k_1),\ldots,v(a_n(x))=v(k_n)\}}\\
&=\frac{(p-1)^{n+1}p^{-\sum_{j=1}^{n}v(k_j)+v(k_{n+1})}p^{-\sum_{j=1}^{n}(v(r_j(k_j))-v(s_j(k_j)))}}{(p-1)^{n}p^{-\sum_{j=1}^{n-1}v(k_j)+v(k_n)}p^{-\sum_{j=1}^{n-1}(v(r_j(k_j))-v(s_j(k_j)))}}\\
&=(p-1)p^{v(k_{n+1})-2v(k_n)-v(r_n(k_n))+v(s_n(k_n))}.
\end{aligned}$$

On the other hand, write

$$\begin{aligned}
A_n&=\{v(a_n(x))=v(k_n),v(a_{n+1}(x))=v(k_{n+1})\},\\
B_n&=\{v(a_n(x))=v(k_n)\}.
\end{aligned}$$

Also by Lemma 3.3, we have

$$\begin{aligned}
&\mathbf{P}\{v(a_{n+1}(x))=v(k_{n+1})\,|\,v(a_n(x))=v(k_n)\}\\
&\quad=\frac{\mathbf{P}\{v(a_n(x))=v(k_n),\,v(a_{n+1}(x))=v(k_{n+1})\}}{\mathbf{P}\{v(a_n(x))=v(k_n)\}}\\
&\quad=\frac{\sum\mathbf{P}(\{v(a_j(x))=v(l_j),\,1\le j\le n-1\}\cap A_n)}{\sum\mathbf{P}(\{v(a_j(x))=v(m_j),\,1\le j\le n-1\}\cap B_n)} \qquad (11)\\
&\quad=\frac{\sum p^{-\sum_{j=1}^{n-1}v(l_j)-v(k_n)}p^{-\sum_{j=1}^{n-1}(v(r_j(l_j))-v(s_j(l_j)))-(v(r_n(k_n))-v(s_n(k_n)))}}{(p-1)^{-1}\sum p^{-\sum_{j=1}^{n-1}v(m_j)+v(k_n)-v(k_{n+1})}p^{-\sum_{j=1}^{n-1}(v(r_j(m_j))-v(s_j(m_j)))}} \qquad (12)\\
&\quad=(p-1)p^{v(k_{n+1})-2v(k_n)-v(r_n(k_n))+v(s_n(k_n))},
\end{aligned}$$

where the summations in the numerators of (11) and (12) are over all $l_1,\ldots,l_{n-1}\in S_p$ satisfying $v(l_1)\le -1$, $v(l_{j+1})\le 2v(l_j)-1+v(r_j(l_j))-v(s_j(l_j))$ for $1\le j\le n-2$ and $v(k_n)\le 2v(l_{n-1})-1+v(r_{n-1}(l_{n-1}))-v(s_{n-1}(l_{n-1}))$, and the summations in the denominators of (11) and (12) are over all $m_1,\ldots,m_{n-1}\in X_p$ satisfying $v(m_1)\le -1$, $v(m_{j+1})\le 2v(m_j)-1+v(r_j(m_j))-v(s_j(m_j))$ for $1\le j\le n-2$ and $v(k_n)\le 2v(m_{n-1})-1+v(r_{n-1}(m_{n-1}))-v(s_{n-1}(m_{n-1}))$. ■

From (2), for any $k,l\in S_p$ satisfying $v(k)=v(l)\le -1$, we have

$$v(r_n(k))=v(r_n(l)),\qquad v(s_n(k))=v(s_n(l)),$$

for any $n \geq 1$. From now on, for any $j \geq 1$, we write $v(r_n(k)) = r(n,j)$ and $v(s_n(k)) = s(n,j)$ if $k \in S_p$ with $v(k) = -j$.

For any $x \in X_p$, let $\{\triangle_n(x) : n \geq 0\}$ be the sequence of random variables such that $\triangle_0(x) = v(a_1(x))$ and $\triangle_n(x) = v(a_{n+1}(x)) - 2v(a_n(x)) - v(r_n(a_n(x))) + v(s_n(a_n(x)))$ for $n \geq 1$.

