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Small values of the Riemann zeta function on the critical line

by

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1. Introduction and main results. The Riemann zeta function satisfies the well-known functional equation $\zeta(s) = \zeta(1-s)\chi(s)$, where

$$\chi(s) = 2^s \pi^{s-1} \sin\left(\tfrac{1}{2}\pi s\right) \Gamma(1-s).$$

The functional equation implies that the values of χ on the critical line $1/2 + i\mathbb{R}$ lie on the unit circle. Given an angle $\phi \in [0, \pi)$, we denote by $t_n(\phi)$, $n = 1, 2, \ldots$, the *generalized Gram points*, the positive roots of the equation

$$e^{2i\phi} = \chi(1/2 + it)$$

in ascending order. These roots correspond to the intersections of the curve $t \mapsto \zeta(1/2+it)$ with straight lines $e^{i\phi}$ through the origin (see [KS]). Of special interest are the intersections with the real line. In this case $\phi = 0$, and the roots are called *Gram's points* [Gr].

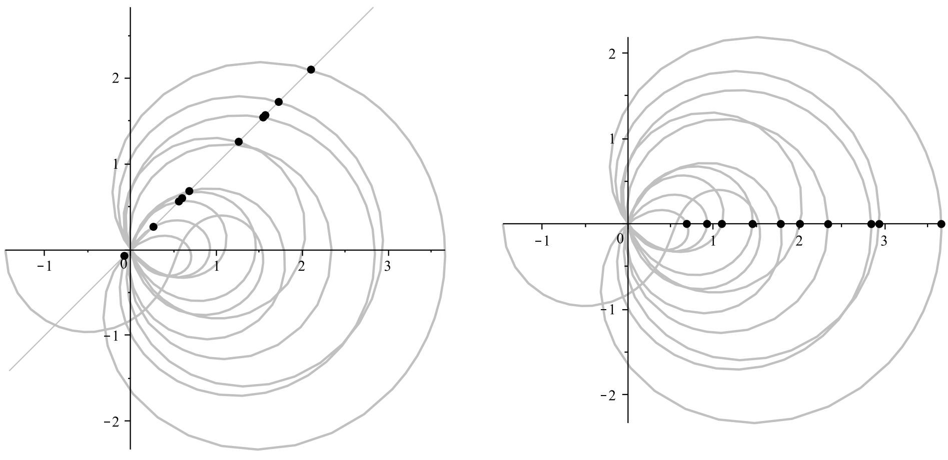

Fig. 1. (a) Values at $t_n(\pi/4)$; (b) values at Gram's points $t_n(0)$.

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Recently the first named author, Korolev and Steuding proved that the negative values (and, more generally, values with any fixed argument) of the Riemann zeta function on the critical line can be arbitrarily large in absolute value [KKS]. In other words, the graph in Figure 1 expands to all directions with the minimum speed of $(\log T)^{5/4}$ for values of $\zeta(1/2+it)$ with $t \in [T, 2T]$.

A natural question to ask is the opposite one: how small the nonzero values of the Riemann zeta function on the critical line can be? Christ states in his thesis [C, p. 56] that there is an interval $A \subset [0, 2\pi)$ of length at least $\pi/4$ such that for every $\theta \in A$ there is a sequence $t_n \in [2, \infty)$ with

$$\zeta(1/2+it_n) \neq 0, \qquad \lim_{n\to\infty} \zeta(1/2+it_n) = 0, \qquad \arg\zeta(1/2+it_n) \equiv \theta \bmod 2\pi.$$

Recently Korolev [Kor] obtained a more precise result. Extending the ideas of Selberg [Se] and Radziwiłł [R] he proved that if $\zeta(1/2+it_n(0)) \neq 0$ (note that $\zeta(1/2+it_n(0)) \in \mathbb{R}$) then

$$\min_{t_n(0)\in[0,T]} |\zeta(1/2+it_n(0))| < \exp\left(-\frac{\log\log T}{\varphi(T)}\right),$$

where $\varphi(T)$ is an unbounded and increasing function as $T \to \infty$.

In this paper we further improve and generalize the results of Christ and Korolev:

Theorem 1.1. *For $\phi \in [0, \pi)$,*

$$\min_{t_n(\phi)\in[0,T]} |\zeta(1/2+it_n(\phi))| < \exp\left(-\left(\frac{1}{\sqrt{6}}+o(1)\right)\sqrt{\frac{\log T}{\log\log T}}\right) \qquad \textit{as } T\to\infty.$$

We prove Theorem 1.1 by considering the following two discrete moments:

$$S_0(T) = \sum_{0<t_n(\phi)\le T} |X(1/2+it_n(\phi))|^2, \tag{1.1}$$

$$S_2(T) = \sum_{0<t_n(\phi)\le T} |\zeta(1/2+it_n(\phi))|^2 |X(1/2+it_n(\phi))|^2, \tag{1.2}$$

where $X(s)$ is a Dirichlet polynomial

$$X(s) = \sum_{n\le X} \frac{x_n}{n^s} \quad \text{with } X \le T. \tag{1.3}$$

It is easy to see that these moments can be used to estimate small values of the Riemann zeta function at Gram points in the following way:

$$\min_{t_n(\phi)\in[0,T]} |\zeta(1/2+it_n(\phi))|^2 \le S_2(T)/S_0(T).$$

In order to minimize the quantity on the right-hand side we use the resonance method introduced by Soundararajan [So], and in order to obtain asymptotic formulas for $S_0(T)$ and $S_2(T)$ we use a method introduced by Kalpokas, Korolev and Steuding [KKS].

Using the same ideas with minor modifications we also obtain a result concerning extremal values of the Riemann zeta function at generalized Gram points:

THEOREM 1.2. *For* $\phi \in [0,\pi)$,

$$\max_{t_n(\phi)\in[0,T]} |\zeta(1/2+it_n(\phi))| > \exp\biggl(\Bigl(\frac{1}{\sqrt{6}}+o(1)\Bigr)\sqrt{\frac{\log T}{\log\log T}}\biggr) \quad \textit{as } T\to\infty.$$

It would be interesting to improve Theorem 1.1 by controlling the sign as well as angle of generalized Gram points. Writing $t_n^+(\phi)$ if $e^{-i\phi}\zeta(1/2+it_n(\phi))\ge 0$, and $t_n^-(\phi)$ if $e^{-i\phi}\zeta(1/2+it_n(\phi))<0$, we would like to find arbitrarily small nonzero $t_n^+(\phi)$ and $t_n^-(\phi)$ values. One possible approach would be to use the inequality

$$\min_{t_n^\pm(\phi)\in[0,T]} |\zeta(1/2+it_n^\pm(\phi))|^2 \le S_2(T)\Bigl(\sum_{0<t_n^\pm(\phi)\le T} |X(1/2+it_n^\pm(\phi))|^2\Bigr)^{-1},$$

and find a lower bound for

$$\sum_{0<t_n^\pm(\phi)\le T} |X(1/2+it_n^\pm(\phi))|^2 = \#\{t_n^\pm(\phi)\mid t_n^\pm(\phi)\in[0,T]\}\sum_{n\le X}\frac{|x_n|^2}{n}+\mathcal{E},$$

where

$$\mathcal{E} = \sum_{n\le X}\frac{x_n}{n}\sum_{m\le X}\frac{\overline{x_m}}{m}\sum_{0<t_n^\pm(\phi)\le T}\Bigl(\frac{m}{n}\Bigr)^{1/2+it_n^\pm(\phi)}.$$

