Metric properties for \( p \)-adic Oppenheim series expansions

by

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1. Introduction. Real numbers have several representations, such as continued fraction expansions, Lüroth series, Engel series, Sylvester series expansions and Cantor infinite products etc. (see [4] and [20]). Analogous to continued fraction expansions, certain types of \( p \)-adic continued fractions have been studied by many mathematicians; see for example, [15], [17], [13] and [14] etc. In [8]–[10], A. Knopfmacher and J. Knopfmacher introduced and studied some properties of various unique \( p \)-adic expansions as sums of reciprocals of \( p \)-adic numbers with \( p \)-adic valuations not less than 1. These expansions, including \( p \)-adic Lüroth series, Engel series, Sylvester series expansions and \( p \)-adic Cantor infinite products, were constructed to be analogous to the so-called Oppenheim series expansions of real numbers discussed in Galambos [4].


The main aim of this paper is to derive metric and asymptotic results for \( p \)-adic Oppenheim series expansions. We generalize the results obtained by A. Knopfmacher and J. Knopfmacher [11] and Grabner and A. Knopfmacher [5] for \( p \)-adic Lüroth and Engel series expansions. Also as special
cases of our results, we give metric results for \( p \)-adic Sylvester series expansions and \( p \)-adic Cantor infinite products. The corresponding results for Oppenheim series expansions of Laurent series have been obtained by Fan and the author [3].

2. The \( p \)-adic Oppenheim series expansions. In order to explain the conclusions, we first fix some notations and describe the \( p \)-adic Oppenheim series expansions to be considered.

Let us give a brief account of \( p \)-adic numbers; more details can be found in the books by Koblitz [12] and Schikhof [19].

Let \( p \) be a fixed prime number. Every non-zero rational number \( A \) can be expressed uniquely in the form \( A = p^a r/s \), where \( (r, p) = (s, p) = 1 \) and \( a \in \mathbb{Z} \). The \( p \)-adic valuation \( | \cdot |_p \) on \( \mathbb{Q} \) is defined to be

\[
|A|_p = p^{-a} \quad \text{if } A \neq 0, \quad |0|_p = 0.
\]

The completion of \( \mathbb{Q} \) with respect to the \( p \)-adic metric \( | \cdot |_p \) gives rise to the field \( \mathbb{Q}_p \). Each element \( A \in \mathbb{Q}_p \) has a unique series representation

\[
A = \sum_{n=m}^{\infty} c_n p^n,
\]

where \( m \in \mathbb{Z} \) and the coefficients \( c_n \) are rational integers satisfying \( 0 \leq c_n \leq p - 1 \) and \( c_m \neq 0 \). The integer \( m \) is called the order of \( A \) and denoted by \( v(A) \), and \( |A|_p = p^{-m} \). The valuation \( | \cdot |_p \) defined on \( \mathbb{Q}_p \) has the properties

\[
|A|_p \geq 0, \quad |A|_p = 0 \text{ if and only if } A = 0, \quad |AB|_p = |A|_p |B|_p, \quad |A + B|_p \leq \max(|A|_p, |B|_p) \quad \text{with equality when } |A|_p \neq |B|_p.
\]

For \( v(A) \), we have

\[
v(0) = \infty, \quad v(AB) = v(A) + v(B), \quad v(A/B) = v(A) - v(B) \quad \text{if } B \neq 0, \quad v(A + B) \geq \min(v(A), v(B)) \quad \text{with equality when } v(A) \neq v(B).
\]

It is well known that the above non-Archimedean valuation leads to an ultrametric distance function \( \rho \), with \( \rho(A, B) = |A - B|_p \), making \( \mathbb{Q}_p \) into a complete metric space with respect to \( \rho \).

**Remark 2.1.** Since the metric \( \rho \) is non-Archimedean, it follows that each point of a disc may be considered its center and thus if two discs intersect, then one contains the other.

For any \( A \in \mathbb{Q}_p \), if \( A = \sum_{n=v(A)}^{\infty} c_n p^n \), we call the finite series \( \langle A \rangle = \sum_{v(A) \leq n \leq 0} c_n p^n \) the fractional part of \( A \). Then \( \langle A \rangle \in S_p \), where we define \( S_p = \{ \langle A \rangle : A \in \mathbb{Q}_p \} \subset \mathbb{Q} \). The set \( S_p \) is multiplicatively but not additively closed. The function \( \langle A \rangle \) and set \( S_p \) have been used in the study of

For any $n \geq 1$, let $r_n, s_n$ be maps from $p^{-1}(S_p \{0\})$ to $\mathbb{Q}\{0\}$ satisfying, for any $a \in p^{-1}(S_p \{0\})$,
\begin{align*}
(1) & \quad 2v(a) - v(s_n(a)) + v(r_n(a)) \leq 0 \quad \text{for any } n \geq 1, \\
(2) & \quad v(r_n(a)) = v(r_n(a')) \quad v(s_n(a)) = v(s_n(a')) \quad \text{if } v(a) = v(a').
\end{align*}

Given any $A \in \mathbb{Q}_p$, note that $\langle A \rangle = a_0 \in S_p$ if and only if $v(A - a_0) \geq 1$. Then define $A_1 = A - a_0$. As in [9], [10], if $A_n \neq 0$ with $v(A_n) \geq 1$ ($n \geq 1$) is already defined, then define the “digit” $a_n = \langle 1/A_n \rangle$ and put
\begin{align*}
A_{n+1} = \left( A_n - \frac{1}{a_n} \right) \frac{s_n(a_n)}{r_n(a_n)}.
\end{align*}

For any $m \geq 1$, if $A_m \neq 0$, by (1) and [10, (2.3)], we have $v(A_m) \geq 1$. If some $A_m = 0$, this recursive process stops. It was shown in [9], [10] that this algorithm leads to a finite or convergent series (relative to $\varrho$), called the $p$-adic Oppenheim series expansion.