Now we prove our key result.

Proposition 3.5. *$\{\triangle_n(x) : n \geq 0\}$ is a sequence of independent and identical distributed random variables, and for any $k \geq 1$,*

$$\mathbf{P}\{x \in S_p : \triangle_n(x) = -k\} = \frac{p-1}{p^k}.$$

*Proof.* For any $n \geq 1$ and $k \geq 1$, by Proposition 3.4,

$$\begin{aligned}
&\mathbf{P}\{x \in X_p : \triangle_n(x) = -k\} \\
&\quad = \sum_{j=1}^{\infty} \mathbf{P}\{\triangle_n(x) = -k \mid v(a_n(x)) = -j\}\mathbf{P}\{v(a_n(x)) = -j\} \\
&\quad = \sum_{j=1}^{\infty} \mathbf{P}\{v(a_{n+1}(x)) - 2v(a_n(x)) - v(r_n(a_n(x))) \\
&\qquad\qquad + v(s_n(a_n(x))) = -k \mid v(a_n(x)) = -j\}\mathbf{P}\{v(a_n(x)) = -j\} \\
&\quad = \sum_{j=1}^{\infty} \mathbf{P}\{v(a_{n+1}(x)) = -2j - k + r(n,j) - s(n,j) \mid v(a_n(x)) = -j\} \\
&\qquad\qquad \times \mathbf{P}\{v(a_n(x)) = -j\} \\
&\quad = \sum_{j=1}^{\infty} \mathbf{P}\{v(a_n(x)) = -j\} \cdot \frac{p-1}{p^k} = \frac{p-1}{p^k}.
\end{aligned}$$

Also it is easy to see that for any $k \geq 1$,

$$\mathbf{P}\{x \in X_p : \triangle_0(x) = -k\} = \frac{p-1}{p^k}.$$

Now we prove that the random variables $\triangle_n(x)$, $n = 0, 1, \ldots$, are independent. For any positive integers $k_1, \ldots, k_{n+1}$,

$$\begin{aligned}
&\mathbf{P}\{x \in X_p : \triangle_0(x) = -k_1,\ \triangle_1(x) = -k_2, \ldots, \triangle_n(x) = -k_{n+1}\} \\
&\qquad = \mathbf{P}\{x \in X_p : v(a_1(x)) = p_1,\ v(a_2(x)) = p_2, \ldots, v(a_{n+1}(x)) = p_{n+1}\},
\end{aligned}$$

where $p_1, \ldots, p_{n+1}$ are defined as follows: $p_1 = -k_1$, and for any $1 \leq j \leq n$,

$$p_{j+1} = 2p_j - k_{j+1} + r(j, -p_j) - s(j, -p_j).$$

By the definition of $\{p_j : 1 \leq j \leq n+1\}$, we have

$$p_{n+1} = \sum_{j=1}^{n} p_j - \sum_{j=1}^{n+1} k_j + \sum_{j=1}^{n} (r(j, -p_j) - s(j, -p_j)).$$

Thus by Lemma 3.3,

$$\begin{aligned}
&\mathbf{P}\{\triangle_0(x) = -k_1,\ \triangle_1(x) = -k_2, \ldots, \triangle_n(x) = -k_{n+1}\} \\
&= (p-1)^{n+1} p^{-\sum_{j=1}^{n} p_j + p_{n+1}} p^{-\sum_{j=1}^{n}(r(j,-p_j)-s(j,-p_j))} = (p-1)^{n+1} p^{-\sum_{j=1}^{n+1} k_j} \\
&= \mathbf{P}\{\triangle_0(x) = -k_1\}\mathbf{P}\{\triangle_1(x) = -k_2\}\ldots\mathbf{P}\{\triangle_n(x) = -k_{n+1}\}. \ \blacksquare
\end{aligned}$$

