ACTA ARITHMETICA
144.2 (2010)

# On the distribution of primitive Pythagorean triangles

by

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**1. Introduction and main results.** A *primitive Pythagorean triangle* is a triple $(a,b,c)$ of natural numbers with $a^2+b^2=c^2$, $a<b$, $\gcd(a,b,c)=1$. For a large real number $x$, let $P(x)$, $A(x)$ and $H(x)$ denote the number of primitive Pythagorean triangles with perimeter, area and hypotenuse less than $x$, respectively. Many authors have studied the asymptotic behavior of $P(x)$, $A(x)$ and $H(x)$.

In 1900, D. N. Lehmer [10] showed that

$$P(x)=\frac{\log 2}{\pi^2}\,x+o(x),\qquad H(x)=\frac{1}{2\pi}\,x+o(x).$$

In 1948, D. H. Lehmer [9] proved

$$P(x)=\frac{\log 2}{\pi^2}\,x+O(x^{1/2}\log x). \tag{1.1}$$

In 1955, J. Lambek and L. Moser [8] obtained (1.1) again and proved

$$H(x)=\frac{1}{2\pi}\,x+O(x^{1/2}\log x) \tag{1.2}$$

and

$$A(x)=cx^{1/2}+O(x^{1/3}), \tag{1.3}$$

where $c=(2\pi^5)^{-1/2}\Gamma^2(1/4)$.

In 1955, R. E. Wild [21] proved

$$A(x)=cx^{1/2}-c'x^{1/3}+R_{\mathrm{area}}(x), \tag{1.4}$$

where

$$c'=\frac{|\zeta(1/3)|(1+2^{-1/3})}{\zeta(3/4)(1+4^{-1/3})}\quad\text{and}\quad R_{\mathrm{area}}(x)=O(x^{1/4}\log x).$$

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2010 *Mathematics Subject Classification*: Primary 11N37; Secondary 11L07.

*Key words and phrases*: primitive Pythagorean triangles, exponential sums, short interval.

DOI: 10.4064/aa144-2-3

In 1980, J. Duttlinger and W. Schwarz [3] proved that there exists a positive constant $\delta > 0$ such that

$$R_{\mathrm{area}}(x) = O(x^{1/4}e^{-\delta \log^{1/2} x}). \tag{1.5}$$

It is difficult to reduce the exponents 1/2, 1/2, 1/4 in the error terms of (1.1), (1.2), and (1.4), since the current technique depends on the strongest estimations of $\sum_{n\le x}\mu(x)$, and the best zero-free regions of the Riemann zeta function so far. Therefore it is natural to search for stronger estimates under the Riemann Hypothesis (RH). The exponent 1/4 in $R_{\mathrm{area}}(x)$ was improved by several authors under RH (see [11], [15], [14], [16], [22]). It is also of interest to consider the distributions of $P(x)$, $A(x)$ and $H(x)$ unconditionally in short intervals.

In this paper, we shall prove the following

THEOREM 1.1. *If RH is true, then for any $\varepsilon > 0$, we have*

$$P(x) = \frac{\log 2}{\pi^2}\, x + O(x^{\frac{5805}{15408}+\varepsilon}). \tag{1.6}$$

THEOREM 1.2. *For any sufficiently small $\varepsilon > 0$ and $x^{\frac{131}{416}+2\varepsilon} < H \le x$, we have*

$$P(x+H) - P(x) = \frac{\log 2}{\pi^2}\, H + O(Hx^{-\varepsilon}). \tag{1.7}$$

THEOREM 1.3. *For any sufficiently small $\varepsilon > 0$ and $x^{\frac{435}{616}+2\varepsilon} < H \le x$, we have*

$$A(x+H) - A(x) = \frac{c}{2}\, Hx^{-1/2} + O(Hx^{-1/2-\varepsilon}), \tag{1.8}$$

*where $c$ is as in* (1.3).

*Notations.* We use $\{t\}$ and $[t]$ to denote the fractional part and the integer part of $t$, respectively, and $\|t\|$ to denote the distance between $t$ and the nearest integer; $\varepsilon$ denotes a small positive constant which may be different at different occurrences; $\mu(n)$ denotes the Möbius function; $e(t) = e^{2\pi it}$; $m \sim M$ means $M < m \le 2M$ and $m \asymp M$ means $c_1M < m \le c_2M$ for some $c_2 > c_1 > 0$.

## 2. Proof of Theorem 1.1

**2.1. Some preliminary lemmas.** The following lemmas will be needed in our proof. Lemma 2.1 is the well-known Euler–Maclaurin summation formula (for example, see Theorem 2.1 of Chapter 2 in [17]). Lemma 2.2 is due to Vaaler [19]. Lemma 2.3 is Theorem 2.2 of Min [12] (see also Lemma 6 of Chapter 1 in [20]). Lemma 2.4 is Theorem 2 of Baker [1] with $(\kappa, \lambda) =$

(1/2, 1/2). Lemma 2.5 is Theorem 3 of Robert and Sargos [18]. Lemmas 2.6–2.8 are Lemma 6, Proposition 1 and Lemma 1 of Fouvry and Iwaniec [5], respectively.

LEMMA 2.1. *Suppose* $f(u)$ *is three times continuously differentiable on* $[a,b]$. *Then*

$$\sum_{a<n\le b} f(n) = \int_a^b f(u)\,du - f(b)\psi(b) + f(a)\psi(a) + \psi_1(b)f'(b) - \psi_1(a)f'(a) - \int_a^b \psi_1(u)f''(u)\,du,$$

*where* $\psi(t)=\{t\}-1/2,\ \psi_1(t)=\{t\}^2/2-\{t\}/2+1/12$.

LEMMA 2.2. *For any* $H_0\ge 2$, *we have*

$$\psi(t) = \sum_{1\le|h|\le H_0} a(h)e(ht) + O\Big(\sum_{0\le h\le H_0} b(h)e(ht)\Big),$$

*where* $a(h)\ll 1/h$ *and* $b(h)\ll 1/H_0$.

