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Explicit estimates for the summatory function of $\Lambda(n)/n$ from the one of $\Lambda(n)$

by

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1. Introduction. We define classically

$$\psi(x) = \sum_{n \le x} \Lambda(n),$$
$$\tilde{\psi}(x) = \sum_{n \le x} \Lambda(n)/n.$$

There has been a good amount of work to find explicit asymptotics for $\psi(x)$ (see for instance [15]–[18] and [13]). The quantity $\tilde{\psi}(x)$ has been much less studied, though [16, Theorem 6] gives an estimate. There has been an attempt in a more general setting in [10], and recent attention has been turned to the Mertens product, as in [2]. The problem here is that one would really want to deduce such an estimate from the ones concerning $\psi(x)$, but such a method is missing. The aim of this paper is to provide a fairly simple roundabout (see Theorem 1.1 below).

Let us note that the prime number theorem in the form $\psi(x) = (1 + o(1))x$ is "elementarily" equivalent to

$$\tilde{\psi}(x) = \log x - \gamma + o(1). \tag{1.1}$$

So in a sense, we are concerned with a quantitative version of this equivalence. A simple integration by parts is *not* enough, as it looses a log-factor. In effect, an estimate of the form $|\psi(x) - x)|/x \le 0.01$ for x large enough transfers into something like $|\tilde{\psi}(x) - \log x + \gamma| \le 0.01 \log x$ which is of no interest. The Landau equivalence theorem can however be made explicit, but does not yield a saving better than $1/\sqrt{\log x}$ in a rough form; allowing a saving of any power of $\log x$ is already theoretically not obvious (see [9] for instance). Here is a conjecture.

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Conjecture (Strong form of Landau equivalence theorem, I). *There exist positive constants c_1 and c_2 such that*

$$|\tilde{\psi}(x) - \log x + \gamma| \le c_1 \max_{x/c_2 < y \le c_2 x} \frac{|\psi(y) - y|}{y} + c_2 x^{-1/4}.$$

Such a conjecture holds (almost trivially) true under the Rieman Hypothesis. The result of [3] indicates that such an inequality does not hold in the case of Beurling generalized integers. Indeed they show that the condition $\psi_{\mathcal{P}}(x) \sim x$ does not ensure that $\tilde{\psi}_{\mathcal{P}}(x) - \log x$ has a limit, with obvious notations.

Let us end this introduction with a remark: in [7], the authors exhibit, under the Riemann Hypothesis, a pseudo-periodic function that (essentially) takes the value $(\tilde{\psi}(e^{-y}) + y)e^{y/2}$ when $y < 0$ and $(\psi(e^y) - e^y)e^{-y/2}$ when $y > 0$. This means that the values of ψ and of $\tilde{\psi}$ may be more closely linked than in the above conjecture.

We are not able to prove our conjecture, but show in Lemma 2.2 that

$$\tilde{\psi}(x) - \log x + \gamma - \frac{\psi(x) - x}{x}$$

is a well-controlled function. Here are some consequences of our formula.

Theorem 1.1. *For $x \ge 8\,950$, we have*

$$\tilde{\psi}(x) = \log x - \gamma + \frac{\psi(x) - x}{x} + \mathcal{O}^*\Big(\frac{1}{2\sqrt{x}}\Big) + \mathcal{O}^*(1.75 \cdot 10^{-12}).$$

Furthermore when $\log x \ge 9270$, we have (with $R = 5.696\,93$)

$$\tilde{\psi}(x) = \log x - \gamma + \frac{\psi(x) - x}{x} + \mathcal{O}^*\Big(\frac{1}{2\sqrt{x}}\Big) + \mathcal{O}^*\Big(\frac{1 + 2\sqrt{(\log x)/R}}{2\pi} \exp\big(-2\sqrt{(\log x)/R}\big)\Big).$$

Corollary. *For $x > 1$, we have*

$$\tilde{\psi}(x) = \log x - \gamma + \mathcal{O}^*(1.833/\log^2 x).$$

Furthermore, for $1 \le x \le 10^{10}$, we have $\tilde{\psi}(x) = \log x - \gamma + \mathcal{O}^*(1.31/\sqrt{x})$.

For $x \ge 23$, we have $\tilde{\psi}(x) = \log x - \gamma + \mathcal{O}^*(0.0067/\log x)$.

As a comparison, [16, Theorem 6] proposes an inequality similar to the last one above, but with $1/2 = 0.5$ instead of 0.0067. No error term with a saving of $1/\log^2 x$ is proposed.