LEMMA 3.6. *For every $n \geq 0$, the random variable $\triangle_n(x)$ has mean value and variance*

$$\mathbf{E}(\triangle_n(x)) = -\frac{p}{p-1}, \qquad \mathbf{Var}(\triangle_n(x)) = \frac{p}{(p-1)^2}.$$

*Proof.* By Proposition 3.5,

$$\mathbf{E}(\triangle_n(x)) = \sum_{k=1}^{\infty} -k\mathbf{P}\{\triangle_n(x) = k\} = \sum_{k=1}^{\infty} -k \cdot \frac{p-1}{p^k} = -\frac{p}{p-1}.$$

Similarly,

$$\mathbf{E}(\triangle_n(x)^2) = \sum_{k=1}^{\infty} (-k)^2 \mathbf{P}\{\triangle_n(x) = -k\} = \sum_{k=1}^{\infty} k^2 \cdot \frac{p-1}{p^k} = \frac{p}{p-1} + \frac{2p}{(p-1)^2},$$

thus

$$\mathbf{Var}(\triangle_n(x)) = \mathbf{E}(\triangle_n(x)^2) - (\mathbf{E}(\triangle_n(x)))^2 = \frac{p}{(p-1)^2}. \ \blacksquare$$

*Proof of Theorem 2.4.* (i) By Proposition 3.5 and Lemma 3.6, $\{\triangle_n(x) : n \geq 0\}$ is a sequence of independent and identical distributed random variables with mean value $-p/(p-1)$ and variance $p/(p-1)^2$. Hence by the central limit theorem (see [1, p. 317, Corollary 2]), we have

$$\lim_{n\to\infty} \mathbf{P}\left\{x \in X_p : \frac{\sum_{j=0}^{n-1} \triangle_j(x) + \frac{p}{p-1} n}{\sqrt{np}/(p-1)} < t\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2}\, du,$$

and thus part (i) of Theorem 2.4 follows.

(ii) By the strong law of large numbers (see [1, p. 125, Corollary 2]), we have, for $\mathbf{P}$-almost all $x \in X_p$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \triangle_j(x) = -\frac{p}{p-1},$$

and the proof of part (ii) of Theorem 2.4 is finished.

(iii) By the iterated logarithm law (see [1, p. 373, Theorem 2]), we have, for $\mathbf{P}$-almost all $x \in X_p$,

$$\limsup_{n\to\infty} \frac{\sum_{j=0}^{n-1} \triangle_j(x) + \frac{p}{p-1} n}{\sqrt{2n \log\log n}} = \frac{\sqrt{p}}{p-1},$$

$$\liminf_{n\to\infty} \frac{\sum_{j=0}^{n-1} \triangle_j(x) + \frac{p}{p-1}\, n}{\sqrt{2n\log\log n}} = -\frac{\sqrt{p}}{p-1},$$

and we finish the proof of part (iii). ∎

We now list some special cases and give applications of Theorem 2.4 to these special expansions. The metric properties of $p$-adic Lüroth series expansions have been discussed in A. Knopfmacher and J. Knopfmacher [11], and Grabner and A. Knopfmacher [5] have investigated the corresponding results for $p$-adic Engel series expansions. It is easy to check that (1) holds in all of the following cases.

Example 1. For any $a \in p^{-1}(S_p\backslash\{0\})$ and any $n \geq 1$, let $s_n(a) = a(a-1)$, $r_n(a) = 1$. Then the algorithm (3) leads the $p$-adic Lüroth series expansion of $x \in X_p$,

$$x = \frac{1}{a_1(x)} + \sum_{n=2}^{\infty} \frac{1}{a_1(x)(a_1(x)-1)\ldots a_{n-1}(x)(a_{n-1}(x)-1)a_n(x)}.$$

In this case, $\triangle_n(x) = v(a_{n+1}(x))$ for any $n \geq 0$. By Theorem 2.4, we have

Corollary 3.7 ([11]). *For $p$-adic Lüroth series expansions*:

(i) $$\lim_{n\to\infty} \mathbf{P}\left\{x \in X_p : \frac{\sum_{j=0}^{n-1} v(a_j(x)) + \frac{p}{p-1}\, n}{\sqrt{np}/(p-1)} < t\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2}\, du.$$

