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On multiple higher Mahler measures and multiple L values

by

YOSHITAKA SASAKI (Osaka)

1. Introduction and notation. For every Laurent polynomial $P \in \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}] \setminus \{0\}$, the (*logarithmic*) *Mahler measure* of P is defined by

$$m(P) := \int_0^1 \cdots \int_0^1 \log |P(e^{2\pi i t_1}, \dots, e^{2\pi i t_n})| \, dt_n \cdots dt_1.$$

It is known that, in some cases, the Mahler measure relates to zeta or L values, for instance, Smyth [6] showed

$$m(1 + X + Y) = m(1 - X - Y) = \frac{3\sqrt{3}}{4\pi} L(2, \chi_{-3}), \tag{1.1}$$

where χ_{-3} is the quadratic character attached to $\mathbb{Q}(\sqrt{-3})$ and $L(*, \chi_{-3})$ is the Dirichlet L series associated with a character χ_{-3}. See [3], [4] and the references therein for other known cases.

Kurokawa, Lalín and Ochiai [5] introduced the *k-higher Mahler measure of* P and the *multiple higher Mahler measure* of $P_1, \dots, P_l$ defined by

$$m_k(P) := \int_0^1 \cdots \int_0^1 \log^k |P(e^{2\pi i t_1}, \dots, e^{2\pi i t_n})| \, dt_n \cdots dt_1$$

and

$$m(P_1, \dots, P_l) := \int_0^1 \cdots \int_0^1 \prod_{q=1}^{l} \log |P_q(e^{2\pi i t_1}, \dots, e^{2\pi i t_n})| \, dt_n \cdots dt_1,$$

respectively. Here, P and P_j's ($j = 1, \dots, l$) are in $\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}] \setminus \{0\}$. In [5], Kurokawa, Lalín and Ochiai evaluated some examples of k-higher Mahler measure and the multiple higher Mahler measure and showed a relation between the multiple higher Mahler measure of $1 - X$ and multiple

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zeta values:

$$m_k(1-X) = (-1)^k k! \sum_{h\geq 1} \frac{1}{2^{2h}} \sum_{\substack{b_1,\dots,b_h\geq 2\\ b_1+\cdots+b_h=k}} \zeta(b_1,\dots,b_h),$$

where $\zeta(b_1,\dots,b_h)$ is the multiple zeta value defined by

$$\zeta(b_1,\dots,b_h) := \sum_{0<n_1<\cdots<n_h} \frac{1}{n_1^{b_1}\cdots n_h^{b_h}}$$

for $(b_1,\dots,b_h)\in\mathbb{N}^h$ with $b_h\geq 2$. They also proved

$$m_2(1+X+Y) = \frac{5\pi^2}{54},$$

which is a generalization of the result (1.1) of Smyth. Further, Akatsuka evaluated $m_k(a-X)$ ($a\in\mathbb{C}$, $|a|<1$), and showed in [1] that $m_k(a-X)$ can be written in terms of multiple polylogarithms.

In this article, we will evaluate the multiple higher Mahler measure $m(1-X-Y_1,\dots,1-X-Y_k)$ and see that those values relate to multiple L values. However the type of multiple L values in our case is slightly different from that of the multiple L values treated by Arakawa and Kaneko [2]. Our result is another generalization of the result (1.1) of Smyth.

2. Statement of result. To state the main theorem, we have to introduce some notation.

We put

$$w_{-3}(n) := \frac{(-1)^n\chi_{-3}(n)}{n}, \qquad \widetilde{w}_{-3}(n) := nw_{-3}(n), \qquad \text{with } \chi_{-3}(n) := \left(\frac{n}{3}\right).$$

Then we define *multiple L values* by

$$L(r_1,\dots,r_k;w_{-3};k,v) = \sum_{0<n_1<\cdots<n_k} \frac{w_{-3}(n_k-n_{k-v})}{n_1^{r_1}\cdots n_k^{r_k}} \tag{2.1}$$

for $v=1,\dots,k-1$, and

$$L(r_1,\dots,r_k;\widetilde{w}_{-3};k) = \sum_{0<n_1<\cdots<n_k} \frac{\widetilde{w}_{-3}(n_k)}{n_1^{r_1}\cdots n_k^{r_k}}. \tag{2.2}$$