LEMMA 2.3. *Let* $A_1,\dots,A_5$ *be absolute positive constants. Suppose* $f(x)$ *and* $g(x)$ *are algebraic functions on* $[a,b]$ *and*

$$\frac{A_1}{R}\le|f''(x)|\le\frac{A_2}{R},\qquad |f'''(x)|\ll\frac{A_3}{RU},\qquad U>1,$$
$$|g(x)|\le A_4G,\qquad |g'(x)|\le A_5GU_1^{-1},\qquad U_1>1.$$

*Suppose* $\alpha\le f'(x)\le\beta$ *for* $x\in[a,b]$. *Then*

$$\sum_{a<n\le b} g(n)e(f(n)) = e^{\pi i/4}\sum_{\alpha\le u\le\beta} b_u\frac{g(n_u)}{\sqrt{f''(n_u)}}e(f(n_u)-un_u) + O(G\log(\beta-\alpha+2)+G(b-a+R)(U^{-1}+U_1^{-1})) + O\bigg(G\min\bigg[\sqrt{R},\max\bigg(\frac{1}{\langle\alpha\rangle},\frac{1}{\langle\beta\rangle}\bigg)\bigg]\bigg),$$

*where* $n_u$ *is the solution of* $f'(n)=u$,

$$\langle t\rangle = \begin{cases} \|t\| & \text{if } t \text{ is not an integer},\\ \beta-\alpha & \text{if } t \text{ is an integer},\end{cases}$$

$$b_u = \begin{cases} 1 & \text{if } \alpha<u<\beta, \text{ or both } \alpha \text{ and } \beta \text{ are not integers},\\ 1/2 & \text{if } u=\alpha \text{ or } \beta \text{ is an integer},\end{cases}$$

$$\sqrt{f''} = \begin{cases} \sqrt{f''} & \text{if } f''>0,\\ i\sqrt{|f''|} & \text{if } f''<0.\end{cases}$$

LEMMA 2.4. *Let* $\alpha,\alpha_1,\alpha_2,z$ *be real numbers such that* $z\alpha\alpha_1\alpha_2\ne 0$ *and* $\alpha\notin\mathbb{N}$. *Let* $M\ge 2,\ M_1\ge 1,\ M_2\ge 1$, *and let* $a_m$ *and* $b_{m_1m_2}$ *be complex* *numbers with* $|a_m| \le 1$, $|b_{m_1m_2}| \le 1$. *Let* $F_1 = |z|M^\alpha M_1^{\alpha_1}M_2^{\alpha_2}$. *If* $F_1 \ge M_1M_2$, *then*

$$\sum_{m\sim M}\sum_{m_1\sim M_1}\sum_{m_2\sim M_2} a_m b_{m_1m_2} e(zm^\alpha m_1^{\alpha_1}m_2^{\alpha_2})$$
$$\ll MM_1M_2\{(M_1M_2)^{-1/2} + F_1^{1/6}M^{-1/3}(M_1M_2)^{-1/6}\}\log(2MM_1M_2).$$

Lemma 2.5. *Let* $\alpha(\alpha-1)\beta\gamma \ne 0$, $X > 1$. *Define*

$$\Big|\sum_{M<m\le 2M} z_m\Big|^* := \max_{M<M_1\le M_2\le 2M}\Big|\sum_{m=M_1}^{M_2} z_m\Big|$$

*and*

$$S := \sum_{h\sim H}\sum_{n\sim N}\Big|\sum_{M<m\le 2M} e\Big(X\,\frac{m^\alpha h^\beta n^\gamma}{M^\alpha H^\beta N^\gamma}\Big)\Big|^*,$$

*where* $H$, $M$, $N$ *are positive integers. Then for any* $\varepsilon > 0$, *we have*

$$S \ll_\varepsilon (HNM)^{1+\varepsilon}\Big(\Big(\frac{X}{HNM^2}\Big)^{1/4} + \frac{1}{M^{1/2}} + \frac{1}{X}\Big).$$

Lemma 2.6. *Let* $0 < L \le N < vN < \lambda L$ *and let* $a_l$ *be complex numbers with* $|a_l| \le 1$. *Then*

$$\sum_{N<n<vN} a_n = \frac{1}{2\pi}\int_{-L}^{L}\Big(\sum_{L<l<\lambda L} a_l l^{-it}\Big)N^{it}(v^{it}-1)t^{-1}\,dt + O(\log(2+L)),$$

*where the constant implied in* $O$ *depends on* $\lambda$ *only.*

Lemma 2.7. *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be two finite sets of real numbers,* $\mathcal{X} \subset [-X, X]$, $\mathcal{Y} \subset [-Y, Y]$. *Then for any complex functions* $u(x)$ *and* $v(y)$,

$$\Big|\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} u(x)v(y)e(xy)\Big|^2$$
$$\le 20(1+XY)\sum_{\substack{x,x'\in\mathcal{X}\\ |x-x'|\le Y^{-1}}}|u(x)u(x')|\sum_{\substack{y,y'\in\mathcal{Y}\\ |y-y'|\le X^{-1}}}|v(y)v(y')|.$$

Lemma 2.8. *Let* $\alpha\beta \ne 0$, $\Delta > 0$, $M \ge 1$ *and* $N \ge 1$. *Let* $\mathcal{A}(M, N; \Delta)$ *be the number of quadruples* $(m, \tilde m, n, \tilde n)$ *such that*

$$\Big|\Big(\frac{\tilde m}{m}\Big)^\alpha - \Big(\frac{\tilde n}{n}\Big)^\beta\Big| < \Delta,$$

*with* $M < m$, $\tilde m \le 2M$ *and* $N < n$, $\tilde n \le 2N$. *Then*

$$\mathcal{A}(M, N; \Delta) \ll MN\log(2MN) + \Delta M^2N^2.$$

**2.2. Reduction of the problem.** We always assume $x > 0$ is a large real number in this paper. Define

$$D_{\text{per}}(x) := \{(\alpha, \beta) \in \mathbb{R}^2 : 2\alpha^2 + 2\alpha\beta \le x,\ 0 < \beta < \alpha\}.$$

Let $L_{\text{per}}(x)$ and $L'_{\text{per}}(x)$ denote the numbers of lattice points and of primitive lattice points inside $D_{\text{per}}(x)$, respectively ("per" comes from "perimeter").

