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On fluctuations in the mean of a sum-of-divisors function

by

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Let P be a prime number. In [2] the authors establish the two-sided Ω-estimate

$$R_P(x) = \Omega_\pm(x\log\log x) \tag{1}$$

for the error term

$$R_P(x) := \sum_{n\le x}\sigma_{(P)}(n) - \left(1-\frac{1}{P}\right)\frac{\pi^2}{12}\,x^2$$

related to the "sum-of-P-prime-divisors" function

$$\sigma_{(P)}(d) := \sum_{d\mid n,\, P\nmid d} d.$$

The object of this note is to establish the more precise estimate

$$\limsup_{x\to\infty}\frac{(-1)^i R_P(x)}{x\log\log x} \ge \frac{P-1}{2(P+1)}\,e^\gamma \qquad (i=0,1), \tag{2}$$

as an application of my general result in [4].

Lemmata 2, 6 and 7 of [2] state respectively that

$$\sum_{n\le x}\frac{\alpha_P(n)}{n} = \log P + O(1/x), \tag{3}$$

where

$$\alpha_P(n) := \begin{cases} 1 & \text{if } P\nmid n,\\ -(P-1) & \text{otherwise,}\end{cases}$$

$$R_P(x)/x - R'_P(x) = O(1), \tag{4}$$

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where

$$R'_P(x) := \sum_{n\le x} \frac{\sigma_{(P)}(n)}{n} - \left(1 - \frac{1}{P}\right)\frac{\pi^2}{6}\, x,$$

and

$$(5)\quad R'_P(x) = -\sum_{d\le y} \frac{\alpha_P(d)}{d}\left\{\frac{x}{d}\right\} + O(1), \qquad \text{uniformly for } x \ge 2,\ y \ge \sqrt{x}.$$

The inequalities (2) will follow from (4) and

$$(6)\qquad \limsup_{x\to\infty} \frac{(-1)^i R'_P(x)}{\log\log x} \ge \frac{P-1}{2(P+1)}\, e^{\gamma} \qquad (i = 0, 1).$$

If we put $y = y(x) = x^{3/4}$ in (5), we see by (3) that Theorem 1 of [4] applies. This yields, also using Lemma 6 of [4] (and with the notation of [2]),

$$(7)\qquad \frac{1}{N}\sum_{n=1}^{N} R'_P(nq+\beta) = \sum_{k\le y(qN+\beta)=:u} \frac{-\alpha_P(k)(q,k)}{k^2}\, \psi\!\left(\frac{\beta}{(q,k)}\right) + O(1),$$

provided $u = o(N)$, where $\psi(t) := \{t\} - 1/2$. Now we put

$$q := \frac{m!}{P^r} = N^{1/4}, \quad \text{where} \quad P^r \,\|\, m!.$$

With the choice $\beta = 0$ we have $u = N^{15/16}$. Noting that $\psi(0) = -1/2$, we write $k = nm$ with $n \mid q$ and $(m, q/n) = 1$ and equation (7) becomes

$$(8)\quad \frac{1}{N}\sum_{n=1}^{N} R'_P(nq)$$
$$= \frac{1}{2}\sum_{n\mid q} \frac{1}{n}\Bigg(\sum_{\substack{m\le u/n \text{ and } P\nmid m\\ p\mid m \Rightarrow p\nmid q/n}} \frac{1}{m^2} - \sum_{\substack{m\le u/n \text{ and } P\mid m\\ p\mid m \Rightarrow p\nmid q/n}} \frac{P-1}{m^2}\Bigg) + O(1).$$

If we let $N \to \infty$, the expression in the large parentheses is

$$(9)\qquad \sum_{\substack{m\le u/n\\ p\mid m\Rightarrow p\nmid q/n}} \frac{1}{m^2} - \frac{1+(P-1)}{P^2} \sum_{\substack{m'\le u/(Pn)\\ p\mid m'\Rightarrow p\nmid q/n}} \frac{1}{m'^2}$$
$$= \left(1-\frac{1}{P}\right) \sum_{\substack{m\ge 1\\ p\mid m\Rightarrow p\nmid q/n}} \frac{1}{m^2} + o(1)$$
$$\ge \left(1-\frac{1}{P}\right) \sum_{i\ge 0} \frac{1}{P^{2i}} + o(1) = \frac{P}{P+1} + o(1),$$

and since $\log m \sim \log\log N$ and $P \nmid q$, we have

$$\sum_{n|q} \frac{1}{n} = \prod_{\substack{p\le m\\ p^{\beta_p}\|q}} \left(1+\frac{1}{p}+\ldots+\frac{1}{p^{\beta_p}}\right) \tag{10}$$
$$\sim \prod_{p\le m,\, p\ne P} \left(1-\frac{1}{p}\right)^{-1} \sim \left(1-\frac{1}{P}\right) e^{\gamma} \log\log N.$$

By using (9) and (10) in (8) we obtain (6) for $i = 0$. The choice $\beta = q-1$ similarly yields (6) for $i = 1$.

REMARK. In [1] the first two authors of [2] prove (1) for $P = 2$, by closely following the long argument in my older paper [3] (which establishes the equivalent of (1) for the error term related to $\sigma(n)$). They state an implied constant of $e^{\gamma}/4$ for both the Ω_+- and Ω_--estimates. But this claim is not substantiated as it depends on an erroneous estimate, in the first displayed formula on page 13 of [1]. Their equality $\sigma(A)/A = e^{\gamma}\log y(1+o(1))$ is not correct, since A (which is our q, y being our m) is not divisible by 2. In fact $\sigma(A)/A = \frac{1}{2}e^{\gamma}\log y(1+o(1))$, and the implied constant obtained once this is amended is only $e^{\gamma}/8$. From (2) with $P = 2$ we have the implied constant $e^{\gamma}/6$, which can also be derived from [1] by noticing that the number $C(A)$ there (which roughly corresponds here to the last sum in (9)) is not only ≥ 1, but $\ge 4/3$, when A is not divisible by 2.

References

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