**Theorem 2.2** ([9], [10]). Every $x \in \mathbb{Q}_p$ has a finite or convergent (relative to $\varrho$) series expansion of the form
\begin{align*}
x = a_0(x) + \frac{1}{a_1(x)} + \sum_{n=1}^{\infty} \frac{r_1(a_1(x)) \ldots r_n(a_n(x))}{s_1(a_1(x)) \ldots s_n(a_n(x))} \frac{1}{a_{n+1}(x)},
\end{align*}
where $a_n(x) \in S_p$, $a_0(x) = \langle x \rangle$, and $v(a_1(x)) \leq 1$, for any $n \geq 1$, 
\begin{align*}
v(a_{n+1}(x)) \leq 2v(a_n(x)) - 1 + v(r_n(a_n(x))) - v(s_n(a_n(x))).
\end{align*}
The expansion is unique for $x$ subject to the above conditions on the “digits” $a_n(x)$.

**Remark 2.3.** The algorithm above is more restricted than the general algorithm described in [10] in order to obtain our metric results. (1) is used to guarantee that $v(a_n) \leq -1$ for any $n \geq 1$ if the process does not stop (see (5)).

Here are some special cases:

- **$p$-adic Lüroth series expansion:** $s_n(a) = a(a - 1)$, $r_n(a) = 1$;
- **$p$-adic Engel expansion:** $s_n(a) = a$, $r_n(a) = 1$;
- **$p$-adic Sylvester expansion:** $s_n(a) = 1$, $r_n(a) = 1$;
- **$p$-adic Cantor infinite product:** $s_n(a) = a$, $r_n(a) = a + 1$.

Let $X_p = p\mathbb{Z}_p$ denote the maximal ideal in the ring $\mathbb{Z}_p$ of all $p$-adic integers, i.e. the set of $p$-adic numbers of order $\geq 0$. Then $X_p$ is compact. For any $A \in X_p$, $v(A) \geq 1$ and from Remark 2.3, $v(A_n) \geq 1$ for any $n \geq 1$ if the process does not stop. Let $\mathbf{P}$ be the probability measure with respect to
Haar measure on $\mathbb{Q}_p$ normalized by $\mathbf{P}(X_p) = 1$. A convenient description of $\mathbf{P}$ on $X_p$ is given in Sprindžuk [21, pp. 67–70]. In particular, $\mathbf{P}(C) = p^{-m}$ for any disc

$$C = C(x, p^{-m-1}) := \{y \in \mathbb{Q}_p : |y - x|_p \leq p^{-m-1}\}$$

of radius $p^{-m-1}$.

For any $x \in X_p$, let $\{\triangle_n(x) : n \geq 0\}$ denote the sequence of random variables such that $\triangle_0(x) = v(a_1(x))$, $\triangle_n(x) = v(a_{n+1}(x)) - 2v(a_n(x)) - v(r_n(a_n(x))) + v(s_n(a_n(x)))$ for $n \geq 1$.

Now we state our main results.

**Theorem 2.4.** For the $p$-adic Oppenheim series expansions described above:

(i) $\lim_{n \to \infty} \mathbf{P}\left\{x \in X_p : \frac{\sum_{j=0}^{n-1} \triangle_j(x) + \frac{p}{p-1} n}{\sqrt{n p/(p - 1)}} < t\right\} = \frac{1}{\sqrt{2\pi}} \int_0^t e^{-u^2/2} du.$

(ii) For $\mathbf{P}$-almost all $x \in X_p$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \triangle_j(x) = -\frac{p}{p - 1}.$$

(iii) For $\mathbf{P}$-almost all $x \in X_p$,

$$\limsup_{n \to \infty} \frac{\sum_{j=0}^{n-1} \triangle_j(x) + \frac{p}{p-1} n}{\sqrt{2n \log \log n}} = \frac{\sqrt{p}}{p - 1},$$

$$\liminf_{n \to \infty} \frac{\sum_{j=0}^{n-1} \triangle_j(x) + \frac{p}{p-1} n}{\sqrt{2n \log \log n}} = -\frac{\sqrt{p}}{p - 1}.$$

Furthermore, we consider the random variables

$$\left| \frac{a_{n+1}(x)s_n(a_n(x))}{a_n(x)^2 r_n(a_n(x))} \right| = p^{-\triangle_n(x)}, \quad n = 1, 2, \ldots$$

In Proposition 3.5, we will show that these are independent and identically distributed with infinite expectation. However, we have the following result:

**Theorem 2.5.** For any fixed $\varepsilon > 0$,

$$\lim_{n \to \infty} \mathbf{P}\left\{x \in X_p : \left| \frac{1}{n \log_p n} \sum_{j=1}^{n} \frac{a_{j+1}(x)s_j(a_j(x))}{a_j(x)^2 r_j(a_j(x))} \right|_p - (p-1) \right| > \varepsilon \right\} = 0,$$

i.e.

$$\frac{1}{n \log_p n} \sum_{j=1}^{n} \left| \frac{a_{j+1}(x)s_j(a_j(x))}{a_j(x)^2 r_j(a_j(x))} \right|_p \to p - 1 \quad \text{in probability.}$$

This paper is organized as follows. In Section 3, we give the proof of Theorem 2.4. Section 4 is devoted to the proof of Theorem 2.5.
3. Proof of Theorem 2.4. In order to prove Theorem 2.4, we need some preliminary results.