LEMMA 2.9. *We have*

$$P(x) = \sum_{k=0}^{\infty} (-1)^k L'_{\text{per}}\left(\frac{x}{2^k}\right), \qquad L'_{\text{per}}(x) = \sum_{m=1}^{\infty} \mu(m) L_{\text{per}}\left(\frac{x}{m^2}\right).$$

*Proof.* This is contained in Lambek and Moser [8]. ■

For $L_{\text{per}}(x)$, we have the following

LEMMA 2.10. *We have*

$$L_{\text{per}}(x) = c_1 x - c_2 x^{1/2} + E_{\text{per}}(x), \tag{2.1}$$

*where*

$$c_1 = \frac{\log 2}{4}, \qquad c_2 = \frac{\sqrt{2}+1}{4}, \tag{2.2}$$

$$E_{\text{per}}(x) = -\sum_{\frac{1}{2}x^{1/2} < d \le \frac{1}{\sqrt{2}}x^{1/2}} \psi\left(\frac{x}{2d}\right) + O(1). \tag{2.3}$$

*Proof.* Let $x' = \frac{1}{2}((2x+1)^{1/2} - 1)$. By the definition of $L_{\text{per}}(x)$, we have

$$\begin{aligned} L_{\text{per}}(x) &= \sum_{\substack{2d^2+2dl \le x \\ l<d}} 1 = \sum_{\substack{2d^2+2dl \le x \\ l \le d}} 1 - \sum_{\substack{2d^2+2dl \le x \\ d=l}} 1 \\ &= \sum_{d \le x'} \sum_{l \le \min(d, \frac{x}{2d} - d)} 1 - \left[\frac{x^{1/2}}{2}\right] \\ &= \sum_{d \le \frac{1}{2}x^{1/2}} d + \sum_{\frac{1}{2}x^{1/2} < d \le x'} \left(\left[\frac{x}{2d}\right] - d\right) - \left[\frac{x^{1/2}}{2}\right] \\ &= \Sigma_1 + 2\Sigma_2 - \Sigma_3 - x^{1/2}/2 + O(1), \end{aligned} \tag{2.4}$$

where

$$\Sigma_1 = \sum_{\frac{1}{2}x^{1/2} < d \le x'} \left[\frac{x}{2d}\right], \qquad \Sigma_2 = \sum_{d \le \frac{1}{2}x^{1/2}} d, \qquad \Sigma_3 = \sum_{d \le x'} d.$$

It is easy to see that

$$\Sigma_2 = \frac{1}{2}\left[\frac{x^{1/2}}{2}\right]\left(\left[\frac{x^{1/2}}{2}\right] + 1\right) = \frac{x}{8} - \frac{x^{1/2}}{2}\psi\left(\frac{x^{1/2}}{2}\right) + O(1). \tag{2.5}$$

Similarly,

$$\Sigma_3 = \frac{x}{4} - \frac{\sqrt{2}}{4}\,x^{1/2} - \frac{\sqrt{2}}{2}\,x^{1/2}\psi(x') + O(1). \tag{2.6}$$

For $\Sigma_1$, we have

$$\begin{aligned}\Sigma_1 &= \sum_{\frac{1}{2}x^{1/2}<d\le x'} \left(\frac{x}{2d} - \psi\left(\frac{x}{2d}\right) - \frac{1}{2}\right) \\ &= \frac{x}{2} \sum_{\frac{1}{2}x^{1/2}<d\le x'} \frac{1}{d} - \sum_{\frac{1}{2}x^{1/2}<d\le x'} \psi\left(\frac{x}{2d}\right) - \frac{1}{2} \sum_{\frac{1}{2}x^{1/2}<d\le x'} 1 \\ &= \frac{x}{2} \sum_{\frac{1}{2}x^{1/2}<d\le x'} \frac{1}{d} - \frac{\sqrt{2}-1}{4}\,x^{1/2} - \sum_{\frac{1}{2}x^{1/2}<d\le \frac{\sqrt{2}}{2}x^{1/2}} \psi\left(\frac{x}{2d}\right) + O(1).\end{aligned} \tag{2.7}$$

By Lemma 2.1 we have

$$\begin{aligned}&\sum_{\frac{1}{2}x^{1/2}<d\le x'} \frac{1}{d} \\ &= \int_{\frac{1}{2}x^{1/2}}^{x'} \frac{1}{t}\,dt - \sqrt{2}\,x^{-1/2}\psi(x') + 2x^{-1/2}\psi\left(\frac{x^{1/2}}{2}\right) + O\left(\frac{1}{x}\right) \\ &= \log\left(\frac{1}{2}((2x+1)^{1/2}-1)\right) - \log\left(\frac{x^{1/2}}{2}\right) - \sqrt{2}\,x^{-1/2}\psi(x') \\ &\quad + 2x^{-1/2}\psi\left(\frac{x^{1/2}}{2}\right) + O\left(\frac{1}{x}\right) \\ &= \frac{\log 2}{2} - \frac{\sqrt{2}}{2}\,x^{-1/2} - \sqrt{2}\,x^{-1/2}\psi(x') + 2x^{-1/2}\psi\left(\frac{x^{1/2}}{2}\right) + O\left(\frac{1}{x}\right).\end{aligned} \tag{2.8}$$