We conjecture that $\mathcal{E}$ is smaller than the main term, but we have not been able to prove this. We can, however, conditionally estimate the main term:

THEOREM 1.3. *If the Riemann hypothesis is true then*

$$\#\{t_n^\pm(\phi)\mid t_n^\pm(\phi)\in[0,T]\} \gg T(\log T)^{-7/2-\epsilon}.$$

2. Technical lemmas. We start with the theorem of Kopetzky [Kop] that establishes an asymptotic formula for the sum of the divisor function over an arithmetic progression:

LEMMA 2.1. *For any constants r and m we have*

$$\sum_{\substack{n\le x\\ n\equiv a \bmod m}} d(n) = \xi_1(a,m)x\log x + \bigl((2C-1)\xi_1(a,m)-2\xi_2(a,m)\bigr)x + O(x^{1/2}),$$

where

$$\xi_1(a,m) = \frac{1}{m^2} \sum_{\substack{\sigma=1 \\ (\sigma,m)|a}}^{m} (\sigma,m),$$

(2.1)

$$\xi_2(a,m) = \frac{1}{m^2} \sum_{\substack{\sigma=1 \\ (\sigma,m)|a}}^{m} (\sigma,m)(C - mC(\sigma,m))$$

and $C(a,m)$ *is defined by*

$$\sum_{\substack{n\le x \\ n\equiv a \bmod m}} \frac{1}{n} = \frac{1}{m}\log x + C(a,m) + O\left(\frac{1}{x}\right). \tag{2.2}$$

Next we consider twisted sums of ξ_1 and ξ_2:

LEMMA 2.2. *Consider the following sums for the quantities* $\xi_1(a,m)$ *and* $\xi_2(a,m)$ *defined in* (2.1)*:*

$$\Sigma_1 = \sum_{a=1}^{m} \exp\left(-2\pi i \frac{ka}{m}\right)\xi_1(a,m), \quad \Sigma_2 = \sum_{a=1}^{m} \exp\left(-2\pi i \frac{ka}{m}\right)\xi_2(a,m).$$

Then

$$\Sigma_1 = \frac{(m,k)}{m} \quad \text{and} \quad \Sigma_2 = \frac{(m,k)}{m}\log\frac{m}{(m,k)}.$$

Proof. Substitute the value of $\xi_1(a,m)$ into Σ_1 and change the order of summation:

$$\begin{aligned}\Sigma_1 &= \frac{1}{m^2}\sum_{a=1}^{m} \exp\left(-2\pi i \frac{ka}{m}\right) \sum_{\substack{\sigma=1 \\ (\sigma,m)|a}}^{m} (\sigma,m) \\ &= \frac{1}{m^2}\sum_{\sigma=1}^{m} (\sigma,m) \sum_{\substack{a=1 \\ (\sigma,m)|a}}^{m} \exp\left(-2\pi i \frac{ka}{m}\right).\end{aligned}$$

The last sum

$$\sum_{\substack{a=1 \\ (\sigma,m)|a}}^{m} \exp\left(-2\pi i \frac{ka}{m}\right) = \sum_{a'=1}^{m/(\sigma,m)} \exp\left(-2\pi i \frac{ka'}{\frac{m}{(\sigma,m)}}\right)$$

is equal to zero unless $\frac{m}{(\sigma,m)} \mid k$, in which case it is equal to $\frac{m}{(\sigma,m)}$. The only values of σ that satisfy this condition are multiples of $\frac{m}{(m,k)}$, and there are (m,k) of them in the interval $[1,m]$. Hence

$$\Sigma_1 = (m,k)/m.$$

Next we evaluate Σ_2. From Lemma 2.1 we substitute the value of $\xi_2(a,m)$ into Σ_2 and change the order of summation. We have

$$\begin{aligned}\Sigma_2 &= \frac{1}{m^2}\sum_{a=1}^{m}\exp\Big(-2\pi i\frac{ka}{m}\Big)\sum_{\substack{\sigma=1\\(\sigma,m)\mid a}}^{m}(\sigma,m)(C_0 - mC(\sigma,m))\\ &= C_0\frac{(m,k)}{m} - \frac{1}{m}\sum_{\sigma=1}^{m}(\sigma,m)C(\sigma,m)\sum_{\substack{a=1\\(\sigma,m)\mid a}}^{m}\exp\Big(-2\pi i\frac{ka}{m}\Big).\end{aligned}$$

Using the same arguments we used for Σ_1 we get

$$\begin{aligned}\Sigma_2 &= C_0\frac{(m,k)}{m} - \frac{1}{m}\sum_{\sigma=1}^{m}C(\sigma,m)(\sigma,m)\sum_{a'=1}^{m/(\sigma,m)}\exp\Big(-2\pi i\frac{ka'}{\frac{m}{(\sigma,m)}}\Big)\\ &= C_0\frac{(m,k)}{m} - \sum_{\substack{\sigma=1\\ \frac{m}{(m,k)}\mid\sigma}}^{m}C(\sigma,m).\end{aligned}$$

By definition of the Euler–Lehmer constants in (2.2) we have

$$\begin{aligned}\sum_{\substack{\sigma=1\\ \frac{m}{(m,k)}\mid\sigma}}^{m}C(\sigma,m) &= \lim_{x\to\infty}\sum_{\sigma'=1}^{(m,k)}\sum_{\substack{n\le x\\ n\equiv\sigma'\frac{m}{(m,k)}\bmod m}}\frac{1}{n} - \frac{(m,k)}{m}\log x\\ &= \lim_{x\to\infty}\sum_{\substack{n\le x\\ \frac{m}{(m,k)}\mid n}}\frac{1}{n} - \frac{(m,k)}{m}\log x\\ &= \lim_{x\to\infty}\sum_{\substack{n\le x\\ n\equiv 0\bmod \frac{m}{(m,k)}}}\frac{1}{n} - \frac{1}{m/(m,k)}\log x = C\Big(0,\frac{m}{(m,k)}\Big).\end{aligned}$$

From Lehmer [L, formula (2)] we have

$$C\Big(0,\frac{m}{(m,k)}\Big) = C_0\frac{(m,k)}{m} - \frac{(m,k)}{m}\log\frac{m}{(m,k)}.$$

Hence,

$$\Sigma_2 = \frac{(m,k)}{m}\log\frac{m}{(m,k)}. \ \blacksquare$$

The following lemma evaluates the key sum that emerges when searching for the asymptotic of $S_2(T)$.

