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The minimal resultant locus

by

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1. Introduction. Let K be a complete, algebraically closed nonarchimedean valued field with absolute value $|\cdot|$ and associated valuation $\mathrm{ord}(\cdot)$. Write $\mathcal{O}$ for the ring of integers of K, $\mathfrak{M}$ for its maximal ideal, and $\widetilde{k} = \mathcal{O}/\mathfrak{M}$ for its residue field.

Let $\varphi(z) \in K(z)$ be a rational function with $\deg(\varphi) = d \geq 2$. There are homogeneous polynomials $F(X,Y), G(X,Y) \in K[X,Y]$ of degree d, having no common factor, such that the map $[X:Y] \mapsto [F(X,Y) : G(X,Y)]$ gives the action of φ on $\mathbb{P}^1$. After scaling F and G, one can arrange that F and G belong to $\mathcal{O}[X,Y]$ and that at least one of their coefficients is a unit in $\mathcal{O}$. Such a pair (F,G) is called a *normalized representation* of φ; it is unique up to scaling by a unit in $\mathcal{O}$. Write $F(X,Y) = f_d X^d + f_{d-1}X^{d-1}Y + \cdots + f_0 Y^d$ and $G(X,Y) = g_d X^d + g_{d-1}X^{d-1}Y + \cdots + g_0 Y^d$. Then the resultant of F and G is

$$\mathrm{Res}(F,G) = \det\left(\begin{bmatrix} f_d & f_{d-1} & \cdots & f_1 & f_0 & & & \\ & f_d & f_{d-1} & \cdots & f_1 & f_0 & & \\ & & & & \vdots & & & \\ & & & f_d & f_{d-1} & \cdots & f_1 & f_0 \\ g_d & g_{d-1} & \cdots & g_1 & g_0 & & & \\ & g_d & g_{d-1} & \cdots & g_1 & g_0 & & \\ & & & & \vdots & & & \\ & & & g_d & g_{d-1} & \cdots & g_1 & g_0 \end{bmatrix}\right). \tag{1.1}$$

Its ord value

$$\mathrm{ordRes}(\varphi) := \mathrm{ord}(\mathrm{Res}(F,G)) \tag{1.2}$$

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is independent of the choice of normalized representation. By construction, it is nonnegative.

If $\gamma \in \mathrm{GL}_2(K)$, the conjugate $\varphi^\gamma = \gamma^{-1} \circ \varphi \circ \gamma$ has its own normalized representation (F_γ, G_γ), and $\mathrm{ordRes}(\varphi^\gamma) := \mathrm{ord}(\mathrm{Res}(F_\gamma, G_\gamma))$ is in general different from $\mathrm{ordRes}(\varphi)$. In this paper we investigate how $\mathrm{ordRes}(\varphi^\gamma)$ changes as γ varies over $\mathrm{GL}_2(K)$. We show that the map $\gamma \mapsto \mathrm{ordRes}(\varphi^\gamma)$ factors through a function $\mathrm{ordRes}_\varphi(\cdot)$ on the Berkovich projective line $\mathbf{P}^1_K$,

$$\mathrm{ordRes}_\varphi : \mathbf{P}^1_K \to [0, \infty],$$

which takes on a minimum value and is continuous, piecewise affine, and convex up on each path in $\mathbf{P}^1_K$. Let the *minimal resultant locus* be the set $\mathrm{MinResLoc}(\varphi)$ where $\mathrm{ordRes}_\varphi(\cdot)$ achieves its minimum. We show that $\mathrm{MinResLoc}(\varphi)$ is either a point or a segment, and that it can be a segment only when $d = \deg(\varphi)$ is odd and φ does not have potential good reduction (see below).

If $H \subset K$ is the minimal field of definition for φ, we obtain an a priori bound of $(d+1)^2$ for the degree of an extension L/H such that there is a $\gamma \in \mathrm{GL}_2(L)$ for which $\mathrm{ordRes}(\varphi^\gamma)$ is minimal. We give an algorithm for computing $\mathrm{MinResLoc}(\varphi)$ which determines the minimal value of $\mathrm{ordRes}_\varphi(\cdot)$, finds a γ for which φ^γ has minimal resultant, and decides whether φ has potential good reduction.

In particular, there always exists a $\gamma \in \mathrm{GL}_2(K)$ for which φ^γ has minimal resultant. This was previously shown by Szpiro, Tepper, and Williams [16], using a moduli-theoretic argument building on work by Levy [10], [11].

Recall that the *reduction* $\widetilde{\varphi}$ is the map $[\widetilde{X} : \widetilde{Y}] \mapsto [\widetilde{F}(\widetilde{X}, \widetilde{Y}) : \widetilde{G}(\widetilde{X}, \widetilde{Y})]$ on $\mathbb{P}^1(\widetilde{k})$ obtained by reducing F and G modulo $\mathfrak{M}$ and eliminating common factors. If $\widetilde{\varphi}$ has degree d, then φ is said to have *good reduction*. Likewise, φ is said to have *potential good reduction* if after a change of coordinates by some $\gamma \in \mathrm{GL}_2(K)$, the map φ^γ has good reduction. It is well known that φ^γ has good reduction if and only if $\mathrm{ordRes}(\varphi^\gamma) = 0$.

Our algorithm (Algorithm A in §5) decides when φ has potential good reduction over the algebraically closed field K. Recently, Benedetto [2] found a faster algorithm which uses different ideas. In [3], he improved the degree bound $(d+1)^2$ given here. When φ is defined over a local field H_v, Bruin and Molnar [6] earlier gave an algorithm which determines whether φ has potential good reduction after conjugation by some $\gamma \in \mathrm{GL}_2(H_v)$.

Recall $\mathbf{P}^1_K$ is a path-connected Hausdorff space containing $\mathbb{P}^1(K)$. By Berkovich's classification theorem (see for example [1, p. 5], points in $\mathbf{P}^1_K$ can be viewed as corresponding to discs in K. There are four kinds of points: type I points are the points of $\mathbb{P}^1(K)$, which we regard as discs of radius 0. Type II and III points correspond to discs $D(a, r) = \{z \in K : |z - a| \le r\}$, with type II points corresponding to discs $D(a, r)$ with r in the value group $|K^\times|$, and type III points corresponding to those with $r \notin |K^\times|$. We write $\zeta_{a,r}$ for the point corresponding to $D(a,r)$. The point $\zeta_G := \zeta_{0,1}$ corresponding to $D(0,1)$ is called the *Gauss point*. Type IV points serve to complete $\mathbf{P}^1_K$; they correspond to (cofinal equivalence classes of) sequences of nested discs with empty intersection. Paths in $\mathbf{P}^1_K$ correspond to ascending or descending chains of discs, or unions of chains sharing an endpoint. For example the path from 0 to 1 in $\mathbf{P}^1_K$ corresponds to the chains $\{D(0,r) : 0 \le r \le 1\}$ and $\{D(1,r) : 1 \ge r \ge 0\}$; here $D(0,1) = D(1,1)$. Topologically, $\mathbf{P}^1_K$ is a tree: there is a unique path $[x,y]$ between any two points $x, y \in \mathbf{P}^1_K$.

The set $\mathbf{H}^1_K = \mathbf{P}^1_K \backslash \mathbb{P}^1(K)$ is called the *Berkovich upper halfspace*; it carries a metric $\rho(x,y)$ called the *logarithmic path distance*, for which the length of the path corresponding to $\{D(a,r) : R_1 \le r \le R_2\}$ is $\log(R_2/R_1)$. There are two natural topologies on $\mathbf{P}^1_K$, called the *weak* and *strong* topologies. The weak topology on $\mathbf{P}^1_K$ is the coarsest one which makes the evaluation functionals $z \mapsto |f(z)|$ continuous for all $f(z) \in K(z)$; under the weak topology, $\mathbf{P}^1_K$ is compact and $\mathbb{P}^1(K)$ is dense in it. The basic open sets for the weak topology are the path-components of $\mathbf{P}^1_K \backslash \{P_1, \ldots, P_n\}$ as $\{P_1, \ldots, P_n\}$ ranges over finite subsets of $\mathbf{H}^1_K$. The strong topology on $\mathbf{P}^1_K$ (which is finer than the weak topology) restricts to the topology on $\mathbf{H}^1_K$ induced by $\rho(x,y)$. The basic open sets for the strong topology are the $\rho(x,y)$-balls in $\mathbf{H}^1_K$, together with the basic open sets from the weak topology. Type II points are dense in $\mathbf{P}^1_K$ for both topologies. The action of φ on $\mathbb{P}^1(K)$ extends functorially to an action on $\mathbf{P}^1_K$ which is continuous for both topologies, and takes points of a given type to points of the same type. Similarly, the action of $\mathrm{GL}_2(K)$ on $\mathbb{P}^1(K)$ extends to an action on $\mathbf{P}^1_K$ which is continuous on $\mathrm{GL}_2(K) \times \mathbf{P}^1_K$ for both topologies, and preserves the type of each point. The action of $\mathrm{GL}_2(K)$ preserves the logarithmic path distance: $\rho(\gamma(x), \gamma(y)) = \rho(x,y)$ for all $x, y \in \mathbf{H}^1_K$ and $\gamma \in \mathrm{GL}_2(K)$. For these facts, see [1]; for additional results about dynamics on $\mathbf{P}^1_K$, see [4], [5], [7], [8], [9], and [12].

Our starting point is the following observation: by standard formulas for the resultant (see for example Silverman [15, Ex. 2.7, p. 75]), for each $\gamma \in \mathrm{GL}_2(K)$ and each $\tau \in K^\times \cdot \mathrm{GL}_2(\mathcal{O})$ we have

$$\mathrm{ordRes}(\varphi^\gamma) = \mathrm{ordRes}(\varphi^{\gamma\tau}).$$

On the other hand, $\mathrm{GL}_2(K)$ acts transitively on type II points, and $K^\times \cdot \mathrm{GL}_2(\mathcal{O})$ is the stabilizer of the Gauss point. Thus the function $\mathrm{ordRes}_\varphi(\cdot)$ on type II points in $\mathbf{P}^1_K$, given by

$$\mathrm{ordRes}_\varphi(\gamma(\zeta_G)) := \mathrm{ordRes}(\varphi^\gamma), \tag{1.3}$$

is well-defined.

Our first main result is

Theorem 1.1. *Let K be a complete, algebraically closed, nonarchimedean valued field. Suppose $\varphi(z) \in K(z)$ has degree $d = \deg(\varphi) \geq 2$. Then the function* $\mathrm{ordRes}_\varphi(\cdot)$ *on type* II *points in $\mathbf{P}^1_K$ extends uniquely to a function* $\mathrm{ordRes}_\varphi : \mathbf{P}^1_K \to [0,\infty]$ *which is finite on $\mathbf{H}^1_K$, takes the value ∞ on $\mathbb{P}^1(K)$, and is continuous with respect to the strong topology. On each path in $\mathbf{P}^1_K$, it is piecewise affine and convex upwards with respect to the logarithmic path distance. It achieves a minimum on $\mathbf{P}^1_K$.*

The set $\mathrm{MinResLoc}(\varphi)$ *where* $\mathrm{ordRes}_\varphi(\cdot)$ *takes on its minimum lies in the ball*

$$\left\{z \in \mathbf{H}^1_K : \rho(\zeta_G, z) \leq \frac{2}{d-1}\,\mathrm{ordRes}(\varphi)\right\}.$$

For each $a \in \mathbb{P}^1(K)$, $\mathrm{MinResLoc}(\varphi)$ *is contained in the tree $\Gamma_{\mathrm{Fix},\varphi^{-1}(a)} \subset \mathbf{P}^1_K$ spanned by the classical fixed points of φ and the preimages of a under φ. It consists of a single type* II *point if d is even, and is a single type* II *point or a segment with type* II *endpoints if d is odd. If the minimum value of* $\mathrm{ordRes}_\varphi(\cdot)$ *is* 0 *(that is, if φ has potential good reduction), then* $\mathrm{MinResLoc}(\varphi)$ *consists of a single point.*

The proof of Theorem 1.1 shows that each affine segment of $\mathrm{ordRes}_\varphi(\cdot)$ has an integer slope $m \equiv d^2 + d \pmod{2d}$ with $-d^2 - d \leq m \leq d^2 + d$, and that breaks between affine segments occur at type II points. The theory has the following applications:

(1) The function $\mathrm{ordRes}_\varphi(\cdot)$ satisfies the principle of steepest descent. This means that the Bruin–Molnar algorithm [6], which is implemented as a recursive search, runs without back-tracking.

(2) If φ is defined over a subfield $H \subset K$, there is an extension L/H with degree $[L : H] \leq (d + 1)^2$ such that $\mathrm{ordRes}(\varphi^\gamma)$ is minimal for some $\gamma \in \mathrm{GL}_2(L)$ (Theorem 3.2). In particular, if φ has potential good reduction, it achieves good reduction over an extension of degree at most $(d + 1)^2$. It follows that if H is Henselian (especially, if H is complete), then the statement "φ has potential good reduction" is first-order in the theory of H, in the sense of mathematical logic.

(3) Suppose H is a number field. An elliptic curve E/H has a global minimal model over H if and only if a certain class $[\mathfrak{a}_E]$ in the ideal class group of $\mathcal{O}_H$, the *Weierstrass class*, is principal. In dynamics, when $\varphi(z)$ is in $H(z)$ and $\deg(\varphi) \geq 2$, Silverman has constructed an analogous ideal class $[\mathfrak{a}_\varphi]$ such that if φ has global minimal model over H, then $[\mathfrak{a}_\varphi]$ is trivial (see [15, Proposition 4.99]). He asks if the converse holds. We show that it can fail by providing examples of number fields H and polynomials $\varphi(z) \in H[z]$ for which $[\mathfrak{a}_\varphi]$ is trivial but φ has no global minimal model over H.

(4) If φ is defined over a subfield $H \subset K$, and φ has potential good reduction, let H_φ be the intersection of all fields L with $H \subset L \subset K$ such

that φ^γ has good reduction for some $\gamma \in \mathrm{GL}_2(L)$ (the "field of moduli for the good reduction problem"). We give examples where $H_\varphi = H$ but φ^γ does not have good reduction for any $\gamma \in \mathrm{GL}_2(H)$. Thus there need not be a unique minimal extension L/H where φ achieves good reduction.

(5) When d is odd and φ does not have potential good reduction, we give examples where the minimal resultant locus is a segment of positive length (Examples 6.5 and 6.7). This means there can be fundamentally different coordinate changes (that is, coordinate changes by γ's belonging to different cosets of $K^\times \cdot \mathrm{GL}_2(\mathcal{O})$) for which φ^γ has minimal resultant. This corresponds to the fact that in moduli theory, when d is odd, φ can be semistable in the sense of geometric invariant theory, without being stable.