Now Lemma 2.10 follows by (2.4)–(2.8). ∎

Lemma 2.10 is important in our proofs of Theorems 1.1 and 1.2. Note that the expression (2.3) of $E_{\mathrm{per}}(x)$ is similar to

$$\Delta(x) = -2 \sum_{d\le x^{1/2}} \psi\left(\frac{x}{d}\right) + O(1),$$

which appears as the error term in the Dirichlet divisor problem. Therefore, many approaches used in the study of $\Delta(x)$ can also be applied to the estimate of $E_{\mathrm{per}}(x)$. The latest result for the upper bound of $\Delta(x)$ reads (see Huxley [6])

$$\Delta(x) \ll x^{\frac{131}{416}}(\log x)^{\frac{26947}{8320}}.$$

Similarly, combining (2.3) with the results of Huxley [6], we immediately get

$$E_{\mathrm{per}}(x) \ll x^{\frac{131}{416}+\varepsilon}. \tag{2.9}$$

Suppose $1 \le y \ll x^{1/2}$ is a parameter to be determined. By Lemma 2.10, we decompose $L'_{\mathrm{per}}(x)$ as

$$\begin{aligned} L'_{\mathrm{per}}(x) &= \sum_{m\le y} \mu(m) L_{\mathrm{per}}\left(\frac{x}{m^2}\right) + \sum_{m>y} \mu(m) L_{\mathrm{per}}\left(\frac{x}{m^2}\right) \\ &= c_1 x \sum_{m\le y} \frac{\mu(m)}{m^2} - c_2 x^{1/2} \sum_{m\le y} \frac{\mu(m)}{m} + S_1 + S_2, \end{aligned} \tag{2.10}$$

where $c_1$, $c_2$ are defined by (2.2), and

$$S_1 = \sum_{m\le y} \mu(m) E_{\mathrm{per}}\left(\frac{x}{m^2}\right), \tag{2.11}$$

$$S_2 = \sum_{m>y} \mu(m) L_{\mathrm{per}}\left(\frac{x}{m^2}\right). \tag{2.12}$$

**2.3. Estimation of $S_2$.** We shall estimate $S_2$ in a standard way. The key step is a familiar contour integration technique in the spirit of Montgomery and Vaughan [13], which is used in many occasions when estimating primitive lattice points (for example, see [7], [16], [23]). We only give an outline of the technique here.

Suppose $s = \sigma + it$. For $\sigma > 1$, define

$$Z(s) := \sum_{n=1}^{\infty} \frac{r(n)}{n^s}, \quad \text{where} \quad r(n) := \sum_{\substack{2d^2+2dl=n \\ l<d}} 1.$$

Obviously, $r(n) \le d(n) \ll n^{\varepsilon}$.

Lemma 2.11. *$Z(s)$ has the following properties:*

(i) *$Z(s)$ has an analytic continuation to $\sigma > 1/4$, which has two simple poles at $s = 1, 1/2$ with residues $c_1$, $-c_2/2$ respectively, where $c_1$, $c_2$ are defined by* (2.2).

(ii) *Suppose $1/4 < \theta < 1/2$ is the smallest $\alpha$ such that $E_{\mathrm{per}}(x) \ll x^{\alpha}$. For any real parameter $T \ge 10$, we have*

$$\int_T^{2T} \left| Z\left(\frac{9+4\theta}{16} + it\right)\right|^2 dt \ll T^{1+\varepsilon}.$$

*Proof.* This can be proved in the same way as Lemma 5.2 of Zhai [22] (quoted as Lemma 4.4 in this paper) with slight modifications only. ∎

To estimate $S_2$, we need the following lemma, the proof of which is contained in Nowak [16].

LEMMA 2.12. *Assume RH. Suppose that for some* $\sigma \ge 1/2,\ T \ge 10$,

$$\int_T^{2T} |Z(\sigma+it)|^2\,dt \ll T^{1+\varepsilon}.$$

*Then*

$$S_2 = c_1 x \sum_{m>y} \frac{\mu(m)}{m^2} + O(x^{\theta+\varepsilon} + x^{\sigma+\varepsilon} y^{1/2-2\sigma}),$$

*where* $\theta$ *is as in Lemma* 2.11(ii).

By (2.9) and Lemmas 2.11, 2.12, we take $\theta = \frac{131}{416} + \varepsilon$, $\sigma = \frac{9+4\theta}{16}$ and $y = x^{\frac{651}{1926}}$, thus

$$\begin{aligned} S_2 &= c_1 x \sum_{m>y} \frac{\mu(m)}{m^2} + O(x^{\theta+\varepsilon} + x^{(9+4\theta)/16+\varepsilon} y^{-(5+4\theta)/8}) \\ &= c_1 x \sum_{m>y} \frac{\mu(m)}{m^2} + O(x^{\frac{5805}{15408}+\varepsilon}). \end{aligned} \tag{2.13}$$

**2.4. Estimation of $S_1$.** By (2.11) and Lemma 2.10,

$$S_1 = -\sum_{m\le y} \mu(m) \sum_{\frac{x^{1/2}}{2m} < u \le \frac{x^{1/2}}{\sqrt{2}m}} \psi\left(\frac{x}{2um^2}\right) + O(y).$$

Our aim is to prove $S_1 \ll x^{\frac{5805}{15408}+\varepsilon}$ for $y = x^{\frac{651}{1926}}$. By Lemma 2.2, we have

$$S_1 \ll |S_1(x,y,H_0)| + |S_2(x,y,H_0)| + x^{1/2+\varepsilon} H_0^{-1} + y,$$

where

$$S_1(x,y,H_0) = \sum_{m\le y} \mu(m) \sum_{h\le H_0} a(h) \sum_{\frac{x^{1/2}}{2m} < u \le \frac{x^{1/2}}{\sqrt{2}m}} e\left(\frac{hx}{2um^2}\right),$$

$$S_2(x,y,H_0) = \sum_{m\le y} \left| \sum_{h\le H_0} b(h) \sum_{\frac{x^{1/2}}{2m} < u \le \frac{x^{1/2}}{\sqrt{2}m}} e\left(\frac{hx}{2um^2}\right)\right|,$$

with $a(h) \ll 1/h$ and $b(h) \ll 1/H_0$. From now on we will take $H_0 = x^{\frac{1899}{15408}}$, thus

$$S_1 \ll |S_1(x,y,H_0)| + |S_2(x,y,H_0)| + x^{\frac{5805}{15408}+\varepsilon}. \tag{2.14}$$