In particular, for the case of $k=2$, we have

$$\begin{aligned} L(r_1,r_2;w_{-3};2,1) &= \sum_{0<n_1<n_2} \frac{(-1)^{n_2-n_1}\chi_{-3}(n_2-n_1)}{n_1^{r_1}n_2^{r_2}(n_2-n_1)} \\ &= \sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty} \frac{\widetilde{w}_{-3}(m_2)}{m_1^{r_1}m_2(m_1+m_2)^{r_2}}, \end{aligned}$$

which is a "Mordell–Tornheim L value" (for instance, see [7]). Therefore, if we put

$$L_{\mathrm{MT},2}(s_1,s_2,s_3;\widetilde{w}_{-3}) := \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\frac{\widetilde{w}_{-3}(n_2)}{n_1^{s_1}n_2^{s_2}(n_1+n_2)^{s_3}},$$

then we can write $L_{\mathrm{MT},2}(r_1,1,r_2;\widetilde{w}_{-3}) = L(r_1,r_2;w_{-3};2,1)$.

THEOREM 2.1. *For any positive integer k, we have*

$$\begin{aligned}(2.3)\qquad & m(1-X-Y_1,\dots,1-X-Y_k)\\ &= \frac{(-1)^k k!}{3}\sum_{h\ge 1}2^{1-2h}\sum_{\substack{b_1,\dots,b_h\ge 2\\ b_1+\cdots+b_h=k}}\zeta(b_1,\dots,b_h)\\ &\quad+\frac{(-1)^k k!\sqrt{3}}{2^k\pi}\Big\{L(\{1\}_{k-1},2;\widetilde{w}_{-3};k)+\sum_{j=1}^{k-1}(T_{1,j}+T_{2,j})\Big\},\end{aligned}$$

where $\{1\}_\mu = (\underbrace{1,\dots,1}_{\mu})$ *is a μ-tuple,*

$$\begin{aligned}(2.4)\qquad T_{1,j} &= \sum_{v=1}^{\kappa_j-1}\sum_{h=1}^{\min\{j,\kappa_j-v\}}\binom{k-v-2h}{j-h}\\ &\times\sum_{(a_1,\dots,a_h)\in\mathcal{S}_h^{k-v}}L(\underbrace{\{1\}_{a_1},2,\dots,\{1\}_{a_h},2}_{k-v-h},\{1\}_v;w_{-3};k-h,v)\end{aligned}$$

and

$$\begin{aligned}(2.5)\qquad T_{2,j} &= \sum_{v=1}^{\kappa_j}\sum_{h=1}^{\min\{j,\kappa_j-v+1\}}\binom{k-v+1-2h}{j-h}\\ &\times\sum_{(a_1,\dots,a_h)\in\mathcal{S}_h^{k-v+1}}L(\underbrace{\{1\}_{a_1},2,\dots,\{1\}_{a_{h-1}},2,\{1\}_{a_h}}_{k-h-v},\{1\}_{v+1};w_{-3};k-h+1,v),\end{aligned}$$

with $\mathcal{S}_h^k := \{(a_1,\dots,a_h)\in\mathbb{N}_0^h \mid a_1+\cdots+a_h = k-2h\}$*; here* $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$ *and* $\kappa_j = k-j$.

REMARK 2.2. When $h=1$, the last sum on the right-hand side of (2.5) is $L(\{1\}_k;w_{-3};k,v)$.