Our second main result concerns the stability, under perturbations of φ, of $\mathrm{ordRes}_\varphi(\cdot)$ and $\mathrm{MinResLoc}(\varphi)$:

THEOREM 1.2. *Let K be a complete, algebraically closed, nonarchimedean valued field. Suppose $\varphi(z), \widehat{\varphi}(z) \in K(z)$ have degree $d \geq 2$, with normalized representations $(F, G), (\widehat{F}, \widehat{G})$ respectively. Write $R = \mathrm{ordRes}(\varphi)$, and let $M > 0$ be arbitrary. If*

$$\min(\mathrm{ord}(\widehat{F} - F), \mathrm{ord}(\widehat{G} - G)) > \max\left(R, \frac{1}{2d}(R + (d^2 + d)M)\right), \tag{1.4}$$

then $\mathrm{ordRes}_{\widehat{\varphi}}(\xi) = \mathrm{ordRes}_\varphi(\xi)$ *for all ξ with $\rho(\zeta_G, \xi) \leq M$. Let*

$$f(d) = \frac{2d^2 + 3d - 1}{2d^2 - 2d},$$

so $f(2) = 3.25$, $f(3) = 2.166\ldots$, and $1 < f(d) < 2$ for $d \geq 4$. If

$$\min(\mathrm{ord}(\widehat{F} - F), \mathrm{ord}(\widehat{G} - G)) > f(d) \cdot \mathrm{ordRes}(\varphi), \tag{1.5}$$

then $\mathrm{MinResLoc}(\widehat{\varphi}) = \mathrm{MinResLoc}(\varphi)$, *and* $\mathrm{ordRes}_{\widehat{\varphi}}(\xi) = \mathrm{ordRes}_\varphi(\xi)$ *for all ξ with $\rho(\zeta_G, \xi) \leq \frac{2}{d-1}$* $\mathrm{ordRes}(\varphi)$.

The structure of the paper is as follows. In Section 2 we prove Theorems 1.1 and 1.2. In Section 3 we study $\mathrm{MinResLoc}(\varphi)$ from a Galois-theoretic viewpoint. In Section 4 we give applications of the theory. In Section 5 we present Algorithm A. In Section 6 we provide examples illustrating some of the possible geometric and dynamical behaviors of $\mathrm{MinResLoc}(\varphi)$. Finally, in Section 7 we prove an analogue of Theorem 1.1 when $d = 1$.

2. Proof of the main theorems. In this section we establish Theorems 1.1 and 1.2. Suppose $\varphi(z) \in K(z)$ has degree d. Then

$$\varphi(z) = \frac{F(z, 1)}{G(z, 1)}$$

where $F(X,Y) = f_d X^d + f_{d-1}X^{d-1}Y + \cdots + f_0 Y^d$ and $G(X,Y) = g_d X^d + g_{d-1}X^{d-1}Y + \cdots + g_0 Y^d$ are homogeneous polynomials in $K[X,Y]$ of degree d with no common factor. The pair (F,G) is called a *representation* of φ; it is unique up to scaling by a nonzero constant. Set $\mathrm{ord}(F) = \min_{0\le i\le d}(\mathrm{ord}(f_i))$ and $\mathrm{ord}(G) = \min_{0\le i\le d}(\mathrm{ord}(g_i))$.

The resultant of F and G is defined by the $2d\times 2d$ determinant in formula (1.1). For any $c \in K^\times$, we have $\mathrm{Res}(cF, cG) = c^{2d}\mathrm{Res}(F,G)$. By choosing c so that $\mathrm{ord}(c) = \min(\mathrm{ord}(F), \mathrm{ord}(G))$ and replacing (F,G) by $(c^{-1}F, c^{-1}G)$ we can assume that

$$\min(\mathrm{ord}(F), \mathrm{ord}(G)) = 0;$$

in this case (F,G) is called a *normalized representation* of φ, and $\mathrm{ordRes}(\varphi)$ is defined to be $\mathrm{ord}(\mathrm{Res}(F,G))$ as in (1.2). Clearly $\mathrm{ordRes}(\varphi)$ is independent of the choice of normalized representation, and $\mathrm{ordRes}(\varphi) \ge 0$.

Whether or not (F,G) is normalized, we have the formula

$$\mathrm{ordRes}(\varphi) = \mathrm{ord}(\mathrm{Res}(F,G)) - 2d\min(\mathrm{ord}(F), \mathrm{ord}(G)). \tag{2.1}$$

Given $\gamma = \left[\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right] \in \mathrm{GL}_2(K)$, let $\mathrm{Adj}(\gamma) = \left[\begin{smallmatrix} D & -B \\ -C & A\end{smallmatrix}\right]$ and define (F^γ, G^γ) by

$$\begin{aligned}\begin{bmatrix} F^\gamma(X,Y) \\ G^\gamma(X,Y)\end{bmatrix} &= \mathrm{Adj}(\gamma)\circ \begin{bmatrix} F \\ G\end{bmatrix} \circ \gamma \begin{bmatrix} X \\ Y\end{bmatrix} \\ &= \begin{bmatrix} DF(AX+BY, CX+DY) - BG(AX+BY, CX+DY) \\ -CF(AX+BY, CX+DY) + AG(AX+BY, CX+DY)\end{bmatrix}.\end{aligned} \tag{2.2}$$

Then (F^γ, G^γ) is a homogeneous representation of φ^γ. It is known (see [15, Ex. 2.7(c), p. 76]) that $\mathrm{Res}(F^\gamma, G^\gamma) = \mathrm{Res}(F,G)\cdot \det(\gamma)^{d^2+d}$, so

$$\begin{aligned}&\mathrm{ordRes}(\varphi^\gamma) \\ &\quad = \mathrm{ordRes}(F,G) + (d^2+d)\,\mathrm{ord}(\det(\gamma)) - 2d\min(\mathrm{ord}(F^\gamma), \mathrm{ord}(G^\gamma)).\end{aligned} \tag{2.3}$$

We will prove Theorems 1.1 and 1.2 after a series of preliminary results. In Theorem 1.1 it is assumed that $d \ge 2$; however, for use in §7, we will develop the theory for $d \ge 1$, and make explicit where $d \ge 2$ is used.

We begin by recalling some facts about the action of $\mathrm{GL}_2(K)$ on $\mathbf{P}^1_K$:

Proposition 2.1. *The natural action of* $\mathrm{GL}_2(K)$ *on* $\mathbb{P}^1(K)$ *extends to an action on* $\mathbf{P}^1_K$ *such that:*

- (A) *The stabilizer of* ζ_G *in* $\mathrm{GL}_2(K)$ *is* $K^\times \cdot \mathrm{GL}_2(\mathcal{O})$.
- (B) *For each* $\gamma \in \mathrm{GL}_2(K)$, $\rho(\gamma(x), \gamma(y)) = \rho(x,y)$ *for all* $x, y \in \mathbf{H}^1_K$.
- (C) *For each* $\gamma \in \mathrm{GL}_2(K)$ *and each path* $[x,y]$, $\gamma([x,y]) = [\gamma(x), \gamma(y)]$.
- (D) *For any triple* (a_0, A, a_1) *where* $a_0, a_1 \in \mathbb{P}^1(K)$, $a_0 \ne a_1$, *and* A *is a type* II *point in* $[a_0, a_1]$, *if* (b_0, B, b_1) *is another triple of the same kind, there is a* $\gamma \in \mathrm{GL}_2(K)$ *such that* $\gamma(a_0) = b_0$, $\gamma(A) = B$, *and* $\gamma(a_1) = b_1$. *In particular,* $\mathrm{GL}_2(K)$ *acts transitively on the type* II *points in* $\mathbf{P}^1_K$.

Proof. As discussed in [1, §2.3], the natural action of any rational function $f(z) \in K(z)$ on $\mathbb{P}^1(K)$ extends uniquely to a continuous action on $\mathbf{P}^1_K$. For part (A), suppose $\gamma \in \mathrm{GL}_2(K)$ stabilizes ζ_G, and let $\gamma(0) = a$, $\gamma(1) = b$, $\gamma(\infty) = c$. By [1, Lemma 2.17], γ has nonconstant reduction, so the reductions $\widetilde{a}$, $\widetilde{b}$, and $\widetilde{c}$ are distinct in $\mathbb{P}^1(\widetilde{k})$. If none of $\widetilde{a}, \widetilde{b}, \widetilde{c}$ is $\widetilde{\infty}$, then

$$\gamma_0(z) = \frac{cz - a(b-c)/(b-a)}{z - (b-c)/(b-a)} \tag{2.4}$$

belongs to $\mathrm{GL}_2(\mathcal{O})$ and satisfies $\gamma_0(0) = a$, $\gamma_0(1) = b$, $\gamma_0(\infty) = c$. If one of the reductions is $\widetilde{\infty}$, by making simple modifications to (2.4) one still finds a $\gamma_0 \in \mathrm{GL}_2(\mathcal{O})$ with $\gamma_0(0) = a$, $\gamma_0(1) = b$, $\gamma_0(\infty) = c$. Since $\gamma_0^{-1} \circ \gamma \in \mathrm{GL}_2(K)$ fixes three points in $\mathbb{P}^1(K)$, it must be a multiple of the identity matrix. Part (B) is [1, Proposition 2.30]. Part (C) follows from the fact that if $\gamma \in \mathrm{GL}_2(K)$, the action of γ on $\mathbf{P}^1_K$ must be bijective and bicontinuous, since $\gamma^{-1} \circ \gamma = \gamma \circ \gamma^{-1} = \mathrm{id}$. Part (D) is [1, Corollary 2.13(B)]. ∎

Lemma 2.2. *For any distinct points $x, y \in \mathbb{P}^1(K)$, the function* $\mathrm{ordRes}_\varphi(\cdot)$ *on type* II *points extends to a continuous function on the path $[x, y]$, which is piecewise affine with respect to the logarithmic path distance, and convex up. The extension is finite on $[x, y] \cap \mathbf{H}^1_K$, and when $d \geq 2$, it is ∞ at x, y.*

If H is a field of definition for φ (so $H(x, y)$ is a field of definition for φ, x, and y), then each affine piece of $\mathrm{ordRes}_\varphi(\cdot)$ *has the form $mt + c$ for some integer m in the range $-d^2 - d \leq m \leq d^2 + d$ satisfying $m \equiv d^2 + d$* $(\mathrm{mod}\ 2d)$, *and some number c in the value group* $\mathrm{ord}(H(x, y)^\times)$, *where t is a parameter measuring the logarithmic path distance along $[x, y]$ (from a given $H(x, y)$-rational type* II *point). There are at most $d+1$ distinct affine pieces, and the breaks between affine pieces occur at type* II *points.*

Proof. Fix $\gamma \in \mathrm{GL}_2(K)$ with $\gamma(0) = x$ and $\gamma(\infty) = y$. The action of $\mathrm{GL}_2(K)$ on $\mathbf{P}^1_K$ takes paths to paths, so $\gamma([0, \infty]) = [x, y]$. The type II points on $[0, \infty]$ are the points $\zeta_{0,|A|}$ corresponding to discs $D(0, |A|)$, as A runs over elements of $K^\times$, and if we put $\mu_A = \left[\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right] \in \mathrm{GL}_2(K)$, then $\zeta_{0,|A|} = \mu_A(\zeta_G)$. Now let $\gamma_A = \gamma \circ \mu_A$. As A varies, the type II points on $[x, y]$ are the points $\gamma(\zeta_{0,|A|}) = \gamma_A(\zeta_G)$, and for all $A, B \in K^\times$ we have

$$\rho(\gamma(\zeta_{0,|A|}), \gamma(\zeta_{0,|B|})) = |\mathrm{ord}(A) - \mathrm{ord}(B)|.$$

Write

$$\begin{aligned} F^\gamma(X, Y) &= a_d X^d + a_{d-1} X^{d-1} Y + \cdots + a_0 Y^d, \\ G^\gamma(X, Y) &= b_d X^d + b_{d-1} X^{d-1} Y + \cdots + b_0 Y^d. \end{aligned} \tag{2.5}$$

Since $\varphi^{\gamma_A} = (\varphi^\gamma)^{\mu_A}$ we have $\left[\begin{smallmatrix} F^{\gamma_A}(X,Y) \\ G^{\gamma_A}(X,Y) \end{smallmatrix}\right] = \left[\begin{smallmatrix} F^{\gamma}(AX,Y) \\ A\, G^{\gamma}(AX,Y) \end{smallmatrix}\right]$; thus

$$\begin{aligned} F^{\gamma_A}(X, Y) &= A^d a_d X^d + A^{d-1} a_{d-1} X^{d-1} Y + \cdots + a_0 Y^d, \\ G^{\gamma_A}(X, Y) &= A^{d+1} b_d X^d + A^d b_{d-1} X^{d-1} Y + \cdots + A b_0 Y^d. \end{aligned} \tag{2.6}$$

Write $Q_A = \gamma_A(\zeta_G)$ and $t = \mathrm{ord}(A)$. Since $\det(\gamma_A) = A\det(\gamma)$, it follows from (2.3) that

$$\begin{aligned}
&\mathrm{ordRes}_\varphi(Q_A) = \mathrm{ordRes}(\varphi^{\gamma_A}) \\
(2.7)\qquad &= \mathrm{ordRes}(F^\gamma, G^\gamma) + (d^2+d)\,\mathrm{ord}(A) \\
&\quad - 2d\min\bigl(\mathrm{ord}(a_0),\dots,\mathrm{ord}(A^d a_d),\mathrm{ord}(Ab_0),\dots,\mathrm{ord}(A^{d+1}b_d)\bigr) \\
(2.8)\qquad &= \max\Bigl(\max_{0\le \ell\le d}\bigl((d^2+d-2d\ell)t + C_\ell\bigr), \\
&\qquad\qquad \max_{0\le \ell\le d}\bigl((d^2+d-2d(\ell+1))t + D_\ell\bigr)\Bigr),
\end{aligned}$$

where $C_\ell = \mathrm{ordRes}(F^\gamma, G^\gamma) - 2d\,\mathrm{ord}(a_\ell)$, $D_\ell = \mathrm{ordRes}(F^\gamma, G^\gamma) - 2d\,\mathrm{ord}(b_\ell)$.

Now let t vary over $\mathbb{R}$. Since the type II points Q_A (which correspond to values of t in the divisible group $\mathrm{ord}(K^\times)$) are dense in $[x,y]$ for the path distance topology, we can use the right side of (2.8) to extend $\mathrm{ordRes}_\varphi(\cdot)$ continuously to $[x,y]$, omitting any terms in (2.8) for which C_ℓ or D_ℓ is $-\infty$ (such terms correspond to coefficients a_ℓ or b_ℓ which are 0). Clearly the extension, being the maximum of finitely many affine functions of t, is piecewise affine and convex upwards. Now suppose $d \ge 2$. Since $F(X,Y)$ and $G(X,Y)$ have no common factors, the same is true for $F^\gamma(X,Y)$ and $G^\gamma(X,Y)$; it follows that at least one of a_0, b_0 is nonzero, and at least one of a_d, b_d is nonzero. The slopes of the corresponding affine functions are d^2+d, d^2-d, $-(d^2-d)$, and $-(d^2+d)$; since $d \ge 2$, these are all nonzero. Thus at least one of the affine functions in (2.8) has positive slope, and at least one has negative slope; this means the extended function $\mathrm{ordRes}_\varphi(\cdot)$ is finite on $[x,y] \cap \mathbf{H}_K^1$, and is ∞ at x and y.