Lemma 2.3. *Let x_n be an arbitrary sequence. Then the sum*

$$S=\sum_{m\le X}\frac{x_m}{m}\sum_{k\le X}x_k\sum_{n\le\frac{Tm}{2\pi k}}d(n)\exp\Big(-2\pi i\frac{nk}{m}\Big)\log\frac{nk}{m}$$

is equal to

$$\bigg(\frac{T}{2\pi}\Big(\log\frac{T}{2\pi}\Big)^2+(2C_0-2)\frac{T}{2\pi}\log\frac{T}{2\pi e}\bigg)\sum_{\substack{m\le X\\ k\le X}}\frac{x_mx_k}{mk}(m,k)$$
$$+\frac{T}{2\pi}\Big(\log\frac{T}{2\pi e}\Big)\sum_{\substack{m\le X\\ k\le X}}\frac{x_mx_k}{mk}(m,k)\log\frac{m}{k}$$
$$-2\frac{T}{2\pi}\Big(\log\frac{T}{2\pi e}\Big)\sum_{\substack{m\le X\\ k\le X}}\frac{x_mx_k}{mk}(m,k)\log\frac{m}{(m,k)}+O(X^2\mathcal{X}_0\mathcal{X}_1T^{1/2}\log T),$$

where $d(n)$ is the divisor function, C_0 is the Euler constant, $\mathcal{X}_0=\max_{n\le X}|x_n|$ and $\mathcal{X}_1=\sum_{n\le X}|x_n|/n$.

Proof. We rewrite S as

$$S=\sum_{m\le X}\frac{x_m}{m}\sum_{k\le X}x_k\sum_{a=1}^{m}\exp\Big(-2\pi i\frac{ka}{m}\Big)\sum_{\substack{1<n\le\frac{Tm}{2\pi k}\\ n\equiv a \bmod m}}d(n)\log\frac{kn}{m}$$
$$=\sum_{m\le X}\frac{x_m}{m}\sum_{k\le X}x_k\sum_{a=1}^{m}\exp\Big(-2\pi i\frac{ka}{m}\Big)\mathcal{S}(a,m,k,T).$$

For the last sum we use Abel's summation. We have

$$\mathcal{S}(a,m,k,T)=\log\frac{T}{2\pi}\sum_{\substack{n\le\frac{Tm}{2\pi k}\\ n\equiv a \bmod m}}d(n)-\int_1^{Tm/2\pi k}\sum_{\substack{n\le u\\ n\equiv a \bmod m}}d(n)\Big(\log\frac{uk}{m}\Big)'\,du$$
$$=A_1-A_2.$$

From Lemma 2.1 we get

$$A_1=\frac{T}{2\pi}\Big(\log\frac{T}{2\pi}\Big)^2\frac{m}{k}\xi_1(a,m)$$
$$+\frac{T}{2\pi}\Big(\log\frac{T}{2\pi}\Big)\Big(\frac{m}{k}\Big(\log\frac{m}{k}\Big)\xi_1(a,m)+\frac{m}{k}\big((2C-1)\xi_1(a,m)-2\xi_2(a,m)\big)\Big)$$
$$+O\Big(\Big(\frac{Tm}{k}\Big)^{1/2}\log T\Big),$$

and after integrating and using Lemma 2.1 again we get

$$A_2 = \xi_1(a,m)\left(\frac{Tm}{2\pi k}\log\frac{Tm}{2\pi k} - \frac{Tm}{2\pi k}\right) + \big((2C-1)\xi_1(a,m) - 2\xi_2(a,m)\big)\frac{Tm}{2\pi k} + O\left(\left(\frac{Tm}{k}\right)^{1/2}\right).$$

Substituting A_1 and A_2 into $\mathcal{S}(a,m,k,T)$ and using the notation of (1.3) we find that the error term of S is bounded by

$$\ll \sum_{m\le X}\frac{x_m}{m}\sum_{k\le X}x_k\sum_{a=1}^{m}\exp\left(-2\pi i\frac{ka}{m}\right)\left(\frac{Tm}{k}\right)^{1/2}\log T \ll X^2\mathcal{X}_0\mathcal{X}_1 T^{1/2}\log T.$$

For the main term we use the values of Σ_1 and Σ_2 calculated in Lemma 2.1. Substituting we see that the contribution of A_1 to S is

$$S_{A_1} = \left(\frac{T}{2\pi}\left(\log\frac{T}{2\pi}\right)^2 + (2C_0-1)\frac{T}{2\pi}\log\frac{T}{2\pi}\right)\sum_{\substack{m\le X\\ k\le X}}\frac{x_m x_k}{mk}(m,k) + \frac{T}{2\pi}\left(\log\frac{T}{2\pi}\right)\sum_{\substack{m\le X\\ k\le X}}\frac{x_m x_k}{mk}(m,k)\log\frac{m}{k} - 2\frac{T}{2\pi}\left(\log\frac{T}{2\pi}\right)\sum_{\substack{m\le X\\ k\le X}}\frac{x_m x_k}{mk}(m,k)\log\frac{m}{(m,k)},$$

and the contribution of A_2 to S is

$$S_{A_2} = \left(\frac{T}{2\pi}\log\frac{T}{2\pi} + (2C_0-2)\frac{T}{2\pi}\right)\sum_{\substack{m\le X\\ k\le X}}\frac{x_m x_k}{mk}(m,k) + \frac{T}{2\pi}\sum_{\substack{m\le X\\ k\le X}}\frac{x_m x_k}{mk}(m,k)\log\frac{m}{k} - 2\frac{T}{2\pi}\sum_{\substack{m\le X\\ k\le X}}\frac{x_m x_k}{mk}(m,k)\log\frac{m}{(m,k)}.$$

The main term of S is equal to $S_{A_1} - S_{A_2}$, hence

$$S = \left(\frac{T}{2\pi}\left(\log\frac{T}{2\pi}\right)^2 + (2C_0-2)\frac{T}{2\pi}\log\frac{T}{2\pi e}\right)\sum_{\substack{m\le X\\ k\le X}}\frac{x_m x_k}{mk}(m,k) + \frac{T}{2\pi}\left(\log\frac{T}{2\pi e}\right)\sum_{\substack{m\le X\\ k\le X}}\frac{x_m x_k}{mk}(m,k)\log\frac{m}{k}$$

$$- 2\frac{T}{2\pi}\left(\log\frac{T}{2\pi e}\right)\sum_{\substack{m\le X\\ k\le X}}\frac{x_m x_k}{mk}(m,k)\log\frac{m}{(m,k)}$$
$$+ O(X^2\mathcal{X}_0\mathcal{X}_1 T^{1/2}\log T). \ \blacksquare$$

The last two lemmas are variations of Gonek's Lemma:

LEMMA 2.4. *For $m = 0, 1, 2, \dots$, A large and $A < r \le B \le 2A$, uniformly for $a \in (1, 2]$,*

$$\int_A^B \left(\frac{t}{2\pi}\right)^{a-1/2} \exp\left(it\log\left(\frac{t}{er}\right)\right)\left(\log\frac{r}{2\pi}\right)^m dt$$
$$= (2\pi)^{1-a} r^a e^{-ir+\pi i/4}\left(\log\frac{r}{2\pi}\right)^m + O(A^{a-1/2}(\log A)^m).$$

Proof. This is a combination of Lemmas 3 and 4 from [G] (in the original paper the remainder term is not uniform in $a > 1$). ■

LEMMA 2.5. *Suppose the series $f(s) = \sum_{n=1}^{\infty}\alpha_n n^{-s}$ converges absolutely for $\Re s > 1$ and $\sum_{n=1}^{\infty}|\alpha_n|n^{-\sigma} \ll (\sigma-1)^{-\gamma}$ for some $\gamma \ge 0$ as $\sigma \to 1+0$. Next, let $X(s)$ be a Dirichlet polynomial as defined in* (1.3). *Then, uniformly for $a \in (1, 2]$,*

$$J = \frac{1}{2\pi i}\int_{a+i}^{a+iT} f(s)X(s)X(1-s)\frac{\chi'}{\chi}(s)\,ds$$
$$= -\frac{T}{2\pi}\left(\log\frac{T}{2\pi e}\right)\sum_{\substack{m\le X\\ mn\le X}}\frac{\alpha_n x_m x_{mn}}{mn} + O\left(\frac{X^a(\log T)^2\mathcal{X}_0^2}{(a-1)^{\gamma+1}}\right),$$

where $\mathcal{X}_0 = \max_{n\le X}|x_n|$ and the implicit constant is absolute.

Proof. This is a simplified version of Lemma 5 from Kalpokas, Korolev and Steuding [KKS] and a variation of Lemma 5.1 from Ng [N]. ■

3. Asymptotic formulas for $S_0(T)$ and $S_2(T)$. In this section we find asymptotic formulas for the discrete moments $S_0(T)$ and $S_2(T)$ defined by (1.1) and (1.2), respectively.

PROPOSITION 3.1 ([KKS, Proposition 9]). *Let $X(s)$ be a Dirichlet polynomial as defined in* (1.3). *Then for any $\phi \in [0, \pi)$, as $T \to \infty$,*

$$S_0(T) = \frac{T}{2\pi}\left(\log\frac{T}{2\pi e}\right)\sum_{n\le X}\frac{|x_n|^2}{n} + O(R_0), \tag{3.1}$$

where

$$R_0 = X\sqrt{T}(\log T)^2 \sum_{n\le X} \frac{|x_n|^2}{n} + X(\log T)^3 \mathcal{X}_0^2,$$

$\mathcal{X}_0 = \max_{n\le X} |x_n|$ *and* $\mathcal{X}_1 = \sum_{n\le X} |x_n|/n$. *All implicit constants are absolute.*

Proposition 3.2. *Let* $X(s)$ *be a Dirichlet polynomial as defined in* (1.3). *Then for any* $\phi \in [0,\pi)$, *as* $T \to \infty$,

$$\begin{aligned}
(3.2)\quad S_2(T) = {} & \left(\frac{T}{2\pi}\left(\log\frac{T}{2\pi}\right)^2 + (2C_0-2)\frac{T}{2\pi}\log\frac{T}{2\pi e}\right) \sum_{\substack{m\le X\\ k\le X}} \frac{x_m x_k}{mk}(m,k) \\
& + \frac{T}{2\pi}\left(\log\frac{T}{2\pi e}\right) \sum_{\substack{m\le X\\ k\le X}} \frac{x_m x_k}{mk}(m,k)\log\frac{m}{k} \\
& - 2\frac{T}{2\pi}\left(\log\frac{T}{2\pi e}\right) \sum_{\substack{m\le X\\ k\le X}} \frac{x_m x_k}{mk}(m,k)\log\frac{m}{(m,k)} \\
& + 2\cos(2\phi)\frac{T}{2\pi}\left(\log\frac{T}{2\pi e}\right) \sum_{\substack{m\le X\\ mk\le X}} \frac{d(n)x_m x_{mk}}{mk} + O(R_2),
\end{aligned}$$

where

$$\begin{aligned}
R_2 = {} & X\mathcal{X}_1^2 T^{1/2}(\log T)^2 + X\mathcal{X}_0^2(\log T)^4 + \mathcal{X}_0\mathcal{X}_1 X T^{1/2}(\log T)^3 \\
& + X^2\mathcal{X}_0\mathcal{X}_1 T^{1/2}\log T,
\end{aligned}$$

$\mathcal{X}_0 = \max_{n\le X} |x_n|$ *and* $\mathcal{X}_1 = \sum_{n\le X} |x_n|/n$. *All implicit constants are absolute.*

Proof. We follow the proof of [KKS, Proposition 10]. We will use the following formulas (see [I, formula 2.17]):

$$(3.3)\quad \chi(\sigma+it) = \left(\frac{|t|}{2\pi}\right)^{1/2-\sigma-it} \exp(i(t+\pi/4))(1+O(|t|^{-1})) \quad \text{for } |t|\ge 1$$

and

$$(3.4)\quad \frac{\chi'}{\chi}(\sigma+it) = -\log\frac{|t|}{2\pi} + O(|t|^{-1}) \quad \text{for } |t|\ge 1.$$

We begin with the estimations

$$|\zeta(1/2+it)| \ll T^{1/6}, \qquad |X(1/2+it)| \le \sqrt{X}\,\mathcal{X}_1;$$

the first one is a well-known bound from zeta function theory (obtained by using van der Corput's method) and the second is straightforward. It is sufficient to obtain (3.2) for the sum over the interval $c < t_n(\phi) \le T$, where $c > 32\pi$ is a large absolute constant.

Next, without loss of generality we may set $T = \frac{1}{2}(t_\nu(\phi) + t_{\nu+1}(\phi))$. Indeed, otherwise we may replace T by the closest value T_1 of such type. Then the error of such replacement in the right-hand side of (3.2) is bounded by

$$\left(\log\frac{T}{2\pi}\right)^{-1}\left(\log\frac{T}{2\pi}\right)\sum_{\substack{m\le X\\ mn\le X}}\frac{|x_m x_{mn}|}{mn} \ll \mathcal{X}_0^2(\log T)^2,$$

where we have used the asymptotic $\frac{T}{2\pi}\log T + O(T)$ for the number of $t_n(\phi) \le T$ (see [KS, Theorem 1]). Since the points $s = 1/2 + it_n(\phi)$ are the roots of the function $\chi(s) - e^{2i\phi}$, the sum in question can be rewritten as a contour integral:

$$\sum_{c<t_n(\phi)\le T} |\zeta(1/2+it_n(\phi))|^2\,|X(1/2+it_n(\phi))|^2$$
$$= \frac{1}{2\pi i}\int_{\Box}\zeta(s)\zeta(1-s)X(s)X(1-s)\frac{\chi'(s)}{\chi(s)-e^{2i\phi}}\,ds;$$

