ACTA ARITHMETICA
144.2 (2010)

M_2-rank differences for overpartitions

by

Jeremy Lovejoy (Paris) and Robert Osburn (Dublin)

In memory of Oliver Atkin

1. Introduction. This is the third and final installment in our series of papers [22, 23] applying the method of Atkin and Swinnerton-Dyer to deduce formulas for rank differences. The *rank* of a partition λ is defined to be the largest part $\ell(\lambda)$ minus the number of parts $n(\lambda)$. Let $N(s,m,n)$ denote the number of partitions of n with rank congruent to s modulo m. Responding to a conjecture of Dyson [14], Atkin and Swinnerton-Dyer proved elegant formulas, in terms of modular functions and generalized Lambert series, for the generating functions for $N(r,\ell,\ell n+d)-N(s,\ell,\ell n+d)$ when $\ell = 5$ and 7. For example, they found [3, Theorem 4]:

$$\sum_{n\geq 0}(N(1,5,5n+2)-N(2,5,5n+2))q^n = \frac{(q^5;q^5)_\infty}{(q^2,q^3;q^5)_\infty} \tag{1.1}$$

and

$$\sum_{n\geq 0}(N(0,5,5n+3)-N(1,5,5n+3))q^n = \frac{q}{(q^5;q^5)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{15n(n+1)/2}}{1-q^{5n+2}} + \frac{(q,q^4,q^5;q^5)_\infty}{(q^2,q^3;q^5)_\infty^2}. \tag{1.2}$$

Here we have employed the standard basic hypergeometric series notation (see [15]),

$$(a_1,\ldots,a_j;q)_n = \prod_{k=0}^{n-1}(1-a_1q^k)\cdots(1-a_jq^k).$$

2010 *Mathematics Subject Classification*: Primary 11P81; Secondary 05A17, 33D15, 11F11.
Key words and phrases: Dyson's rank, rank differences, generalized Lambert series, modular functions, mock theta functions, overpartitions, M_2-rank.

DOI: 10.4064/aa144-2-8

We follow the custom of dropping the "$;q$" unless the base is something other than q.

The rank of a partition studied by Atkin and Swinnerton-Dyer is now understood to be a special case of a more general rank which is defined on overpartition pairs [6, 21]. Recall that an *overpartition* of n is a partition of n where we may overline the first occurrence of a part, while an *overpartition pair* (λ, μ) of n is a pair of overpartitions where the sum of all of the parts is n. The rank of an overpartition pair (λ, μ) is

$$\ell((\lambda,\mu)) - n(\lambda) - \overline{n}(\mu) - \chi((\lambda,\mu)), \tag{1.3}$$

where $\overline{n}(\cdot)$ is the number of overlined parts only and $\chi((\lambda,\mu))$ is defined to be 1 if the largest part of (λ,μ) occurs only non-overlined and only in μ, and 0 otherwise.

When μ is empty and λ has no overlined parts, (1.3) becomes the rank of a partition. In addition to this rank, three other special cases of (1.3) have turned out to be of particular interest: the rank of an overpartition, the M_2-rank of a partition without repeated odd parts, and the M_2-rank of an overpartition. For more on these three ranks and their generating functions, see [4, 5, 7, 8, 9, 10, 11, 13, 19, 20, 22, 23]. In [22] and [23], we applied the method of Atkin and Swinnerton-Dyer to find formulas like (1.1) and (1.2) for certain rank differences for overpartitions and M_2-rank differences for partitions without repeated odd parts. Here we complete the picture by doing the same for M_2-rank differences for overpartitions.

This rank arises by replacing parts m in the first component of an overpartition pair (λ,μ) by $2m$ and parts m in the second component by $2m-1$. From (1.3), the *M_2-rank* of the resulting overpartition λ is

$$M_2\text{-rank}(\lambda) := \lceil \ell(\lambda)/2 \rceil - n(\lambda) + n(\lambda_o) - \chi(\lambda),$$

where λ_o is the subpartition consisting of the odd non-overlined parts, and $\chi(\lambda) = 1$ if the largest part of λ is odd and non-overlined and $\chi(\lambda) = 0$ otherwise. For example, the M_2-rank of the overpartition $5+\overline{4}+4+\overline{3}+1+1$ is $3-6+3-1=-1$.

Let $\overline{N}_2(s,\ell,n)$ denote the number of overpartitions of n whose M_2-rank is congruent to s modulo ℓ. Using the notation

$$R_{st}(d) = \sum_{n\geq 0} (\overline{N}_2(s,\ell,\ell n+d) - \overline{N}_2(t,\ell,\ell n+d))q^n, \tag{1.4}$$

where the prime ℓ will always be clear, our main results are summarized in Theorems 1.1 and 1.2 below.

THEOREM 1.1. *For $\ell = 3$, we have*

$$R_{01}(0) = -1 + \frac{(-q)_\infty (q^3;q^3)_\infty^2}{(q)_\infty (-q^3;q^3)_\infty^2}, \tag{1.5}$$

$$(1.6) \qquad R_{01}(1) = \frac{2(q^3;q^3)_\infty(q^6;q^6)_\infty}{(q)_\infty},$$

$$(1.7) \qquad R_{01}(2) = \frac{4(q^6;q^6)_\infty^4}{(q^2;q^2)_\infty(q^3;q^3)_\infty^2} + \frac{6q(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{3n^2+6n}}{1-q^{6n+2}}.$$

THEOREM 1.2. *For $\ell = 5$, we have*

$$(1.8) \qquad R_{12}(0) = \frac{10q^2(q^{10};q^{10})_\infty^4(q,q^2,q^8,q^9;q^{10})_\infty}{(q)_\infty^3(q;q^2)_\infty} + \frac{2q(q^5;q^5)_\infty}{(q;q^2)_\infty^5(q^3,q^4,q^6,q^7;q^{10})_\infty},$$

$$(1.9) \qquad R_{12}(1) = \frac{-6q^3(-q^5;q^5)_\infty}{(q^5;q^5)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{5n^2+10n}}{1-q^{10n+4}} + \frac{4q(q^{10};q^{10})_\infty^2}{(q;q^2)_\infty^5(q^4,q^6;q^{10})_\infty(q^5;q^5)_\infty} + \frac{20q^3(q^{10};q^{10})_\infty^7(q,q^9;q^{10})_\infty^2(q^2,q^8;q^{10})_\infty^3}{(q)_\infty^4(q^5;q^5)_\infty^2},$$

$$(1.10) \qquad R_{12}(2) = \frac{10q(q^{10};q^{10})_\infty^3}{(q)_\infty^2(q^3,q^7;q^{10})_\infty^3(q,q^2,q^8,q^9;q^{10})_\infty},$$

$$(1.11) \qquad R_{12}(3) = \frac{10(q^{10};q^{10})_\infty^3}{(q)_\infty^2(q^3,q^4,q^6,q^7;q^{10})_\infty(q,q^9;q^{10})_\infty^3} - \frac{8(q^5;q^5)_\infty}{(q;q^2)_\infty^5(q^2,q^3,q^7,q^8;q^{10})_\infty},$$

$$(1.12) \qquad R_{12}(4) = \frac{-2q(-q^5;q^5)_\infty}{(q^5;q^5)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{5n^2+10n}}{1-q^{10n+2}} + \frac{4(q^2;q^2)_\infty}{(q;q^2)_\infty^4(q^2,q^3,q^7,q^8;q^{10})_\infty^3},$$

$$(1.13) \qquad R_{02}(0) = -1 + \frac{(q^5;q^5)_\infty^6}{(q;q^2)_\infty^6(q^3,q^4,q^6,q^7;q^{10})_\infty(q^{10};q^{10})_\infty^5} + \frac{q^2(q)_\infty(q^{10};q^{10})_\infty}{(q^3,q^4,q^6,q^7;q^{10})_\infty^3(q^5;q^5)_\infty} + \frac{4q(q^5;q^5)_\infty^2(q,q^9;q^{10})_\infty}{(q;q^2)_\infty^6(q^4,q^6,q^{10};q^{10})_\infty} - \frac{10q^2(q^{10};q^{10})_\infty^3}{(q)_\infty^2(q^2,q^8;q^{10})_\infty(q^3,q^7;q^{10})_\infty^4},$$