Let H be a field of definition for φ. Then $F(X,Y)$, $G(X,Y)$ can be taken to be rational over H, and γ can be taken to be rational over $H(x,y)$; if this is the case, then $a_0,\dots,a_d,b_0,\dots,b_d$ and $\det(\gamma)$ will also be rational over $H(x,y)$. Comparing (2.7) and (2.8), we see that each affine piece of $\mathrm{ordRes}_\varphi(\cdot)$ has the form $mt+c$, where m is an integer in the range $-d^2-d \le m \le d^2+d$ satisfying $m \equiv d^2+d \pmod{2d}$, and c belongs to the value group $\mathrm{ord}(H(x,y)^\times)$. If two of the affine functions in (2.8) have the same slope, only one will contribute to $\mathrm{ordRes}_\varphi(\cdot)$. There are $d+1$ possible slopes, so $\mathrm{ordRes}_\varphi(\cdot)$ has at most $d+1$ affine pieces on $[x,y]$.

Finally, suppose $m_i t + c_i$ and $m_j t + c_j$ are consecutive affine pieces. Their intersection occurs at

$$t = t_{ij} = -\frac{c_j - c_i}{m_j - m_i}, \tag{2.9}$$

which belongs to $\mathrm{ord}(K^\times)$; thus the breaks between affine pieces occur at type II points. Indeed, $m = m_j - m_i$ is a nonzero integer satisfying $m \equiv 0 \pmod{2d}$ and $|m| \le 2d(d+1)$, and by (2.7) and (2.8) we find that

$c_j - c_i \in 2d \cdot \mathrm{ord}(H(x,y)^\times)$. Thus t_{ij} actually belongs to the divisible hull of $\mathrm{ord}(H(x,y)^\times)$, with denominator taken from $\{1, 2, \ldots, d+1\}$. ∎

Proposition 2.3. *There is a unique extension of* $\mathrm{ordRes}_\varphi(\cdot)$ *on type* II *points to a function* $\mathrm{ordRes}_\varphi : \mathbf{P}^1_K \to [0,\infty]$ *which agrees with the one given in Lemma 2.2 on paths with endpoints in* $\mathbb{P}^1(K)$*, and is continuous on* $\mathbf{H}^1_K$ *for the strong topology. When* $d = 1$*, the extension is continuous with respect to the strong topology at each* $x \in \mathbf{H}^1_K$*, and at each* $x \in \mathbb{P}^1(K)$ *where* $\mathrm{ordRes}_\varphi(x) = \infty$*. When* $d \geq 2$*, it is continuous with respect to the strong topology at each* $x \in \mathbf{P}^1_K$*. The extension is finite on* $\mathbf{H}^1_K$*, and when* $d \geq 2$*, it takes the value* ∞ *at each* $x \in \mathbb{P}^1(K)$*.*

On each path in $\mathbf{P}^1_K$*, the extension is convex upwards and piecewise affine with respect to* $\rho(x,y)$*; moreover, the slope of each affine piece is an integer* $m \equiv d^2 + d \pmod{2d}$ *with* $-d^2 - d \leq m \leq d^2 + d$*, the breaks between affine pieces occur at type* II *points, and there are at most* $d+1$ *distinct affine pieces. In particular, on* $\mathbf{H}^1_K$*, the extension is Lipschitz continuous with respect to* $\rho(x,y)$ *with Lipschitz constant* $d^2 + d$*.*

Proof. Given two paths $[x_1, y_1]$, $[x_2, y_2]$ with endpoints in $\mathbb{P}^1(K)$, the extensions of $\mathrm{ordRes}_\varphi(\cdot)$ to $[x_1, y_1]$ and $[x_2, y_2]$ given by Lemma 2.2 are consistent on $[x_1, y_1] \cap [x_2, y_2]$, since type II points are dense in the intersection if it is nonempty, and the extension to each path is continuous. If $P \in \mathbf{P}^1_K$ is of type I, II, or III, define $\mathrm{ordRes}_\varphi(P)$ to be the value of the extension given by Lemma 2.2, on any path $[x,y]$ containing P with endpoints in $\mathbb{P}^1(K)$. In this way, we obtain a well-defined function $\mathrm{ordRes}_\varphi(\cdot)$ on the points of type I, II, and III. When $d \geq 2$, Lemma 2.2 shows that $\mathrm{ordRes}_\varphi(x) = \infty$ for each $x \in \mathbb{P}^1(K)$.

We next show that there is a unique continuous extension of $\mathrm{ordRes}_\varphi(\cdot)$ to type IV points. Since any pair of type II points belongs to a path with endpoints in $\mathbb{P}^1(K)$, Lemma 2.2 shows that for all type II points x, y we have

$$|\mathrm{ordRes}_\varphi(x) - \mathrm{ordRes}_\varphi(y)| \leq (d^2 + d) \cdot \rho(x,y).$$

Since each point of type IV is at finite logarithmic path distance from ζ_G, and type II points are dense in $\mathbf{H}^1_K$ with respect to $\rho(x,y)$, there is a unique extension of $\mathrm{ordRes}_\varphi(\cdot)$ to $\mathbf{H}^1_K$ which is Lipschitz continuous with respect to $\rho(x,y)$, with Lipschitz constant $d^2 + d$. Since $\mathrm{ordRes}_\varphi(x) \geq 0$ on type II points, $\mathrm{ordRes}_\varphi(z) \geq 0$ for all $z \in \mathbf{P}^1_K$.

Since each segment $[u,v]$ with type II endpoints is contained in a path $[x,y]$ with type I endpoints, the restriction of $\mathrm{ordRes}_\varphi(\cdot)$ to $[u,v]$ is piecewise affine and convex upwards with respect to the logarithmic path distance, with at most $d+1$ affine pieces, and slopes $m \equiv d^2 + d \pmod{2d}$ where $-d^2 - d \leq m \leq d^2 + d$; the breaks between affine pieces occur at type II points. These same properties must hold for $\mathrm{ordRes}_\varphi(\cdot)$ on an arbitrary path

$[z,w]$ in $\mathbf{P}^1_K$, since the interior of the path can be exhausted by an increasing sequence of segments with type II endpoints, and the number of affine pieces on each such segment is uniformly bounded.

To complete the proof, it suffices to show that $\mathrm{ordRes}_\varphi(\cdot)$ is continuous with respect to the strong topology at each type I point x where $\mathrm{ordRes}_\varphi(x) = \infty$. Fix $y \in \mathbb{P}^1(K)$ with $y \neq x$, and consider the path $[x,y]$. For each $P \in [x,y] \cap \mathbf{H}^1_K$, let $U_x(P)$ be the component of $\mathbf{P}^1_K \backslash \{P\}$ containing x. As $P \to x$, the sets $U_x(P)$ form a basis for the neighborhoods of x in the strong topology. We claim that for each $M \in \mathbb{R}$, there is a P_M such that $\mathrm{ordRes}_\varphi(z) > M$ for all $z \in U_x(P_M)$. To see this, note that since $\mathrm{ordRes}_\varphi(P)$ increases to ∞ as $P \to x$ along $[x,y]$, there is a P_M such that $\mathrm{ordRes}_\varphi(P_M) > M$ and $\mathrm{ordRes}_\varphi(\cdot)$ is increasing on $[P_M, x]$. Let $z \in U_x(P_M)$ be arbitrary. The path $[P_M, z]$ shares an initial segment with $[P_M, x]$, and $\mathrm{ordRes}_\varphi(\cdot)$ is increasing along that initial segment. Since $\mathrm{ordRes}_\varphi(\cdot)$ is convex up on $[P_M, z]$, we have $\mathrm{ordRes}_\varphi(z) > \mathrm{ordRes}_\varphi(P_M) > M$. ∎

For each $Q \in \mathbf{P}^1_K$, we call paths $[Q,x]$ and $[Q,y]$ emanating from Q *equivalent* if they share an initial segment. The *tangent space* T_Q is the set of equivalence classes of paths emanating from Q; these classes are called *directions*. The directions at Q are in 1-1 correspondence with the components of $\mathbf{P}^1_K \backslash \{Q\}$. If Q is of type I or IV, T_Q has one element; if Q is of type III, T_Q has two elements; and if Q is of type II, T_Q is infinite. Given $\beta \neq Q$, we will write $\vec{v}_\beta \in T_Q$ for the direction containing $[Q,\beta]$, or $\vec{v}_{Q,\beta}$ if it is necessary to specify Q.

Recall that $\widetilde{k} = \mathcal{O}/\mathfrak{M}$ is the residue field of K. When $Q = \zeta_G$, the components of $\mathbf{P}^1_K \backslash \{\zeta_Q\}$ correspond to elements of $\mathbb{P}^1(\widetilde{k})$; thus the directions in T_{ζ_G} are $\vec{v}_\infty$ and the $\vec{v}_\beta$ for $\beta \in \mathcal{O}$, where $\vec{v}_{\beta_1} = \vec{v}_{\beta_2}$ iff $\beta_1 \equiv \beta_2 \pmod{\mathfrak{M}}$. For an arbitrary type II point Q, we can write $Q = \gamma(\zeta_G)$ for some $\gamma \in \mathrm{GL}_2(K)$; since γ takes paths to paths, it induces a 1-1 correspondence $\gamma_* : T_{\zeta_G} \to T_Q$ with $\gamma_*(\vec{v}_\beta) = \vec{v}_{\gamma(\beta)} \in T_Q$. Hence the directions in T_Q are $\vec{v}_{\gamma(\infty)}$ and the $\vec{v}_{\gamma(\beta)}$ for $\beta \in \mathcal{O}$, where again $\vec{v}_{\gamma(\beta_1)} = \vec{v}_{\gamma(\beta_2)}$ iff $\beta_1 \equiv \beta_2 \pmod{\mathfrak{M}}$.

We will say $\mathrm{ordRes}_\varphi(\cdot)$ is *locally decreasing* (resp. *locally constant*, resp. *increasing*) in a direction $\vec{v} \in T_Q$ if it is initially decreasing (resp. constant, resp. increasing) along $[Q,\beta]$ for some (hence every) path with $\vec{v} = \vec{v}_\beta$. A crucial observation is that since $\mathrm{ordRes}_\varphi(\cdot)$ is convex upward, at each point Q there can be at most one direction in which $\mathrm{ordRes}_\varphi(\cdot)$ is locally decreasing: thus, $\mathrm{ordRes}_\varphi(\cdot)$ satisfies the principle of steepest descent. Likewise, if it is locally constant in some direction at Q, it must be locally constant or increasing in every other direction. If it is locally increasing in some direction at Q, by convexity it must be increasing along every path $[Q,\beta]$ in that direction, so we need not distinguish between *locally increasing* and *increasing*.

When Q is of type II, we will now give necessary and sufficient conditions for $\mathrm{ordRes}_\varphi(\cdot)$ to be locally decreasing, locally constant, or increasing in a given direction. Suppose $Q = \gamma(\zeta_G)$ where $\gamma \in \mathrm{GL}_2(K)$; let (F^γ, G^γ) be the representation of φ^γ from (2.2). By replacing γ with $c\gamma$ for an appropriate $c \in K^\times$ (which does not change the action of γ), we can assume (F^γ, G^γ) is normalized. As in (2.5), write

$$\begin{aligned} F^\gamma(X,Y) &= a_d X^d + a_{d-1}X^{d-1}Y + \cdots + a_0 Y^d, \\ G^\gamma(X,Y) &= b_d X^d + b_{d-1}X^{d-1}Y + \cdots + b_0 Y^d. \end{aligned} \tag{2.10}$$

For each $\beta \in \mathcal{O}$, the map $\nu^\beta := \left[\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right] \in \mathrm{GL}_2(\mathcal{O})$ stabilizes ζ_G and takes the path $[0,\infty]$ to $[\beta,\infty]$. Write $\gamma^\beta = \gamma \circ \nu^\beta$; then $\gamma^\beta(\zeta_G) = Q$, and since $\varphi^{\gamma^\beta} = (\varphi^\gamma)^{\nu^\beta}$, it follows that the pair $(F^{\gamma^\beta}, G^{\gamma^\beta})$ given by

$$\begin{aligned} \begin{bmatrix} F^{\gamma^\beta}(X,Y) \\ G^{\gamma^\beta}(X,Y) \end{bmatrix} &= \mathrm{Adj}(\nu^\beta) \circ \begin{bmatrix} F^\gamma \\ G^\gamma \end{bmatrix} \circ \nu^\beta \circ \begin{bmatrix} X \\ Y \end{bmatrix} \\ &= \begin{bmatrix} F^\gamma(X+\beta Y, Y) - \beta G^\gamma(X+\beta Y, Y) \\ G^\gamma(X+\beta Y, Y) \end{bmatrix} \end{aligned}$$

is another representation of φ at Q. It is normalized since $\nu^\beta \in \mathrm{GL}_2(\mathcal{O})$. Write

$$\begin{aligned} F^{\gamma^\beta}(X,Y) &= a_d(\beta) X^d + a_{d-1}(\beta)X^{d-1}Y + \cdots + a_0(\beta) Y^d, \\ G^{\gamma^\beta}(X,Y) &= b_d(\beta) X^d + b_{d-1}(\beta)X^{d-1}Y + \cdots + b_0(\beta) Y^d. \end{aligned} \tag{2.11}$$

LEMMA 2.4. *Let Q be of type* II; *suppose $Q = \gamma(\zeta_G)$ where $\gamma \in \mathrm{GL}_2(K)$ is such that (F^γ, G^γ) is normalized. Let $\vec{v} \in T_Q$.*

(A) *If $\vec{v} = \vec{v}_{Q,\gamma(0)}$, then*
- $\mathrm{ordRes}_\varphi(\cdot)$ *is locally decreasing in the direction $\vec{v}$ iff*
$$\begin{cases} \mathrm{ord}(a_\ell) > 0 \ \textit{when}\ 0 \le \ell \le (d+1)/2, \\ \mathrm{ord}(b_\ell) > 0 \ \textit{when}\ 0 \le \ell \le (d-1)/2; \end{cases}$$
- $\mathrm{ordRes}_\varphi(\cdot)$ *is locally constant in the direction $\vec{v}$ iff d is odd and*
$$\begin{cases} \mathrm{ord}(a_{(d+1)/2}) = 0 \ \textit{or}\ \mathrm{ord}(b_{(d-1)/2}) = 0, \\ \mathrm{ord}(a_\ell) > 0 \ \textit{when}\ 0 \le \ell < (d+1)/2, \\ \mathrm{ord}(b_\ell) > 0 \ \textit{when}\ 0 \le \ell < (d-1)/2; \end{cases}$$
- $\mathrm{ordRes}_\varphi(\cdot)$ *is increasing in the direction $\vec{v}$ otherwise.*

(B) *If $\vec{v} = \vec{v}_{Q,\gamma(\beta)}$ for some $\beta \in \mathcal{O}$, the same criteria as in case* (A) *hold, provided the a_ℓ and b_ℓ are replaced with the $a_\ell(\beta)$ and $b_\ell(\beta)$.*

(C) *If $\vec{v} = \vec{v}_{Q,\gamma(\infty)}$, then*

- $\mathrm{ordRes}_\varphi(\cdot)$ *is locally decreasing in the direction* $\vec{v}$ *iff*

$$\begin{cases} \mathrm{ord}(a_\ell) > 0 \text{ when } (d+1)/2 \le \ell \le d, \\ \mathrm{ord}(b_\ell) > 0 \text{ when } (d-1)/2 \le \ell \le d; \end{cases}$$

- $\mathrm{ordRes}_\varphi(\cdot)$ *is locally constant in the direction* $\vec{v}$ *iff* d *is odd and*

$$\begin{cases} \mathrm{ord}(a_{(d+1)/2}) = 0 \text{ or } \mathrm{ord}(b_{(d-1)/2}) = 0, \\ \mathrm{ord}(a_\ell) > 0 \text{ when } (d+1)/2 < \ell \le d, \\ \mathrm{ord}(b_\ell) > 0 \text{ when } (d-1)/2 < \ell \le d; \end{cases}$$

- $\mathrm{ordRes}_\varphi(\cdot)$ *is increasing in the direction* $\vec{v}$ *otherwise.*

Proof. Note that $\gamma([0,\zeta_G]) = [\gamma(0), Q]$ and $\gamma([\zeta_G,\infty]) = [Q, \gamma(\infty)]$. We will establish the criteria when $\vec{v} = \vec{v}_{Q,\gamma(0)}$ and $\vec{v} = \vec{v}_{Q,\gamma(\infty)}$ using formula (2.8) and the normalized representation (F^γ, G^γ). Since $\gamma^\beta([0,\zeta_G]) = [\gamma(\beta), Q]$, the criteria for the directions $\vec{v}_{Q,\gamma(\beta)}$ with $\beta \in \mathcal{O}$ follow by applying the same arguments to $(F^{\gamma^\beta}, G^{\gamma^\beta})$.