here $\Box$ stands for the counterclockwise oriented rectangular contour with vertices $a+ic$, $a+iT$, $1-a+iT$, $1-a+ic$, where $a = 1+(\log T)^{-1}$. Let $\mathcal{I}_1$ and $\mathcal{I}_3$ be the integrals over the right and left sides of the contour, and $\mathcal{I}_2$ and $\mathcal{I}_4$ be the integrals over the top and bottom edges. We may assume the constant c is so large that

$$|\chi(a+it)| = \left(\frac{t}{2\pi}\right)^{1/2-a}(1+O(t^{-1})) \le 2\left(\frac{t}{2\pi}\right)^{-1/2} < 1/2$$

for any $t > c$. By (3.3), (3.4) and geometric progression, for $\sigma > 1/2$,

$$\frac{1}{\chi(s)-e^{2i\phi}} = \frac{-e^{-2i\phi}}{1-e^{-2i\phi}\chi(s)} = -e^{-2i\phi}\left(1+\sum_{k=1}^{\infty}e^{-2ki\phi}\chi(s)^k\right). \tag{3.5}$$

In view of (3.5) we have

$$\mathcal{I}_1 = \frac{1}{2\pi i}\int_{a+ic}^{a+iT}\zeta(s)\zeta(1-s)X(s)X(1-s)\frac{\chi'(s)}{\chi(s)-e^{2i\phi}}\,ds$$
$$= -\frac{e^{-2i\phi}}{2\pi i}\int_{a+ic}^{a+iT}\frac{\zeta(s)^2}{\chi(s)}X(s)X(1-s)\chi'(s)\Big(1+\sum_{k=1}^{\infty}e^{-2ik\phi}\chi(s)^k\Big)\,ds$$
$$= -e^{-2i\phi}(j_1+j_2),$$

where

$$j_1 = \frac{1}{2\pi i} \int_{a+ic}^{a+iT} \zeta(s)^2 X(s) X(1-s) \frac{\chi'}{\chi}(s)\, ds,$$

$$j_2 = \frac{1}{2\pi i} \int_{a+ic}^{a+iT} \zeta(s)^2 X(s) X(1-s) \frac{\chi'}{\chi}(s) \sum_{k=1}^{\infty} e^{-2ik\phi} \chi(s)^k\, ds.$$

We observe that for $s = a + it$,

$$|X(a+it)| \le \sum_{n \le X} \frac{|x_n|}{n^a} \le \mathcal{X}_1, \qquad |X(1-a-it)| \le \sum_{m \le X} \frac{m^a |x_m|}{m} \ll X\mathcal{X}_1,$$

$$\left| \sum_{k=1}^{\infty} e^{-2ik\phi} \chi(a+it)^k \right| \le 2\left(\frac{t}{2\pi}\right)^{-1/2} \sum_{k=0}^{\infty} \frac{1}{2^k} \ll t^{-1/2}.$$

Thus, we have

$$|j_2| \ll \zeta(a) X \mathcal{X}_1 \mathcal{X}_1 \int_c^T \frac{\log t\, dt}{\sqrt{t}} \ll X\sqrt{T} (\log T)^2 \mathcal{X}_1^2.$$

Applying Lemma 2.5 to j_1 we get

$$\mathcal{I}_1 = e^{-2i\phi} \frac{T}{2\pi} \left(\log \frac{T}{2\pi e}\right) \sum_{\substack{m \le X \\ mn \le X}} \frac{d(n) x_m x_{mn}}{mn} + O\big(X\sqrt{T}(\log T)^2 \mathcal{X}_1^2 + X(\log T)^4 \mathcal{X}_0^2\big).$$

In a similar way we may compute $\mathcal{I}_3$. First we observe that

$$\mathcal{I}_3 = -\frac{1}{2\pi} \int_c^T \zeta(1-(a-it)) \zeta(a-it) X(1-(a-it)) X(a-it) \times \frac{\chi'(1-(a-it))}{\chi(1-(a-it)) - e^{2i\phi}}\, dt.$$

We define $X_1(s) = \sum_{n \le X} \overline{x_n} n^{-s}$ and take the conjugate of $\mathcal{I}_3$. This in combination with $\overline{X}(s) = X_1(\overline{s})$ yields

$$\overline{\mathcal{I}}_3 = -\frac{1}{2\pi} \int_c^T \zeta(1-(a+it)) \zeta(a+it) X_1(1-(a+it)) X_1(a+it) \times \frac{\chi'(1-(a+it))}{\chi(1-(a+it)) - e^{-2i\phi}}\, dt.$$

Hence

$$\overline{\mathcal{I}}_3 = -\frac{1}{2\pi i} \int_{a+ic}^{a+iT} \zeta(1-s) \zeta(s) X_1(1-s) X_1(s) \frac{\chi'(1-s)}{\chi(1-s) - e^{-2i\phi}}\, ds.$$

In view of $\zeta(1-s)=\zeta(s)\chi(1-s)$, $\chi(1-s)\chi(s)=1$, (3.4) and (3.5) we find that

$$\begin{aligned}
\overline{\mathcal{I}}_3 = -\frac{1}{2\pi i}\int\limits_{a+ic}^{a+iT} \zeta(s)^2\chi(1-s)X_1(1-s)X_1(s)\frac{\chi'}{\chi}(s)& \\
\times\Big(1+\sum_{k=1}^{\infty} e^{-2ik\phi}\chi(s)^k\Big)\,ds& \\
= -\frac{1}{2\pi i}\int\limits_{a+ic}^{a+iT} \zeta(s)^2\chi(1-s)X_1(1-s)X_1(s)\frac{\chi'}{\chi}(s)\,ds& \\
- e^{-2i\phi}\frac{1}{2\pi i}\int\limits_{a+ic}^{a+iT} \zeta(s)^2X_1(1-s)X_1(s)\frac{\chi'}{\chi}(s)\,ds& \\
- \frac{1}{2\pi i}\int\limits_{a+ic}^{a+iT} \zeta(s)^2X_1(1-s)X_1(s)\frac{\chi'}{\chi}(s)\sum_{k=1}^{\infty} e^{-2ik\phi}\chi(s)^k\,ds& \\
= j_3+j_4+j_5.&
\end{aligned}$$

The expressions for j_4 and j_5 can be obtained by replacing $X(s)$ with $X_1(s)$ and $X(1-s)$ with $X_1(1-s)$ in the expressions for j_1 and j_2. Applying Lemma 2.5 to j_4 and estimating j_5 similarly to j_2, we get

$$\begin{aligned}
j_4+j_5 = e^{-2i\phi}\frac{T}{2\pi}\Big(\log\frac{T}{2\pi e}\Big)\sum_{\substack{m\le X\\ mn\le X}}\frac{d(n)\overline{x}_m\overline{x}_{mn}}{mn}& \\
+O\big(X\sqrt{T}(\log T)^2\mathcal{X}_1^2+X(\log T)^4\mathcal{X}_0^2\big).&
\end{aligned}$$