(ii) *For $\mathbf{P}$-almost all $x \in X_p$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} v(a_j(x)) = -\frac{p}{p-1}.$$

(iii) *For $\mathbf{P}$-almost all $x \in X_p$,*

$$\limsup_{n\to\infty} \frac{\sum_{j=0}^{n-1} v(a_j(x)) + \frac{p}{p-1}\, n}{\sqrt{2n\log\log n}} = \frac{\sqrt{p}}{p-1},$$

$$\liminf_{n\to\infty} \frac{\sum_{j=0}^{n-1} v(a_j(x)) + \frac{p}{p-1}\, n}{\sqrt{2n\log\log n}} = -\frac{\sqrt{p}}{p-1}.$$

Example 2. For any $a \in p^{-1}(S_p\backslash\{0\})$ and any $n \geq 1$, let $s_n(a) = a$ and $r_n(a) = 1$. Using the algorithm (3), we get the $p$-adic Engel series expansion of $x \in X_p$,

$$x = \sum_{n=1}^{\infty} \frac{1}{a_1(x)\ldots a_n(x)}.$$

Now $\triangle_0(x) = v(a_1(x))$, and $\triangle_n(x) = v(a_{n+1}(x)) - v(a_n(x))$ for any $n \geq 1$. By Theorem 2.4, we have

COROLLARY 3.8 ([5]). *For p-adic Engel series expansions*:

(i) $$\lim_{n\to\infty} \mathbf{P}\left\{x \in X_p : \frac{v(a_n(x)) + \frac{p}{p-1}\, n}{\sqrt{np}/(p-1)} < t\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2}\, du.$$

(ii) *For* $\mathbf{P}$*-almost all* $x \in X_p$,

$$\lim_{n\to\infty} \frac{1}{n}\, v(a_n(x)) = -\frac{p}{p-1}.$$

(iii) *For* $\mathbf{P}$*-almost all* $x \in X_p$,

$$\limsup_{n\to\infty} \frac{v(a_n(x)) + \frac{p}{p-1}\, n}{\sqrt{2n \log\log n}} = \frac{\sqrt{p}}{p-1},$$

$$\liminf_{n\to\infty} \frac{v(a_n(x)) + \frac{p}{p-1}\, n}{\sqrt{2n \log\log n}} = -\frac{\sqrt{p}}{p-1}.$$

EXAMPLE 3. For any $a \in p^{-1}(S_p \backslash \{0\})$ and any $n \geq 1$, let $s_n(a) = 1$ and $r_n(a) = 1$ for all $n \geq 1$. The algorithm (3) yields the $p$-adic Sylvester series expansion of $x \in X_p$,

$$x = \sum_{n=1}^{\infty} \frac{1}{a_n(x)}.$$

Here $\triangle_0(x) = v(a_1(x))$, and $\triangle_n(x) = v(a_{n+1}(x)) - 2v(a_n(x))$ for any $n \geq 1$. From Theorem 2.4, we have

COROLLARY 3.9. *For p-adic Sylvester series expansions*:

(i) $$\lim_{n\to\infty} \mathbf{P}\left\{x \in X_p : \frac{v(a_n(x)) - \sum_{j=1}^{n-1} v(a_j(x)) + \frac{p}{p-1}\, n}{\sqrt{np}/(p-1)} < t\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2}\, du.$$

(ii) *For* $\mathbf{P}$*-almost all* $x \in X_p$,

$$\lim_{n\to\infty} \frac{1}{n}\Big(v(a_n(x)) - \sum_{j=1}^{n-1} v(a_j(x))\Big) = -\frac{p}{p-1}.$$

(iii) *For* $\mathbf{P}$*-almost all* $x \in X_p$,

$$\limsup_{n\to\infty} \frac{v(a_n(x)) - \sum_{j=1}^{n-1} v(a_j(x)) + \frac{p}{p-1}\, n}{\sqrt{2n \log\log n}} = \frac{\sqrt{p}}{p-1},$$

$$\liminf_{n\to\infty} \frac{v(a_n(x)) - \sum_{j=1}^{n-1} v(a_j(x)) + \frac{p}{p-1}\, n}{\sqrt{2n \log\log n}} = -\frac{\sqrt{p}}{p-1}.$$