COROLLARY 2.3. *If we put $k=2$ in Theorem 2.1, then*

$$\begin{aligned}& m(1-X-Y,1-X-Z)\\ &\qquad= \frac{1}{3}\zeta(2)+\frac{\sqrt{3}}{2\pi}\{L(1,2;\widetilde{w}_{-3};2)+L_{\mathrm{MT},2}(1,1,1;\widetilde{w}_{-3})\}.\end{aligned}$$

3. Preliminary lemmas. The following two lemmas are important and essential tools for evaluating Mahler measures.

LEMMA 3.1 (Jensen's formula). *For $\alpha \in \mathbb{C} \setminus \{0\}$, we have*

$$\int_0^1 \log|\alpha - e^{2\pi it}|\,dt = \log\max\{1, |\alpha|\}.$$

LEMMA 3.2 (Kurokawa, Lalín and Ochiai [5]). *Let $x = e^{2\pi it}$ with $0 < t < 1$. Then*

$$\log^k|1-x| = \frac{(-1)^k k!}{2^k}\sum_{j=0}^{k} \mathrm{Li}_{\underbrace{1,\ldots,1}_{j}}(x)\,\mathrm{Li}_{\underbrace{1,\ldots,1}_{k-j}}(x^{-1}), \tag{3.1}$$

where

$$\mathrm{Li}_{r_1,\ldots,r_k}(x) := \sum_{0<n_1<\cdots<n_k} \frac{x^{n_k}}{n_1^{r_1}\cdots n_k^{r_k}}$$

is the multiple polylogarithm.

4. Proof of Theorem 2.1. From the definition of the multiple higher Mahler measure and Lemma 3.1, we have

$$\begin{aligned} &m(1-X-Y_1,\ldots,1-X-Y_k) \\ &\quad= \int_0^1 \Big\{\prod_{q=1}^{k}\int_0^1 (\log|1-e^{2\pi it}-e^{2\pi it_q}|)\,dt_q\Big\}\,dt \\ &\quad= \int_0^1 (\log\max\{1,|1-e^{2\pi it}|\})^k\,dt = \int_{1/6}^{5/6} \log^k|1-e^{2\pi it}|\,dt. \end{aligned} \tag{4.1}$$

The right-hand side of (3.1) can be split as

$$\mathrm{Li}_{\underbrace{1,\ldots,1}_{k}}(x) + \mathrm{Li}_{\underbrace{1,\ldots,1}_{k}}(x^{-1}) + \sum_{j=1}^{k-1}\{D_j + 2\Re S_j(x)\} \tag{4.2}$$

for $x = e^{2\pi it}$ $(0<t<1)$, where

$$D_j := \sum_{\substack{0<l_1<\cdots<l_{j-1}<u \\ 0<n_1<\cdots<n_{k-j-1}<u}} \frac{1}{l_1\cdots l_{j-1}n_1\cdots n_{k-j-1}u^2},$$

$$S_j(x) := \sum_{\substack{0<l_1<\cdots<l_j \\ 0<n_1<\cdots<n_{k-j} \\ l_j<n_{k-j}}} \frac{x^{n_{k-j}-l_j}}{l_1\cdots l_j n_1\cdots n_{k-j}}.$$

We can easily evaluate

$$\int_{1/6}^{5/6} \mathrm{Li}_{\underbrace{1,\ldots,1}_{k}}(e^{2\pi it})\,dt = \frac{\sqrt{3}}{2\pi} L(\{1\}_{k-1}, 2; \widetilde{w}_{-3}; k) \tag{4.3}$$

and

$$\int_{1/6}^{5/6} D_j\,dt = \frac{2}{3} D_j,$$

since $\chi_{-3}(n) = 2\sin(2\pi n/3)/\sqrt{3}$. It is shown in [5] that

$$\sum_{j=1}^{k-1} D_j = \sum_{h=1}^{k-1} 2^{k-2h} \sum_{(a_1,\ldots,a_h)\in\mathcal{S}_h^k} \zeta(a_h+2,\ldots,a_1+2). \tag{4.4}$$

Note that the sum on the right-hand side of (4.4) vanishes for $h > [k/2]$. Hence

$$\sum_{j=1}^{k-1} \int_{1/6}^{5/6} D_j\,dt = \frac{2}{3} \sum_{h=1}^{k-1} 2^{k-2h} \sum_{(a_1,\ldots,a_h)\in\mathcal{S}_h^k} \zeta(a_h+2,\ldots,a_1+2). \tag{4.5}$$