**Lemma 3.1.** For any \( k_1, \ldots, k_n \in S_p \) satisfying
\[
v(k_1) \leq -1, \quad v(k_{j+1}) \leq 2v(k_j) - 1 + v(r_j(k_j)) - v(s_j(k_j)), \quad 1 \leq j \leq n - 1,
\]
we have
\[
P\{x \in X_p : a_1(x) = k_1, \ldots, a_n(x) = k_n\} = p^{-\sum_{j=1}^{n-1}(v(r_j(k_j)) - v(s_j(k_j))) + 2v(k_n)}.
\]

**Proof.** From (5), we have
\[
v\left( \frac{r_1(k_1) \ldots r_n(k_n)}{s_1(k_1) \ldots s_n(k_n)} \frac{1}{a_{n+1}(x)} \right) \geq v\left( \frac{r_1(k_1) \ldots r_{n-1}(k_{n-1})}{s_1(k_1) \ldots s_{n-1}(k_{n-1})} \frac{1}{a_n(x)} \right) + 1.
\]
Thus by Theorem 2.2, \( \{x \in X_p : a_1(x) = k_1, \ldots, a_n(x) = k_n\} \) is a disc with center at
\[
\frac{1}{k_1} + \sum_{j=2}^{n} \frac{r_1(k_1) \ldots r_{j-1}(k_{j-1})}{s_1(k_1) \ldots s_{j-1}(k_{j-1})} \frac{1}{k_j}
\]
and diameter
\[
p^{-\sum_{j=1}^{n-1}(v(r_j(k_j)) - v(s_j(k_j))) + 2v(k_n) - 1}.
\]
Thus
\[
P\{x \in X_p : a_1(x) = k_1, \ldots, a_n(x) = k_n\} = p^{-\sum_{j=1}^{n-1}(v(r_j(k_j)) - v(s_j(k_j))) + 2v(k_n)}.
\]

**Proposition 3.2.** For any \( k_1, \ldots, k_{n+1} \in S_p \) satisfying
\[
v(k_1) \leq -1, \quad v(k_{j+1}) \leq 2v(k_j) - 1 + v(r_j(k_j)) - v(s_j(k_j)), \quad 1 \leq j \leq n,
\]
we have
\[
P\{a_{n+1}(x) = k_{n+1} \mid a_n(x) = k_n\}
\]
\[
= P\{a_{n+1}(x) = k_{n+1} \mid a_1(x) = k_1, \ldots, a_n(x) = k_n\}
\]
\[
= \left| \frac{r_n(k_n)}{s_n(k_n)} \right| \frac{|k_n|^2}{p |k_{n+1}|^2},
\]
i.e. \( \{a_n(x) : n \geq 1\} \) forms a Markov chain with transition probabilities,
\[
P\{a_{n+1}(x) = l_{n+1} \mid a_n(x) = l_n\} = \left| \frac{r_n(l_n)}{s_n(l_n)} \right| \frac{|l_n|^2}{p |l_{n+1}|^2}
\]
if \( v(l_{n+1}) \leq 2v(l_n) - 1 + v(r_n(l_n)) - v(s_n(l_n)) \), and 0 otherwise.

**Proof.** By Lemma 3.1,
\[
P\{a_{n+1}(x) = k_{n+1} \mid a_n(x) = k_n, \ldots, a_1(x) = k_1\}
\]
\[
= \frac{P\{a_1(x) = k_1, \ldots, a_n(x) = k_n, a_{n+1}(x) = k_{n+1}\}}{P\{a_1(x) = k_1, \ldots, a_{n-1}(x) = k_{n-1}, a_n(x) = k_n\}}
\]
\[
= \frac{p^{-\sum_{j=1}^{n}(v(r_j(k_j)) - v(s_j(k_j))) + 2v(k_{n+1})}}{p^{-\sum_{j=1}^{n-1}(v(r_j(k_j)) - v(s_j(k_j))) + 2v(k_n)}}
\]
\[
= \left| \frac{r_n(k_n)}{s_n(k_n)} \right| \frac{|k_n|^2}{p |k_{n+1}|^2}.
\]
On the other hand,
\[
\mathbf{P}\{a_{n+1}(x) = k_{n+1} | a_n(x) = k_n\} = \frac{\mathbf{P}\{a_n(x) = k_n, a_{n+1}(x) = k_{n+1}\}}{\mathbf{P}\{a_n(x) = k_n\}}
\]
\[
= \sum \mathbf{P}\{a_j(x) = l_j, 1 \leq j \leq n - 1, a_n(x) = k_n, a_{n+1}(x) = k_{n+1}\}
\sum \mathbf{P}\{a_j(x) = m_j, 1 \leq j \leq n - 1, a_n(x) = k_n\}
\]
\[
(6) = \sum \frac{p - \sum_{j=1}^{n-1} (v(r_j(l)) - v(s_j(l))) - (v(r_n(k_n)) - v(s_n(k_n))) + 2v(k_{n+1})}{p - \sum_{j=1}^{n-1} (v(r_j(l)) - v(s_j(l))) + 2v(k_n)}
\]
\[
= \frac{\left| r_n(k_n) \right|^2}{\left| s_n(k_n) \right| p \left| k_{n+1} \right|^2},
\]
where the summations in the numerators of (6) and (7) are over all \(l_1, \ldots, l_{n-1} \in S_p\) satisfying \(v(l_1) \leq -1, v(l_{j+1}) \leq 2v(l_j) - 1 + v(r_j(l_j)) - v(s_j(l_j))\) for \(1 \leq j \leq n - 2\) and \(v(k_n) \leq 2v(l_{n-1}) - 1 + v(r_n(l_{n-1})) - v(s_n(l_{n-1}))\), and the summations in the denominators of (6) and (7) are over all \(m_1, \ldots, m_{n-1} \in S_p\) satisfying \(v(m_1) \leq -1, v(m_{j+1}) \leq 2v(m_j) - 1 + v(r_j(m_j)) - v(s_j(m_j))\) for \(1 \leq j \leq n - 2\) and \(v(k_n) \leq 2v(m_{n-1}) - 1 + v(r_{n-1}(m_{n-1})) - v(s_{n-1}(m_{n-1}))\).