Using the same notation as in (2.7) and (2.8), for each $A \in K^\times$ put $Q_A = \gamma(\zeta_{0,|A|}) = \gamma_A(\zeta_G)$. Making the constants C_ℓ, D_ℓ in (2.8) explicit, we have

$$\begin{aligned} (2.12)\qquad & \mathrm{ordRes}_\varphi(Q_A) - \mathrm{ordRes}_\varphi(Q) \\ &= \max\Big(\max_{0\le\ell\le d} \big((d^2+d-2d\ell)t - 2d\,\mathrm{ord}(a_\ell)\big), \\ &\qquad \max_{0\le\ell\le d} \big((d^2+d-2d(\ell+1))t - 2d\,\mathrm{ord}(b_\ell)\big)\Big), \end{aligned}$$

where $t = \mathrm{ord}(A)$. By assumption $\mathrm{ord}(a_\ell), \mathrm{ord}(b_\ell) \ge 0$ for each ℓ, and some $\mathrm{ord}(a_\ell)$ or $\mathrm{ord}(b_\ell)$ is 0. If $t = 0$, then $Q_A = Q$ and both sides of (2.12) are 0.

Values of $t > 0$ correspond to points in the direction $\vec{v}_{Q,\gamma(0)}$. For small positive t, the right side of (2.12) will be negative if and only if each of the affine functions in (2.12) with a nonnegative slope has a negative constant term. Hence $\mathrm{ordRes}_\varphi(\cdot)$ is locally decreasing in the direction $\vec{v}_{\gamma(0)} \in T_Q$ if and only if $\mathrm{ord}(a_\ell) > 0$ for each ℓ such that $d^2+d-2d\ell \ge 0$, and $\mathrm{ord}(b_\ell) > 0$ for each ℓ such that $d^2+d-2d(\ell+1) \ge 0$. Similarly $\mathrm{ordRes}_\varphi(\cdot)$ is locally constant in the direction $\vec{v}_{Q,\gamma(0)}$ if and only if at least one of the affine functions with slope 0 has constant term 0, and each of the affine functions with positive slope has a negative constant term. This happens if and only if d is odd, either $\mathrm{ord}(a_{(d+1)/2}) = 0$ or $\mathrm{ord}(b_{(d-1)/2}) = 0$, $\mathrm{ord}(a_\ell) > 0$ for each ℓ such that $d^2+d-2d\ell > 0$, and $\mathrm{ord}(b_\ell) > 0$ for each ℓ such that $d^2+d-2d(\ell+1) > 0$.

Values of $t < 0$ correspond to points in the direction $\vec{v}_{\gamma(\infty)}$ at Q. For small negative t, the right side of (2.12) will be negative if and only if each of the affine functions in (2.12) with a nonpositive slope has a negative constant term. Hence $\mathrm{ordRes}_\varphi(\cdot)$ is locally decreasing in the direction $\vec{v}_{Q,\gamma(\infty)}$ if and only if $\mathrm{ord}(a_\ell) > 0$ for each ℓ such that $d^2 + d - 2d\ell \le 0$, and $\mathrm{ord}(b_\ell) > 0$ for each ℓ such that $d^2+d-2d(\ell+1) \le 0$. Similarly $\mathrm{ordRes}_\varphi(\cdot)$ is locally constant in the direction $\vec{v}_{Q,\gamma(\infty)}$ if and only if d is odd, either $\mathrm{ord}(a_{(d+1)/2}) = 0$ or $\mathrm{ord}(b_{(d-1)/2}) = 0$, $\mathrm{ord}(a_\ell) > 0$ for each ℓ such that $d^2 + d - 2d\ell < 0$, and $\mathrm{ord}(b_\ell) > 0$ for each ℓ such that $d^2 + d - 2d(\ell + 1) < 0$. ∎

LEMMA 2.5. *If $d \ge 2$ is even, then* $\mathrm{ordRes}_\varphi(\cdot)$ *is never locally constant. If $d \ge 3$ is odd, then at each $Q \in \mathbf{P}^1_K$, there are at most two directions in T_Q where* $\mathrm{ordRes}_\varphi(\cdot)$ *is locally constant.*

Proof. If $d \ge 2$ is even, then on any path the slope of each affine piece of $\mathrm{ordRes}_\varphi(\cdot)$ is an integer $m \equiv d^2 + d \pmod{2d}$, hence is nonzero.

Suppose $d \ge 3$ is odd. If $Q \in \mathbf{P}^1_K$ is of type I, III, or IV, then there are at most two directions in T_Q, so trivially there are at most two directions in T_Q in which $\mathrm{ordRes}_\varphi(\cdot)$ is locally constant. Let Q be a type II point with at least two distinct directions where $\mathrm{ordRes}_\varphi(\cdot)$ is locally constant, say $\vec{v}_\alpha$ and $\vec{v}_\beta$. Take any $\gamma \in \mathrm{GL}_2(K)$ with $Q = \gamma(\zeta_G)$. After replacing γ with $\gamma\tau$ for a suitable $\tau \in \mathrm{GL}_2(\mathcal{O})$, we can assume that $\vec{v}_\alpha = \vec{v}_{Q,\gamma(0)}$ and $\vec{v}_\beta = \vec{v}_{Q,\gamma(\infty)}$. Also, after replacing γ with $c\gamma$ for a suitable $c \in K^\times$, we can assume that (F^γ, G^γ) is a normalized representation of φ. Write $F^\gamma(X, Y) = a_d X^d + a_{d-1}X^{d-1}Y + \cdots + a_0 Y^d$, $G^\gamma(X, Y) = b_d X^d + b_{d-1}X^{d-1}Y + \cdots + b_0 Y^d$. By Lemma 2.4, if we put $D = (d+1)/2$ and $E = (d-1)/2$, then $\mathrm{ord}(a_\ell) > 0$ for all $\ell \ne D$, $\mathrm{ord}(b_\ell) > 0$ for all $\ell \ne E$, and either $\mathrm{ord}(a_D) = 0$ or $\mathrm{ord}(b_E) = 0$. Since $d \ge 3$, we have $D, E \ge 1$.

First suppose $\mathrm{ord}(b_E) = 0$; then $G^\gamma(X, Y) \equiv b_E X^E Y^{d-E} \pmod{\mathfrak{M}}$, so for each $\beta \in \mathcal{O}$,

$$G^{\gamma^\beta}(X, Y) := G^\gamma(X + \beta Y, Y) \equiv b_E(X + \beta Y)^E Y^{d-E} \pmod{\mathfrak{M}}.$$

Comparing this with (2.11) shows $b_0(\beta) \equiv b_E\beta^E \pmod{\mathfrak{M}}$. If $\beta \not\equiv 0 \pmod{\mathfrak{M}}$, this means $\mathrm{ord}(b_0(\beta)) = 0$, so the criterion for local constancy in Lemma 2.4(B) is not met for the direction $\vec{v}_{Q,\gamma(\beta)}$. Thus $\vec{v}_{Q,\gamma(0)}$ and $\vec{v}_{Q,\gamma(\infty)}$ are the only directions in which $\mathrm{ordRes}_\varphi(\cdot)$ is locally constant.

Next suppose $\mathrm{ord}(b_E) > 0$, so necessarily $\mathrm{ord}(a_D) = 0$. Then $G^\gamma(X, Y) \equiv 0 \pmod{\mathfrak{M}}$ and $F^\gamma(X, Y) \equiv a_D X^D Y^{d-D} \pmod{\mathfrak{M}}$, so for each $\beta \in \mathcal{O}$,

$$\begin{aligned} F^{\gamma^\beta}(X, Y) &:= F^\gamma(X + \beta Y, Y) - \beta G^\gamma(X + \beta Y, Y) \\ &\equiv a_D(X + \beta Y)^D Y^{d-D} \pmod{\mathfrak{M}}. \end{aligned}$$

Comparing this with (2.11) shows $a_0(\beta) \equiv a_D\beta^D \pmod{\mathfrak{M}}$. When $\beta \not\equiv 0 \pmod{\mathfrak{M}}$, this means $\mathrm{ord}(a_0(\beta)) = 0$, so the criterion for local constancy in Lemma 2.4(B) is not met for the direction $\vec{v}_{Q,\gamma(\beta)}$, and again $\vec{v}_{Q,\gamma(0)}$ and $\vec{v}_{Q,\gamma(\infty)}$ are the only directions where $\mathrm{ordRes}_\varphi(\cdot)$ can be locally constant. ∎

Remark. By a similar argument, one can show that at any type II point there can be at most one direction in which $\mathrm{ordRes}_\varphi(\cdot)$ is locally decreasing, without using convexity.

Our next goal is to show that $\mathrm{ordRes}_\varphi(\cdot)$ is strictly increasing as one moves away from the tree $\Gamma_{\mathrm{Fix},\varphi^{-1}(\infty)}$ in $\mathbf{P}^1_K$ spanned by the fixed points and the poles of φ. This means that $\mathrm{ordRes}_\varphi(\cdot)$ achieves a minimum on $\mathbf{P}^1_K$, and shows that the locus $\mathrm{MinResLoc}(\varphi)$ where it takes on its minimum is contained in that tree.

Two main facts underlie this. The first is that the group of affine transformations $\mathrm{Aff}_2(K) = \{az + b : a \in K^\times,\ b \in K\}$, corresponding to matrices $\left[\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right] \in \mathrm{GL}_2(K)$, acts transitively on type II points. Indeed, if Q corresponds to a disc $D(b, r)$ with $r \in |K^\times|$, and $|a| = r$, then $\gamma(z) = az + b$ takes ζ_G to Q. The second is that the fixed points of φ are equivariant under $\mathrm{GL}_2(K)$, and the poles are equivariant under $\mathrm{Aff}_2(K)$: for each $\gamma \in \mathrm{GL}_2(K)$, Δ is a fixed point of φ iff $\gamma^{-1}(\Delta)$ is a fixed point of φ^γ; and for each $\gamma \in \mathrm{Aff}_2(K)$, δ is a pole of φ iff $\gamma^{-1}(\delta)$ is a pole of φ^γ.

Lemma 2.6. *If $d \geq 2$, the set of poles and fixed points of φ in $\mathbb{P}^1(K)$ contains at least two distinct elements.*

Proof. Using the representation $(F(X,Y), G(X,Y))$ for $\varphi(z)$, the fixed points correspond to nontrivial solutions of $YF(X,Y) - XG(X,Y) = 0$ and the poles correspond to nontrivial solutions of $G(X,Y) = 0$.

Suppose all the poles and fixed points of φ occur at a single point $\alpha \in \mathbb{P}^1(K)$. If $\alpha = \infty$, there are $C, D \in K^\times$ such that $G(X,Y) = CY^d$ and $YF(X,Y) - XG(X,Y) = DY^{d+1}$. Solving, we see that $YF(X,Y) = DY^{d+1} + CXY^d$. Since $d \geq 2$, this contradicts the fact that $F(X,Y)$ and $G(X,Y)$ have no common factors. If $\alpha \in K$, there are $C, D \in K^\times$ such that $G(X,Y) = C(X - \alpha Y)^d$ and $YF(X,Y) - XG(X,Y) = D(X - \alpha Y)^{d+1}$. In this case $YF(X,Y) = D(X - \alpha Y)^{d+1} + CX(X - \alpha Y)^d$, which again contradicts the fact that $F(X,Y)$ and $G(X,Y)$ have no common factors. ∎

Proposition 2.7. *If $d \geq 2$, the function* $\mathrm{ordRes}_\varphi(\cdot)$ *is strictly increasing as one moves away from the tree $\Gamma_{\mathrm{Fix},\varphi^{-1}(\infty)}$ in $\mathbf{P}^1_K$ spanned by the fixed points and poles of φ in $\mathbb{P}^1(K)$.*

Proof. Write $\Gamma = \Gamma_{\mathrm{Fix},\varphi^{-1}(\infty)}$ for short. Branches off Γ in $\mathbf{P}^1_K$ can only occur at type II points. By the convexity of $\mathrm{ordRes}_\varphi(\cdot)$, it suffices to show that at each type II point $Q \in \Gamma$, $\mathrm{ordRes}_\varphi(\cdot)$ is increasing in each direction $\vec{v} \in T_Q$ which points away from Γ.

Fix a type II point $Q \in \Gamma$, and let $\vec{v} \in T_Q$ be a direction away from Γ. Let $\gamma \in \mathrm{Aff}_2(K)$ be such that $\gamma(\zeta_G) = Q$. If $\vec{v} = \vec{v}_{Q,\infty}$, then $\gamma_*(\vec{v}_{\zeta_G,\infty}) = \vec{v}$. If $\vec{v} \neq \vec{v}_{Q,\infty}$, there is some $\beta \in \mathcal{O}$ such that $\gamma_*(\vec{v}_{\zeta_G,\beta}) = \vec{v}$, and after replacing γ with $\gamma^\beta = \gamma \circ \nu^\beta$ we can assume that $\gamma_*(\vec{v}_{\zeta_G,0}) = \vec{v}$. Finally, by replacing

γ with $c\gamma$ for some $c \in K^\times$, we can assume that the representation (F^γ, G^γ) of φ^γ is normalized.