We shall only prove $S_1(x,y,H_0) \ll x^{\frac{5805}{15408}+\varepsilon}$. The proof of $S_2(x,y,H_0) \ll x^{\frac{5805}{15408}+\varepsilon}$ is similar and easier. By Lemma 2.3,

$$S_1(x,y,H_0) \ll x^{1/4}\Big|\sum_{m\le y}\mu(m)\sum_{h\le H_0} ha(h)\sum_{h\le v\le 2h}\frac{1}{m^{1/2}h^{3/4}v^{3/4}}\,e\bigg(\frac{\sqrt{2}\,x^{1/2}h^{1/2}v^{1/2}}{m}\bigg)\Big| + x^{\frac{5805}{15408}}.$$

Let $n = hv$. By a splitting argument and partial summation, we have

$$x^{-\varepsilon}S_1(x,y,H_0) \ll x^{1/4}\sup_{\substack{1\ll M\ll y\\ 1\ll N\ll H_0^2}} M^{-1/2}N^{-3/4}|S_1^*(x,M,N)| + x^{\frac{5805}{15408}}, \tag{2.15}$$

where

$$S_1^*(x,M,N) = \sum_{m\sim M}\mu(m)\sum_{n\sim N} a_n e\bigg(\frac{\sqrt{2}\,x^{1/2}n^{1/2}}{m}\bigg)$$

with $|a_n| \le 1$.

We shall estimate $M^{-1/2}N^{-3/4}S_1^*(x,M,N)$ in three cases.

Case 1: $M \le x^{3/11}$, $N \ll H_0^2$. Note that $S_1^*(x,M,N)$ is the same exponential sum as in [2] apart from the constant $\sqrt{2}$. We use the result of [2] directly to obtain

$$x^{-\varepsilon}M^{-1/2}N^{-3/4}|S_1^*(x,M,N)| \ll x^{5/44}, \tag{2.16}$$

for $M \ll x^{3/11}$ and $N \ll x^{3/11}$. This estimate is acceptable for us, since $H_0^2 \ll x^{3/11}$ and $\frac{1}{4}+\frac{5}{44} < \frac{5805}{15408}$.

Case 2: $x^{3/11} < M \ll y$, $x^{-\frac{1953}{3852}}M^2 \le N \ll H_0^2$. By Lemma 2.4 (take $m_1 = 1, m_2 = n$),

$$x^{-\varepsilon}M^{-1/2}N^{-3/4}|S_1^*(x,M,N)| \ll M^{1/2}N^{-1/4} + x^{1/12}N^{1/6} \ll x^{\frac{1953}{15408}}.$$

Case 3: $x^{3/11} < M \ll y, N \ll x^{-\frac{1953}{3852}}M^2$. By the skillful decomposition due to Montgomery and Vaughan [13] and a splitting argument, we can decompose $S_1^*(x,M,N)$ into at most $O(\log M)$ sums of the following two forms:

$$\Sigma_1 = \sum_{n\sim N} a_n \sum_{k\sim K} b_k \sum_{r\sim Mk^{-1}} e\bigg(\frac{\sqrt{2}\,x^{1/2}n^{1/2}}{kr}\bigg), \qquad K \ll M^{1/3},$$

$$\Sigma_2 = \sum_{n\sim N} a_n \sum_{k\sim K} b_k \sum_{r\sim Mk^{-1}} c_r e\bigg(\frac{\sqrt{2}\,x^{1/2}n^{1/2}}{kr}\bigg), \qquad M^{1/3} \ll K \ll M^{1/2},$$

where $b_k \ll M^\varepsilon$ and $c_r \ll M^\varepsilon$. Applying Lemma 2.5 to $\Sigma_1$ with $X$, $M$, $H$, $N$ replaced by $x^{1/2}M^{-1}N^{1/2}$, $MK^{-1}$, $K$, $N$, respectively, we get

$$\begin{aligned}(2.17)\qquad & x^{-\varepsilon}M^{-1/2}N^{-3/4}\Sigma_1 \\ &\ll x^{1/8}M^{-1/4}N^{1/8}K^{1/4} + N^{1/4}K^{1/2} + x^{-1/2}M^{3/2}N^{-1/4} \\ &\ll x^{\frac{1899}{30816}}M^{1/12} + x^{-\frac{1953}{15408}}M^{2/3} + x^{-1/2}M^{3/2} \ll x^{\frac{1953}{15408}}.\end{aligned}$$

Applying Lemma 2.6 to the sum over $r$ in $\Sigma_2$, we get

$$\begin{aligned}\Sigma_2 \ll \int_{-\frac{1}{2}MK^{-1}}^{\frac{1}{2}MK^{-1}} &\Big| t^{-1}(2^{it}-1)\sum_{n\sim N} a_n \sum_{k\sim K} b_k (Mk^{-1})^{it} \\ &\times \sum_{\frac{1}{2}Mk^{-1}<r<2Mk^{-1}} c_r r^{it} e\bigg(\frac{\sqrt{2}\,x^{1/2}n^{1/2}}{kr}\bigg)\Big|\,dt + O(NK\log x).\end{aligned}$$

Applying Lemma 2.7 to the three-dimensional exponential sum in the integral with $\mathcal{X} = \sqrt{2}\,x^{1/2}r^{-1}$ and $\mathcal{Y} = n^{1/2}k^{-1}$, we get

$$(2.18)\qquad x^{-\varepsilon}\Sigma_2 \ll x^{1/4}N^{1/4}M^{-1/2}\mathcal{A}_1^{1/2}\mathcal{A}_2^{1/2} + NK,$$

where

$$\mathcal{A}_1 = \sum_{\substack{|x^{1/2}r_1^{-1} - x^{1/2}r_2^{-1}| \ll N^{-1/2}K \\ r_1,r_2\sim MK^{-1}}} 1, \qquad \mathcal{A}_2 = \sum_{\substack{|n_1^{1/2}k_1^{-1} - n_2^{1/2}k_2^{-1}| \ll x^{-1/2}MK^{-1} \\ r_1,r_2\sim MK^{-1} \\ k_1,k_2\sim K}} 1.$$