$$
\begin{aligned}
(1.14)\qquad R_{02}(1) = {} & \frac{2q^3(-q^5;q^5)_\infty}{(q^5;q^5)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{5n^2+10n}}{1-q^{10n+4}} \\
& + \frac{2(q^5;q^5)_\infty^4}{(q;q^2)_\infty^6(q^4,q^6;q^{10})_\infty(q^{10};q^{10})_\infty^3} \\
& + \frac{2q^2(q)_\infty(q^{10};q^{10})_\infty^3}{(q^3,q^7;q^{10})_\infty^2(q^4,q^6;q^{10})_\infty^3(q^5;q^5)_\infty^3},
\end{aligned}
$$

$$
\begin{aligned}
(1.15)\qquad R_{02}(2) = {} & \frac{4(q^3,q^7;q^{10})_\infty(q^5;q^5)_\infty^2}{(q;q^2)_\infty^6(q^4,q^6,q^{10};q^{10})_\infty} \\
& - \frac{10q(q^{10};q^{10})_\infty^3}{(q)_\infty^2(q^3,q^7;q^{10})_\infty^3(q,q^2,q^8,q^9;q^{10})_\infty},
\end{aligned}
$$

$$
(1.16)\qquad R_{02}(3) = \frac{4(q^5;q^5)_\infty}{(q;q^2)_\infty^5(q^2,q^3,q^7,q^8;q^{10})_\infty},
$$

$$
\begin{aligned}
(1.17)\qquad R_{02}(4) = {} & \frac{4q(-q^5;q^5)_\infty}{(q^5;q^5)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{5n^2+10n}}{1-q^{10n+2}} \\
& - \frac{10(q^{10};q^{10})_\infty^3(q^5;q^5)_\infty(q^4,q^6;q^{10})_\infty}{(q,q^9;q^{10})_\infty(q)_\infty^3(q^3,q^7;q^{10})_\infty} \\
& - \frac{2(q^{10};q^{10})_\infty(-q^2,-q^3;q^5)_\infty}{q(-q,-q^4;q^5)_\infty(q,q^2,q^8,q^9;q^{10})_\infty} \\
& + \frac{2(q^5;q^5)_\infty^2(q^2,q^8;q^{10})_\infty^5(q^{10};q^{10})_\infty^2}{q(q;q^2)_\infty^4(q,q^9;q^{10})_\infty^4(q^2;q^2)_\infty^3}.
\end{aligned}
$$

The method of Atkin and Swinnerton-Dyer may be generally described as regarding groups of identities as equalities between polynomials of degree $\ell-1$ in q whose coefficients are power series in q^ℓ. Specifically, we first consider the expression

$$
(1.18)\qquad \sum_{n=0}^{\infty}\{\overline{N}_2(s,\ell,n)-\overline{N}_2(t,\ell,n)\}q^n\frac{(q)_\infty}{2(-q)_\infty}.
$$

By (2.4), (2.5), and (4.3), we write (1.18) as a polynomial in q whose coefficients are power series in q^ℓ. We then alternatively express (1.18) in the same manner using the formulas in Theorem 1.2 and equation (4.4) or (4.5). Finally, we use the theory of modular forms to show that these two resulting polynomials are the same for each pair of values of s and t.

Some comments are in order here. First, if the number of overpartitions of n with M_2-rank m is denoted by $\overline{N}_2(m,n)$, then $\overline{N}_2(m,n)=\overline{N}_2(-m,n)$ (see (2.3)). Hence the values of s and t considered in Theorems 1.1 and 1.2 are sufficient to find any rank difference generating function $R_{st}(d)$ for $\ell=3,5$. Second, the formulas in Theorems 1.1 and 1.2 are somewhat more complicated than the ones in [3], [22], and [23]. When there are exactly two infinite products, we have verified using Euler's algorithm [1, p. 98, Ex. 2] that they cannot be reduced to one product. However, in (1.13) and (1.17) we cannot rule out the possibility of a simpler expression.

Finally, the formulas for $R_{01}(0)$ and $R_{02}(1)$ when $\ell = 3$ match those for the classical rank differences for overpartitions [22, eqs. (1.1) and (1.2)]. In other words, letting $\overline{N}(s,m,n)$ denote the number of overpartitions whose rank is s modulo m, we have

$$\overline{N}_2(0,3,3n+d) - \overline{N}_2(1,3,3n+d) = \overline{N}(0,3,3n+d) - \overline{N}(1,3,3n+d)$$

for $n \geq 0$ and $d = 0$ or 1. When $d = 2$ it turns out that the generating function for the difference of the rank differences is proportional to the third order mock theta function

$$\omega(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2}.$$

Corollary 1.3. *We have*

$$\begin{aligned} 6\omega(q) = &\sum_{n\geq 0} (\overline{N}_2(0,3,3n+2) - \overline{N}_2(1,3,3n+2))q^n \\ &- \sum_{n\geq 0} (\overline{N}(0,3,3n+2) - \overline{N}(1,3,3n+2))q^n. \end{aligned}$$

This is not the first time that mock theta functions have appeared in relation to rank differences. Andrews and Garvan [2, Section 4] and Hickerson [17, Section 5] have already shown that certain fifth and seventh order mock theta functions can be expressed in terms of rank differences of Atkin and Swinnerton-Dyer. Some tenth order mock theta functions are also rank differences. Specifically, using identities for the tenth order mock theta functions $\phi(q)$ and $\psi(q)$ on pages 533–534 of [12], combined with identities (1.9), (1.11), and (1.14) of [22], we have

$$2\phi(q) = \sum_{n\geq 0} (\overline{N}(0,5,5n+1) - \overline{N}(2,5,5n+1))q^n,$$

$$2\psi(q) = \sum_{n\geq 0} (\overline{N}(0,5,5n+4) + \overline{N}(1,5,5n+4) - 2\overline{N}(2,5,5n+4))q^{n+1}.$$

In general, the generalized Lambert series (i.e., Lerch sums) which arise in the study of rank differences are known to be building blocks of mock theta functions [26].

The paper is organized as follows. In Section 2 we collect some basic definitions, notations and generating functions. In Section 3 we prove two key q-series identities relating generalized Lambert series to infinite products, and in Section 4 we give the proofs of Theorems 1.1 and 1.2. In Section 5, we prove Corollary 1.3.

2. Preliminaries. We begin by introducing some notation and definitions, essentially following [3]. With $y = q^\ell$, let

$$r_s(d) := \sum_{n=0}^{\infty} \overline{N}_2(s, \ell, \ell n + d) y^n, \qquad r_{st}(d) := r_s(d) - r_t(d).$$

Thus we have

$$\sum_{n=0}^{\infty} \overline{N}_2(s, \ell, n) q^n = \sum_{d=0}^{\ell-1} r_s(d) q^d.$$

To abbreviate the sums occurring in Theorems 1.1 and 1.2, we define

$$\Sigma(z, \zeta, q) := \sum_{n \in \mathbb{Z}} \frac{(-1)^n \zeta^{2n} q^{n^2+2n}}{1 - z^2 q^{2n}}.$$

Henceforth we assume that a is not a multiple of ℓ. We write

$$\Sigma(a, b) := \Sigma(y^a, y^b, y^\ell) = \sum_{n \in \mathbb{Z}} \frac{(-1)^n y^{2bn+\ell n(n+2)}}{1 - y^{2\ell n + 2a}},$$

$$\Sigma(0, b) := {\sum_{n \in \mathbb{Z}}}' \frac{(-1)^n y^{2bn+\ell n(n+2)}}{1 - y^{2\ell n}},$$

where the prime means that the term corresponding to $n = 0$ is omitted.