First suppose $\vec{v} = \gamma_*(\vec{v}_{\zeta_G,\infty}) = \vec{v}_{\gamma(\infty)}$. By the equivariance of poles and fixed points under $\mathrm{Aff}_2(K)$, φ^γ has no poles or fixed points in the direction $\vec{v}_\infty$ at ζ_G. As in (2.5), write $F^\gamma(X,Y) = a_d X^d + a_{d-1}X^{d-1}Y + \cdots + a_0 Y^d$, $G^\gamma(X,Y) = b_d X^d + b_{d-1}X^{d-1}Y + \cdots + b_0 Y^d$. By hypothesis the poles δ_i of φ^γ all belong to $\mathcal{O}$, so we can factor $G^\gamma(X,Y) = b_d \cdot \prod_{i=1}^d (X - \delta_i Y)$ where $|\delta_i| \le 1$ for each i. Expanding this and comparing coefficients shows that $\max(|b_d|, |b_{d-1}|, \ldots, |b_0|) = |b_d|$. Likewise, the fixed points Δ_i of φ^γ all belong to $\mathcal{O}$. Since the fixed points are the zeros of

$$\begin{aligned} &YF^\gamma(X,Y) - XG^\gamma(X,Y) \\ &\qquad = a_d X^{d+1} + (a_{d-1} - b_d)X^d Y + \cdots + (a_0 - b_1)XY^d - b_0 Y^{d+1}, \end{aligned}$$

we can write $XF^\gamma(X,Y) - XG^\gamma(X,Y) = a_d \prod_{i=1}^{d+1}(X - \Delta_i Y)$. Expanding and comparing coefficients shows $\max(|a_d|, |a_{d-1} - b_d|, \ldots, |a_0 - b_1|, |b_0|) = |a_d|$. However, it is an easy consequence of the ultrametric inequality that

$$\begin{aligned} \max(|a_d|, |a_{d-1} - b_d|, \ldots, |a_0 - b_1|, |b_0|, |b_d|, |b_{d-1}|, \ldots, |b_0|) \\ = \max(|a_d|, |a_{d-1}|, \ldots, |a_0|, |b_d|, |b_{d-1}|, \ldots, |b_0|). \end{aligned} \tag{2.13}$$

Thus $\max(|a_d|, |a_{d-1}|, \ldots, |a_0|, |b_d|, |b_{d-1}|, \ldots, |b_0|) = \max(|a_d|, |b_d|)$. Since (F^γ, G^γ) is normalized, it follows that $\mathrm{ord}(a_d) = 0$ or $\mathrm{ord}(b_d) = 0$. By Lemma 2.4, $\mathrm{ordRes}_\varphi(\cdot)$ cannot be decreasing or constant in the direction $\vec{v} = \vec{v}_{\gamma(\infty)}$, so it must be increasing.

Next suppose $\vec{v} = \gamma_*(\vec{v}_{\zeta_G,0}) = \vec{v}_{\gamma(0)}$. In this case φ^γ has no poles or fixed points in the direction $\vec{v}_0$ at ζ_G. As before, write $F^\gamma(X,Y) = a_d X^d + a_{d-1}X^{d-1}Y + \cdots + a_0 Y^d$, $G^\gamma(X,Y) = b_d X^d + b_{d-1}X^{d-1}Y + \cdots + b_0 Y^d$. By hypothesis the poles of φ^γ belong to $(K \setminus \mathfrak{M}) \cup \{\infty\}$, so we can factor $G^\gamma(X,Y) = CY^m \cdot \prod_{i=1}^{d-m}(X - \delta_i Y)$ for some $C \in K^\times$, where m is the number of poles of φ^γ at ∞ and $|\delta_i| \ge 1$ for $i = 1, \ldots, d-m$. Expanding and comparing coefficients shows that $|b_0| = \max(|b_d|, |b_{d-1}|, \ldots, |b_0|)$. Likewise, the fixed points of φ^γ all belong to $(K \setminus \mathfrak{M}) \cup \{\infty\}$, so for some $D \in K^\times$ we can write $YF^\gamma(X,Y) - XG^\gamma(X,Y) = D \cdot Y^n \prod_{i=1}^{d-n}(X - \Delta_i Y)$ where n is the number of fixed points of φ^γ at ∞, and $|\Delta_i| \ge 1$ for $i = 1, \ldots, d-n$. Expanding and comparing coefficients shows that $|b_0| = \max(|a_d|, |a_{d-1} - b_d|, \ldots, |a_0 - b_1|, |b_0|)$. Using (2.13) we see that $|b_0| = \max(|a_d|, |a_{d-1}|, \ldots, |a_0|, |b_d|, |b_{d-1}|, \ldots, |b_0|)$. Since (F^γ, G^γ) is normalized, it must be the case that $\mathrm{ord}(b_0) = 0$. By Lemma 2.4, $\mathrm{ordRes}_\varphi(\cdot)$ cannot be locally decreasing or constant in the direction $\vec{v} = \vec{v}_{\gamma(0)}$, so it must be increasing. ∎

Proposition 2.8. *Suppose $d \ge 2$. Given a point $x \in \mathbb{P}^1(K)$, let ξ be the unique point in $[\zeta_G, x]$ such that $\rho(\zeta_G, \xi) = \frac{2}{d-1}\,\mathrm{ordRes}(\varphi)$. Then $\mathrm{ordRes}_\varphi(\cdot)$ is increasing along $[\xi, x]$ as one moves away from ζ_G.*

Proof. There is a $\gamma \in \mathrm{GL}_2(\mathcal{O})$ such that $\gamma(0) = x$. Since $\gamma \in \mathrm{GL}_2(\mathcal{O})$, it fixes ζ_G. Thus $\mathrm{ordRes}_{\varphi^\gamma}(\zeta_G) = \mathrm{ordRes}(\varphi^\gamma) = \mathrm{ordRes}(\varphi)$. Let (F^γ, G^γ) be a representation of φ^γ as in (2.5); after scaling (F^γ, G^γ), we can assume it is normalized. At least one of the coefficients a_0, b_0 in F^γ, G^γ must be nonzero. Expanding the determinant (1.1) for $\mathrm{Res}(F^\gamma, G^\gamma)$ using its last column, one sees that $\min(\mathrm{ord}(a_0), \mathrm{ord}(b_0)) \le \mathrm{ordRes}(\varphi)$. Similarly, $\min(\mathrm{ord}(a_d), \mathrm{ord}(b_d)) \le \mathrm{ordRes}(\varphi)$.

Given $A \in K^\times$, define $Q_A = \zeta_{0,|A|}$ and let $(F^{\gamma_A}, G^{\gamma_A})$ be as in (2.6). By (2.7), (2.8), and the discussion above,

$$\begin{aligned}
(2.14)\qquad \mathrm{ordRes}_{\varphi^\gamma}(Q_A) - \mathrm{ordRes}(\varphi) &= (d^2+d)\,\mathrm{ord}(A) \\
&\quad -2d\min\bigl(\mathrm{ord}(a_0), \ldots, \mathrm{ord}(A^d a_d), \mathrm{ord}(Ab_0), \ldots, \mathrm{ord}(A^{d+1}b_d)\bigr) \\
&\ge \max\bigl(-2d\,\mathrm{ord}(a_0) + (d^2+d)\,\mathrm{ord}(A), -2d\,\mathrm{ord}(b_0) + (d^2-d)\,\mathrm{ord}(A), \\
&\qquad -2d\,\mathrm{ord}(a_d) + (d-d^2)\,\mathrm{ord}(A), -2d\,\mathrm{ord}(b_d) + (-d-d^2)\,\mathrm{ord}(A)\bigr) \\
&\ge -2d\,\mathrm{ordRes}(\varphi) + \max\bigl((d^2-d)\,\mathrm{ord}(A), (d-d^2)\,\mathrm{ord}(A)\bigr).
\end{aligned}$$

The minimum value of $\mathrm{ordRes}_{\varphi^\gamma}(\cdot) - \mathrm{ordRes}(\varphi)$ is ≤ 0, since $\mathrm{ordRes}_{\varphi^\gamma}(\zeta_G) - \mathrm{ordRes}(\varphi) = 0$. Type II points in $[\zeta_G, x]$ correspond to values $A \in K^\times$ with $\mathrm{ord}(A) \ge 0$, and for $\mathrm{ord}(A) \ge 0$ the right side of (2.14) is nonpositive precisely when

$$(2.15)\qquad 0 \le \mathrm{ord}(A) \le \frac{2}{d-1}\,\mathrm{ordRes}(\varphi).$$

By convexity, $\mathrm{ordRes}_{\varphi^\gamma}(Q_A)$ must be increasing with $\mathrm{ord}(A)$ for $\mathrm{ord}(A) > \frac{2}{d-1}\,\mathrm{ordRes}(\varphi)$. Since $\mathrm{ordRes}_\varphi(\gamma(Q_A)) = \mathrm{ordRes}_{\varphi^\gamma}(Q_A)$, we are done. ∎

Proof of Theorem 1.1. Assume $d \ge 2$. By Proposition 2.3, the function $\mathrm{ordRes}_\varphi(\cdot)$ on type II points extends to a function $\mathrm{ordRes}_\varphi : \mathbf{P}^1_K \to [0, \infty]$ which is continuous with respect to the strong topology, finite on $\mathbf{H}^1_K$, identically ∞ on $\mathbb{P}^1(K)$, and piecewise affine and convex upwards with respect to $\rho(x,y)$ on each path.

By Lemma 2.6, the tree $\Gamma = \Gamma_{\mathrm{Fix},\varphi^{-1}(\infty)}(\varphi)$ spanned by the poles and fixed points of φ is nontrivial, and by Proposition 2.7, $\mathrm{ordRes}_\varphi(\cdot)$ is strictly increasing as one moves away from Γ. It follows that $\mathrm{ordRes}_\varphi(\cdot)$ takes on a minimum value on $\mathbf{P}^1_K$, and that the set $\mathrm{MinResLoc}(\varphi)$ where the minimum is achieved is a compact connected subset of $\Gamma \cap \mathbf{H}^1_K$.

On any path the slopes of $\mathrm{ordRes}_\varphi(\cdot)$ are integers $m \equiv d^2 + d \pmod{2d}$. If d is even, then $d^2 + d \equiv d \pmod{2d}$, so none of the slopes are 0. Since the breaks between affine pieces occur at type II points, $\mathrm{MinResLoc}(\varphi)$ consists of a single type II point. If d is odd, then $d^2 + d \equiv 0 \pmod{2d}$. By Lemma 2.5, at each $Q \in \mathrm{MinResLoc}(\varphi)$ there are at most two directions where $\mathrm{ordRes}_\varphi(\cdot)$ is constant. Thus $\mathrm{MinResLoc}(\varphi)$ is either a single type II point, or a segment with type II endpoints.

For each $\gamma \in \mathrm{GL}_2(K)$, it is easy to see that for all $Q \in \mathbf{P}^1_K$,

$$\operatorname{ordRes}_{\varphi^\gamma}(Q) = \operatorname{ordRes}_\varphi(\gamma(Q)). \tag{2.16}$$

By continuity it suffices to check this for type II points. Suppose $Q = \tau(\zeta_G)$ for some $\tau \in \mathrm{GL}_2(K)$. Then

$$\operatorname{ordRes}_{\varphi^\gamma}(Q) = \operatorname{ordRes}_{\varphi^\gamma}(\tau(\zeta_G)) = \operatorname{ordRes}_\varphi(\gamma(\tau(\zeta_G))) = \operatorname{ordRes}_\varphi(\gamma(Q)).$$

It follows from (2.16) that for each $a \in \mathbb{P}^1(K)$, $\operatorname{MinResLoc}(\varphi)$ is contained in the tree $\Gamma_{\mathrm{Fix},\varphi^{-1}(a)}$ spanned by the fixed points of φ and the preimages $\{z \in \mathbb{P}^1(K) : \varphi(z) = a\}$. Indeed, given $a \in \mathbb{P}^1(K)$, choose $\gamma \in \mathrm{GL}_2(K)$ with $\gamma(\infty) = a$. By (2.16) one has

$$\operatorname{MinResLoc}(\varphi) = \gamma(\operatorname{MinResLoc}(\varphi^\gamma)).$$

By Proposition 2.7 applied to φ^γ, $\operatorname{MinResLoc}(\varphi^\gamma) \subseteq \Gamma_{\mathrm{Fix},(\varphi^\gamma)^{-1}(\infty)}$. Thus $\operatorname{MinResLoc}(\varphi)$ is contained in $\gamma(\Gamma_{\mathrm{Fix},(\varphi^\gamma)^{-1}(\infty)})$. By equivariance, Q is a fixed point of φ^γ if and only if $\gamma(Q)$ is a fixed point of φ, and P is a pole of φ^γ if and only if $\varphi(\gamma(P)) = a$. Hence $\gamma(\Gamma_{\mathrm{Fix},(\varphi^\gamma)^{-1}(\infty)}) = \Gamma_{\mathrm{Fix},\varphi^{-1}(a)}$.

We next show $\operatorname{MinResLoc}(\varphi) \subseteq \big\{z \in \mathbf{H}^1_K : \rho(\zeta_G, z) \le \frac{2}{d-1} \operatorname{ordRes}(\varphi)\big\}$. Fix z with $\rho(\zeta_G, z) > \frac{2}{d-1} \operatorname{ordRes}(\varphi)$. Let ξ be the unique point on $[\zeta_G, z]$ with $\rho(\zeta_G, \xi) = \frac{2}{d-1} \operatorname{ordRes}(\varphi)$; then ξ is of type II. Let $x \in \mathbb{P}^1(K)$ be a type I point in the same direction from ξ as z. By Proposition 2.8, $\operatorname{ordRes}_\varphi(\cdot)$ is increasing along $[\xi, x]$. Since $[\xi, z]$ and $[\xi, x]$ share an initial segment, by convexity $\operatorname{ordRes}_\varphi(\cdot)$ is increasing along $[\xi, z]$. Thus $z \notin \operatorname{MinResLoc}(\varphi)$.

The final assertion in Theorem 1.1 reformulates a result of Favre and Rivera–Letelier [9, Theorem E]. Suppose the minimal value of $\operatorname{ordRes}_\varphi(\cdot)$ is 0. By the results above, there is a type II point $\xi \in \operatorname{MinResLoc}(\varphi)$ where $\operatorname{ordRes}_\varphi(\xi) = 0$. Let $\gamma \in \mathrm{GL}_2(K)$ be such that $\gamma(\zeta_G) = \xi$. Then $\operatorname{ordRes}(\varphi^\gamma) = 0$, so φ^γ has good reduction. Since $d \ge 2$, by [9, Theorem E], or [1, Proposition 10.5], ξ is the unique point where φ achieves good reduction. Thus $\operatorname{MinResLoc}(\varphi) = \{\xi\}$. ∎

Proof of Theorem 1.2. Since $\operatorname{ordRes}_\varphi(\cdot)$ and $\operatorname{ordRes}_{\widehat{\varphi}}(\cdot)$ are continuous for the strong topology, to prove the first assertion it suffices to show that if (1.4) holds, then $\operatorname{ordRes}_\varphi(\xi) = \operatorname{ordRes}_{\widehat{\varphi}}(\xi)$ for all type II points ξ with $\rho(\zeta_G, \xi) \le M$. If $\xi = \zeta_{a,r}$, then the path from ζ_G to ξ goes from ζ_G to $\zeta_{0,T}$ where $T = \max(1, |a|)$, and then from $\zeta_{0,T} = \zeta_{a,T}$ to ξ. Hence if $A, B \in K^\times$ are such that $|A| = 1/T$ and $|B| = r/T$, then $\rho(\zeta_G, \xi) = \operatorname{ord}(A \cdot B)$ and $\xi = \gamma(\zeta_G)$, where

$$\gamma = \begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} \cdot \begin{bmatrix} B & aA \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} B & aA \\ 0 & A \end{bmatrix} \in \mathrm{GL}_2(K) \cap M_2(\mathcal{O}).$$

By (2.3) we have

$$\text{(2.17)}\quad \operatorname{ordRes}_\varphi(\xi) = \operatorname{ordRes}(F,G) + (d^2+d)(\operatorname{ord}(A\cdot B)) - 2d\min(\operatorname{ord}(F^\gamma),\operatorname{ord}(G^\gamma))$$

where F^γ and G^γ are given by (2.2). A similar formula holds for $\operatorname{ordRes}_{\widehat{\varphi}}(\xi)$. Since $\operatorname{ordRes}_\varphi(\xi) \ge 0$, $\operatorname{ordRes}(F,G) = R$, and $\operatorname{ord}(A\cdot B) \le M$, we conclude from (2.17) that

$$\min(\operatorname{ord}(F^\gamma),\operatorname{ord}(G^\gamma)) \le \frac{1}{2d}(R + (d^2+d)M).$$

Now (1.4) shows that $\min(\operatorname{ord}(F^\gamma),\operatorname{ord}(G^\gamma)) = \min(\operatorname{ord}(\widehat{F}^\gamma),\operatorname{ord}(\widehat{G}^\gamma))$ and that $\operatorname{ordRes}(F,G) = \operatorname{ordRes}(\widehat{F},\widehat{G}) = R$. Hence $\operatorname{ordRes}_\varphi(\xi) = \operatorname{ordRes}_{\widehat{\varphi}}(\xi)$.