For $\mathcal{A}_1$ we have

$$\begin{aligned}(2.19)\qquad \mathcal{A}_1 &\ll \sum_{\substack{|r_1-r_2| \ll x^{-1/2}M^2N^{-1/2}K^{-1} \\ r_1,r_2\sim MK^{-1}}} 1 \\ &\ll MK^{-1}(1 + x^{-1/2}M^2N^{-1/2}K^{-1}).\end{aligned}$$

Applying Lemma 2.8 to $\mathcal{A}_2$, we have

$$(2.20)\qquad \mathcal{A}_2 \ll NK\log(NK) + (x^{-1/2}MN^{-1/2})N^2K^2.$$

Combining (2.19), (2.20) with (2.18) yields

$$\begin{aligned}& x^{-\varepsilon}M^{-1/2}N^{-3/4}\Sigma_2 \\ &\quad \ll x^{-1/4}M + N^{1/4}K^{1/2} + M^{1/2}N^{-1/4}K^{-1/2} + x^{1/4}M^{-1/2} \\ &\quad \ll x^{-1/4}M + x^{-\frac{1953}{15408}}M^{3/4} + M^{1/3} + x^{\frac{1}{4}}M^{-1/2} \ll x^{\frac{1953}{15408}}.\end{aligned}$$

Now $S_1 \ll x^{\frac{5805}{15408}+\varepsilon}$ follows by combining the estimates in Cases 1–3 with (2.14) and (2.15).

**2.5. Proof of Theorem 1.1.** Under the assumption of RH, we have

$$c_3 x^{1/2} \sum_{m\le y} \frac{\mu(m)}{m} \ll x^{1/2+\varepsilon} y^{-1/2} \ll x^{\frac{5805}{15408}}.$$

Combining this and the estimates for $S_1, S_2$ with (2.10), we get

$$L'_{\mathrm{per}}(x) = c_1 \zeta(2)^{-1} x + O(x^{\frac{5805}{15408}+\varepsilon}). \tag{2.21}$$

Theorem 1.1 follows immediately from (2.21) and Lemma 2.9.

**3. Proof of Theorem 1.2.** By (1.1), it is easy to see that (1.7) holds for $x^{1/2+2\varepsilon} < H \le x$. Hence we only need to prove (1.7) for $x^{\frac{131}{416}+2\varepsilon} < H \le x^{1/2+2\varepsilon}$. By Lemma 2.9, we write

$$\begin{aligned} L'_{\mathrm{per}}(x+H) - L'_{\mathrm{per}}(x) &= \sum_{m=1}^{\infty} \mu(m)\Big(L_{\mathrm{per}}\Big(\frac{x+H}{m^2}\Big) - L_{\mathrm{per}}\Big(\frac{x}{m^2}\Big)\Big) \\ &= \sum_{m\le x^\varepsilon} + \sum_{m>x^\varepsilon}, \end{aligned}$$

say. By Lemma 2.10 and (2.9), we have

$$\sum_{m\le x^\varepsilon} = c_1\zeta(2)^{-1} H + O(x^{\frac{131}{416}+\varepsilon}).$$

To estimate $\sum_{m>x^\varepsilon}$, we need the following lemma which is contained in the proof of Theorem 1 of Filaseta and Trifonov [4].

Lemma 3.1. *For any integer $k \ge 1$, we have*

$$\sum_{\substack{x\le nm^k<x+y \\ m>x^\varepsilon}} 1 \ll y x^{-\varepsilon/2} + x^{1/(2k+1)+\varepsilon}.$$

Now we estimate $\sum_{m>x^\varepsilon}$. Note that $L_{\mathrm{per}}(x) = \sum_{n\le x} r(n)$. Hence by Lemma 3.1,

$$\sum_{m>x^\varepsilon} = \sum_{\substack{x\le nm^2<x+H \\ m>x^\varepsilon}} r(n)\mu(m) \ll x^{\varepsilon^2} \sum_{\substack{x\le nm^2<x+H \\ m>x^\varepsilon}} 1 \ll Hx^{-\varepsilon} + x^{1/5+3\varepsilon},$$

where we have used the estimate $r(n) \ll n^\varepsilon$. By the above arguments, we get

$$L'_{\mathrm{per}}(x+H) - L'_{\mathrm{per}}(x) = c_1\zeta(2)^{-1} H + O(Hx^{-\varepsilon} + x^{\frac{131}{416}+\varepsilon}).$$

This together with Lemma 2.9 yields Theorem 1.2.

**4. Proof of Theorem 1.3.** Define

$$D_{\mathrm{area}}(x) := \{(\alpha,\beta)\in\mathbb{R}^2 : \alpha\beta(\alpha^2-\beta^2) < x,\ 0<\beta<\alpha\}.$$

Let $L_{\mathrm{area}}(x)$ and $L'_{\mathrm{area}}(x)$ denote the number of lattice points and primitive lattice points inside $D_{\mathrm{area}}(x)$, respectively. For $s=\sigma+it$, $\sigma>1$, define

$$f_1(k;y):=\sum_{\substack{d^4mn(m^2-n^2)=k\\ d>y}}\mu(d),\qquad f_2(k):=\sum_{mn(m^2-n^2)=k}1,$$

and

$$F_1(s;y):=\sum_{k=1}^{\infty}\frac{f_1(k;y)}{k^s},\qquad F_2(s):=\sum_{k=1}^{\infty}\frac{f_2(k)}{k^s},\qquad F(s;y):=\sum_{d>y}\frac{\mu(d)}{d^{4s}}.$$

Obviously

$$F_1(s;y)=F_2(s)F(s;y). \tag{4.1}$$

To prove Theorem 1.3 we need the following lemmas. Lemma 4.1 is due to Lambek and Moser [8]. Lemma 4.2 is due to Nowak [16, p. 176]. Lemmas 4.3 and 4.4 are (4.7) and Lemma 5.2 of Zhai [22], respectively. Lemma 4.5 is a well-known result on the mean value of Dirichlet polynomial (for example, see Theorem 2.2 of Chapter 29 in [17] with $q=1$).