To abbreviate the products occurring in Theorems 1.1 and 1.2, we define

$$P(z, q) := \prod_{r=1}^{\infty} (1 - zq^{r-1})(1 - z^{-1}q^r), \qquad P(0) := \prod_{r=1}^{\infty} (1 - y^{2\ell r}).$$

We also have the relations

$$P(z^{-1}q, q) = P(z, q), \tag{2.1}$$

$$P(zq, q) = -z^{-1} P(z, q). \tag{2.2}$$

In [20], it is shown that the two-variable generating function for $\overline{N}_2(m, n)$ is

$$\sum_{n=0}^{\infty} \overline{N}_2(m, n) q^n = \frac{2(-q)_\infty}{(q)_\infty} \sum_{n=1}^{\infty} (-1)^{n+1} q^{n^2+2|m|n} \frac{1 - q^{2n}}{1 + q^{2n}}. \tag{2.3}$$

From this we may easily deduce that the generating function for $\overline{N}_2(s, m, n)$ is

$$\sum_{n=0}^{\infty} \overline{N}_2(s, m, n) q^n = \frac{2(-q)_\infty}{(q)_\infty} {\sum_{n \in \mathbb{Z}}}' \frac{(-1)^n q^{n^2+2n}(q^{2sn} + q^{2(m-s)n})}{(1 + q^{2n})(1 - q^{2mn})}. \tag{2.4}$$

Hence it will be beneficial to consider sums of the form

$$\overline{S}_2(b) := \sum_{n\in\mathbb{Z}}{}' \frac{(-1)^n q^{n^2+2bn}}{1-q^{2\ell n}}. \tag{2.5}$$

We will require the relation

$$\overline{S}_2(b) = -\overline{S}_2(\ell - b), \tag{2.6}$$

which follows from the substitution $n \mapsto -n$ in (2.5). We shall also exploit the fact that the functions $\overline{S}_2(\ell)$ are essentially infinite products.

LEMMA 2.1. *We have*

$$\overline{S}_2(\ell) = \frac{-(q)_\infty}{2(-q)_\infty} + \frac{1}{2}.$$

Proof. Use the relation (2.6) to compute $-2\overline{S}_2(\ell)$; then apply the case $z=-1$ of Jacobi's triple product identity,

$$\sum_{n\in\mathbb{Z}} z^n q^{n^2} = (-zq, -q/z, q^2; q^2)_\infty. \ \blacksquare \tag{2.7}$$

3. Two lemmas. The proofs of Theorems 1.1 and 1.2 will follow from identities which relate the sums $\Sigma(a,b)$ to the products $P(z,q)$. The key steps are the two lemmas below. The first is equation (5.4) of [9].

LEMMA 3.1. *We have*

$$\begin{aligned}(3.1)\quad & \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2+2n}\left[\frac{\zeta^{-2n}}{1-z^2\zeta^{-2}q^{2n}} + \frac{\zeta^{2n+4}}{1-z^2\zeta^2 q^{2n}}\right] \\ &= \frac{-2(\zeta^4, q^2\zeta^{-4}; q^2)_\infty (-q)_\infty^2}{(-\zeta^2, -q\zeta^{-2})_\infty (\zeta^{-2}, q^2\zeta^2; q^2)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n \frac{q^{n^2+2n}}{1-z^2q^{2n}} \\ &\quad + \frac{(-z^2, -qz^{-2})_\infty (\zeta^4, q^2\zeta^{-4}, \zeta^2, q^2\zeta^{-2}; q^2)_\infty (q^2;q^2)_\infty^2}{(-\zeta^2, -q\zeta^{-2})_\infty (z^2\zeta^2, q^2z^{-2}\zeta^{-2}, z^2\zeta^{-2}, q^2z^{-2}\zeta^2, z^2, z^{-2}q^2; q^2)_\infty}.\end{aligned}$$

We now specialize Lemma 3.1 to the case $\zeta = y^a$, $z = y^b$, and $q = y^\ell$:

$$\begin{aligned}(3.2)\quad & y^{4a}\Sigma(a+b,a) + \Sigma(b-a,-a) + \frac{P(y^{4a}, y^{2\ell})P(-1, y^\ell)}{P(-y^{2a}, y^\ell)P(y^{-2a}, y^{2\ell})}\Sigma(b,0) \\ & - \frac{P(-y^{2b}, y^\ell)P(y^{4a}, y^{2\ell})P(y^{2a}, y^{2\ell})P(0)^2}{P(y^{2b+2a}, y^{2\ell})P(y^{2b-2a}, y^{2\ell})P(-y^{2a}, y^\ell)P(y^{2b}, y^{2\ell})} = 0.\end{aligned}$$

We define

$$\begin{aligned} g(z,q) := & -\frac{P(z^4, q^2)P(-1, q)}{P(-z^2, q)P(z^2q^2, q^2)}\Sigma(z,1,q) - z^4\Sigma(z^2, z, q) \\ & - \sum_{n=-\infty}^{\infty}{}' \frac{(-1)^n z^{-2n} q^{n(n+2)}}{1-q^{2n}}\end{aligned}$$

and

$$g(a) := g(y^a, y^\ell) = -\frac{P(y^{4a}, y^{2\ell})P(-1, y^\ell)}{P(-y^{2a}, y^\ell)P(y^{-2a}, y^{2\ell})}\Sigma(a, 0) - y^{4a}\Sigma(2a, a) - \Sigma(0, -a). \tag{3.3}$$

The second key lemma is the following.

LEMMA 3.2. *We have*

$$2g(z, q) - g(z^2, q) + \frac{1}{2} = \frac{P(z^6, q^2)^2(q^2; q^2)_\infty^2}{P(z^2, q^2)^2P(z^8, q^2)} - \frac{P(z^2, q)^2P(z^4, q)(q)_\infty^2}{P(-z^2, q)^2P(-z^4, q)P(-1, q)} \tag{3.4}$$

and

$$g(z, q) + g(z^{-1}q, q) = 0. \tag{3.5}$$

Proof. We first require a short computation involving $\Sigma(z, \zeta, q)$. Note that

$$\begin{aligned} z^2\Sigma(z, \zeta, q) + q\zeta^2\Sigma(zq, \zeta, q) &= \sum_{n=-\infty}^{\infty}(-1)^n\frac{z^2\zeta^{2n}q^{n(n+2)}}{1 - z^2q^{2n}} + \sum_{n=-\infty}^{\infty}(-1)^n\frac{\zeta^{2n+2}q^{n(n+2)+1}}{1 - z^2q^{2n+2}} \\ &= -\sum_{n=-\infty}^{\infty}(-1)^n\zeta^{2n}q^{n^2} \end{aligned} \tag{3.6}$$

upon writing $n - 1$ for n in the second sum of the first equation. Taking $\zeta = 1$ yields

$$z^2\Sigma(z, 1, q) + q\Sigma(zq, 1, q) = -\sum_{n=-\infty}^{\infty}(-1)^nq^{n^2}. \tag{3.7}$$

Now write $g(z, q)$ in the form

$$g(z, q) = f_1(z) - f_2(z) - f_3(z)$$

where

$$\begin{aligned} f_1(z) &:= -\frac{P(z^4, q^2)P(-1, q)}{P(-z^2, q)P(z^2q^2, q^2)}\Sigma(z, 1, q), \\ f_2(z) &:= z^4\Sigma(z^2, z, q), \\ f_3(z) &:= \sideset{}{'}\sum_{n=-\infty}^{\infty}\frac{(-1)^nz^{-2n}q^{n(n+2)}}{1 - q^{2n}}. \end{aligned}$$

By (2.1), (2.2), and (3.7),

$$f_1(zq) - f_1(z) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \frac{P(z^4, q^2)P(-1, q)}{P(-z^2, q)P(z^2, q^2)}. \tag{3.8}$$

A similar argument to (3.6) yields

$$f_2(zq) - f_2(z) = \sum_{n=-\infty}^{\infty} (-1)^n z^{2n} q^{n^2}, \tag{3.9}$$

$$f_3(zq) - f_3(z) = -1 + \sum_{n=-\infty}^{\infty} (-1)^n z^{-2n} q^{n^2}. \tag{3.10}$$

Adding (3.9) and (3.10), then subtracting from (3.8) gives

$$g(z, q) - g(zq, q) = -1. \tag{3.11}$$

Here we have used the identity

$$\begin{aligned} &\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \frac{P(z^4, q^2)P(-1, q)}{P(-z^2, q)P(z^2, q^2)} \\ &\qquad = \sum_{n=-\infty}^{\infty} (-1)^n z^{2n} q^{n^2} + \sum_{n=-\infty}^{\infty} (-1)^n z^{-2n} q^{n^2}, \end{aligned} \tag{3.12}$$

which follows from the triple product identity (2.7) after writing $-n$ for n in the second sum. If we now define

$$\begin{aligned} f(z) := 2g(z, q) - g(z^2, q) + \frac{1}{2} - \frac{P(z^6, q^2)^2 (q^2; q^2)_\infty^2}{P(z^2, q^2)^2 P(z^8, q^2)} \\ + \frac{P(z^2, q)^2 P(z^4, q)(q)_\infty^2}{P(-z^2, q)^2 P(-z^4, q)P(-1, q)}, \end{aligned}$$

then from (2.1), (2.2), and (3.11), one can verify that

$$f(zq) - f(z) = 0. \tag{3.13}$$

Now, it follows from a routine complex analytic argument similar to the proof of Lemma 4.2 in [22] (see also Lemma 2 in [3]) that $f(z) = 0$. This proves (3.4).