The second assertion follows by taking $M = \frac{2}{d-1}R + \varepsilon$ in (1.4), with $\varepsilon > 0$ small, and using Theorem 1.1. ∎

3. Rationality considerations. Let H be a subfield of K. Throughout this section, assume $\varphi(z) \in H(z)$. We first show that $\operatorname{MinResLoc}(\varphi)$ contains a type II point rational over an extension L/K with $[L:K] \le (d+1)^2$. We then consider the action of the group of continuous automorphisms $\operatorname{Aut}^c(K/H)$ on $\mathbf{P}^1_K$, and show that $\operatorname{MinResLoc}(\varphi)$ always contains at least one type II point fixed by $\operatorname{Aut}^c(K/H)$. Finally, when $K = \mathbb{C}_v$ and $H = H_v$ is a local field, we show that if $\operatorname{MinResLoc}(\varphi)$ does not contain points rational over H_v, then any extension L/H_v such that $\operatorname{MinResLoc}(\varphi)$ contains L-rational points must be ramified over H_v.

We begin by distinguishing two notions of H-rationality for points of $\mathbf{P}^1_K$, the first of which is more restrictive:

Definition 1. A point $Q \in \mathbf{P}^1_K$ is *H-rational* if it is a type I point in $\mathbb{P}^1(H)$ or if it is a type II point such that $Q = \gamma(\zeta_G)$ for some $\gamma \in \operatorname{GL}_2(H)$. A point $Q \in \mathbf{P}^1_K$ is *weakly H-rational* if it belongs to the tree spanned by $\mathbb{P}^1(K)$ in $\mathbf{P}^1_K$.

If Q is H-rational, it is weakly H-rational; however it can be weakly H-rational without being H-rational. Weakly H-rational points are necessarily of type I, II, or III.

Proposition 3.1. *A type* II *point Q is H-rational if and only if it has the form $\zeta_{a,r}$ with $a \in H$ and $r \in |H^\times|$. A type* II *or type* III *point is weakly H-rational if and only if it has the form $\zeta_{a,r}$ for some $a \in H$.*

Proof. Let Q be an H-rational type II point, and suppose that $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \operatorname{GL}_2(H)$ is such that $Q = \gamma(\zeta_G)$. Write $\mathcal{O}_H$ for the ring of integers of H. Multiplying γ on the right by $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ interchanges its columns, so without loss of generality we can assume $|c| \le |d|$. Then, multiplying γ on the right by $\left[\begin{smallmatrix} 1 & 0 \\ -c/d & 1 \end{smallmatrix}\right] \in \operatorname{GL}_2(\mathcal{O}_H)$ brings it to the form $\left[\begin{smallmatrix} a_1 & b_1 \\ 0 & d_1 \end{smallmatrix}\right]$. Since $\operatorname{GL}_2(\mathcal{O}_H)$

stabilizes ζ_G, Q corresponds to the disc $D(b_1/d_1, |a_1/d_1|)$. Conversely, if Q corresponds to a disc $D(a,r)$ with $a \in H$ and $r \in |H^\times|$, take $b \in H^\times$ with $|b| = r$ and set $\gamma = \left[\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right] \in \mathrm{GL}_2(H)$; then $Q = \gamma(\zeta_G)$.

If Q is a weakly H-rational point of type II or III, there are points $a_1, a_2 \in \mathbb{P}^1(H)$ such that $Q \in [a_1, a_2]$. At least one of a_1, a_2, say a_1, does not belong to the direction $\vec{v}_\infty \in T_Q$. Then there is an $r \in \mathbb{R}$ such that Q corresponds to $D(a_1, r)$. ∎

Define $M = \min_{Q \in \mathbf{P}^1_K}(\mathrm{ordRes}_\varphi(Q)) = \min_{\gamma \in \mathrm{GL}_2(K)}(\mathrm{ordRes}(\varphi^\gamma))$.

THEOREM 3.2. *Let H be a subfield of K, and let $\varphi(z) \in H(z)$ have degree $d \geq 2$. Then there is an extension L/H in K with $[L:H] \leq (d+1)^2$ such that* MinResLoc(φ) *contains a type* II *point Q rational over L. Equivalently,* $\mathrm{ordRes}(\varphi^\gamma) = M$ *for some $\gamma \in \mathrm{GL}_2(L)$.*

Proof. Write $a = \varphi(\infty) \in \mathbb{P}^1(H)$. Let $F_1, \ldots, F_{d+1}$ be the fixed points of φ, and let $A_1, \ldots, A_d$ be the preimages of a under φ, listed with multiplicity. Without loss of generality we can assume that $A_1 = \infty$. By Theorem 1.1, MinResLoc(φ) is contained in the tree $\Gamma_{\mathrm{Fix}, \varphi^{-1}(a)}$, which is the union of the paths $[F_i, \infty]$ and $[A_j, \infty]$ for $i = 1, \ldots, d+1$, $j = 2, \ldots, d$. Let Q be an endpoint of MinResLoc(φ), and let $P \in \{F_1, \ldots, F_{d+1}, A_2, \ldots, A_d\}$ be such that $Q \in [P, \infty]$. Put $L_0 = H(P)$. We have $[H(F_i):H] \leq d+1$ for each i, and $[H(A_j):H] \leq d-1$ for each j, so $[L_0:H] \leq d+1$. Fix $\gamma \in \mathrm{GL}_2(L_0)$ with $\gamma(0) = P$ and $\gamma(\infty) = \infty$, and let $Q_0 = \gamma^{-1}(Q) \in [0, \infty]$. Since Q is an endpoint of MinResLoc(φ), the restriction of $\mathrm{ordRes}_{\varphi^\gamma}(\cdot)$ to $[0,\infty]$ has a break between affine pieces at Q_0. Hence the discussion after formula (2.9) shows there are an $\alpha \in L_0^\times$ and an integer e with $1 \leq e \leq d+1$ such that $Q_0 = \zeta_{0,|\alpha|^{1/e}}$. Set $L = L_0(\alpha^{1/e})$. Then Q is rational over L, and $[L:H] \leq (d+1)^2$. ∎

Let $\mathrm{Aut}^c(K/H)$ be the group of continuous automorphisms of K fixing H. The action of $\mathrm{Aut}^c(K/H)$ on $\mathbb{P}^1(K)$ extends to an action on $\mathbf{P}^1_K$ which preserves the type of each point. The action can be described as follows: For points of type II or III, if $\sigma \in \mathrm{Aut}^c(K/H)$ and Q corresponds to a disc $D(b,r)$, then $\sigma(Q)$ corresponds to $D(\sigma(b), r)$. The image disc is well-defined, since for any $b' \in K$ with $D(b', r) = D(b, r)$ we have $|\sigma(b') - \sigma(b)| = |b' - b| \leq r$. For points of type IV, if Q corresponds to a sequence of nested discs $\{D(a_i, r_i)\}_{i \geq 0}$, then $\sigma(Q)$ corresponds to the sequence of nested discs $\{D(\sigma(a_i), r_i)\}_{i \geq 0}$. Although this description of the action depends on the choice of a system of coordinates for $\mathbf{P}^1_K$, the action is canonical:

PROPOSITION 3.3. *For all $\varphi(z) \in K(z)$, $\sigma \in \mathrm{Aut}^c(K/H)$, and $Q \in \mathbf{P}^1_K$, we have $\sigma(\varphi(Q)) = (\sigma(\varphi))(\sigma(Q))$. In particular, $\sigma(\gamma(Q)) = \gamma(\sigma(Q))$ for all $\gamma \in \mathrm{GL}_2(H)$ and $\sigma \in \mathrm{Aut}^c(K/H)$. Thus the action of $\mathrm{Aut}^c(K/H)$ on $\mathbf{P}^1_K$ is independent of H-rational changes of coordinates.*

Proof. Given $\varphi(z) \in K(z)$ and $\sigma \in \mathrm{Aut}^c(K/H)$, if Q is of type I, the assertion is clear. If Q is of type II and corresponds to a disc $D(b,r)$, the assertion follows from the case of type I points and the description of the action of φ on generic type I points in $D(b,r)$ given in [1, Proposition 2.18]. Finally, if Q is of type III or IV, the assertion follows from the case of type II points and continuity. ∎

If a point of $\mathbf{P}^1_K$ is weakly rational over H, clearly it is fixed by each σ in $\mathrm{Aut}^c(K/H)$. However, the converse is not true: there can be type II points in $\mathbf{P}^1_K$ outside the tree spanned by $\mathbb{P}^1(H)$, which are fixed by $\mathrm{Aut}^c(K/H)$. For example, if $H = \mathbb{Q}_2$ and $K = \mathbb{C}_2$, then $\zeta_{i,1/2}$ is fixed by $\mathrm{Aut}^c(\mathbb{C}_2/\mathbb{Q}_2)$ since $|\sigma(i) - i| \le 1/2$ for each $\sigma \in \mathrm{Aut}^c(\mathbb{C}_2/\mathbb{Q}_2)$. However $D(i, 1/2) \cap \mathbb{Q}_2$ is empty: $|x - i| \ge 1/\sqrt{2}$ for each $x \in \mathbb{Q}_2$. Thus $\zeta_{i,1/2}$ is not in the tree spanned by $\mathbb{P}^1(\mathbb{Q}_2)$. It would be interesting to know how far outside the tree spanned by $\mathbb{P}^1(H)$, points fixed by $\mathrm{Aut}^c(K/H)$ can lie.

The action of $\sigma \in \mathrm{Aut}^c(K/H)$ on $\mathbf{P}^1_K$ is continuous for the strong topology: indeed, the description of the action shows that for all $x, y \in \mathbf{H}^1_K$, one has $\rho(\sigma(x), \sigma(y)) = \rho(x, y)$. It follows that σ takes paths to paths: if $[x, y]$ is a path with endpoints in $\mathbf{H}^1_K$, then for each $Q \in \mathbf{H}^1_K$ we have $Q \in [x, y]$ iff $\rho(x, y) = \rho(x, Q) + \rho(Q, y)$; thus $Q \in [x, y]$ iff $\sigma(Q) \in [\sigma(x), \sigma(y)]$. If $[x, y]$ has one or both endpoints in $\mathbb{P}^1(K)$, it can be exhausted by paths with endpoints in $\mathbf{H}^1_K$, so we still have $\sigma([x, y]) = [\sigma(x), \sigma(y)]$.

We will say that a subset $X \subset \mathbf{P}^1_K$ is *stable under* $\mathrm{Aut}^c(K/H)$ if $\sigma(x) \in X$ for each $x \in X$ and $\sigma \in \mathrm{Aut}^c(K/H)$; and X is *pointwise fixed by* $\mathrm{Aut}^c(K/H)$ if $\sigma(x) = x$ for each $x \in X$ and $\sigma \in \mathrm{Aut}^c(K/H)$.

PROPOSITION 3.4. *Suppose* $\varphi(z) \in H(z)$ *and* $d \ge 2$. *Then* $\mathrm{MinResLoc}(\varphi)$ *is stable under* $\mathrm{Aut}^c(K/H)$, *and it contains at least one point fixed by* $\mathrm{Aut}^c(K/H)$. *However, it need not be pointwise fixed by* $\mathrm{Aut}^c(K/H)$, *and it need not contain weakly* H*-rational points.*

Proof. If φ is rational over H, then $\mathrm{ordRes}_\varphi(\sigma(Q)) = \mathrm{ordRes}_\varphi(Q)$ for all $\sigma \in \mathrm{Aut}^c(K/H)$ and all $Q \in \mathbf{P}^1_K$. Thus, $\mathrm{MinResLoc}(\varphi)$ is stable under $\mathrm{Aut}^c(K/H)$. To see that $\mathrm{MinResLoc}(\varphi)$ always contains at least one point fixed by $\mathrm{Aut}^c(K/H)$, note that if $\mathrm{MinResLoc}(\varphi)$ consists of a single point, $\mathrm{Aut}^c(K/H)$ fixes that point. On the other hand, if $\mathrm{MinResLoc}(\varphi)$ is a segment, then since $\mathrm{Aut}^c(K/H)$ preserves path distances, each $\sigma \in \mathrm{Aut}^c(K/H)$ must either leave $\mathrm{MinResLoc}(\varphi)$ pointwise fixed, or flip it end-to-end; in either case σ fixes the midpoint of $\mathrm{MinResLoc}(\varphi)$.

Example 6.2 below, with $\varphi(z) = (z^2 - z)/(2z)$ and $K = \mathbb{C}_2$ and $H = \mathbb{Q}_2$, shows that $\mathrm{MinResLoc}(\varphi)$ can be pointwise fixed by $\mathrm{Aut}^c(K/H)$ without meeting the tree spanned by $\mathbb{P}^1(H)$: $\mathrm{MinResLoc}(\varphi) = \{\zeta_{i,1/2}\}$ and, as shown above, $\zeta_{i,1/2}$ does not belong to the tree spanned by $\mathbb{P}^1(H)$.

It is also possible for $\mathrm{MinResLoc}(\varphi)$ to be a segment "orthogonal to" the tree spanned by $\mathbb{P}^1(H)$: take $K = \mathbb{C}_p$ and $H = \mathbb{Q}_p$ with p odd. If $a \in \mathbb{Z}_p^\times$ is a non-square unit, and $u = \sqrt{a}$, then the function $\varphi_1(z) = \frac{-z^3+(4p^n+1)u^2z}{(4p^n-1)z^2+u^2}$ from Example 6.5 below is $\mathbb{Q}_p$-rational. The segment $[\zeta_{-u,1/p^n}, \zeta_{u,1/p^n}]$ is its minimal resultant locus, and this segment meets the tree spanned by $\mathbb{P}^1(\mathbb{Q}_p)$ only at the $\mathbb{Q}_p$-rational type II point ζ_G.