Notation. We use the classical counting function

$$N(T) = \sum_{\substack{\rho \\ 0 < \gamma \le T}} 1, \tag{1.2}$$

where $\rho = \beta + i\gamma$ is a zero of the Riemann zeta-function. Furthermore, by $f(x) = \mathcal{O}^*(g(x))$ we mean $|f(x)| \le g(x)$.

The computations required have been done via Pari/GP (see [12]).

2. An explicit formula. We will need [14, Lemma 4]:

LEMMA 2.1. *Let g be a continuously differentiable function on $[a,b]$ with $2 \le a \le b < \infty$. Then*

$$\int_a^b \psi(t)g(t)\,dt = \int_a^b tg(t)\,dt - \sum_\rho \int_a^b \frac{t^\rho}{\rho} g(t)\,dt + \int_a^b \big(\log 2\pi - \tfrac{1}{2}\log(1-t^{-2})\big)g(t)\,dt.$$

Here is our main formula.

LEMMA 2.2. *For $x \ge 1$, we have*

$$\tilde{\psi}(x) = \log x - \gamma + \frac{\psi(x) - x}{x} + \sum_\rho \frac{x^{\rho-1}}{\rho(\rho-1)} + \frac{B(x)}{x}.$$

where the sum is over the zeroes ρ of the Riemann zeta function that lie in the critical strip $0 < \Im s < 1$ (the so-called nontrivial zeroes) and $B(x)$ is the bounded function given by

$$B(x) = \tfrac{1}{2} + \log 2\pi - \tfrac{1}{2}(x-1)\log(1-x^{-1}).$$

The main point of the lemma is that the sum over the zeroes is uniformly convergent, a feature not shared by the explicit formulae for $\psi(x)$ or $\tilde{\psi}(x)$. In fact, the main difficulty is carried by the term $(\psi(x) - x)/x$.

Proof. We simply proceed by integration by parts:

$$\tilde{\psi}(x) = \int_1^x \psi(t)\,\frac{dt}{t^2} + \frac{\psi(x)}{x} = \log x - \gamma + \int_x^\infty (\psi(t) - t)\,\frac{dt}{t^2} + \frac{\psi(x) - x}{x}.$$

Note that the existence of the integral requires a strong enough form of the equivalence between $\psi(t)$ and t. Next we apply the explicit formula of Lemma 2.1 to get

$$\int_x^Y (\psi(t) - t)\,\frac{dt}{t^2} = -\sum_\rho \int_x^Y \frac{t^{\rho-2}\,dt}{\rho} + \int_x^Y \big(\log 2\pi - \tfrac{1}{2}\log(1-t^{-2})\big)\,\frac{dt}{t^2}$$
$$= -\sum_\rho \frac{Y^{\rho-1} - x^{\rho-1}}{\rho(\rho-1)} + \int_x^Y \big(\log 2\pi - \tfrac{1}{2}\log(1-t^{-2})\big)\,\frac{dt}{t^2}.$$

Since (1.1) is known to hold, and $\sum_\rho 1/|\rho(\rho-1)|$ is convergent, we can send Y to infinity and get

$$\int_x^Y (\psi(t)-t)\,\frac{dt}{t^2} = \sum_\rho \frac{x^{\rho-1}}{\rho(\rho-1)} + \int_x^\infty \big(\log 2\pi - \tfrac12 \log(1-t^{-2})\big)\,\frac{dt}{t^2}. \ ■$$

3. Known bounds on $\psi(x)$. In [13], we find that

$$|\psi(x)-x| \le \sqrt{x} \quad (8 \le x \le 10^{10}). \tag{3.1}$$

If we change $\sqrt{x}$ to $\sqrt{2x}$, this is valid from $x=1$ onwards. Furthermore

$$|\psi(x)-x| \le 0.8\sqrt{x} \quad (1\,500 \le x \le 10^{10}). \tag{3.2}$$

By [4, Théorème 1.3] improving on [18, Theorem 7], we have

$$|\psi(x)-x| \le 0.0065\,x/\log x \quad (x \ge \exp(22)). \tag{3.3}$$

We readily extend this estimate to $x \ge 3\,430\,190$ by using (3.1), and then to $x \ge 1\,514\,928$ by direct inspection.

We quote [4, Théorème 1.4] and [5, Theorem 5.2]:

$$|\vartheta(x)-x| \le 3.965\,x/\log^2 x \quad (x > 2), \tag{3.4}$$

$$|\vartheta(x)-x| \le 515\,x/\log^3 x \quad (x > 2), \tag{3.5}$$

In fact [5, Theorem 5.2] proposes the constant 21 instead of 515 in this inequality, but this preprint has not been published. We will not use this bound but take this opportunity to record this fact.