To prove (3.5), it suffices to show, after (3.11),

$$g(z^{-1}, q) + g(z, q) = -1. \tag{3.14}$$

Note that

$$\begin{aligned} &z^2 \Sigma(z, 1, q) + z^{-2} \Sigma(z^{-1}, 1, q) \\ &= z^2 \sum_{n=-\infty}^{\infty} (-1)^n \frac{q^{n(n+2)}}{1 - z^2 q^{2n}} - \sum_{n=-\infty}^{\infty} (-1)^n \frac{q^{n^2}}{1 - z^2 q^{2n}} = - \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \end{aligned} \tag{3.15}$$

where we have written $-n$ for n in the second sum in the first equation. Thus, by (2.1), (2.2), and (3.15), we have

$$f_1(z) + f_1(z^{-1}) = -\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \frac{P(z^4, q^2)P(-1, q)}{P(-z^2, q)P(z^2, q^2)}. \tag{3.16}$$

Again, a similar argument to (3.15) gives

$$f_2(z) + f_2(z^{-1}) = -\sum_{n=-\infty}^{\infty} (-1)^n z^{2n} q^{n^2}, \tag{3.17}$$

$$f_3(z) + f_3(z^{-1}) = 1 - \sum_{n=-\infty}^{\infty} (-1)^n z^{-2n} q^{n^2}. \tag{3.18}$$

Adding (3.17) and (3.18), then subtracting from (3.16) yields (3.14). Here we have again used (3.12). ∎

Letting $z = y^a$ and $q = y^\ell$ in Lemma 3.2, we get

$$\begin{aligned} 2g(a) - g(2a) + \frac{1}{2} &= \frac{P(y^{6a}, y^{2\ell})^2 P(0)^2}{P(y^{2a}, y^{2\ell})^2 P(y^{8a}, y^{2\ell})} \\ &\quad - \frac{P(y^{2a}, y^\ell)^2 P(y^{4a}, y^\ell)(y^\ell; y^\ell)_\infty^2}{P(-y^{2a}, y^\ell)^2 P(-y^{4a}, y^\ell) P(-1, y^\ell)} \end{aligned} \tag{3.19}$$

and

$$g(a) + g(\ell - a) = 0. \tag{3.20}$$

These two identities will be of key importance in the next section.

4. Proofs of Theorems 1.1 and 1.2. We now compute the sums $\overline{S}_2(\ell - m)$. The reason for this choice is two-fold. First, we would like to obtain as simple an expression as possible in the final formulation (4.3). Secondly, to prove Theorem 1.1, we only need to compute $\overline{S}_2(1)$, whereas to prove Theorem 1.2, we need $\overline{S}_2(1)$ and $\overline{S}_2(3)$. The former yields $\overline{S}_2(2)$ while the latter in turn yields $\overline{S}_2(4)$ via (2.6). For $\ell = 3$, we can choose $m = 1$, and for $\ell = 5$, $m = 1$ and $m = 2$ respectively. As this point, we follow the idea of Section 6 in [3]. Namely, we write

$$n = \ell r + m + b, \tag{4.1}$$

where $-\infty < r < \infty$. The idea is to simplify the exponent of q in $\overline{S}_2(\ell - m)$. Thus

$$2\ell n - 2mn + n^2 = \ell^2 r(r + 2) + 2b\ell r + (b + m)(b - m + 2\ell).$$

We now substitute (4.1) into (2.5) and let b take the values 0, $\pm a$, and $\pm m$. Here a runs through $1, 2, \ldots, (\ell - 1)/2$ where the value $a \equiv \pm m \bmod \ell$ is omitted. As in [3], we use the notation $\sum_a''$ to denote the sum over these values of a. We thus obtain

$$\overline{S}_2(\ell-m) = \sum_{n=-\infty}^{\infty}{}' (-1)^n \frac{q^{2(\ell-m)n+n^2}}{1-y^{2n}}$$
$$= \sum_b \sum_{r=-\infty}^{\infty}{}' (-1)^{r+m+b} q^{(b+m)(b-m+2\ell)} \frac{y^{\ell r(r+2)+2br}}{1-y^{2\ell r+2m+2b}},$$

where b takes the values 0, $\pm a$, and $\pm m$ and the term corresponding to $r=0$ and $b=-m$ is omitted. Thus

$$\begin{aligned}(4.2)\qquad & \overline{S}_2(\ell-m) \\ &= (-1)^m q^{m(2\ell-m)} \Sigma(m,0) + \Sigma(0,-m) + y^{4m}\Sigma(2m,m) \\ &\quad + \sum_a{}'' (-1)^{m+a} q^{(a+m)(a-m+2\ell)} \{\Sigma(m+a,a) + y^{-4a}\Sigma(m-a,-a)\}.\end{aligned}$$

Here the first three terms arise from taking $b=0$, $-m$, and m respectively. We now can use (3.2) to simplify this expression. By taking $b=m$ and dividing by y^{4a} in (3.2), the sum of the two terms inside the curly brackets becomes

$$\begin{aligned}& - y^{-4a} \frac{P(y^{4a},y^{2\ell})P(-1,y^{\ell})}{P(-y^{2a},y^{\ell})P(y^{-2a},y^{2\ell})} \Sigma(m,0) \\ & + y^{-4a} \frac{P(-y^{2m},y^{\ell})P(y^{4a},y^{2\ell})P(y^{2a},y^{2\ell})P(0)^2}{P(y^{2a+2m},y^{2\ell})P(y^{2m-2a},y^{2\ell})P(-y^{2a},y^{\ell})P(y^{2m},y^{2\ell})}.\end{aligned}$$

Similarly, if we take $a=m$ in (3.3), then the sum of the second and third terms in (4.2) is

$$-\frac{P(y^{4m},y^{2\ell})P(-1,y^{\ell})}{P(-y^{2m},y^{\ell})P(y^{-2m},y^{2\ell})} \Sigma(m,0) - g(m).$$

In total, we have

$$\begin{aligned}(4.3)\qquad & \overline{S}_2(\ell-m) \\ &= -g(m) + \sum_a{}'' \bigg\{ (-1)^{m+a} q^{(a+m)(a-m+2\ell)} y^{-4a} \\ &\qquad \times \frac{P(-y^{2m},y^{\ell})P(y^{4a},y^{2\ell})P(y^{2a},y^{2\ell})P(0)^2}{P(y^{2a+2m},y^{2\ell})P(y^{2m-2a},y^{2\ell})P(-y^{2a},y^{\ell})P(y^{2m},y^{2\ell})} \bigg\} \\ &\quad + \Sigma(m,0) \bigg\{ (-1)^m q^{m(2\ell-m)} - \frac{P(y^{4m},y^{2\ell})P(-1,y^{\ell})}{P(-y^{2m},y^{\ell})P(y^{-2m},y^{2\ell})} \\ &\quad - \sum_a{}'' (-1)^{m+a} q^{(a+m)(a-m+2\ell)} y^{-4a} \frac{P(y^{4a},y^{2\ell})P(-1,y^{\ell})}{P(-y^{2a},y^{\ell})} P(y^{-2a},y^{2\ell}) \bigg\}.\end{aligned}$$

We can simplify some of the terms appearing in (4.3) as we are interested in certain values of ℓ, m, and a. To this end, we prove the following result. Let $\{\quad\}$ denote the coefficient of $\Sigma(m,0)$ in (4.3).