Similarly,

$$\begin{aligned}\int_{1/6}^{5/6} &\Re S_j(e^{2\pi it})\,dt \\ &= \frac{\sqrt{3}}{2\pi} \sum_{\substack{0<l_1<\cdots<l_j \\ 0<n_1<\cdots<n_{k-j} \\ l_j<n_{k-j}}} \frac{(-1)^{n_{k-j}-l_j}\chi_{-3}(n_{k-j}-l_j)}{l_1\cdots l_j n_1\cdots n_{k-j}(n_{k-j}-l_j)} = \frac{\sqrt{3}}{2\pi}\widetilde{S}_j.\end{aligned}$$

We split $\widetilde{S}_j$ as follows:

$$\begin{aligned}\widetilde{S}_j &= \sum_{v=1}^{\kappa_j-1} \sum_{\substack{0<l_1<\cdots<l_j \\ 0<n_1<\cdots<n_{\kappa_j} \\ l_j=n_{\kappa_j-v}}} + \sum_{v=1}^{\kappa_j} \sum_{\substack{0<l_1<\cdots<l_j \\ 0<n_1<\cdots<n_{\kappa_j} \\ n_{\kappa_j-v}<l_j<n_{\kappa_j-v+1}}} \\ &= T_{1,j} + T_{2,j},\end{aligned} \tag{4.6}$$

where $\kappa_j := k-j$. To calculate $T_{1,j}$, we consider the following finite sum:

$$U_{j,v} := \sum_{\substack{0<l_1<\cdots<l_j \\ 0<n_1<\cdots<n_{\kappa_j-v}<n_{\kappa_j-v+1} \\ l_j=n_{\kappa_j-v}}} \frac{1}{l_1\cdots l_j n_1\cdots n_{\kappa_j-v}}.$$

By an argument similar to that of [5], we have

$$U_{j,v} = \sum_{h=1}^{\min\{j,\kappa_j-v\}} \binom{k-v-2h}{j-h} \times \sum_{(a_1,\dots,a_h)\in\mathcal{S}_h^{k-v}} \zeta_{n_{\kappa_j-v+1}}(\{1\}_{a_1},2,\dots,\{1\}_{a_h},2),$$

where

$$\zeta_N(r_1,\dots,r_k) := \sum_{0<n_1<\cdots<n_k<N} \frac{1}{n_1^{r_1}\cdots n_k^{r_k}}$$

is a finite sum. Consequently, we obtain

$$(4.7)\quad T_{1,j} = \sum_{v=1}^{\kappa_j-1} \sum_{h=1}^{\min\{j,\kappa_j-v\}} \binom{k-v-2h}{j-h} \times \sum_{(a_1,\dots,a_h)\in\mathcal{S}_h^{k-v}} L(\underbrace{\{1\}_{a_1},2,\dots,\{1\}_{a_h},2}_{k-v-h},\{1\}_v;w_{-3};k-h,v).$$

Similarly,

$$(4.8)\quad T_{2,j} = \sum_{v=1}^{\kappa_j} \sum_{h=1}^{\min\{j,\kappa_j-v+1\}} \binom{k-v+1-2h}{j-h} \times \sum_{(a_1,\dots,a_h)\in\mathcal{S}_h^{k-v+1}} L(\underbrace{\{1\}_{a_1},2,\dots,\{1\}_{a_{h-1}},2,\{1\}_{a_h}}_{k-h-v},\{1\}_{v+1};w_{-3};k-h+1,v).$$

Combining (4.3), (4.5), (4.7) and (4.8), we have Theorem 2.1.

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Yoshitaka Sasaki
Interdisciplinary Graduate School of Science and Engineering
Kinki University
Higashi-Osaka, Osaka 577-8502, Japan
E-mail: sasaki@alice.math.kindai.ac.jp

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