We go from ϑ to ψ by using [17, Theorem 6]

$$0 \le \psi(x)-\vartheta(x) \le 1.0012\sqrt{x} + 3x^{1/3} \quad (x > 0). \tag{3.6}$$

Lemma 3.1. *For $x \ge 7\,105\,266$, we have*

$$|\psi(x)-x|/x \le 0.000\,213.$$

Proof. We start with the estimate from [17, (4.1)]:

$$|\psi(x)-x|/x \le 0.000\,213 \quad (x \ge 10^{10}). \tag{3.7}$$

We extend it to $x \ge 14\,500\,000$ by using (3.1). We conclude by direct inspection. ■

Lemma 3.2. *For $x > 1$, we have*

$$|\psi(x)-x| \le 1.830\,x/\log^2 x, \qquad |\psi(x)-x| \le 516\,x/\log^3 x.$$

Proof. Indeed, we readily find that

$$\begin{aligned}|\psi(x)-x|\frac{(\log x)^2}{x} &\le |\psi(x)-\vartheta(x)|\frac{(\log x)^2}{x} + |\vartheta(x)-x|\frac{(\log x)^2}{x}\\ &\le \min\bigg(\frac{1.0012(\log x)^2}{\sqrt{x}} + \frac{3(\log x)^2}{x^{2/3}} + \frac{515}{\log x},\ 0.0065\log x\bigg),\end{aligned}$$

which is not more than 1.830, on using the first estimate for $x \geq \exp(281.5)$ and the second one for the smaller values. We prove the second estimate in the same way:

$$|\psi(x)-x|\frac{(\log x)^3}{x} \leq |\psi(x)-\vartheta(x)|\frac{(\log x)^3}{x} + |\vartheta(x)-x|\frac{(\log x)^3}{x}$$
$$\leq \min\left(\frac{1.0012(\log x)^3}{\sqrt{x}} + \frac{3(\log x)^3}{x^{2/3}} + 515,\ 0.0065\log^2 x\right),$$

which is not more than 516, on using again the first estimate for $x \geq \exp(281.5)$ and the second one for the smaller values. For lower x, we first use

$$|\psi(x)-x| \leq (\log^2 x/(1.830\sqrt{x}))1.830\,x/\log^2 x,$$

which extends our bound till $x \geq 55$. A very primitive GP script shows that

$$|\psi(x)-x| \leq 1.417\,x/\log^2 x \qquad (1 \leq x \leq 10^5).$$

We proceed similarly for the bound with $\log^3 x$. ∎

4. Lemmas on the zeroes.

We quote from [14]:

Lemma 4.1. *If T is a real number $\geq 10^3$ then*

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + \mathcal{O}^*\left(0.67\log\frac{T}{2\pi}\right).$$

This is a version of Theorem 19 of [15], relying on [1].

Lemma 4.2. *For $T \geq 10^3$, we have*

$$\sum_{\gamma\geq T_0} 1/\gamma^2 \leq \frac{\log(T/(2\pi))}{2\pi T} + 0.67\,\frac{2\log(T/(2\pi))+1/2}{T^2}.$$

Proof. We denote by S the sum to be evaluated and we simply use integration by parts:

$$S = 2\int_T^\infty \frac{N(t)-N(T)}{t^3}\,dt$$
$$\leq \frac{2}{(2\pi)^2}\int_{T/(2\pi)}^\infty \frac{u\log u - u + \frac{7}{8} + 0.67\log u}{u^3}\,du$$
$$- \frac{\frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} - 0.67\log\frac{T}{2\pi}}{T^2}$$
$$\leq \frac{\log(T/(2\pi))}{2\pi T} + 0.67\,\frac{2\log(T/(2\pi))+1/2}{T^2},$$

and the lemma follows readily. ∎

Lemma 4.3. *We have* $\sum_\rho 1/|\rho(\rho-1)| \le 0.047$, *where* ρ *ranges over all nontrivial zeroes of* ζ.

In particular, we do not impose $\Im\rho > 0$. We prove this lemma by using the file of the first 10^5 zeroes provided by Odlyzko [11].

We in fact used zeroes only up to height 10 000 and ran the computations using 28 digits precision on GP/Pari. Note that when $\Im\rho = 1/2$, we have $\rho(\rho-1) = -|\rho|^2$. Truncation of the imaginary parts only increases the sum, while the high enough precision takes care of the machine error. The restricted sum is about 0.023 02 (with $\Im\rho > 0$). We next use Lemma 4.2 to handle the tail of the series. We finally double the value to remove the condition $\Im\rho > 0$, and round the value up.