PROPOSITION 4.1. *If $\ell = 3$ and $m = 1$, then*

$$\{\quad\} = -q^5 \frac{(q)_\infty(-q^9;q^9)_\infty}{(-q)_\infty(q^9;q^9)_\infty}.$$

If $\ell = 5$, $m = 2$, and $a = 1$, then

$$\{\quad\} = q^{16} \frac{(q)_\infty(-q^{25};q^{25})_\infty}{(-q)_\infty(q^{25};q^{25})_\infty}.$$

If $\ell = 5$, $m = 1$, $a = 2$, then

$$\{\quad\} = -q^9 \frac{(q)_\infty(-q^{25};q^{25})_\infty}{(-q)_\infty(q^{25};q^{25})_\infty}.$$

Proof. This is a straightforward application of the identities

$$\frac{(q)_\infty}{(-q)_\infty} = \frac{(q^9;q^9)_\infty}{(-q^9;q^9)_\infty} - 2q(q^3,q^{15},q^{18};q^{18})_\infty, \tag{4.4}$$

$$\frac{(q)_\infty}{(-q)_\infty} = \frac{(q^{25};q^{25})_\infty}{(-q^{25};q^{25})_\infty} - 2q(q^{15},q^{35},q^{50};q^{50})_\infty + 2q^4(q^5,q^{45},q^{50};q^{50})_\infty. \tag{4.5}$$

These are Lemma 3.1 in [22]. ∎

We are now in a position to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. By (2.4)–(2.6), we have

$$\sum_{n=0}^{\infty}\{\overline{N}_2(0,3,n) - \overline{N}_2(1,3,n)\}q^n \frac{(q)_\infty}{2(-q)_\infty} = 3\overline{S}_2(1) + \overline{S}_2(3). \tag{4.6}$$

By (2.1), (2.2), (4.3), and Proposition 4.1,

$$\overline{S}_2(1) = g(1) + q^2 y\Sigma(1,0)\frac{(q)_\infty(-q^9;q^9)_\infty}{(-q)_\infty(q^9;q^9)_\infty}. \tag{4.7}$$

By Lemma 2.1,

$$\overline{S}_2(3) = \frac{-(q)_\infty}{2(-q)_\infty} + \frac{1}{2}. \tag{4.8}$$

We have

$$3g(1) + 3q^2 y\Sigma(1,0)\frac{(q)_\infty(-q^9;q^9)_\infty}{(-q)_\infty(q^9;q^9)_\infty} - \frac{(q)_\infty}{2(-q)_\infty} + \frac{1}{2}$$
$$= \{r_{01}(0)q^0 + r_{01}(1)q + r_{01}(2)q^2\}\frac{(q)_\infty}{2(-q)_\infty}.$$

We now multiply the right hand side of the above expression using (4.4) and the $R_{01}(d)$ from Theorem 1.2 (recall that $r_{01}(d)$ is just $R_{01}(d)$ with q replaced by q^3). We then equate coefficients of powers of q and verify the resulting identities. For q^1 and q^2 the resulting equation follows easily upon cancelling factors in infinite products. For q^0 we obtain

$$3g(1)+\frac{1}{2}=\frac{(-q^3;q^3)_\infty(q^9;q^9)_\infty^3}{2(q^3;q^3)_\infty(-q^9;q^9)_\infty^3}-4y\frac{(q^{18};q^{18})_\infty^4(q^3,q^{15},q^{18};q^{18})_\infty}{(q^6;q^6)_\infty(q^9;q^9)_\infty^2}.$$

Appealing to (3.19) and (3.20) and then replacing q by $q^{1/3}$, we see that

$$\frac{P(q^2,q^3)^2P(q^4,q^3)(q^3;q^3)_\infty^2}{P(-q^2,q^3)^2P(-q^4,q^3)P(-1,q^3)}+\frac{(-q;q)_\infty(q^3;q^3)_\infty^3}{2(q;q)_\infty(-q^3;q^3)_\infty^3}$$
$$=4y\frac{(q^6;q^6)_\infty^4(q,q^5,q^6;q^6)_\infty}{(q^2;q^2)_\infty(q^3;q^3)_\infty^2}.$$

After making a common denominator on the left and simplifying, this equation may be verified using the case $(z,\zeta,t,q)=(-q^2,q^2,-1,q^3)$ of the addition theorem [3, eq. (3.7)],

$$P^2(z,q)P(\zeta t,q)P(\zeta/t,q)-P^2(\zeta,q)P(zt,q)P(z/t,q)$$
$$+(\zeta/t)P^2(t,q)P(z\zeta,q)P(z/\zeta,q)=0.$$

This completes the proof of Theorem 1.1. ■

Proof of Theorem 1.2. We begin with the rank differences $R_{12}(d)$. By (2.4)–(2.6),

$$\sum_{n=0}^{\infty}\{\overline{N}_2(1,5,n)-\overline{N}_2(2,5,n)\}q^n\frac{(q)_\infty}{2(-q)_\infty}=-\overline{S}_2(1)-3\overline{S}_2(3),\tag{4.9}$$

and by (2.1), (2.2), (4.3), and Proposition 4.1,

$$\overline{S}(1)=g(1)+q^4y\Sigma(1,0)\frac{(q)_\infty(-q^{25};q^{25})_\infty}{(-q)_\infty(q^{25};q^{25})_\infty}-q^3\frac{(q^{50};q^{50})_\infty^2(-q^{10},-q^{15};q^{25})_\infty}{(q^{10},q^{40};q^{50})_\infty(-q^5,-q^{20};q^{25})_\infty}\tag{4.10}$$

and

$$\overline{S}(3)=-g(2)+qy^3\Sigma(2,0)\frac{(q)_\infty(-q^{25};q^{25})_\infty}{(-q)_\infty(q^{25};q^{25})_\infty}-q^2y\frac{(q^{50};q^{50})_\infty^2(-q^5,-q^{20};q^{25})_\infty}{(q^{20},q^{30};q^{50})_\infty(-q^{10},-q^{15};q^{25})_\infty}.\tag{4.11}$$

By (4.9)–(4.11), we have

$$
\begin{aligned}
&- g(1) - q^4 y \Sigma(1,0) \frac{(q)_\infty (-q^{25};q^{25})_\infty}{(-q)_\infty (q^{25};q^{25})_\infty} + q^3 \frac{(q^{50};q^{50})_\infty^2 (-q^{10},-q^{15};q^{25})_\infty}{(q^{10},q^{40};q^{50})_\infty (-q^5,-q^{20};q^{25})_\infty} \\
&\qquad + 3g(2) - 3qy^3 \Sigma(2,0) \frac{(q)_\infty (-q^{25};q^{25})_\infty}{(-q)_\infty (q^{25};q^{25})_\infty} \\
&\qquad + 3q^2 y \frac{(q^{50};q^{50})_\infty^2 (-q^5,-q^{20};q^{25})_\infty}{(q^{20},q^{30};q^{50})_\infty (-q^{10},-q^{15};q^{25})_\infty} \\
&\quad = \{r_{12}(0)q^0 + r_{12}(1)q + r_{12}(2)q^2 + r_{12}(3)q^3 + r_{12}(4)q^4\} \frac{(q)_\infty}{2(-q)_\infty}.
\end{aligned}
$$