Next we show that \(\{v(a_n(x)) : n \geq 1\}\) forms a Markov chain.

**Lemma 3.3.** For any \(k_1, \ldots, k_n \in S_p\) as in Lemma 3.1, we have
\[
\mathbf{P}\{x \in X_p : v(a_1(x)) = v(k_1), \ldots, v(a_n(x)) = v(k_n)\} = (p-1)^n p^{-\sum_{j=1}^{n-1} v(k_j)+v(k_n)} p^{-\sum_{j=1}^{n-1} (v(r_j(k)) - v(s_j(k)))}.
\]

**Proof.** By Lemma 3.1 and (2),
\[
\mathbf{P}\{x \in X_p : v(a_1(x)) = v(k_1), \ldots, v(a_n(x)) = v(k_n)\}
\]
\[
= \sum \mathbf{P}\{a_1(x) = l_1, \ldots, a_n(x) = l_n\}
\]
\[
= \sum \frac{p - \sum_{j=1}^{n-1} (v(r_j(l)) - v(s_j(l))) + 2v(l_n)}{p - \sum_{j=1}^{n-1} (v(r_j(l)) - v(s_j(l))) + 2v(l_n)}
\]
\[
= \frac{\left| r_n(k_n) \right|^2}{\left| s_n(k_n) \right| p \left| k_n \right|^2},
\]
where the summations in (8), (9) and (10) are over all \(l_1, \ldots, l_n \in S_p\) such that \(v(l_j) = v(k_j), 1 \leq j \leq n\).
Proposition 3.4. For any $k_1, \ldots, k_{n+1} \in S_p$ as in Proposition 3.2, we have
\[
P\{v(a_{n+1}(x)) = v(k_{n+1}) \mid v(a_n(x)) = v(k_n)\} = \frac{P\{v(a_{n+1}(x)) = v(k_{n+1}) \mid v(a_1(x)) = v(k_1), \ldots, v(a_n(x)) = v(k_n)\}}{\big| P\{v(a_1(x)) = v(k_1), \ldots, v(a_n(x)) = v(k_n)\} \big|} = (p-1)p^{v(k_{n+1})-2v(k_n)-v(r_n(k_n))+v(s_n(k_n))}.
\]

Proof. By Lemma 3.3,
\[
P\{v(a_{n+1}(x)) = v(k_{n+1}) \mid v(a_n(x)) = v(k_n), v(a_{n+1}(x)) = v(k_{n+1})\} = \frac{P\{v(a_1(x)) = v(k_1), \ldots, v(a_{n+1}(x)) = v(k_{n+1})\}}{\big| P\{v(a_1(x)) = v(k_1), \ldots, v(a_n(x)) = v(k_n)\} \big|} = (p-1)^{n+1}p^{-\sum_{j=1}^{n} v(k_j)+v(k_{n+1})}p^{-\sum_{j=1}^{n} (v(r_j(k_j))-v(s_j(k_j)))}
\]
\[
= (p-1)p^{v(k_{n+1})-2v(k_n)-v(r_n(k_n))+v(s_n(k_n))}.
\]