EXAMPLE 4. Let $s_n(a) = a$ and $r_n(a) = a + 1$ for any $a \in p^{-1}(S_p \backslash \{0\})$ and any $n \geq 1$. By the algorithm (3), we get the $p$-adic Cantor infinite product of $x \in X_p$,

$$1 + x = \prod_{n=1}^{\infty} \left(1 + \frac{1}{a_n(x)}\right).$$

Here $\triangle_0(x) = v(a_1(x))$, and $\triangle_n(x) = v(a_{n+1}(x)) - 2v(a_n(x))$ for any $n \geq 1$. From Theorem 2.4, we have

COROLLARY 3.10. *For p-adic Cantor infinite products*:

(i) $$\lim_{n\to\infty} \mathbf{P}\left\{x \in X_p : \frac{v(a_n(x)) - \sum_{j=1}^{n-1} v(a_j(x)) + \frac{p}{p-1}\, n}{\sqrt{np}/(p-1)} < t\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2}\, du.$$

(ii) *For* $\mathbf{P}$*-almost all* $x \in X_p$,

$$\lim_{n\to\infty} \frac{1}{n}\left(v(a_n(x)) - \sum_{j=1}^{n-1} v(a_j(x))\right) = -\frac{p}{p-1}.$$

(iii) *For* $\mathbf{P}$*-almost all* $x \in X_p$,

$$\limsup_{n\to\infty} \frac{v(a_n(x)) - \sum_{j=1}^{n-1} v(a_j(x)) + \frac{p}{p-1}\, n}{\sqrt{2n \log\log n}} = \frac{\sqrt{p}}{p-1},$$

$$\liminf_{n\to\infty} \frac{v(a_n(x)) - \sum_{j=1}^{n-1} v(a_j(x)) + \frac{p}{p-1}\, n}{\sqrt{2n \log\log n}} = -\frac{\sqrt{p}}{p-1}.$$

EXAMPLE 5. The $p$-adic modified Engel series expansion of $x \in X_p$ is obtained from the algorithm (3) by taking $s_n(a) = a - 1$ and $r_n(a) = 1$ for all $n \geq 1$ and all $a \in p^{-1}(S_p \backslash \{0\})$,

$$x = \sum_{n=1}^{\infty} \frac{1}{(a_1(x) - 1) \ldots (a_{n-1}(x) - 1) a_n(x)}.$$

For this expansion, $\triangle_0(x) = v(a_1(x))$, and $\triangle_n(x) = v(a_{n+1}(x)) - v(a_n(x))$ for any $n \geq 1$. By Theorem 2.4, we have

COROLLARY 3.11. *For p-adic modified Engel series expansions*:

(i) $$\lim_{n\to\infty} \mathbf{P}\left\{x \in X_p : \frac{v(a_n(x)) + \frac{p}{p-1}\, n}{\sqrt{np}/(p-1)} < t\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2}\, du.$$

(ii) *For* $\mathbf{P}$*-almost all* $x \in X_p$,

$$\lim_{n\to\infty} \frac{1}{n} v(a_n(x)) = -\frac{p}{p-1}.$$

(iii) *For* $\mathbf{P}$*-almost all* $x \in X_p$,

$$\limsup_{n\to\infty} \frac{v(a_n(x)) + \frac{p}{p-1}\, n}{\sqrt{2n \log\log n}} = \frac{\sqrt{p}}{p-1},$$

$$\liminf_{n\to\infty} \frac{v(a_n(x)) + \frac{p}{p-1}\, n}{\sqrt{2n \log\log n}} = -\frac{\sqrt{p}}{p-1}.$$

**4. Proof of Theorem 2.5.** In this section, we use Proposition 3.5 and the central idea in the proof of Theorem 5 in [11] to prove Theorem 2.5.