Likewise, the function $\varphi_2(z) = \frac{-z^3+(4p^n+1)u^2pz}{(4p^n-1)z^2+pu^2}$ from Example 6.5 is $\mathbb{Q}_p$-rational. Its minimal resultant locus meets the tree spanned by $\mathbb{P}^1(\mathbb{Q}_p)$ at $\zeta_{0,p^{-1/2}}$, but that point is not $\mathbb{Q}_p$-rational because its radius does not belong to $|\mathbb{Q}_2^\times|$. In both examples, each $\sigma \in \mathrm{Aut}^c(K/H)$ with $\sigma(\sqrt{a}) = -\sqrt{a}$ flips $\mathrm{MinResLoc}(\varphi)$ end-to-end; the midpoint of $\mathrm{MinResLoc}(\varphi)$ is the only point fixed by $\mathrm{Aut}^c(K/H)$. ∎

If Q is a type II point rational over H, the action of $\mathrm{Aut}^c(K/H)$ on $\mathbf{P}^1_K$ induces an action of $\mathrm{Aut}^c(K/H)$ on the tangent space T_Q, which takes the class of a path $[Q,x]$ to the class of $[Q,\sigma(x)]$. This is well-defined, since if x and x' belong to the same tangent direction at Q, then the paths $[Q,x]$ and $[Q,x']$ share an initial segment; thus $[Q,\sigma(x)]$ and $[Q,\sigma(x')]$ share an initial segment as well.

Now consider the arithmetic case, where $H = H_v$ is a local field and $K = \mathbb{C}_v$ is the completion of an algebraic closure of H_v. In this case, the H-rational type II points are discrete in $\mathbf{H}^1_K$ for the strong topology, and the subtree of $\mathbf{P}^1_K$ spanned by $\mathbb{P}^1(H)$ is branched at precisely the H-rational type II points.

The following proposition shows that if φ is rational over H_v, and if $P \notin \mathrm{MinResLoc}(\varphi)$ is a type II point rational over H_v, then $\mathrm{MinResLoc}(\varphi)$ lies in a tangent direction at P containing points of $\mathbb{P}^1(H_v)$.

PROPOSITION 3.5. *Suppose H_v is a local field and φ is rational over H_v. Let P be an H_v-rational type* II *point not contained in* $\mathrm{MinResLoc}(\varphi)$*. Then* $\mathrm{MinResLoc}(\varphi)$ *lies in a tangent direction at P coming from the tree spanned by $\mathbb{P}^1(H_v)$.*

Proof. Note that since $\mathrm{MinResLoc}(\varphi)$ is stable under $\mathrm{Aut}^c(K/H_v)$ and is connected, the tangent direction at P that it lies in must be fixed by $\mathrm{Aut}^c(K/H_v)$. If H_v has residue field $\mathbb{F}_q$, then T_P is parametrized by $\mathbb{P}^1(\overline{\mathbb{F}_q})$, and the tangent directions at P fixed by $\mathrm{Aut}^c(K/H_v)$ correspond to the points of $\mathbb{P}^1(\mathbb{F}_q)$. These are precisely the tangent directions at P coming from the tree spanned by $\mathbb{P}^1(H_v)$. ∎

COROLLARY 3.6. *Suppose $H = H_v$ is a local field and $K = \mathbb{C}_v$, and that $\varphi(z) \in H_v(z)$ has degree $d \geq 2$. If* $\mathrm{MinResLoc}(\varphi)$ *contains no points of*

$\mathbf{P}^1_K$ *rational over* H_v, *then each extension* L/H_v *such that* MinResLoc(φ) *contains an* L*-rational point is ramified over* H_v.

Proof. Suppose that MinResLoc(φ) has no points rational over H_v. Let $Q \in$ MinResLoc(φ) be a type II point rational over an extension L/H_v, and let P be the point of the tree Γ_{H_v} spanned by $\mathbb{P}^1(H_v)$ closest to Q (possibly $Q = P$). Necessarily P is of type II. Furthermore, P cannot be rational over H_v, since if it were, then $Q \neq P$, and Proposition 3.5 shows Q would lie in a direction $\vec{v} \in T_P$ coming from $\mathbb{P}^1(H_v)$, contradicting that P is the point of Γ_{H_v} nearest Q.

If $Q = P$, then under Berkovich's classification theorem, Q corresponds to a disc $D(a, r)$ where $a \in H_v$ and $r \in |L^\times| \backslash |H_v^\times|$. Clearly L is ramified over H_v in this case.

If $Q \neq P$, then Q corresponds to a disc $D(a, r)$ where $a \in L$ and $r \in |L^\times|$, and P corresponds to a disc $D(b, R)$ where $b \in H_v$. As shown above, P is not rational over H_v, so $R \notin |H_v^\times|$. We cannot have $b \in D(a, r)$ since otherwise Q would belong to the tree spanned by $\mathbb{P}^1(H_v)$, contradicting our assumption that $Q \neq P$. It follows that P is the point where the paths $[a, \infty]$ and $[b, \infty]$ meet, so $R = |b - a|$. Since both a, b are in L, we have $R \in |L^\times|$. Thus, again L/H_v is ramified. ∎

4. Applications. In this section we give some applications of the theory developed above. We show that the function $\mathrm{ordRes}_\varphi(\cdot)$ satisfies the principle of steepest descent, that over a Henselian ground field the property that φ has potential good reduction is first-order in the sense of mathematical logic, we give a negative answer to a question of Silverman concerning the existence of global minimal models for rational functions φ defined over number fields, and we show that φ need not achieve its minimal resultant over the field of moduli for the minimal resultant problem.

The principle of steepest descent. In Theorem 1.1 it was shown that $\mathrm{ordRes}_\varphi(\cdot)$ is convex upwards. This means that $\mathrm{ordRes}_\varphi(\cdot)$ satisfies the principle of steepest descent: at each $Q \in \mathbb{H}^1_K$ there is at most one direction in which $\mathrm{ordRes}_\varphi(\cdot)$ is decreasing, and by following the path of steepest descent one will eventually reach the global minimum.

When φ is defined over a local field H_v, this yields a simple algorithm for finding the minimum value of $\mathrm{ordRes}(\varphi^\gamma)$ under H_v-rational changes of coordinates: starting at ζ_G, move in the direction of steepest descent, stepping from one H_v-rational type II point Q to the next, until there are no adjacent H_v-rational points where the value of $\mathrm{ordRes}_\varphi(\cdot)$ is smaller; at such a point the H_v-minimum has been reached, and Proposition 3.1 provides a $\gamma \in \mathrm{GL}_2(H_v)$ for which $\mathrm{ordRes}(\varphi^\gamma)$ is minimal.

Theorem 1.1 gives a bound for the number of steps needed by this algorithm. If $\mathrm{ordRes}_\varphi(\cdot)$ is not minimal at a given Q, Proposition 3.5 shows that the direction of steepest descent is rational over H_v, and Lemma 2.4 gives a criterion identifying that direction. Proposition 3.5 and Lemma 2.4 also provide a criterion telling whether the minimum value of $\mathrm{ordRes}_\varphi(\cdot)$ on H_v-rational points is the global minimum on $\mathbf{P}^1_K$: at a point Q where the H_v-minimum occurs, if there is no H_v-rational direction where $\mathrm{ordRes}_\varphi(\cdot)$ is decreasing, the H_v-minimum is the global minimum; otherwise it is not.

The algorithm described above is essentially a geometric version of the Bruin–Molnar algorithm [6]: Bruin and Molnar construct a bounded subset S of $\mathrm{GL}_2(H)$ such that the H_v-minimum of $\mathrm{ordRes}(\varphi^\gamma)$ is achieved by some $\gamma \in S$, and they find the minimum value by means of a recursive search. The convexity of $\mathrm{ordRes}_\varphi(\cdot)$ means the Bruin–Molnar algorithm runs without any back-tracking.

Over a Henselian field, potential good reduction is first-order. Let H be a subfield of K such that $\varphi(z) \in H(z)$. It follows from Theorem 3.2 that if H is Henselian (in particular, if H is a local field), the property that φ has potential good reduction is first-order in the theory of H, in the sense of mathematical logic:

PROPOSITION 4.1. *For each $d \geq 2$, there is a first order formula*

$$\mathcal{F}_d(f_0, \ldots, f_d, g_0, \ldots, g_d)$$

in the language of valued fields such that if H is a Henselian nonarchimedean valued field, and if $\varphi(z) = (f_d a^d + \cdots + f_0)/(g_d z^d + \cdots + g_0) \in H(z)$, then φ has potential good reduction if and only if $H \models \mathcal{F}_d(f_0, \ldots, f_d, g_0, \ldots, g_d)$.

Proof. Note that φ has potential good reduction if and only if there is a $\gamma \in \mathrm{GL}_2(K)$ such that $\mathrm{ordRes}(\varphi^\gamma) = 0$. By Theorem 3.2, there is an extension L/H with $[L : H] \leq (d+1)^2$ for which there is a $\gamma \in \mathrm{GL}_2(L)$ with $Q = \gamma(\zeta_G) \in \mathrm{MinResLoc}(\varphi)$. Hence φ has potential good reduction if and only if $\mathrm{ordRes}(\varphi^\gamma) = 0$ for this γ.

Since H is Henselian, for each finite extension $H(\beta)/H$ there is a unique extension of the valuation $\mathrm{ord}(\cdot)$ on H to a valuation on $H(\beta)$, given by $\mathrm{ord}_{H(\beta)}(z) = (1/m)\,\mathrm{ord}(N_{H(\beta)/H}(z))$ for $z \in H(\beta)$, where $[H(\beta) : H] = m$. If $z = a_0 + a_1\beta + \cdots + a_{m-1}\beta^{m-1}$ with $a_0, \ldots, a_{m-1} \in H$, then $N_{H(\beta)/H}(z)$ is a universal polynomial in the a_i and the coefficients of the minimal polynomial of β over H. Write (F, G) for the natural representation of φ.

Let $\mathcal{F}_{d,0}(f_0, \ldots, f_d, g_0, \ldots, g_d)$ be the formula "$\mathrm{Res}(F, G) \neq 0$", and for each $m = 1, \ldots, (d+1)^2$, let $F_{d,m}(f_0, \ldots, f_d, g_0, \ldots, g_d)$ be the formula

"There exist $a_1, \dots, a_m \in H$ with $h_m(x) = x^m + a_1 x^{m-1} + \cdots + a_m$ irreducible over H, and there exist a root β of $h_m(x)$ and numbers $a, b, c, d \in H(\beta)$ with $ad - bc \neq 0$, such that for $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, we have $\mathrm{ord}_{H(\beta)}(\mathrm{Res}(F^\gamma, G^\gamma)) - 2d \min(\mathrm{ord}_{H(\beta)}(F^\gamma), \mathrm{ord}_{H(\beta)}(G^\gamma)) = 0$".

Then we can take $\mathcal{F}_d$ to be $\mathcal{F}_{d,0} \wedge (\mathcal{F}_{d,1} \vee \cdots \vee \mathcal{F}_{d,(d+1)^2})$. ■

An answer to a question of Silverman. Let H be a number field, and suppose $\varphi(z) \in H(z)$ has degree $d \geq 2$. In [15, §4.11] Silverman asks when it is possible to choose an "optimal" integral representation for φ over H, analogous to a global minimal Weierstrass model for an elliptic curve, and he provides a necessary condition for such a representation to exist.

Here we show that Silverman's necessary condition is not sufficient: there is an obstruction coming from the ideal class group of H. Let $\mathcal{O}_H$ be the ring of integers of H. Given a nonarchimedean place v of H, let H_v be the completion of H at v, $\mathcal{O}_v$ the valuation ring of H_v, and π_v a generator for the maximal ideal of $\mathcal{O}_v$. Let $\mathbb{C}_v$ be the completion of the algebraic closure of H_v. Write $\mathrm{ord}_v(\cdot)$ for the valuation on $\mathbb{C}_v$ normalized so that $\mathrm{ord}_v(\pi_v) = 1$, and $\mathrm{ordRes}_v(\varphi)$ and $\mathrm{ordRes}_{\varphi,v}(\cdot)$ for the functions previously denoted $\mathrm{ordRes}(\varphi)$ and $\mathrm{ordRes}_\varphi(\cdot)$. In this way the theory developed in §2 and §3 is applicable for each nonarchimedean place v of H.

A representation (F, G) of φ with $F(X,Y), G(X,Y) \in H[X,Y]$ will be called a *representation of φ over H*; such a pair is unique up to scaling by $H^\times$. One can always arrange that $F, G \in \mathcal{O}_H[X,Y]$; in that case, the representation is called *integral.* Silverman defines the "global minimal resultant" of φ to be the ideal

$$\mathfrak{R}_\varphi = \prod_{\mathfrak{p}} \mathfrak{p}^{\varepsilon_{\mathfrak{p}}(\varphi)}, \tag{4.1}$$

where for each prime $\mathfrak{p} = \mathfrak{p}_v$ of $\mathcal{O}_H$,

$$\varepsilon_{\mathfrak{p}}(\varphi) = \min_{\substack{\gamma \in \mathrm{GL}_2(H_v) \\ \mathrm{ord}_{\mathfrak{p}}(F^\gamma, G^\gamma) \geq 0}} \mathrm{Res}(F^\gamma, G^\gamma) \geq 0.$$

Here the product in (4.1) is finite, since for a given representation (F, G) of φ over H,

$$\mathrm{ord}_{\mathfrak{p}}(F^\gamma, G^\gamma) \geq 0 \quad \text{and} \quad \mathrm{ord}_{\mathfrak{p}}(\mathrm{Res}(F, G)) = 0$$

for all but finitely many $\mathfrak{p}$. We will say that φ has a *global minimal model over H* if for some $\gamma \in \mathrm{GL}_2(H)$ the function φ^γ has an integral representation (F^γ, G^γ) over H such that

$$\mathrm{ord}_{\mathfrak{p}}(\mathrm{Res}(F^\gamma, G^\gamma)) = \varepsilon_{\mathfrak{p}}(\varphi) \quad \text{for each prime } \mathfrak{p} \text{ of } \mathcal{O}_H.$$

Given a representation (F,G) for φ over H, in [15, Proposition 4.99] Silverman shows there is a fractional ideal $\mathfrak{a}_{F,G}$ of H such that

$$\mathfrak{R}_\varphi = \begin{cases} \mathfrak{a}_{F,G}^{2d} \cdot (\operatorname{Res}(F,G)) & \text{if } d \text{ is odd,} \\ \mathfrak{a}_{F,G}^{d} \cdot (\operatorname{Res}(F,G)) & \text{if } d \text{ is even.} \end{cases}$$

Let $I(H)$ be the group of fractional ideals of H, and $P(H)$ the group of principal fractional ideals. Silverman shows that if d is odd, the ideal class $[\mathfrak{a}_\varphi] := [\mathfrak{a}_{F,G}] \in I(H)/P(H)$ is independent of the choice of (F,G), while if d is even, the refined ideal class $[\mathfrak{a}_\varphi] := [\mathfrak{a}_{F,G}] \in I(H)/\{(\alpha)^2 : (\alpha) \in P(H)\}$ is independent of the choice of (F,G). He calls $[\mathfrak{a}_\varphi]$ the *Weierstrass class of φ over H*.