Lemma 4.1. *We have*

$$A(x)=\sum_{k=0}^{\infty}(-1)^kL'_{\mathrm{area}}\Big(\frac{x}{4^k}\Big),\quad \textit{where}\quad L'_{\mathrm{area}}(x)=\sum_{m=1}^{\infty}\mu(m)L_{\mathrm{area}}\Big(\frac{x}{m^4}\Big).$$

Lemma 4.2. *We have*

$$L_{\mathrm{area}}(x)=c_3x^{1/2}-c_4x^{1/3}+F_{\mathrm{area}}(x),$$

*where*

$$c_3=\frac{\Gamma^2(1/4)}{4(2\pi)^{1/2}},\qquad c_4=|\zeta(1/3)|(1+2^{-1/3}),$$
$$F_{\mathrm{area}}(x)\ll x^{\frac{23}{146}}(\log x)^{\frac{315}{146}}.$$

Lemma 4.3. *For $1\le y\ll x^{1/4}$, we have*

$$\sum_{d\le y}\Big|F_{\mathrm{area}}\Big(\frac{x}{d^4}\Big)\Big|\ll x^{\frac{127}{616}}\log^{\frac{963}{308}}x+x^{1/8}y^{1/2}\log^4 x.$$

Lemma 4.4. *$F_2(s)$ has the following properties:*

(i) *$F_2(s)$ has an analytic continuation to $\sigma>1/8$. It has two simple poles at $s=1/2,\ 1/3$ with residues $c_3/2,\ -c_4/3$, respectively, where $c_3$, $c_4$ are defined in Lemma 4.2.*
(ii) *$F_2(\sigma+it)\ll\min\big(\log|t|,\frac{2}{2\sigma-1}\big)$ for $\sigma\ge 1/2$ and $|t|\ge 2$.*
(iii) *$F_2(\sigma+it)\ll|t|^{(4-8\sigma)/3}\log t$ uniformly for $1/8<\sigma_1\le\sigma\le 1/2$ and $|t|\ge 2$.*

(iv) *For any $T \geq 10$, we have*

$$\int_{T}^{2T} \left| F_2\left(\frac{24}{73} + it\right)\right|^2 dt \ll T \log^7 T.$$

LEMMA 4.5. *For any arithmetic function $a : \mathbb{N} \to \mathbb{C}$, we have*

$$\int_{-T}^{T} \left| \sum_{n \leq x} \frac{a(n)}{n^{it}} \right|^2 dt \ll \sum_{n \leq x} (T+n)|a(n)|^2.$$

From (1.4) and (1.5), it is easy to see that (1.8) holds for $x^{3/4+\varepsilon} < H \leq x$, hence we only need to prove (1.8) for $x^{\frac{435}{616}+2\varepsilon} < H \leq x^{3/4+\varepsilon}$. Lemma 4.1 gives

$$L'_{\mathrm{area}}(x+H) - L'_{\mathrm{area}}(x) = \sum_{d=1}^{\infty} \mu(d) \left( L_{\mathrm{area}}\left(\frac{x+H}{d^4}\right) - L_{\mathrm{area}}\left(\frac{x}{d^4}\right)\right). \tag{4.2}$$

Suppose $x^{\varepsilon} < y \ll x^{1/4}$ is a parameter to be determined. We divide the sum over $d$ into two sums,

$$L'_{\mathrm{area}}(x+H) - L'_{\mathrm{area}}(x) = \sum_{d \leq y} + \sum_{d > y}, \tag{4.3}$$

say. By Lemmas 4.2 and 4.3,

$$\begin{aligned} \sum_{d \leq y} = {} & c_3((x+H)^{1/2} - x^{1/2}) \sum_{d \leq y} \frac{\mu(d)}{d^2} \\ & - c_4((x+H)^{1/3} - x^{1/3}) \sum_{d \leq y} \frac{\mu(d)}{d^{4/3}} \\ & + O(x^{\frac{127}{616}+\varepsilon} + x^{1/8+\varepsilon} y^{1/2}). \end{aligned} \tag{4.4}$$

By the definition of $L_{\mathrm{area}}(x)$, we have

$$\begin{aligned} \sum_{d > y} &= \sum_{d > y} \mu(d) \sum_{\frac{x}{d^4} < mn(m^2-n^2) < \frac{x+H}{d^4}} 1 \\ &= \sum_{\substack{x < d^4 mn(m^2-n^2) \leq x+H \\ d > y}} \mu(d) = \sum_{x < k \leq x+H} f_1(k; y). \end{aligned} \tag{4.5}$$

By Perron's formula,

$$\sum_{x < k \leq x+H} f_1(k; y) = \frac{1}{2\pi i} \int_{1+\varepsilon-ix}^{1+\varepsilon+ix} F_1(s; y) \frac{(x+H)^s - x^s}{s} ds + O(x^{\varepsilon}).$$

Move the line of integration above to $\Re s = \sigma_0 = \frac{24}{73}$. By the residue theorem, we have

$$\sum_{x<k\le x+H} f_1(k;y) = \mathrm{Res}_{s=1/2} + \mathrm{Res}_{s=1/3} + O(x^{\varepsilon}) \tag{4.6}$$
$$+ \frac{1}{2\pi i}\Big(\int_{\sigma_0-ix}^{\sigma_0+ix} - \int_{1+\varepsilon+ix}^{\sigma_0+ix} - \int_{\sigma_0-ix}^{1+\varepsilon-ix}\Big) F_2(s)F(s;y)\,\frac{(x+H)^s - x^s}{s}\,ds$$

where $\mathrm{Res}_{s=1/2,1/3}$ are the residues of $F_2(s)F(s;y)\frac{(x+H)^s-x^s}{s}$ at $s = 1/2, 1/3$, respectively. By Lemma 4.4(i),

$$\mathrm{Res}_{s=1/2} + \mathrm{Res}_{s=1/3} = c_3((x+H)^{1/2} - x^{1/2}) \sum_{n>y} \frac{\mu(d)}{d^2} \tag{4.7}$$
$$- c_4((x+H)^{1/3} - x^{1/3}) \sum_{n>y} \frac{\mu(d)}{d^{4/3}}.$$

