On fluctuations in the mean of a sum-of-divisors function

by

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Let $P$ be a prime number. In [2] the authors establish the two-sided $\Omega$-estimate

$$R_P(x) = \Omega_{\pm}(x \log \log x)$$

for the error term

$$R_P(x) := \sum_{n \leq x} \sigma_{(P)}(n) - \left(1 - \frac{1}{P}\right) \frac{\pi^2}{12} x^2$$

related to the “sum-of-$P$-prime-divisors” function

$$\sigma_{(P)}(d) := \sum_{d | n, P \nmid d} d.$$ 

The object of this note is to establish the more precise estimate

$$\limsup_{x \to \infty} \frac{(-1)^i R_P(x)}{x \log \log x} \geq \frac{P - 1}{2(P + 1)} e^{\gamma} \quad (i = 0, 1),$$

as an application of my general result in [4].

Lemmata 2, 6 and 7 of [2] state respectively that

$$\sum_{n \leq x} \frac{\alpha_P(n)}{n} = \log P + O(1/x),$$

where

$$\alpha_P(n) := \begin{cases} 1 & \text{if } P \nmid n, \\ -(P - 1) & \text{otherwise,} \end{cases}$$

$$R_P(x)/x - R'_P(x) = O(1),$$

2000 Mathematics Subject Classification: Primary 11N37; Secondary 11N64.
where
\[ R'_P(x) := \sum_{n \leq x} \frac{\sigma_P(n)}{n} - \left(1 - \frac{1}{P}\right) \frac{\pi^2}{6} x, \]
and
\[ (5) \quad R'_P(x) = - \sum_{d \leq y} \frac{\alpha_P(d)}{d} \left\{ \frac{x}{d} \right\} + O(1), \quad \text{uniformly for } x \geq 2, \ y \geq \sqrt{x}. \]
The inequalities (2) will follow from (4) and
\[ (6) \quad \limsup_{x \to \infty} \frac{(-1)^i R'_P(x)}{\log \log x} \geq \frac{P - 1}{2(P + 1)} e^\gamma \quad (i = 0, 1). \]
If we put \( y = y(x) = x^{3/4} \) in (5), we see by (3) that Theorem 1 of [4] applies. This yields, also using Lemma 6 of [4] (and with the notation of [2]),
\[ (7) \quad \frac{1}{N} \sum_{n=1}^{N} R'_P(nq + \beta) = \sum_{k \leq y(qN+\beta) =: u} \frac{-\alpha_P(k)(q,k)}{k^2} \psi\left(\frac{\beta}{(q,k)}\right) + O(1), \]
provided \( u = o(N) \), where \( \psi(t) := \{ t \} - 1/2 \). Now we put
\[ q := \frac{m!}{P^{r'}} = N^{1/4}, \quad \text{where } \ P^r \parallel m!. \]
With the choice \( \beta = 0 \) we have \( u = N^{15/16} \). Noting that \( \psi(0) = -1/2 \), we write \( k = nm \) with \( n \parallel q \) and \( (m, q/n) = 1 \) and equation (7) becomes
\[ (8) \quad \frac{1}{N} \sum_{n=1}^{N} R'_P(nq) = \frac{1}{2} \sum_{n \parallel q} \frac{1}{n} \left( \sum_{m \leq u/n \text{ and } P|m} \frac{1}{m^2} - \sum_{m \leq u/n \text{ and } P|m} \frac{P - 1}{m^2} \right) + O(1). \]
If we let \( N \to \infty \), the expression in the large parentheses is
\[ (9) \quad \sum_{m \leq u/n \atop p|m} \frac{1}{m^2} - \frac{1 + (P - 1)}{P^2} \sum_{m' \leq u/(Pn) \atop p|m' \Rightarrow p|q/n} \frac{1}{m'^2} \]
\[ = \left(1 - \frac{1}{P}\right) \sum_{m \geq 1 \atop p|m} \frac{1}{m^2} + o(1) \]
\[ \geq \left(1 - \frac{1}{P}\right) \sum_{i \geq 0} \frac{1}{P^{2i}} + o(1) = \frac{P}{P+1} + o(1), \]
and since \( \log m \sim \log \log N \) and \( P \nmid q \), we have

\[
\sum_{n \mid q} \frac{1}{n} = \prod_{p \leq m, p \nmid q} \left( 1 + \frac{1}{p} + \ldots + \frac{1}{p^{\beta_p}} \right)
\]

\[
\sim \prod_{p \leq m, p \neq P} \left( 1 - \frac{1}{p} \right)^{-1} \sim \left( 1 - \frac{1}{P} \right) e^\gamma \log \log N.
\]

By using (9) and (10) in (8) we obtain (6) for \( i = 0 \). The choice \( \beta = q - 1 \) similarly yields (6) for \( i = 1 \).

**Remark.** In [1] the first two authors of [2] prove (1) for \( P = 2 \), by closely following the long argument in my older paper [3] (which establishes the equivalent of (1) for the error term related to \( \sigma(n) \)). They state an implied constant of \( e^\gamma/4 \) for both the \( \Omega_+ \)- and \( \Omega_- \)-estimates. But this claim is not substantiated as it depends on an erroneous estimate, in the first displayed formula on page 13 of [1]. Their equality \( \sigma(A)/A = e^\gamma \log y(1 + o(1)) \) is not correct, since \( A \) (which is our \( q \), \( y \) being our \( m \)) is not divisible by 2. In fact \( \sigma(A)/A = \frac{1}{2} e^\gamma \log y(1 + o(1)) \), and the implied constant obtained once this is amended is only \( e^\gamma/8 \). From (2) with \( P = 2 \) we have the implied constant \( e^\gamma/6 \), which can also be derived from [1] by noticing that the number \( C(A) \) there (which roughly corresponds here to the last sum in (9)) is not only \( \geq 1 \), but \( \geq 4/3 \), when \( A \) is not divisible by 2.

**References**


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Received on 28.11.2002
and in revised form on 30.12.2003 (4406)