We now multiply the right hand side of the above expression using (4.5) and the $R_{12}(d)$ from Theorem 1.2, and equate coefficients of powers of q. The coefficients of q^0, q^1, q^2, q^3, q^4 give us, respectively,

$$
\begin{aligned}
(4.12)\qquad & 3g(2) - g(1) \\
&= 5y^2 \frac{(q^{50};q^{50})_\infty^4 (q^5,q^{10},q^{40},q^{45};q^{50})_\infty (q^{25};q^{25})_\infty}{(q^5;q^5)_\infty^3 (q^5;q^{10})_\infty (-q^{25};q^{25})_\infty} \\
&\quad + y \frac{(q^{25};q^{25})_\infty^2}{(q^5;q^{10})_\infty^5 (q^{15},q^{20},q^{30},q^{35};q^{50})_\infty (-q^{25};q^{25})_\infty} \\
&\quad + 4y^2 \frac{(q^{50};q^{50})_\infty^3 (q^5,q^{45};q^{50})_\infty}{(q^5;q^{10})_\infty^5 (q^{20},q^{30};q^{50})_\infty (q^{25};q^{25})_\infty} \\
&\quad + 20y^4 \frac{(q^{50};q^{50})_\infty^7 (q^5,q^{45};q^{50})_\infty^2 (q^{10},q^{40};q^{50})_\infty^3 (q^5,q^{45},q^{50};q^{50})_\infty}{(q^5;q^5)_\infty^4 (q^{25};q^{25})_\infty^2} \\
&\quad - 4y \frac{(q^{10};q^{10})_\infty (q^{15},q^{35},q^{50};q^{50})_\infty}{(q^5;q^{10})_\infty^4 (q^{10},q^{15},q^{35},q^{40};q^{50})_\infty^3},
\end{aligned}
$$

$$
\begin{aligned}
(4.13)\qquad & \frac{(q^5,q^{10},q^{40},q^{45};q^{50})_\infty (q^{15},q^{35},q^{50};q^{50})_\infty}{(q^5;q^5)_\infty (q^5;q^{10})_\infty} \\
&\qquad = y \frac{(q^{50};q^{50})_\infty^2 (q^5,q^{45};q^{50})_\infty^2 (q^{10},q^{40};q^{50})_\infty^3}{(q^5;q^5)_\infty^2} \\
&\qquad\quad + \frac{1}{(q^{15},q^{35};q^{50})_\infty^3 (q^{10},q^{40};q^{50})_\infty},
\end{aligned}
$$

$$
\begin{aligned}
(4.14)\qquad & 3 \frac{(-q^5,-q^{20};q^{25})_\infty (q^{50};q^{50})_\infty^2}{(q^{20},q^{30};q^{50})_\infty (-q^{10},-q^{15};q^{25})_\infty} \\
&= -4 \frac{(q^{50};q^{50})_\infty^2 (q^{15},q^{35},q^{50};q^{50})_\infty}{(q^5;q^{10})_\infty^5 (q^{20},q^{30};q^{50})_\infty (q^{25};q^{25})_\infty} \\
&\quad - 20y^2 \frac{(q^{50};q^{50})_\infty^7 (q^5,q^{45};q^{50})_\infty^2 (q^{10},q^{40};q^{50})_\infty^3 (q^{15},q^{35},q^{50};q^{50})_\infty}{(q^5;q^5)_\infty^4 (q^{25};q^{25})_\infty^2}
\end{aligned}
$$

$$+5\frac{(q^{50};q^{50})_\infty^3(q^{25};q^{25})_\infty}{(q^5;q^5)_\infty^2(q^{15},q^{35};q^{50})_\infty^3(q^5,q^{10},q^{40},q^{45};q^{50})_\infty(-q^{25};q^{25})_\infty}$$
$$+10\frac{(q^{50};q^{50})_\infty^3(q^5,q^{45},q^{50};q^{50})_\infty}{(q^5;q^5)_\infty^2(q^5,q^{45};q^{50})_\infty^3(q^{15},q^{20},q^{30},q^{35};q^{50})_\infty}$$
$$-8\frac{(q^{25};q^{25})_\infty(q^5,q^{45},q^{50};q^{50})_\infty}{(q^5;q^{10})_\infty^5(q^{10},q^{15},q^{35},q^{40};q^{50})_\infty},$$

$$\frac{(-q^{10},-q^{15};q^{25})_\infty(q^{50};q^{50})_\infty^2}{(q^{10},q^{40};q^{50})_\infty(-q^5,-q^{20};q^{25})_\infty} \tag{4.15}$$
$$=-10y\frac{(q^{50};q^{50})_\infty^3(q^{15},q^{35},q^{50};q^{50})_\infty}{(q^5;q^5)_\infty^2(q^{15},q^{35};q^{50})_\infty^3(q^5,q^{10},q^{40},q^{45};q^{50})_\infty}$$
$$+5\frac{(q^{50};q^{50})_\infty^3(q^{25};q^{25})_\infty}{(q^5;q^5)_\infty^2(q^5,q^{45};q^{50})_\infty^3(q^{15},q^{20},q^{30},q^{35};q^{50})_\infty(-q^{25};q^{25})_\infty}$$
$$-4\frac{(q^{25};q^{25})_\infty^2}{(q^5;q^{10})_\infty^5(q^{10},q^{15},q^{35},q^{40};q^{50})_\infty(-q^{25};q^{25})_\infty}$$
$$+4y\frac{(q^{10};q^{10})_\infty(q^5,q^{45},q^{50};q^{50})_\infty}{(q^5;q^{10})_\infty^4(q^{10},q^{15},q^{35},q^{40};q^{50})_\infty^3},$$

$$0=5y^2\frac{(q^{50};q^{50})_\infty^4(q^5,q^{10},q^{40},q^{45};q^{50})_\infty(q^5,q^{45},q^{50};q^{50})_\infty}{(q^5;q^5)_\infty^3(q^5;q^{10})_\infty} \tag{4.16}$$
$$+y\frac{(q^{25};q^{25})_\infty(q^5,q^{45},q^{50};q^{50})_\infty}{(q^5;q^{10})_\infty^5(q^{15},q^{20},q^{30},q^{35};q^{50})_\infty}$$
$$-5\frac{(q^{50};q^{50})_\infty^3(q^{15},q^{35},q^{50};q^{50})_\infty}{(q^5;q^5)_\infty^2(q^5,q^{45};q^{50})_\infty^3(q^{15},q^{20},q^{30},q^{35};q^{50})_\infty}$$
$$+4\frac{(q^{25};q^{25})_\infty(q^{50};q^{50})_\infty}{(q^5;q^{10})_\infty^5(q^{10},q^{40};q^{50})_\infty}$$
$$+\frac{(q^{10};q^{10})_\infty(q^{25};q^{25})_\infty}{(q^5;q^{10})_\infty^4(q^{10},q^{15},q^{35},q^{40};q^{50})_\infty^3(-q^{25};q^{25})_\infty}.$$

While we cannot rule out the possibility that (4.12)–(4.16) could be proven with clever applications of identities involving infinite products, we verify them using standard computational techniques from the theory of modular forms, which we briefly summarize. First, divide each identity by one of its terms to put it in the form

$$1=\sum F_i, \tag{4.17}$$

where each F_i can be expressed in terms of generalized η-products [24]. (For (4.12) we first use (3.19) and (3.20) to write the left hand side as a sum of four infinite products.) By equations (11) and (12) in [24], one establishes that each of the products F_i is a modular function on some congruence subgroup. (In our case, after letting $q = q^{1/5}$, the functions were always on $\Gamma_1(10)$.) Using Theorem 4 in [24], we then determine the order of F_i at the cusps and multiply both sides of (4.17) by an appropriate power of the Delta function, $\Delta^k(z)$, so that each $\Delta^k F_i$ is holomorphic at the cusps. Verifying the identity up to q^T, where T is the dimension of the appropriate space, is then sufficient to confirm its truth. (In our case, $T = \frac{k(10)^2}{12}\prod_{p|10}(1-1/p^2) = 72k$.)