We also know, thanks to [6], that the zeroes ρ in the critical strip and satisfying $|\Im\rho| \le 2.44\cdot 10^{12} = T_0$ are all on the line $\Re\rho = 1/2$. We handle zeros with large imaginary part by using the following theorem from [8].

Lemma 4.4. *Every zero* $\rho = \beta + i\gamma$ *of* ζ *in the strip* $0 < \beta < 1$ *and with* $\gamma \ge 10$ *satisfies*

$$\beta \le 1 - \varphi(\gamma) = 1 - 1/(R\log\gamma), \qquad R = 5.696\,93.$$

5. Proof of Theorem 1.1. We start with Lemma 2.2. Let us set

$$J(x) = \sum_\rho \frac{x^{\rho-1}}{\rho(\rho-1)}. \tag{5.1}$$

By considering the symmetry $\rho \mapsto 1-\rho$, we get (remember that no zero of ζ lies on the segment $[0,1]$)

$$J(x) = \sum_{\substack{\rho,\\ \Im\rho>0}} \frac{x^{\rho-1} + x^{-\rho}}{\rho(\rho-1)}.$$

We are ready to majorize $J(x)$:

$$\begin{aligned} J(x) &\le \sum_{|\gamma|\le T_0} \frac{x^{-1/2}}{|\rho|^2} + \sum_{\gamma>T_0}\left(\frac{x^{-1/2}}{|\rho(\rho-1)|} + \frac{x^{-\varphi(\gamma)}}{|\rho(\rho-1)|}\right) \\ &\le \frac{0.047}{\sqrt{x}} + \sum_{\gamma>T_0}\frac{x^{-\varphi(\gamma)}}{\gamma^2}. \end{aligned}$$

We first bound $x^{-\varphi(\gamma)}$ by 1 and get, by Lemma 4.2,

$$J(x) \le \frac{0.047}{\sqrt{x}} + \frac{\log(T_0/(2\pi))}{2\pi T_0}\left(1 + 1.36\,\frac{2\pi}{T_0}\right) \le \frac{0.047}{\sqrt{x}} + 1.75\cdot 10^{-12}.$$

This proves the first part of Theorem 1.1. For large x, we can take advantage of the zero free region. We set $\varphi_2(\gamma) = x^{-\varphi(\gamma)}/\gamma^2$ and get

$$\begin{aligned} J(x) &\le \frac{0.047}{\sqrt{x}} - \int_{T_0}^{\infty} (N(t) - N(T_0))\varphi_2'(t)\,dt \\ &\le \frac{0.047}{\sqrt{x}} - \int_{T_0}^{\infty} (N^*(t) - N(T_0))\varphi_2'(t)\,dt - \int_{T_0}^{\infty} (N(t) - N^*(t))\varphi_2'(t)\,dt \\ &\le \frac{0.047}{\sqrt{x}} + (N^*(T_0) - N(T_0))\varphi_2(T_0) \\ &\quad + \int_{T_0}^{\infty} N^*(t)'\varphi_2(t)\,dt - \int_{T_0}^{\infty} (N(t) - N^*(t))\varphi_2'(t)\,dt \\ &\le \frac{0.047}{\sqrt{x}} + 3\cdot 10^{-24}x^{-\varphi(T_0)} + \int_{T_0}^{\infty} \frac{x^{-\varphi(t)}\log(t/(2\pi))\,dt}{2\pi t^2} \\ &\quad + \int_{T_0}^{\infty} \left|\frac{\log x}{2R} - \log^2 t\right| \frac{2x^{-\varphi(t)}\log(t/(2\pi))\,dt}{t^3\log^2 t}. \end{aligned}$$

We now assume $\log x \ge 2R\log^2 T_0$ and infer the bound

$$\begin{aligned} J(x) &\le \frac{0.047}{\sqrt{x}} + 6\cdot 10^{-24}x^{-\varphi(T_0)} \\ &\quad + \int_{T_0}^{\infty} \frac{x^{-\varphi(t)}\log(t/(2\pi))\,dt}{2\pi t^2} + 0.67\int_{T_0}^{\infty} \frac{x^{-\varphi(t)}\,dt}{t^3} \\ &\le \frac{0.047}{\sqrt{x}} + 6\cdot 10^{-24}x^{-\varphi(T_0)} + \int_{T_0}^{\infty} \frac{x^{-\varphi(t)}\log(t/6.25)\,dt}{2\pi t^2} \\ &\le \frac{0.047}{\sqrt{x}} + \int_{T_0}^{\infty} \frac{x^{-\varphi(t)}\log t\,dt}{2\pi t^2}. \end{aligned}$$