By Lemma 4.4(ii) & (iii), we easily get

$$\frac{1}{2\pi i}\Big(\int_{1+\varepsilon-ix}^{\sigma_0-ix} + \int_{\sigma_0+ix}^{1+\varepsilon+ix}\Big) \ll x^{\frac{127}{616}+\varepsilon}. \tag{4.8}$$

Now we only need to estimate

$$\frac{1}{2\pi i}\int_{\sigma_0-ix}^{\sigma_0+ix} \ll \Big|\int_0^x F_2(\sigma_0+it)F(\sigma_0+it;y)\,\frac{(x+H)^{\sigma_0+it} - x^{\sigma_0+it}}{\sigma_0+it}\,dt\Big|$$
$$\ll \Big|\int_0^y\Big| + \Big|\int_y^x\Big|,$$

say. By Cauchy's equality, Lemma 4.4(iv) and Lemma 4.5, we have

$$\Big|\int_0^y\Big| = \Big|\int_0^y F_2(\sigma_0+it)F(\sigma_0+it;y)\Big(\int_x^{x+H} u^{\sigma_0-1+it}\,du\Big)\,dt\Big|$$
$$\ll \int_x^{x+H} u^{\sigma_0-1}\,du\,\Big|\int_0^y F_2(\sigma_0+it)F(\sigma_0+it;y)\,dt\Big|$$
$$\ll ((x+H)^{\sigma_0} - x^{\sigma_0})\Big(\int_0^y |F_2(\sigma_0+it)|^2\,dt\Big)^{1/2}\Big(\int_0^y |F(\sigma_0+it;y)|^2\,dt\Big)^{1/2}$$
$$\ll Hx^{\sigma_0-1}y^{1/2}\Big(\sum_{d>y}(y+d)d^{-8\sigma_0}\Big)^{1/2} \ll Hx^{\sigma_0-1}y^{3/2-4\sigma_0}$$

and

$$\Big|\int_y^x\Big| = \Big|\int_y^x F_2(\sigma_0+it)F(\sigma_0+it;y)\,\frac{(x+H)^{\sigma_0+it} - x^{\sigma_0+it}}{\sigma_0+it}\,dt\Big|$$
$$\ll x^{\sigma_0}\max_{y\ll M\ll x} M^{-1}\Big(\int_M^{2M} |F_2(\sigma_0+it)|^2 dt\Big)^{1/2}\Big(\int_M^{2M} |F(\sigma_0+it;y)|^2\,dt\Big)^{1/2}$$

$$\ll x^{\sigma_0+\varepsilon} \max_{y\ll M\ll x} M^{-1/2}\Big(\sum_{d>y}(M+d)d^{-8\sigma_0}\Big)^{1/2}$$
$$\ll x^{\sigma_0+\varepsilon} \max_{y\ll M\ll x} (y^{1/2-4\sigma_0}+M^{-1/2}y^{1-4\sigma_0}) \ll x^{\sigma_0+\varepsilon}y^{1/2-4\sigma_0}.$$

Thus

$$\frac{1}{2\pi i}\int\limits_{\sigma_0-ix}^{\sigma_0+ix} \ll Hx^{\sigma_0-1}y^{3/2-4\sigma_0}+x^{\sigma_0+\varepsilon}y^{1/2-4\sigma_0}. \tag{4.9}$$

Combining (4.6)–(4.9) with (4.5), we get

$$\begin{aligned}\sum_{d>y} &= c_3((x+H)^{1/2}-x^{1/2})\sum_{n>y}\frac{\mu(d)}{d^2}\\ &\quad - c_4((x+H)^{1/3}-x^{1/3})\sum_{n>y}\frac{\mu(d)}{d^{4/3}}\\ &\quad + O(x^{\frac{127}{616}+\varepsilon}+Hx^{\sigma_0-1}y^{3/2-4\sigma_0}+x^{\sigma_0+\varepsilon}y^{1/2-4\sigma_0}).\end{aligned} \tag{4.10}$$

This together with (4.3), (4.4) yields

$$\begin{aligned}&L'_{\text{area}}(x+H)-L'_{\text{area}}(x)\\ &\quad = c_3\zeta(2)^{-1}((x+H)^{1/2}-x^{1/2})-c_4\zeta(4/3)^{-1}((x+H)^{1/3}-x^{1/3})\\ &\qquad + O(x^{\frac{127}{616}+\varepsilon}+x^{1/8+\varepsilon}y^{1/2}+Hx^{\sigma_0-1}y^{3/2-4\sigma_0}+x^{\sigma_0+\varepsilon}y^{1/2-4\sigma_0})\\ &\quad = \frac{c_3}{2}\zeta(2)^{-1}Hx^{-1/2}+O(x^{\frac{127}{616}+\varepsilon})+O(x^{1/8+\varepsilon}y^{1/2}+x^{\sigma_0+\varepsilon}y^{1/2-4\sigma_0})\\ &\qquad + O(Hx^{\sigma_0-1+\varepsilon}y^{3/2-4\sigma_0}+Hx^{-2/3}).\end{aligned}$$

Take $y = x^{1/4-1/32\sigma_0}$. On recalling $\sigma_0 = \frac{24}{73}$, it is easy to check that the second $O$-term is $\ll x^{\frac{127}{616}+\varepsilon}$ and the third $O$-term is $\ll Hx^{-1/2-\varepsilon}$. Thus

$$\begin{aligned}&L'_{\text{area}}(x+H)-L'_{\text{area}}(x)\\ &\qquad = \frac{c_3}{2}\zeta(2)^{-1}Hx^{-1/2}+O(Hx^{-1/2-\varepsilon}+x^{\frac{127}{616}+\varepsilon}).\end{aligned} \tag{4.11}$$

Now Theorem 1.3 follows from (4.11) and Lemma 4.1.

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*Received on 22.5.2009* (6035)