Next we evaluate j_3. By (3.3) and (3.4) we have

$$\begin{aligned}
j_3 &= -\frac{1}{2\pi i}\int\limits_{a+ic}^{a+iT} \zeta(s)^2\chi(1-s)X_1(1-s)X_1(s)\frac{\chi'}{\chi}(s)\,ds \\
&= \sum_{n=1}^{\infty}\frac{d(n)}{n^a}\sum_{m\le X}\frac{\overline{x}_m}{m^{1-a}}\sum_{k\le X}\frac{\overline{x}_k}{k^a}\frac{1}{2\pi}\int\limits_{c}^{T}\chi(1-a-it)\frac{\chi'}{\chi}(a+it)\Big(\frac{m}{kn}\Big)^{it}dt \\
&= \sum_{n=1}^{\infty}\frac{d(n)}{n^a}\sum_{m\le X}\frac{\overline{x}_m}{m^{1-a}}\sum_{k\le X}\frac{\overline{x}_k}{k^a} \\
&\quad\times\frac{e^{\pi i/4}}{2\pi}\int\limits_{c}^{T}\Big(\frac{t}{2\pi}\Big)^{a-1/2}\exp\Big(it\log\frac{tm}{2\pi ekn}\Big)\Big(\log\frac{t}{2\pi}\Big)\Big(\frac{m}{kn}\Big)^{it}dt \\
&\quad+O(\mathcal{X}_0\mathcal{X}_1XT^{1/2}(\log T)^3).
\end{aligned}$$

An application of Gonek's Lemma (Lemma 2.4) to j_3 shows that

$$j_3 = \sum_{m\le X} \frac{\overline{x}_m}{m} \sum_{k\le X} \overline{x}_k \sum_{c<n\le \frac{Tm}{2\pi k}} d(n) e^{-2\pi i \frac{nk}{m}} \log \frac{nk}{m} + O(\mathcal{X}_0\mathcal{X}_1 X T^{1/2} (\log T)^3)$$

$$= \sum_{m\le X} \frac{\overline{x}_m}{m} \sum_{k\le X} \overline{x}_k \sum_{n\le \frac{Tm}{2\pi k}} d(n) e^{-2\pi i \frac{nk}{m}} \log \frac{nk}{m}$$

$$+ O\big(\mathcal{X}_0\mathcal{X}_1 X T^{1/2} (\log T)^3 + X\mathcal{X}_0\mathcal{X}_1 \log X\big).$$

We notice that the sum of j_3 is the same as the sum in Lemma 2.3.

In order to estimate $\mathcal{I}_2$ we first note that the following inequalities hold along the line segment of integration:

$$|\zeta(s)\zeta(1-s)| \ll \sqrt{T}\log T,$$

$$|X(s)| \le \sum_{n\le X} \frac{|x_n|}{n} n^{1-\sigma} \ll X^{1-\sigma}\mathcal{X}_1,$$

$$|X(1-s)| \le \sum_{n\le X} \frac{|x_n|}{n} n^{\sigma} \ll X^{\sigma}\mathcal{X}_1,$$

$$|\zeta(s)\zeta(1-s)X(s)X(1-s)| \ll \mathcal{X}_1^2 X\sqrt{T}\log T.$$

Next, by (3.4) we get

$$\frac{\chi'(s)}{\chi(s)-e^{2i\phi}} = \frac{\chi'(s)}{\chi(s)}\left(1+\frac{e^{2i\phi}}{\chi(s)-e^{2i\phi}}\right) \ll \log T,$$

hence

$$\mathcal{I}_2 \ll X\sqrt{T}(\log T)^2\mathcal{X}_1^2.$$

The integral $\mathcal{I}_4$ can be estimated in a similar way. We refer the reader to [KKS, proof of Proposition 10] for more details. ∎

4. Small values at Gram's points. In order to prove Theorem 1.1 we have to minimize the right-hand side of the inequality

$$\min_{t_n(\phi)\le T} |\zeta(1/2+it_n(\phi))|^2 \le S_2/S_0.$$

Let $L = \sqrt{(1-\delta)\log X \log\log X}$, where X is a sufficiently large parameter and $\delta = \delta(X)$ is a function sufficiently slowly converging to 0 (as $X \to \infty$), to be chosen later. Following Soundararajan [So], we define $x_n = n^{1/2}\mu(n)f(n)$, where f is the multiplicative function such that $f(p^k) = 0$ for all primes p and positive integers $k \ge 2$,

$$f(p) = \frac{L}{\sqrt{p}\log p}$$

for all primes p satisfying $L^2 \le p \le \exp((\log L)^2)$, and $f(p) = 0$ for all other primes. We observe that for the quantities defined by

$$\mathcal{X}_0 = \max_{n \le X} |x_n|, \qquad \mathcal{X}_1 = \sum_{n \le X} \frac{|x_n|}{n}$$

we have

$$\mathcal{X}_0 = \max_{n \le X} \sqrt{n}\, f(n) \le L^m \prod_{j=1}^{m} \frac{1}{\log p_j},$$

where $p_1, \ldots, p_m$ are the least distinct m prime numbers in $[L^2, \exp((\log L)^2)]$ for which $n = p_1 \ldots p_m \le X$. Since $X \ge n \ge L^{2m}$, we have $L^m \le X^{1/2}$ and $\mathcal{X}_0 < L^m \le X^{1/2}$. Moreover, since $f(n) \le 1$ for any n, we find

$$\mathcal{X}_1 = \sum_{n \le X} \frac{f(n)}{\sqrt{n}} \le \sum_{n \le X} \frac{1}{\sqrt{n}} \ll X^{1/2}$$

as well as

$$\mathcal{X}_2 = \sum_{n \le X} \frac{|x_n|^2}{n} = \sum_{n \le X} f(n)^2 = \sum_{\substack{n = p_1 \ldots p_m \le X \\ L^2 < p_1 \ldots p_m \le e^{L^2}}} \frac{L^{2m}}{(p_1(\log p_1) \ldots p_m(\log p_m))^2}$$

$$\le \prod_{L^2 < p \le e^{L^2}} \left(1 + \frac{L^2}{p^2(\log p)^2}\right) < \exp\left(L^2 \sum_{p > L^2} \frac{1}{p^2(\log p)^2}\right) < e.$$