On the other hand, write
\[
A_n = \{v(a_n(x)) = v(k_n), v(a_{n+1}(x)) = v(k_{n+1})\},
B_n = \{v(a_n(x)) = v(k_n)\}.
\]
Also by Lemma 3.3, we have
\[
P\{v(a_{n+1}(x)) = v(k_{n+1}) \mid v(a_n(x)) = v(k_n)\} = \frac{\sum P\{\{v(a_j(x)) = v(l_j), 1 \leq j \leq n-1\} \cap A_n\}}{\sum P\{\{v(a_j(x)) = v(m_j), 1 \leq j \leq n-1\} \cap B_n\}}
\]
\[
(11) = \frac{\sum p^{v(l_j)-v(k_n)}p^{v(s_j(l_j))}(v(r_j(l_j))-v(s_j(l_j)))+(v(r_j(k_n))-v(s_j(k_n)))}}{(p-1)^{-1}\sum p^{-\sum_{j=1}^{n-1} v(m_j)+v(k_n)-v(k_{n+1})}p^{-\sum_{j=1}^{n-1} (v(r_j(m_j))-v(s_j(m_j)))}}
\]
\[
(12) = (p-1)p^{v(k_{n+1})-2v(k_n)-v(r_n(k_n))+v(s_n(k_n))},
\]
where the summations in the numerators of (11) and (12) are over all $l_1, \ldots, l_{n-1} \in S_p$ satisfying $v(l_1) \leq -1$, $v(l_{j+1}) \leq 2v(l_j) - 1 + v(r_j(l_j))-v(s_j(l_j))$ for $1 \leq j \leq n-2$ and $v(k_n) \leq 2v(l_{n-1}) - 1 + v(r_{n-1}(l_{n-1}))-v(s_{n-1}(l_{n-1}))$, and the summations in the denominators of (11) and (12) are over all $m_1, \ldots, m_{n-1} \in X_p$ satisfying $v(m_1) \leq -1$, $v(m_{j+1}) \leq 2v(m_j) - 1 + v(r_j(m_j))-v(s_j(m_j))$ for $1 \leq j \leq n-2$ and $v(k_n) \leq 2v(m_{n-1}) - 1 + v(r_{n-1}(m_{n-1}))-v(s_{n-1}(m_{n-1})).$ ⬤

From (2), for any $k, l \in S_p$ satisfying $v(k) = v(l) \leq -1$, we have
\[
v(r_n(k)) = v(r_n(l)), \quad v(s_n(k)) = v(s_n(l)),
\]
for any \( n \geq 1 \). From now on, for any \( j \geq 1 \), we write \( v(r_n(k)) = r(n, j) \) and \( v(s_n(k)) = s(n, j) \) if \( k \in S_p \) with \( v(k) = -j \).

For any \( x \in X_p \), let \( \{ \triangle_n(x) : n \geq 0 \} \) be the sequence of random variables such that \( \triangle_0(x) = v(a_1(x)) \) and \( \triangle_n(x) = v(a_{n+1}(x)) - 2v(a_n(x)) - v(r_n(a_n(x))) + v(s_n(a_n(x))) \) for \( n \geq 1 \).

Now we prove our key result.

**Proposition 3.5.** \( \{ \triangle_n(x) : n \geq 0 \} \) is a sequence of independent and identically distributed random variables, and for any \( k \geq 1 \),

\[
\Pr\{ x \in S_p : \triangle_n(x) = -k \} = \frac{p-1}{p^k}.
\]

**Proof.** For any \( n \geq 1 \) and \( k \geq 1 \), by Proposition 3.4,

\[
\Pr\{ x \in X_p : \triangle_n(x) = -k \}
= \sum_{j=1}^{\infty} \Pr\{ \triangle_n(x) = -k \mid v(a_n(x)) = -j \} \Pr\{ v(a_n(x)) = -j \}
= \sum_{j=1}^{\infty} \Pr\{ v(a_{n+1}(x)) - 2v(a_n(x)) - v(r_n(a_n(x))) + v(s_n(a_n(x))) = -k \mid v(a_n(x)) = -j \} \Pr\{ v(a_n(x)) = -j \}
= \sum_{j=1}^{\infty} \Pr\{ v(a_{n+1}(x)) = -2j - k + r(n, j) - s(n, j) \mid v(a_n(x)) = -j \}
\times \Pr\{ v(a_n(x)) = -j \}
= \sum_{j=1}^{\infty} \Pr\{ v(a_n(x)) = -j \} \cdot \frac{p-1}{p^k} = \frac{p-1}{p^k}.
\]

Also it is easy to see that for any \( k \geq 1 \),

\[
\Pr\{ x \in X_p : \triangle_0(x) = -k \} = \frac{p-1}{p^k}.
\]

Now we prove that the random variables \( \triangle_n(x) \), \( n = 0, 1, \ldots \), are independent. For any positive integers \( k_1, \ldots, k_{n+1} \),

\[
\Pr\{ x \in X_p : \triangle_0(x) = -k_1, \triangle_1(x) = -k_2, \ldots, \triangle_n(x) = -k_{n+1} \}
= \Pr\{ x \in X_p : v(a_1(x)) = p_1, v(a_2(x)) = p_2, \ldots, v(a_{n+1}(x)) = p_{n+1} \},
\]

where \( p_1, \ldots, p_{n+1} \) are defined as follows: \( p_1 = -k_1 \), and for any \( 1 \leq j \leq n \),

\[ p_{j+1} = 2p_j - k_{j+1} + r(j, -p_j) - s(j, -p_j). \]

By the definition of \( \{ p_j : 1 \leq j \leq n + 1 \} \), we have

\[ p_{n+1} = \sum_{j=1}^{n} p_j - \sum_{j=1}^{n+1} k_j + \sum_{j=1}^{n} (r(j, -p_j) - s(j, -p_j)). \]
Thus by Lemma 3.3,
\[ P\{\triangle_0(x) = -k_1, \, \triangle_1(x) = -k_2, \ldots, \, \triangle_n(x) = -k_{n+1}\} = (p-1)^{n+1} p^{-\sum_{j=1}^n p_j + p_{n+1}} p^{-\sum_{j=1}^n (r(j, -p_j) - s(j, -p_j))} = (p-1)^{n+1} p^{-\sum_{j=1}^{n+1} k_j} = P\{\triangle_0(x) = -k_1\} P\{\triangle_1(x) = -k_2\} \ldots P\{\triangle_n(x) = -k_{n+1}\}. \]