*Proof of Theorem 2.5.* By Proposition 3.5, the random variables

$$\left|\frac{a_{k+1}(x)s_k(a_k(x))}{a_k(x)^2 r_k(a_k(x))}\right|_p = p^{-\triangle_k(x)}, \qquad k = 1, 2, \ldots,$$

are independent and identically distributed, and it is easy to check that $p^{-\triangle_k(x)}$ has infinite expectation. For any $k \le n$, define

$$U_k(x) = \left|\frac{a_{k+1}(x)s_k(a_k(x))}{a_k(x)^2 r_k(a_k(x))}\right|_p, \qquad V_k(x) = 0$$
$$\text{if } \left|\frac{a_{k+1}(x)s_k(a_k(x))}{a_k(x)^2 r_k(a_k(x))}\right|_p \le n \log_p n,$$

$$V_k(x) = \left|\frac{a_{k+1}(x)s_k(a_k(x))}{a_k(x)^2 r_k(a_k(x))}\right|_p, \qquad U_k(x) = 0$$
$$\text{if } \left|\frac{a_{k+1}(x)s_k(a_k(x))}{a_k(x)^2 r_k(a_k(x))}\right|_p > n \log_q n.$$

Then

$$\begin{aligned}
&\mathbf{P}\left\{x \in X_p : \left|\frac{1}{n \log_q n} \sum_{j=1}^{n} \left|\frac{a_{j+1}(x)s_j(a_j(x))}{a_j(x)^2 r_j(a_j(x))}\right|_p - (p-1)\right| > \varepsilon\right\} \\
&\quad \le \mathbf{P}\{x \in X_p : |U_1(x) + \ldots + U_n(x) - (p-1)n \log_p n| > \varepsilon n \log_p n\} \\
&\qquad + \mathbf{P}\{x \in X_p : V_1(x) + \ldots + V_n(x) \ne 0\}.
\end{aligned}$$

By Proposition 3.5,

$$\begin{aligned}
&\mathbf{P}\{x \in X_p : V_1(x) + \ldots + V_n(x) \ne 0\} \\
&\qquad \le n\mathbf{P}\left\{x \in X_p : \left|\frac{a_2(x)s_1(a_1(x))}{a_1(x)^2 r_1(a_1(x))}\right|_p > n \log_p n\right\} \\
&\qquad = n \sum_{k\,:\,p^k > n \log_p n} (p-1)p^{-k} \le \frac{p}{\log_p n} = o(1).
\end{aligned}$$

Also by Proposition 3.5, we have

$$\mathbf{E}(U_1(x)+\ldots+U_n(x)) = n\mathbf{E}(U_1(x)),$$
$$\mathbf{Var}(U_1(x)+\ldots+U_n(x)) = n\mathbf{Var}(U_1(x)),$$

where

$$\begin{aligned}\mathbf{E}(U_1(x)) &= \sum_{p^k \le n\log_q n} p^k \mathbf{P}(\triangle_1(x) = -k)\\ &= \sum_{p^k \le n\log_p n} p^{-k}(p-1)p^k = (p-1)\log_p([n\log_p n]),\\ \mathbf{Var}(U_1(x)) \le \mathbf{E}(U_1(x)^2) &= \sum_{p^k \le n\log_p n} (p-1)p^k < pn\log_p n.\end{aligned}$$

Chebyshev's inequality then yields

$$\begin{aligned}&\mathbf{P}\{x \in X_p : |U_1(x)+\ldots+U_n(x) - n\mathbf{E}(U_1(x))| > \varepsilon n\mathbf{E}(U_1(x))\}\\ &\qquad \le \frac{n\mathbf{Var}(U_1(x))}{(\varepsilon n\mathbf{E}(U_1(x)))^2} < \frac{pn^2\log_p n}{(\varepsilon(p-1)n\log([n\log_p n]))^2} = o(1).\end{aligned}$$

Since $\mathbf{E}(U_1(x)) \sim (p-1)\log_p n$ as $n \to \infty$, Theorem 2.5 follows. ∎

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