We now turn to the rank differences $R_{02}(d)$, proceeding as above. Again by (2.4)–(2.6), we have

$$(4.18)\qquad \sum_{n=0}^{\infty}\{\overline{N}_2(0,5,n)-\overline{N}_2(2,5,n)\}q^n\frac{(q)_\infty}{2(-q)_\infty} = \overline{S}_2(5)+2\overline{S}_2(1)+\overline{S}_2(3).$$

By Lemma 2.1 (with $\ell = 5$), (4.18), (4.10), and (4.11),

$$\begin{aligned}
&\frac{-(q)_\infty}{2(-q)_\infty}+\frac{1}{2}+2g(1)+2q^4y\Sigma(1,0)\frac{(q)_\infty(-q^{25};q^{25})_\infty}{(-q)_\infty(q^{25};q^{25})_\infty}\\
&\quad -2q^3\frac{(q^{50};q^{50})_\infty^2(-q^{10},-q^{15};q^{25})_\infty}{(q^{10},q^{40};q^{50})_\infty(-q^5,-q^{20};q^{25})_\infty}-g(2)\\
&\quad +qy^3\Sigma(2,0)\frac{(q)_\infty(-q^{25};q^{25})_\infty}{(-q)_\infty(q^{25};q^{25})_\infty}-q^2y\frac{(q^{50};q^{50})_\infty^2(-q^5,-q^{20};q^{25})_\infty}{(q^{20},q^{30};q^{50})_\infty(-q^{10},-q^{15};q^{25})_\infty}\\
&\ = \{r_{02}(0)q^0+r_{02}(1)q+r_{02}(2)q^2+r_{02}(3)q^3+r_{02}(4)q^4\}\frac{(q)_\infty}{2(-q)_\infty}.
\end{aligned}$$

Again, equating coefficients of powers of q yields the following identities:

$$\begin{aligned}
(4.19)\qquad &\tfrac{1}{2}+2g(1)-g(2)\\
&= \frac{(q^{25};q^{25})_\infty^7}{2(q^5;q^{10})_\infty^6(q^{15},q^{20},q^{30},q^{35};q^{50})_\infty(q^{50};q^{50})_\infty^5(-q^{25};q^{25})_\infty}\\
&\quad+\frac{y^2(q^5;q^5)_\infty(q^{50};q^{50})_\infty(q^{25};q^{25})_\infty}{2(q^{15},q^{20},q^{30},q^{35};q^{50})_\infty^3(q^{25};q^{25})_\infty(-q^{25};q^{25})_\infty}\\
&\quad+\frac{2y(q^{25};q^{25})_\infty^3(q^5,q^{45};q^{50})_\infty}{(q^5;q^{10})_\infty^6(q^{20},q^{30},q^{50};q^{50})_\infty}\\
&\quad-\frac{5y^2(q^{50};q^{50})_\infty^3(q^{25};q^{25})_\infty}{(q^5;q^5)_\infty^2(q^{10},q^{40};q^{50})_\infty(q^{15},q^{35};q^{50})_\infty^4(-q^{25};q^{25})_\infty}\\
&\quad+\frac{2y^3(q^5;q^5)_\infty(q^{50};q^{50})_\infty^3(q^5,q^{45},q^{50};q^{50})_\infty}{(q^{15},q^{35};q^{50})_\infty^2(q^{20},q^{30};q^{50})_\infty^3(q^{25};q^{25})_\infty^3}\\
&\quad+\frac{10y(q^{50};q^{50})_\infty^4(q^{25};q^{25})_\infty(q^{20},q^{30};q^{50})_\infty}{(q^5,q^{45};q^{50})_\infty(q^5;q^5)_\infty^3}
\end{aligned}$$

$$+\frac{2(q^{50};q^{50})_\infty(-q^{10},-q^{15};q^{25})_\infty(q^{15},q^{35},q^{50};q^{50})_\infty}{(-q^5,-q^{20};q^{25})_\infty(q^5,q^{10},q^{40},q^{45};q^{50})_\infty}$$
$$-\frac{2(q^{25};q^{25})^2_\infty(q^{10},q^{40};q^{50})^5_\infty(q^{50};q^{50})^2_\infty(q^{15},q^{35},q^{50};q^{50})_\infty}{(q^5;q^{10})^4_\infty(q^5,q^{45};q^{50})^4_\infty(q^{10};q^{10})^3_\infty}$$
$$+\frac{2y(q^{25};q^{25})^4_\infty(q^5,q^{45};q^{50})_\infty}{(q^5;q^{10})^6_\infty(q^{20},q^{30};q^{50})_\infty(q^{50};q^{50})^2_\infty},$$

$$0=0, \tag{4.20}$$

$$2\frac{y^2(q^5;q^5)_\infty(q^{50};q^{50})^4_\infty}{(q^{15},q^{35};q^{50})_\infty(q^{20},q^{30};q^{50})^3_\infty(q^{25};q^{25})^3_\infty} \tag{4.21}$$
$$=\frac{y(q^{50};q^{50})^2_\infty(-q^5,-q^{20};q^{25})_\infty}{(-q^{10},-q^{15};q^{25})_\infty(q^{20},q^{30};q^{50})_\infty}$$
$$+\frac{4y(q^{25};q^{25})_\infty(q^5,q^{45},q^{50};q^{50})_\infty}{(q^5;q^{10})^5_\infty(q^{10},q^{15},q^{35},q^{40};q^{50})_\infty}$$
$$-\frac{5y(q^{50};q^{50})^3_\infty(q^{25};q^{25})_\infty}{(q^5;q^5)^2_\infty(q^{15},q^{35};q^{50})^3_\infty(q^5,q^{10},q^{40},q^{45};q^{50})_\infty(-q^{25};q^{25})_\infty},$$

$$4\frac{(q^{15},q^{35};q^{50})^2_\infty(q^{25};q^{25})^2_\infty}{(q^5;q^{10})^6_\infty(q^{20},q^{30};q^{50})_\infty} \tag{4.22}$$
$$=\frac{2(q^{25};q^{25})^2_\infty}{(q^5;q^{10})^5_\infty(q^{10},q^{15},q^{35},q^{40};q^{50})_\infty(-q^{25};q^{25})_\infty}$$
$$+\frac{2(q^{25};q^{25})^2_\infty(q^{10},q^{40};q^{50})^5_\infty(q^{50};q^{50})^2_\infty(q^5,q^{45},q^{50};q^{50})_\infty}{(q^5;q^{10})^4_\infty(q^5,q^{45};q^{50})^4_\infty(q^{10};q^{10})^3_\infty},$$

$$\frac{(q^{25};q^{25})^6_\infty(q^5,q^{45},q^{50};q^{50})_\infty}{(q^5;q^{10})^6_\infty(q^{15},q^{20},q^{30},q^{35};q^{50})_\infty(q^{50};q^{50})^5_\infty} \tag{4.23}$$
$$=\frac{-y^2(q^5;q^5)_\infty(q^{50};q^{50})_\infty(q^5,q^{45},q^{50};q^{50})_\infty}{(q^{15},q^{20},q^{30},q^{35};q^{50})^3_\infty(q^{25};q^{25})_\infty}$$
$$-\frac{4y(q^{25};q^{25})^2_\infty(q^5,q^{45};q^{50})_\infty(q^5,q^{45},q^{50};q^{50})_\infty}{(q^5;q^{10})^6_\infty(q^{20},q^{30},q^{50};q^{50})_\infty}$$
$$+\frac{10y^2(q^{50};q^{50})^3_\infty(q^5,q^{45},q^{50};q^{50})_\infty}{(q^5;q^5)^2_\infty(q^{10},q^{40};q^{50})_\infty(q^{15},q^{35};q^{50})^4_\infty}$$
$$+\frac{4(q^{25};q^{25})_\infty(q^{15},q^{35},q^{50};q^{50})_\infty}{(q^5;q^{10})^5_\infty(q^{10},q^{15},q^{35},q^{40};q^{50})_\infty}$$
$$+\frac{5(q^{50};q^{50})^3_\infty(q^{25};q^{25})^2_\infty(q^{20},q^{30};q^{50})_\infty}{(q^5,q^{45};q^{50})_\infty(q^5;q^5)^3_\infty(q^{15},q^{35};q^{50})_\infty(-q^{25};q^{25})_\infty}$$

$$+\frac{(q^{50};q^{50})_\infty(-q^{10},-q^{15};q^{25})_\infty(q^{25};q^{25})_\infty}{y(-q^5,-q^{20};q^{25})_\infty(q^5,q^{10},q^{40},q^{45};q^{50})_\infty(-q^{25};q^{25})_\infty}$$
$$-\frac{(q^{25};q^{25})^3_\infty(q^{10},q^{40};q^{50})^5_\infty(q^{50};q^{50})^2_\infty}{y(q^5;q^{10})^4_\infty(q^5,q^{45};q^{50})^4_\infty(q^{10};q^{10})^3_\infty(-q^{25};q^{25})_\infty}.$$