**Lemma 3.6.** For every \( n \geq 0 \), the random variable \( \triangle_n(x) \) has mean value and variance
\[
\mathbb{E}(\triangle_n(x)) = -\frac{p}{p-1}, \quad \text{Var}(\triangle_n(x)) = \frac{p}{(p-1)^2}.
\]

**Proof.** By Proposition 3.5,
\[
\mathbb{E}(\triangle_n(x)) = \sum_{k=1}^\infty -k P\{\triangle_n(x) = k\} = \sum_{k=1}^\infty -k \cdot \frac{p-1}{p^k} = -\frac{p}{p-1}.
\]
Similarly,
\[
\mathbb{E}(\triangle_n(x)^2) = \sum_{k=1}^\infty (-k)^2 P\{\triangle_n(x) = -k\} = \sum_{k=1}^\infty k^2 \cdot \frac{p-1}{p^k} = \frac{p}{p-1} + \frac{2p}{(p-1)^2},
\]
thus
\[
\text{Var}(\triangle_n(x)) = \mathbb{E}(\triangle_n(x)^2) - (\mathbb{E}(\triangle_n(x)))^2 = \frac{p}{(p-1)^2}.
\]

**Proof of Theorem 2.4.** (i) By Proposition 3.5 and Lemma 3.6, \( \{\triangle_n(x) : n \geq 0\} \) is a sequence of independent and identical distributed random variables with mean value \(-p/(p-1)\) and variance \(p/(p-1)^2\). Hence by the central limit theorem (see [1, p. 317, Corollary 2]), we have
\[
\lim_{n \to \infty} P\left\{ x \in X_p : \frac{\sum_{j=0}^{n-1} \triangle_j(x) + \frac{p-1}{p} n}{\sqrt{np/(p-1)}} < t \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-u^2/2} du,
\]
and thus part (i) of Theorem 2.4 follows.

(ii) By the strong law of large numbers (see [1, p. 125, Corollary 2]), we have, for \( P \)-almost all \( x \in X_p \),
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \triangle_j(x) = -\frac{p}{p-1},
\]
and the proof of part (ii) of Theorem 2.4 is finished.

(iii) By the iterated logarithm law (see [1, p. 373, Theorem 2]), we have, for \( P \)-almost all \( x \in X_p \),
\[
\limsup_{n \to \infty} \frac{\sum_{j=0}^{n-1} \triangle_j(x) + \frac{p}{p-1} n}{\sqrt{2n \log \log n}} = \frac{\sqrt{p}}{p-1},
\]
\[
\lim \inf_{n \to \infty} \frac{\sum_{j=0}^{n-1} \triangle_j(x) + \frac{p}{p-1} n}{\sqrt{2n \log \log n}} = -\frac{\sqrt{p}}{p-1},
\]
and we finish the proof of part (iii). □

We now list some special cases and give applications of Theorem 2.4 to these special expansions. The metric properties of \(p\)-adic Lüroth series expansions have been discussed in A. Knopfmacher and J. Knopfmacher [11], and Grabner and A. Knopfmacher [5] have investigated the corresponding results for \(p\)-adic Engel series expansions. It is easy to check that (1) holds in all of the following cases.

**Example 1.** For any \(a \in p^{-1}(S_p \setminus \{0\})\) and any \(n \geq 1\), let \(s_n(a) = a(a-1)\), \(r_n(a) = 1\). Then the algorithm (3) leads the \(p\)-adic Lüroth series expansion of \(x \in X_p\),

\[
x = \frac{1}{a_1(x)} + \sum_{n=2}^{\infty} \frac{1}{a_1(x)(a_1(x)-1)\ldots a_{n-1}(x)(a_{n-1}(x)-1)a_n(x)}.
\]

In this case, \(\triangle_n(x) = v(a_{n+1}(x))\) for any \(n \geq 0\). By Theorem 2.4, we have

**Corollary 3.7 ([11]).** For \(p\)-adic Lüroth series expansions:

(i) \(\lim_{n \to \infty} \mathbb{P}\left\{ x \in X_p : \frac{\sum_{j=0}^{n-1} v(a_j(x)) + \frac{p}{p-1} n}{\sqrt{n p/(p-1)}} < t \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} du.\)

(ii) For \(\mathbb{P}\)-almost all \(x \in X_p\),

\[\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} v(a_j(x)) = -\frac{p}{p-1}.\]

(iii) For \(\mathbb{P}\)-almost all \(x \in X_p\),

\[
\lim \sup_{n \to \infty} \frac{\sum_{j=0}^{n-1} v(a_j(x)) + \frac{p}{p-1} n}{\sqrt{2n \log \log n}} = \frac{\sqrt{p}}{p-1},
\]

\[
\lim \inf_{n \to \infty} \frac{\sum_{j=0}^{n-1} v(a_j(x)) + \frac{p}{p-1} n}{\sqrt{2n \log \log n}} = -\frac{\sqrt{p}}{p-1}.
\]