Inserting these bounds into the asymptotic formulas of Propositions 3.1 and 3.2 we get

$$S_0(T) = \frac{T}{2\pi}\left(\log \frac{T}{2\pi e}\right) \sum_{n \le X} |f(n)|^2 + O\big(XT^{1/2}(\log T)^2 + X^2(\log T)^3\big)$$

and

$$\begin{aligned} S_2(T) = {} & \left(\frac{T}{2\pi}\left(\log \frac{T}{2\pi}\right)^2 + (2C_0 - 2)\frac{T}{2\pi} \log \frac{T}{2\pi e}\right) \\ & \times \sum_{\substack{m \le X \\ k \le X}} \frac{\mu(m) f(m) \mu(k) f(k)}{m^{1/2} k^{1/2}} (m,k) \\ (*) \qquad & + \frac{T}{2\pi}\left(\log \frac{T}{2\pi e}\right) \sum_{\substack{m \le X \\ k \le X}} \frac{\mu(m) f(m) \mu(k) f(k)}{m^{1/2} k^{1/2}} (m,k) \log \frac{m}{k} \\ & - 2\frac{T}{2\pi}\left(\log \frac{T}{2\pi e}\right) \sum_{\substack{m \le X \\ k \le X}} \frac{\mu(m) f(m) \mu(k) f(k)}{m^{1/2} k^{1/2}} (m,k) \log \frac{m}{(m,k)} \end{aligned}$$

$$+ 2\cos(2\phi)\frac{T}{2\pi}\left(\log\frac{T}{2\pi e}\right)\sum_{\substack{m\le X\\ mk\le X}}\frac{d(k)\mu(m)f(m)\mu(mk)f(mk)}{k^{1/2}}$$
$$+ O\big(X^3T^{1/2}\log T + X^2T^{1/2}(\log T)^3\big).$$

We choose $X = T^{1/6-\epsilon}$ and apply the following estimates (note that $(*)$ is equal to zero by symmetry):

LEMMA 4.1.

$$\sum_{\substack{m\le X\\ k\le X}}\frac{\mu(m)f(m)\mu(k)f(k)}{m^{1/2}k^{1/2}}(m,k) = (1+o(1))\prod_p\left(1+f(p)^2-2\frac{f(p)}{p^{1/2}}\right), \tag{4.1}$$

$$\left|\sum_{\substack{m\le X\\ k\le X}}\frac{\mu(m)f(m)\mu(k)f(k)}{m^{1/2}k^{1/2}}(m,k)\log\frac{m}{(m,k)}\right| \ll \prod_p\left(1+f(p)^2-2\frac{f(p)}{p^{1/2}}\right)(\log X)^{1/2+\varepsilon}, \tag{4.2}$$

$$\sum_{\substack{m\le X\\ mk\le X}}\frac{d(k)\mu(m)f(m)\mu(mk)f(mk)}{k^{1/2}} = (1+o(1))\prod_p\left(1+f(p)^2-2\frac{f(p)}{p^{1/2}}\right), \tag{4.3}$$

$$\sum_{m\le X} f(m)^2 = (1+o(1))\prod_p(1+f(p)^2). \tag{4.4}$$

Substituting these estimates we get

$$S_2(T) \ll T(\log T)^2\prod_p\left(1+f(p)^2-2\frac{f(p)}{p^{1/2}}\right),$$
$$S_0(T) \gg T(\log T)\prod_p(1+f(p)^2),$$

and so

$$\frac{S_2}{S_0} \ll \exp\left(\log\log T - 2\sum_p\frac{f(p)}{p^{1/2}}\right) = \exp\left(-(2+o(1))\sqrt{\frac{\log X}{\log\log X}}\right),$$

proving Theorem 1.1.

We now prove each inequality of Lemma 4.1 separately.

Proof of (4.1). Summing over $b=(m,k)$ we get

$$\sum_{k,m\le X}\frac{\mu(m)f(m)\mu(k)f(k)}{m^{1/2}k^{1/2}}(m,k)=\sum_{b\le X}\sum_{\substack{m,k\le X/b\\(m,k)=1\\(b,mk)=1}}\frac{f(b)^2\mu(m)f(m)\mu(k)f(k)}{m^{1/2}k^{1/2}}$$
$$=\sum_{b\le X}f(b)^2\sum_{(l,b)=1}\frac{\mu(l)f(l)d(l)}{l^{1/2}}+E_1$$
$$=\sum_{b}f(b)^2\sum_{(l,b)=1}\frac{\mu(l)f(l)d(l)}{l^{1/2}}+E_2+E_1$$
$$=\prod_p\Big(1+f(p)^2-2\frac{f(p)}{p^{1/2}}\Big)+E_2+E_1.$$

The error terms E_1 and E_2 can be bounded in absolute value using Ranking's trick (see [So, p. 5]):

$$|E_1|\ll\sum_b f(b)^2\sum_{\substack{(l,b)=1\\l>X/b}}\frac{f(l)d(l)}{l^{1/2}}$$
$$\le\sum_b f(b)^2\Big(\frac{b}{X}\Big)^{\alpha}\sum_{(l,b)=1}\frac{l^{\alpha}f(l)d(l)}{l^{1/2}}$$
$$=\frac{1}{X^{\alpha}}\prod_p(1+p^{\alpha}f(p)^2+2f(p)p^{\alpha-1/2}),$$
$$|E_2|=\Big|\sum_{b>X}f(b)^2\sum_{(l,b)=1}\frac{\mu(l)f(l)d(l)}{l^{1/2}}\Big|$$
$$\le\frac{1}{X^{\alpha}}\prod_p(1+p^{\alpha}f^2(p)+2f(p)p^{\alpha-1/2}).$$

It remains to show that the error terms are smaller than the main term. For that we take $\alpha=1/(\log L)^3$ as in [So, proof of Lemma 6] and estimate the logarithm of the error term divided by the main term:

$$\log\Big(\frac{1}{X^{\alpha}}\prod_p(1+p^{\alpha}f(p)^2+2f(p)p^{\alpha-1/2})\Big)-\log\Big(\prod_p\Big(1+f(p)^2-2\frac{f(p)}{p^{1/2}}\Big)\Big)\tag{4.5}$$
$$\le-\alpha\log X+\sum_p(p^{\alpha}-1)f(p)^2+\sum_p 4f(p)p^{\alpha-1/2}+O\Big(\sum_p p^{2\alpha}f(p)^4\Big).$$

Using the Prime Number Theorem and Abel's summation we get

$$\sum_p (p^\alpha - 1) f(p)^2 = (1+o(1)) \frac{L^2}{2(\log L)^4},$$

$$\sum_p f(p) p^{\alpha - 1/2} \ll \frac{L}{\log L},$$

$$\sum_p p^{2\alpha} f(p)^4 \ll \frac{L^2}{(\log L)^5}.$$

The largest of the sums $\sum_p (p^\alpha - 1) f(p)^2$ is equal to $(1 - \delta + o(1))\alpha \log X$, so we can make (4.5) diverge to $-\infty$ by taking δ to be sufficiently slowly convergent to 0. ∎