These equations were verified using modular forms as with (4.12)–(4.16) above. ∎

5. Proof of Corollary 1.3. We first recall the two "universal mock theta functions" (see Section 6 in [16])

$$g_2(x,q) := \frac{(-q)_\infty}{(q)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{n(n+1)}}{1-xq^n},$$
$$g_3(x,q) := \frac{1}{(q)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{3n(n+1)/2}}{1-xq^n}.$$

Watson [25, p. 66] showed that

$$\omega(q) = g_3(q,q^2). \tag{5.1}$$

By Theorem 1.1 in [18],

$$xg_2(x,q) = \frac{\eta^4(2\tau)}{\eta^2(\tau)\vartheta(2\alpha;2\tau)} + xq^{-1/4}\mu(2\alpha,\tau;2\tau). \tag{5.2}$$

Here

$$\mu(u,v;\tau) := \frac{a^{1/2}}{\vartheta(v;\tau)}\sum_{n\in\mathbb{Z}}\frac{(-b)^n q^{n(n+1)/2}}{1-aq^n}$$

and $\vartheta(v;\tau)$ is the classical theta series with product representation

$$\vartheta(v;\tau) = q^{1/8}b^{-1/2}\prod_{n=1}^{\infty}(1-q^n)(1-bq^{n-1})(1-b^{-1}q^n),$$

where $x=e^{2\pi i\alpha}$, $q=e^{2\pi i\tau}$, $a=e^{2\pi iu}$, and $b=e^{2\pi iv}$. Now by Theorem 1.1, Theorem 1.1 in [22] and (5.2), we have

$$\begin{aligned}(5.3)\quad &\sum_{n\ge0}(\overline{N}_2(0,3,3n+2)-\overline{N}_2(1,3,3n+2))q^n\\ &\quad-\sum_{n\ge0}(\overline{N}(0,3,3n+2)-\overline{N}(1,3,3n+2))q^n\\ &=\frac{6q(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{3n^2+6n}}{1-q^{6n+2}}+\frac{6(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{3n^2+3n}}{1-q^{3n+1}}\\ &=6q^{-3/4}\mu(2\tau,3\tau;6\tau)+6g_2(q,q^3)=12g_2(q,q^3)-\frac{6\eta^4(6\tau)}{q\eta^2(3\tau)\vartheta(2\tau;6\tau)}.\end{aligned}$$

Identity (6.1) in [16] states that

$$(5.4) \qquad g_3(x^4, q^4) = \frac{qg_2(x^6q, q^6)}{x^2} + \frac{x^2g_2(x^6q^{-1}, q^6)}{q} - \frac{x^2(q^2;q^2)_\infty^3(q^{12};q^{12})_\infty j(x^2q,q^2)j(x^{12}q^6,q^{12})}{q(q^4;q^4)_\infty(q^6;q^6)_\infty^2 j(x^4,q^2)j(x^6q^{-1},q^2)},$$

where $j(x,q) := (x)_\infty(q/x)_\infty(q)_\infty$. Letting $q \to q^{1/2}$, $x \to q^{1/4}$ in (5.4) and using the fact that $g_2(q^2, q^3) = g_2(q, q^3)$ gives us

$$(5.5) \qquad g_3(q, q^2) = 2g_2(q, q^3) - \frac{(q^6;q^6)_\infty^4}{(q^2;q^2)_\infty(q^3;q^3)_\infty^2}.$$

Thus the result follows after substituting (5.5) into (5.3) and using (5.1). ∎

Acknowledgements. The authors would like to thank Kathrin Bringmann for her comments regarding Lemma 3.2. The first author was partially supported by the Agence Nationale de Recherche ANR-08-JCJC-0011. The second author was partially funded by Science Foundation Ireland 08/RFP/MTH1081.

References

[1] G. E. Andrews, *The Theory of Partitions*, Cambridge Univ. Press, Cambridge, 1998.

[2] G. E. Andrews and F. G. Garvan, *Ramanujan's "lost" notebook VI: The mock theta conjectures*, Adv. Math. 73 (1989), 242–255.

[3] A. O. L. Atkin and H. P. F. Swinnerton-Dyer, *Some properties of partitions*, Proc. London Math. Soc. 4 (1954), 84–106.

[4] A. Berkovich and F. G. Garvan, *Some observations on Dyson's new symmetries of partitions*, J. Combin. Theory Ser. A 100 (2002), 61–93.

[5] K. Bringmann and J. Lovejoy, *Dyson's rank, overpartitions, and weak Maass forms*, Int. Math. Res. Notices 2007, no. 19, art. ID rnm063.

[6] —, —, *Rank and congruences for overpartition pairs*, Int. J. Number Theory 4 (2008), 303–322.

[7] —, —, *Overpartitions and class numbers of binary quadratic forms*, Proc. Nat. Acad. Sci. USA 106 (2009), 5513–5516.

[8] K. Bringmann, J. Lovejoy and R. Osburn, *Rank and crank moments for overpartitions*, J. Number Theory 129 (2009), 1758–1772.

[9] —, —, —, *Automorphic properties of generating functions for generalized rank moments and Durfee symbols*, Int. Math. Res. Notices 2010, no. 2, 238–260.

[10] K. Bringmann, K. Ono and R. Rhoades, *Eulerian series as modular forms*, J. Amer. Math. Soc. 21 (2008), 1085–1104.

[11] K. Bringmann and S. Zwegers, *Rank-crank type PDE's and non-holomorphic Jacobi forms*, Math. Res. Lett., to appear.

[12] Y.-S. Choi, *Tenth order mock theta functions in Ramanujan's lost notebook*, Invent. Math. 136 (1999), 497–569.

[13] M. Dewar, *The nonholomorphic parts of certain weak Maass forms*, J. Number Theory 130 (2010), 559–573.

[14] F. J. Dyson, *Some guesses in the theory of partitions*, Eureka (Cambridge) 8 (1944), 10–15.

[15] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Cambridge Univ. Press, Cambridge, 1990.

[16] B. Gordon and R. McIntosh, *A survey of classical mock theta functions*, Ramanujan J., to appear.

[17] D. Hickerson, *On the seventh order mock theta functions*, Invent. Math. 94 (1988), 661–677.

[18] S. Y. Kang, *Mock Jacobi forms in basic hypergeometric series*, Compos. Math. 145 (2009), 553–565.

[19] J. Lovejoy, *Rank and conjugation for the Frobenius representation of an overpartition*, Ann. Combin. 9 (2005), 321–334.

[20] —, *Rank and conjugation for a second Frobenius representation of an overpartition*, ibid. 12 (2008), 101–113.

[21] J. Lovejoy and O. Mallet, *Overpartition pairs and two classes of basic hypergeometric series*, Adv. Math. 217 (2008), 386–418.

[22] J. Lovejoy and R. Osburn, *Rank differences for overpartitions*, Quart. J. Math. (Oxford) 59 (2008), 257–273.

[23] —, —, *M_2-rank differences for partitions without repeated odd parts*, J. Théor. Nombres Bordeaux 21 (2009), 313–334.

[24] S. Robins, *Generalized Dedekind η-products*, in: The Rademacher Legacy to Mathematics (University Park, PA, 1992), Contemp. Math. 166, Amer. Math. Soc., Providence, RI, 1994, 119–128.

[25] G. N. Watson, *The final problem: an account of the mock theta functions*, J. London Math. Soc. 11 (1936), 55–80.

[26] S. Zwegers, *Mock theta functions*, Ph.D. thesis, Utrecht Univ., 2002.

Jeremy Lovejoy
CNRS, LIAFA
Université Denis Diderot–Paris 7
Case 7014
75205 Paris Cedex 13, France
E-mail: lovejoy@liafa.jussieu.fr

Robert Osburn
School of Mathematical Sciences
University College Dublin
Belfield, Dublin 4, Ireland
E-mail: robert.osburn@ucd.ie

Received on 14.8.2009 (6122)