**Example 2.** For any \(a \in p^{-1}(S_p \setminus \{0\})\) and any \(n \geq 1\), let \(s_n(a) = a\) and \(r_n(a) = 1\). Using the algorithm (3), we get the \(p\)-adic Engel series expansion of \(x \in X_p\),

\[
x = \sum_{n=1}^{\infty} \frac{1}{a_1(x)\ldots a_n(x)}.
\]

Now \(\triangle_0(x) = v(a_1(x))\), and \(\triangle_n(x) = v(a_{n+1}(x)) - v(a_n(x))\) for any \(n \geq 1\). By Theorem 2.4, we have
Corollary 3.8 ([5]). For $p$-adic Engel series expansions:

(i) $\lim_{n \to \infty} P\left\{ x \in X_p : \frac{v(a_n(x)) + \frac{p}{p-1} n}{\sqrt{np}/(p-1)} < t \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} du.

(ii) For $P$-almost all $x \in X_p$, $\lim_{n \to \infty} \frac{1}{n} v(a_n(x)) = -\frac{p}{p-1}$.

(iii) For $P$-almost all $x \in X_p$,

$$\limsup_{n \to \infty} \frac{v(a_n(x)) + \frac{p}{p-1} n}{\sqrt{2n \log \log n}} = \frac{\sqrt{p}}{p-1},$$

$$\liminf_{n \to \infty} \frac{v(a_n(x)) + \frac{p}{p-1} n}{\sqrt{2n \log \log n}} = -\frac{\sqrt{p}}{p-1}.$$

Example 3. For any $a \in p^{-1}(S_p \setminus \{0\})$ and any $n \geq 1$, let $s_n(a) = 1$ and $r_n(a) = 1$ for all $n \geq 1$. The algorithm (3) yields the $p$-adic Sylvester series expansion of $x \in X_p$,

$$x = \sum_{n=1}^{\infty} \frac{1}{a_n(x)}.$$  

Here $\Delta_0(x) = v(a_1(x))$, and $\Delta_n(x) = v(a_{n+1}(x)) - 2v(a_n(x))$ for any $n \geq 1$. From Theorem 2.4, we have

Corollary 3.9. For $p$-adic Sylvester series expansions:

(i) $\lim_{n \to \infty} P\left\{ x \in X_p : \frac{v(a_n(x)) - \sum_{j=1}^{n-1} v(a_j(x)) + \frac{p}{p-1} n}{\sqrt{np}/(p-1)} < t \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} du.$

(ii) For $P$-almost all $x \in X_p$, $\lim_{n \to \infty} \frac{1}{n} \left( v(a_n(x)) - \sum_{j=1}^{n-1} v(a_j(x)) \right) = -\frac{p}{p-1}$.

(iii) For $P$-almost all $x \in X_p$,

$$\limsup_{n \to \infty} \frac{v(a_n(x)) - \sum_{j=1}^{n-1} v(a_j(x)) + \frac{p}{p-1} n}{\sqrt{2n \log \log n}} = \frac{\sqrt{p}}{p-1},$$

$$\liminf_{n \to \infty} \frac{v(a_n(x)) - \sum_{j=1}^{n-1} v(a_j(x)) + \frac{p}{p-1} n}{\sqrt{2n \log \log n}} = -\frac{\sqrt{p}}{p-1}.$$

Example 4. Let $s_n(a) = a$ and $r_n(a) = a + 1$ for any $a \in p^{-1}(S_p \setminus \{0\})$ and any $n \geq 1$. By the algorithm (3), we get the $p$-adic Cantor infinite
product of \( x \in X_p \),
\[
1 + x = \prod_{n=1}^{\infty} \left( 1 + \frac{1}{a_n(x)} \right).
\]
Here \( \triangle_0(x) = v(a_1(x)) \), and \( \triangle_n(x) = v(a_{n+1}(x)) - 2v(a_n(x)) \) for any \( n \geq 1 \).

From Theorem 2.4, we have

**Corollary 3.10.** For \( p \)-adic Cantor infinite products:

(i) \[
\lim_{n \to \infty} P \left\{ x \in X_p : \frac{v(a_n(x)) - \sum_{j=1}^{n-1} v(a_j(x)) + \frac{p}{p-1} n}{\sqrt{n^p/(p-1)}} < t \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} \, du.
\]

(ii) For \( P \)-almost all \( x \in X_p \),
\[
\lim_{n \to \infty} \frac{1}{n} \left( v(a_n(x)) - \sum_{j=1}^{n-1} v(a_j(x)) \right) = -\frac{p}{p-1}.
\]

(iii) For \( P \)-almost all \( x \in X_p \),
\[
\limsup_{n \to \infty} \frac{v(a_n(x)) - \sum_{j=1}^{n-1} v(a_j(x)) + \frac{p}{p-1} n}{\sqrt{2n \log \log n}} = \frac{\sqrt{p}}{p-1},
\]
\[
\liminf_{n \to \infty} \frac{v(a_n(x)) - \sum_{j=1}^{n-1} v(a_j(x)) + \frac{p}{p-1} n}{\sqrt{2n \log \log n}} = -\frac{\sqrt{p}}{p-1}.
\]