Proof of (4.2). We rewrite the sum similarly to the proof above:

$$\begin{aligned}
\sum_{m,k \le X} & \frac{\mu(m) f(m) \mu(k) f(k)}{m^{1/2} k^{1/2}} (m,k) \log \frac{m}{(m,k)} \\
&= \sum_{b \le X} f(b)^2 \sum_{\substack{m,k \le X/b \\ (m,k)=1 \\ (b,mk)=1}} \frac{\mu(m) f(m) \mu(k) f(k)}{m^{1/2} k^{1/2}} \sum_{p \mid m} \log p \\
&= \sum_{p \le X} (\log p) \sum_{\substack{b \le X \\ p \nmid b}} f(b)^2 \sum_{\substack{(m,k)=1 \\ (b,mk)=1 \\ p \mid m}} \frac{\mu(m) f(m) \mu(k) f(k)}{m^{1/2} k^{1/2}} + E_1 \\
&= \sum_{p \le X} \frac{f(p) \log p}{p^{1/2}} \sum_{\substack{b \le X \\ p \nmid b}} f(b)^2 \prod_{q \nmid pb} \left(1 - 2\frac{f(q)}{q^{1/2}}\right) + E_1.
\end{aligned}$$

Since for $q > L^2$ we have $1 > 2f(q)/q^{1/2}$, the above summands are positive and the sum increases if we extend the summation range over all b:

$$\le \sum_{p \le X} \frac{f(p) \log p}{p^{1/2}} \prod_q \left(1 + f(q)^2 - \frac{2f(q)}{q^{1/2}}\right) + E_1.$$

Note that

$$\sum_{p \le X} \frac{f(p) \log p}{p^{1/2}} \le L \sum_{p \le X} \frac{1}{p} \ll (\log X)^{1/2+\varepsilon},$$

so it remains to estimate the error term:

$$|E_1| = \left| \sum_{p \le X} \frac{f(p) \log p}{p^{1/2}} \sum_{\substack{b \le X \\ p \nmid b}} f(b)^2 \sum_{\substack{k > X/b \text{ or } m > X/bp \\ (m,k)=1 \\ (mk,pb)=1}} \frac{\mu(m) f(m) \mu(k) f(k)}{m^{1/2} k^{1/2}} \right|$$

$$\ll \sum_{p\le X} \frac{f(p)\log p}{p^{1/2}} \sum_b f(b)^2 \sum_{l>X/bp} \frac{f(l)d(l)}{l^{1/2}}$$
$$\ll \sum_{p\le X} \frac{p^\alpha f(p)\log p}{p^{1/2}} \frac{1}{X^\alpha} \prod_q (1+q^\alpha f(q)^2 + 2f(q)q^{\alpha-1/2}).$$

Note that for $p \le \exp((\log L)^2)$ we have $p^\alpha \ll 1$, so the ratio between the error term and the main term is essentially the same as in the proof of (4.1) and tends to 0. ∎

Proof of (4.3). Note that $f(mk)\neq 0 \;\Rightarrow\; \mu(mk)=\mu(m)\mu(k)$, so the sum in question is equal to

$$\sum_{mk\le X} \frac{d(k)f(m)\mu(k)f(mk)}{k^{1/2}} = \sum_{m\le X} f(m)^2 \sum_{(k,m)=1} \frac{d(k)\mu(k)f(k)}{k^{1/2}} + E_1$$
$$= \prod_p \Big(1+f(p)^2 - 2\frac{f(p)}{p^{1/2}}\Big) + E_2 + E_1,$$

and the error terms can be bounded in the same way as in the proof of (4.1). ∎

Proof of (4.4). Using Rankin's trick we write

$$\sum_{n\le X} f(n)^2 = \prod_p (1+f(p)^2) + O\Big(\frac{1}{X^\alpha}\prod_p (1+f(p)^2p^\alpha)\Big).$$

The logarithm of the ratio of the error term to the main term is equal to

$$-\alpha\log X + \sum_p (p^\alpha - 1)f(p)^2 + O\Big(\sum_p p^{2\alpha}f(p)^4\Big),$$

which again tends to $-\infty$ as in the proof of (4.1). ∎

5. Large values at Gram's points. The proof of Theorem 1.2 follows the lines of the proof of Theorem 1.1, so we only give a short outline here. We use the following inequality to get a lower bound for large values of the zeta function:

$$\max_{t_n(\phi)\ge T} |\zeta(1/2+it_n(\phi))|^2 \ge S_2/S_0.$$

The expressions of S_2 and S_0 are the same, but instead of $x_n = n^{1/2}\mu(n)f(n)$ we take $x_n = n^{1/2}f(n)$ and get the following equivalent to Lemma 4.1:

LEMMA 5.1.

$$\sum_{\substack{m\le X\\ k\le X}} \frac{f(m)f(k)}{m^{1/2}k^{1/2}}(m,k) = (1+o(1))\prod_p \Big(1+f(p)^2 + 2\frac{f(p)}{p^{1/2}}\Big),$$

$$\sum_{\substack{m\le X\\ k\le X}} \frac{f(m)f(k)}{m^{1/2}k^{1/2}}(m,k)\log\biggl(\frac{(m,k)}{k}\biggr) \ll \prod_p\biggl(1+f(p)^2+2\frac{f(p)}{p^{1/2}}\biggr)(\log X)^{1/2+\varepsilon},$$

$$\sum_{\substack{m\le X\\ mk\le X}} \frac{d(k)f(m)f(mk)}{k^{1/2}} = (1+o(1))\prod_p\biggl(1+f(p)^2+2\frac{f(p)}{p^{1/2}}\biggr).$$

Substituting these estimates we get

$$S_2(T) \gg T(\log T)^2\prod_p\biggl(1+f(p)^2+2\frac{f(p)}{p^{1/2}}\biggr),$$
$$S_0(T) \ll \prod_p(1+f(p)^2),$$

and so

$$\frac{S_2}{S_0} \gg \exp\biggl(\log\log T + 2\sum_p \frac{f(p)}{p^{1/2}}\biggr) = \exp\biggl((2+o(1))\sqrt{\frac{\log X}{\log\log X}}\biggr).$$

6. Proof of Theorem 1.3. We prove Theorem 1.3 by estimating the third and sixth discrete moments of the Riemann zeta function and using the Cauchy–Schwarz inequality:

$$\sum_{t_n^{\pm}(\phi)\le T} 1 \ge \frac{(\sum_{t_n^{\pm}(\phi)\le T}|\zeta(1/2+it_n^{\pm}(\phi))|^3)^2}{\sum_{t_n(\phi)\le T}|\zeta(1/2+it_n(\phi))|^6}.$$

From [KS] we have

$$\sum_{t_n^{\pm}(\phi)\le T} |\zeta(1/2+it_n^{\pm}(\phi))|^3 \gg T(\log T)^{13/4},$$

and from [CK] under the Riemann Hypothesis we have

$$\sum_{t_n(\phi)\le T} |\zeta(1/2+it_n(\phi))|^6 \ll T(\log T)^{10+\epsilon},$$

hence

$$\#\{t_n^{\pm}(\phi) : t_n^{\pm}(\phi)\in[0,T]\} \gg T(\log T)^{-7/2-\epsilon},$$

as required.

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