Define

$$I = \int_{T_0}^{\infty} \exp\left(-\frac{\log x}{R\log t} - \log t\right)\frac{\log t\,dt}{2\pi t} = \int_{\log T_0}^{\infty} \exp\left(-\frac{\log x}{Ru} - u\right)\frac{u\,du}{2\pi},$$

and set

$$\frac{\log x}{Ru} + u = v,$$

which gets solved in $(u^2 - uv + (\log x)/R = 0)$

$$2u = v \pm \sqrt{v^2 - 4(\log x)/R}.$$

We further get

$$4u\,du = \big(v \pm \sqrt{v^2 - 4(\log x)/R}\big)\left(1 \pm \frac{v}{\sqrt{v^2 - 4(\log x)/R}}\right) dv$$
$$= \left(v \pm \sqrt{v^2 - 4(\log x)/R} \pm \frac{v^2}{\sqrt{v^2 - 4(\log x)/R}} + v\right) dv$$
$$= \left(2v \pm \frac{2v^2 - 4(\log x)/R}{\sqrt{v^2 - 4(\log x)/R}}\right) dv$$

so that I gets rewritten as

$$I = \int_{2\sqrt{(\log x)/R}}^{\infty} e^{-v}\left(v + \frac{v^2 - 2(\log x)/R}{\sqrt{v^2 - 4(\log x)/R}}\right)\frac{dv}{4\pi}$$
$$+ \int_{2\sqrt{(\log x)/R}}^{\frac{\log x}{R\log T_0}+\log T_0} e^{-v}\left(v - \frac{v^2 - 2(\log x)/R}{\sqrt{v^2 - 4(\log x)/R}}\right)\frac{dv}{4\pi},$$

which yields

$$I \le \int_{2\sqrt{(\log x)/R}}^{\infty} ve^{-v}\,\frac{dv}{2\pi} = \frac{1 + 2\sqrt{(\log x)/R}}{2\pi}\exp\big(-2\sqrt{(\log x)/R}\big).$$

It is then immediate to conclude the proof of Theorem 1.1.

6. Proof of the Corollary. When $\log x \le 2R\log^2 T_0$, but $x \ge 10^{10}$, we use Lemma 3.2 to get

$$|\tilde{\psi}(x) - \log x + \gamma|\log^2 x \le 1.830 + \frac{\log^2 x}{2\sqrt{x}} + 1.68\cdot 10^{-12}\log^2 x \le 1.833.$$

When $8\,950 \le x \le 10^{10}$, we have

$$|\tilde{\psi}(x) - \log x + \gamma|\log^2 x \le \frac{1.3\log^2 x}{\sqrt{x}} + 1.68\cdot 10^{-12}\log^2 x \le 1.14.$$

When $\log x \ge 2R\log^2 T_0$, the bound becomes

$$1.830 + \frac{1 + 2\sqrt{(\log x)/R}}{2\pi}\exp\big(-2\sqrt{(\log x)/R}\big)\log^2 x \le 1.832.$$

We complete the proof by direct inspection. For the limited range bound, we write

$$|\tilde{\psi}(x) - \log x + \gamma|\sqrt{x} \le 1.3 + 1.68\cdot 10^{-12}\sqrt{x} \le 1.31$$

when $x \ge 8\,950$. We again conclude by direct inspection.

When $\log x \le 2R\log^2 T_0$, but $x \ge 10^{10}$, we have

$$|\tilde{\psi}(x) - \log x + \gamma| \log x \le 0.0065 + \frac{\log x}{2\sqrt{x}} + 1.68 \cdot 10^{-12} \log x \le 0.0067.$$

When $8\,950 \le x \le 10^{10}$, we have

$$|\tilde{\psi}(x) - \log x + \gamma| \log x \le \frac{1.3 \log x}{\sqrt{x}} + 1.68 \cdot 10^{-12} \log x \le 0.0003.$$

When $\log x \ge 2R\log^2 T_0$, the bound becomes

$$0.0065 + \frac{1 + 2\sqrt{(\log x)/R}}{2\pi} \exp\big(-2\sqrt{(\log x)/R}\big) \log^2 x \le 0.0066.$$

We complete the proof by direct inspection.

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