**Example 5.** The \( p \)-adic modified Engel series expansion of \( x \in X_p \) is obtained from the algorithm (3) by taking \( s_n(a) = a - 1 \) and \( r_n(a) = 1 \) for all \( n \geq 1 \) and all \( a \in p^{-1}(S_p \setminus \{0\}) \),
\[
x = \sum_{n=1}^{\infty} \frac{1}{(a_1(x) - 1) \cdots (a_{n-1}(x) - 1) a_n(x)}.
\]
For this expansion, \( \triangle_0(x) = v(a_1(x)) \), and \( \triangle_n(x) = v(a_{n+1}(x)) - v(a_n(x)) \) for any \( n \geq 1 \). By Theorem 2.4, we have

**Corollary 3.11.** For \( p \)-adic modified Engel series expansions:

(i) \[
\lim_{n \to \infty} P \left\{ x \in X_p : \frac{v(a_n(x)) + \frac{p}{p-1} n}{\sqrt{n^p/(p-1)}} < t \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} \, du.
\]

(ii) For \( P \)-almost all \( x \in X_p \),
\[
\lim_{n \to \infty} \frac{1}{n} v(a_n(x)) = -\frac{p}{p-1}.
\]
(iii) For $\mathbb{P}$-almost all $x \in X_p$,
\[
\limsup_{n \to \infty} \frac{v(a_n(x)) + \frac{p}{p-1} n}{\sqrt{2n \log \log n}} = \frac{\sqrt{p}}{p-1},
\]
\[
\liminf_{n \to \infty} \frac{v(a_n(x)) + \frac{p}{p-1} n}{\sqrt{2n \log \log n}} = -\frac{\sqrt{p}}{p-1}.
\]

4. Proof of Theorem 2.5. In this section, we use Proposition 3.5 and the central idea in the proof of Theorem 5 in [11] to prove Theorem 2.5.

Proof of Theorem 2.5. By Proposition 3.5, the random variables
\[
\left| \frac{a_{k+1}(x) s_k(a_k(x))}{a_k(x)^2 r_k(a_k(x))} \right|_p = p^{-\Delta_k(x)}, \quad k = 1, 2, \ldots,
\]
are independent and identically distributed, and it is easy to check that $p^{-\Delta_k(x)}$ has infinite expectation. For any $k \leq n$, define
\[
U_k(x) = \left| \frac{a_{k+1}(x) s_k(a_k(x))}{a_k(x)^2 r_k(a_k(x))} \right|_p, \quad V_k(x) = 0
\]
if
\[
\left| \frac{a_{k+1}(x) s_k(a_k(x))}{a_k(x)^2 r_k(a_k(x))} \right|_p \leq n \log_p n,
\]
and
\[
V_k(x) = \left| \frac{a_{k+1}(x) s_k(a_k(x))}{a_k(x)^2 r_k(a_k(x))} \right|_p, \quad U_k(x) = 0
\]
if
\[
\left| \frac{a_{k+1}(x) s_k(a_k(x))}{a_k(x)^2 r_k(a_k(x))} \right|_p > n \log_q n.
\]

Then
\[
\mathbb{P}\left\{ x \in X_p : \left| \frac{1}{n \log_q n} \sum_{j=1}^{n} \left| \frac{a_{j+1}(x) s_j(a_j(x))}{a_j(x)^2 r_j(a_j(x))} \right|_p - (p-1) \right| > \varepsilon \right\}
\]
\[
\leq \mathbb{P}\{ x \in X_p : |U_1(x) + \ldots + U_n(x) - (p-1)n \log_p n| > \varepsilon n \log_p n \}
+ \mathbb{P}\{ x \in X_p : V_1(x) + \ldots + V_n(x) \neq 0 \}.
\]

By Proposition 3.5,
\[
\mathbb{P}\{ x \in X_p : V_1(x) + \ldots + V_n(x) \neq 0 \}
\]
\[
\leq n \mathbb{P}\left\{ x \in X_p : \left| \frac{a_2(x) s_1(a_1(x))}{a_1(x)^2 r_1(a_1(x))} \right|_p > n \log_p n \right\}
\]
\[
= n \sum_{k : p^k > n \log_p n} (p-1)p^{-k} \leq \frac{p}{ \log_p n} = o(1).
\]
Also by Proposition 3.5, we have
\[
E(U_1(x) + \ldots + U_n(x)) = nE(U_1(x)),
\]
\[
\text{Var}(U_1(x) + \ldots + U_n(x)) = n\text{Var}(U_1(x)),
\]
where
\[
E(U_1(x)) = \sum_{p^k \leq n \log_p n} p^k P(\Delta_1(x) = -k)
= \sum_{p^k \leq n \log_p p} p^{-k}(p-1)p^k = (p-1)\log_p([n \log_p n]),
\]
\[
\text{Var}(U_1(x)) \leq E(U_1(x))^2 = \sum_{p^k \leq n \log_p p} (p-1)p^k < pn \log_p n.
\]
Chebyshev’s inequality then yields
\[
P\{x \in X_p : |U_1(x) + \ldots + U_n(x) - nE(U_1(x))| > \varepsilon nE(U_1(x))\} \leq \frac{n\text{Var}(U_1(x))}{(\varepsilon nE(U_1(x)))^2} \leq \frac{pn^2 \log_p n}{(\varepsilon(p-1)n \log([n \log_p n]))^2} = o(1).
\]
Since \(E(U_1(x)) \sim (p-1)\log_p n\) as \(n \to \infty\), Theorem 2.5 follows.

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References


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