In [15, Proposition 4.100] Silverman shows that if φ has a global minimal model over H, then the Weierstrass class $[\mathfrak{a}_\varphi]$ is trivial. In [15, Exercise 4.46] he asks:

(a) When $H = \mathbb{Q}$, does every $\varphi(z) \in \mathbb{Q}(z)$ of degree $d \geq 2$ have a global minimal model over $\mathbb{Q}$?

(b) When H is an arbitrary number field and $\varphi(z) \in H(z)$ has degree $d \geq 2$, if S is a finite set of primes of $\mathcal{O}_H$ such that the localization $\mathcal{O}_{H,S}$ is a principal ideal domain, does φ have a *global S-minimal model*? In other words, is there a $\gamma \in \mathrm{GL}_2(H)$ such that φ^γ has a representation (F^γ, G^γ) with $F^\gamma(X,Y), G^\gamma(X,Y) \in \mathcal{O}_{H,S}[X,Y]$, satisfying

$$\operatorname{ord}_\mathfrak{p}(\operatorname{Res}(F^\gamma, G^\gamma)) = \varepsilon_\mathfrak{p}(\varphi) \quad \text{for each prime } \mathfrak{p} \notin S?$$

(c) When H is an arbitrary number field and $\varphi(z) \in H(z)$ has degree $d \geq 2$, if the Weierstrass class $[\mathfrak{a}_\varphi]$ is trivial, does φ have a global minimal model over H?

As has already been noted by Bruin and Molnar [6], the answer to the first two questions is "Yes". This follows from the Strong Approximation Theorem and the fact that the subgroup $\mathrm{Aff}_2(K) \subset \mathrm{GL}_2(K)$ acts transitively on the type II points in $\mathbf{P}^1_K$. Indeed, in (b), let $\widetilde{S} \supseteq S$ be a finite set of primes such that φ has good reduction outside $\widetilde{S}$. For each prime $\mathfrak{p} = \mathfrak{p}_v \in \widetilde{S}$, choose a $\gamma_\mathfrak{p} \in \mathrm{GL}_2(H)$ such that $\operatorname{ordRes}_v(\varphi^{\gamma_\mathfrak{p}}) = \varepsilon_\mathfrak{p}$ and put $\xi_\mathfrak{p} = \gamma_\mathfrak{p}(\zeta_G)$. By Proposition 3.1, $\xi_\mathfrak{p} \in \mathbf{P}^1_K$ is rational over H; thus there exist $a_\mathfrak{p}, b_\mathfrak{p} \in H$ with $a_\mathfrak{p} \neq 0$ such that $\xi_\mathfrak{p} = \zeta_{b_\mathfrak{p}, |a_\mathfrak{p}|_v}$. Since $\mathcal{O}_{H,S}$ is a PID, there is an $a \in H$ such that $\operatorname{ord}_\mathfrak{p}((a)) = \operatorname{ord}_\mathfrak{p}((a_\mathfrak{p}))$ for each $\mathfrak{p} \in \widetilde{S}$ and $\operatorname{ord}_\mathfrak{p}((a)) = 0$ for each $\mathfrak{p} \notin \widetilde{S}$. By the Strong Approximation Theorem there is a $b \in H$ such that $\operatorname{ord}_\mathfrak{p}(b - b_\mathfrak{p}) > \operatorname{ord}(a_\mathfrak{p})$ for each $\mathfrak{p} \in \widetilde{S}$ and $\operatorname{ord}_\mathfrak{p}(b) = 0$ for each $\mathfrak{p} \notin \widetilde{S}$. Set $\gamma = \left[\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right]$; then $\gamma(\zeta_G) = \xi_\mathfrak{p}$ for each $\mathfrak{p} \in \widetilde{S}$, and $\gamma(\zeta_G) = \zeta_G$ for each $\mathfrak{p} \notin \widetilde{S}$, so $\operatorname{ordRes}_v(\varphi^\gamma) = \varepsilon_{\mathfrak{p}_v}$ for each prime $\mathfrak{p}_v$. Let (F^γ, G^γ) be a representation of

φ^γ over H; since $\mathcal{O}_{H,S}$ is a PID, we can assume (F^γ, G^γ) has been scaled so that $\min(\mathrm{ord}_{\mathfrak{p}}(F^\gamma), \mathrm{ord}_{\mathfrak{p}}(G^\gamma)) = 0$ for each $\mathfrak{p} \notin S$. Then F^γ, G^γ are defined over $\mathcal{O}_{H,S}$, and $\mathrm{ord}_{\mathfrak{p}}(\mathrm{Res}(F^\gamma, G^\gamma)) = \varepsilon_{\mathfrak{p}}$ for each $\mathfrak{p} \notin S$, so (F^γ, G^γ) is a global S-minimal model.

The answer to question (c) is "No" in general. The underlying reason for this is a disconnect between the values of $\mathrm{ordRes}_v(\cdot)$ and the points at which they are taken. To obtain counterexamples, consider polynomials of the form $\varphi(z) = z^d + c$ with $d \geq 2$, $c \in H$. For a given prime $\mathfrak{p} = \mathfrak{p}_v$ of $\mathcal{O}_H$, if $\mathrm{ord}_v(c) \geq 0$, then $\varphi(z)$ has good reduction at $\mathfrak{p}$. Suppose that $\mathrm{ord}_v(c) < 0$. Then $\mathrm{ordRes}_v(\varphi) = -2d\,\mathrm{ord}_v(c)$. Computing $\mathrm{ordRes}_{\varphi,v}(\cdot)$ on the path $[0,\infty] \subset \mathbf{P}^1_{K,v}$, we find that for each $A \in \mathbb{C}_v^\times$,

$$\mathrm{ordRes}_{\varphi,v}(\zeta_{0,|A|_v}) = \max\bigl((d-d^2)\,\mathrm{ord}_v(A), -2d\,\mathrm{ord}_v(c) + (d+d^2)\,\mathrm{ord}_v(A)\bigr).$$

This is minimal when $\mathrm{ord}_v(A) = (1/d)\,\mathrm{ord}_v(c)$. If $(1/d)\,\mathrm{ord}_v(c)$ is not an integer, by convexity the least value of $\mathrm{ordRes}_{\varphi,v}(\cdot)$ on H-rational points in $\mathbf{P}^1_{K,w}$ occurs when $\mathrm{ord}_v(A)$ is one of the two integers adjacent to $(1/d)\,\mathrm{ord}_v(c)$.

For a counterexample when d is odd, take $\varphi(z) = z^5 + 1/(1+4\sqrt{-5})$, so $d = 5$ and $c = 1/(1+4\sqrt{-5})$, with $H = \mathbb{Q}(\sqrt{-5})$. The field H has class number 2. The ideal $\mathfrak{p} = \mathfrak{p}_v = (3, 1+\sqrt{-5})$ in $\mathcal{O}_H$ is one of the primes containing (3); it is not principal, but $\mathfrak{p}^4 = (1+4\sqrt{-5})$, so $\mathrm{ord}_v(c) = -4$.

Clearly $\varphi(z)$ has good reduction at all primes other than $\mathfrak{p}$. The least value of $\mathrm{ordRes}_{\varphi,v}(\cdot)$ on H-rational points occurs only when $\mathrm{ord}_v(A) = -1$, and one has

$$20 = \mathrm{ordRes}_{\varphi,v}(\zeta_{0,|A|_v}) < \mathrm{ordRes}_v(\varphi) = 40.$$

The integral representation (F,G) with $F(X,Y) = X^5/c + Y^5$ and $G(X,Y) = Y^5/c$ satisfies $(\mathrm{Res}(F,G)) = \mathfrak{p}^{40}$, while $\mathfrak{R}_\varphi = \mathfrak{p}^{20}$. Since $\mathfrak{R}_\varphi = \mathfrak{a}_{F,G}^{10} \cdot (\mathrm{Res}(F,G))$, it follows that $\mathfrak{a}_{F,G} = \mathfrak{p}^{-2} = (1/(2-\sqrt{-5}))$. Thus the class $[\mathfrak{a}_\varphi]$ is trivial. However, there is no $\gamma \in \mathrm{GL}_2(H)$ for which $\mathrm{ordRes}_v(\varphi^\gamma) = \mathfrak{R}_\varphi$. If there were, in $\mathbf{P}^1_{K,w}$ we would have $\gamma(\zeta_G) = \zeta_{0,3}$, while for each finite place $w \neq v$, in $\mathbf{H}^1_{K,w}$ we would have $\gamma(\zeta_G) = \zeta_G$. By the proof of Proposition 3.1, this would mean that $\mathrm{ord}_v(\det(\gamma)) = -1$ and $\mathrm{ord}_w(\det(\gamma)) = 0$ for all $w \neq v$, so $(\det(\gamma)) = \mathfrak{p}^{-1}$. This is a contradiction since $\mathfrak{p}^{-1}$ is not principal.

For a counterexample when d is even, take $\varphi(z) = z^4 + 1/(19+4\sqrt{-23})$, so $d = 4$ and $c = 1/(19+4\sqrt{-23})$, with $H = \mathbb{Q}(\sqrt{-23})$. The field H has class number 3. The ideal $\mathfrak{p} = \mathfrak{p}_v = (3, (1+\sqrt{-23})/2)$ in $\mathcal{O}_H$ is one of the primes containing (3); it is not principal, but $\mathfrak{p}^3 = (2-\sqrt{-23})$ and $\mathfrak{p}^6 = (19+4\sqrt{-23})$, so $\mathrm{ord}_v(c) = -6$.

Clearly $\varphi(z)$ has good reduction at all primes other than $\mathfrak{p}$. The least value of $\mathrm{ordRes}_{\varphi,v}(\cdot)$ on H-rational points occurs only when $\mathrm{ord}_v(A) = -2$, and one has

$$24 = \mathrm{ordRes}_{\varphi,v}(\zeta_{0,|A|_v}) < \mathrm{ordRes}_v(\varphi) = 48.$$

The normalized representation (F,G) with $F(X,Y) = X^4/c + Y^4$ and $G(X,Y) = Y^4/c$ satisfies $(\mathrm{Res}(F,G)) = \mathfrak{p}^{48}$, while $\mathfrak{R}_\varphi = \mathfrak{p}^{24}$. Since $\mathfrak{R}_\varphi = \mathfrak{a}_{F,G}^4 \cdot (\mathrm{Res}(F,G))$, it follows that $\mathfrak{a}_{F,G} = \mathfrak{p}^{-6} = (1/(2-\sqrt{-23}))^2$. Thus the class $[\mathfrak{a}_\varphi]$ is trivial. However, there is no $\gamma \in \mathrm{GL}_2(H)$ for which $\mathrm{ordRes}_v(\varphi^\gamma) = \mathfrak{R}_\varphi$. If there were, we would have $\mathrm{ord}_v(\det(\gamma)) = -2$ and $\mathrm{ord}_w(\det(\gamma)) = 0$ for all $w \neq v$, so $(\det(\gamma)) = \mathfrak{p}^{-2}$. This is impossible since $\mathfrak{p}^{-2}$ is not principal.

Failure to achieve the minimal resultant over the field of moduli. Suppose $\varphi(z) \in H(z)$, where $H \subset K$. Let M be the value of $\mathrm{ordRes}_\varphi(\cdot)$ on $\mathrm{MinResLoc}(\varphi)$, and let $\mathcal{F}_H(\varphi)$ be the set of fields L with $H \subseteq L \subseteq K$ for which there is some $\gamma \in \mathrm{GL}_2(L)$ such that $\mathrm{ordRes}(\varphi^\gamma) = M$. When $\mathrm{MinResLoc}(\varphi) = \{Q\}$ consists of a single point, $\mathcal{F}_H(\varphi)$ is the set of fields $H \subseteq L \subseteq K$ such that there is some $\gamma \in \mathrm{GL}_2(L)$ with $\gamma(\zeta_G) = Q$. The *field of moduli* for the minimal resultant problem is

$$H_\varphi := \bigcap_{L \in \mathcal{F}_H(\varphi)} L.$$

One can ask if there is a $\gamma \in \mathrm{GL}_2(H_\varphi)$ for which $\mathrm{ordRes}(\varphi^\gamma)$ is minimal. The answer is clearly "Yes" if $\mathrm{MinResLoc}(\varphi)$ contains an H-rational point, but in general it is "No":

PROPOSITION 4.2. *Suppose $\varphi(z) \in H(z)$, and that $H_\varphi \supseteq H$ is the field of moduli for the minimal resultant problem. There may be no $\gamma \in \mathrm{GL}_2(H_\varphi)$ for which* $\mathrm{ordRes}(\varphi^\gamma) = M$*, even when φ has potential good reduction and has a trivial automorphism group.*

Proof. In Example 6.1 below, take $d = p$ where p is an odd prime, with $\varphi(z) = (z^p - p)/z^{p-1}$, $K = \mathbb{C}_p$, and $H = \mathbb{Q}_p$. Then $\mathrm{ord}(\mathrm{Res}(\varphi)) = p-1$ and $\mathrm{MinResLoc}(\varphi) = \{Q\}$ where $Q = \zeta_{0,p^{1/p}}$, and φ achieves good reduction at Q. Note that Q is rational over an extension L/K if and only if the value group of L contains $p^{1/p}$. In particular, Q is rational over $L_1 = \mathbb{Q}_p(\sqrt[p]{p})$ and over $L_2 = \mathbb{Q}_p(\zeta_p \sqrt[p]{p})$ where ζ_p is any primitive pth root of unity. One easily sees that $L_1 \cap L_2 = \mathbb{Q}_p$ (otherwise $L_1 = L_2$, since both extensions have degree p; but then $\zeta_p \in L_1$, so $p-1 = [\mathbb{Q}_p(\zeta_p) : \mathbb{Q}_p]$ divides $[L_1 : \mathbb{Q}_p] = p$). Thus $H_\varphi = \mathbb{Q}_p$. However, $p^{1/p}$ is not in the value group of $\mathbb{Q}_p^\times$, so by Proposition 3.1 there can be no $\gamma \in \mathrm{GL}_2(\mathbb{Q}_p)$ with $\gamma(\zeta_G) = Q$.

The function $\varphi(z) = (z^p - p)/z^{p-1}$ has automorphisms; indeed, for each $(p-1)$st root of unity ξ, the map $\gamma(z) = \xi \cdot x$ satisfies $\varphi^\gamma = \varphi$. However, we can easily modify φ to destroy these automorphisms. Define

$$\widehat{\varphi}(z) = \frac{z^p + p^{3(p-1)}z - p}{z^{p-1}}.$$

Clearly $\widehat{\varphi}(z)$ is rational over $\mathbb{Q}_p$. In Theorem 1.2 we have $f(d) < 3$, so $\mathrm{MinResLoc}(\widehat{\varphi}) = \mathrm{MinResLoc}(\varphi) = \{\zeta_{0,p^{1/p}}\}$ and $\widehat{\varphi}$ has potential good reduction at $\zeta_{0,p^{1/p}}$. By the same argument as above, $H_{\widehat{\varphi}} = \mathbb{Q}_p$ and there is no $\gamma \in \mathrm{GL}_2(\mathbb{Q}_p)$ with $\gamma(\zeta_G) = Q$. However, $\widehat{\varphi}$ has no nontrivial automorphisms: If $\widehat{\varphi}^\gamma = \widehat{\varphi}$ for some $\gamma \in \mathrm{GL}_2(\mathbb{C}_p)$, then γ permutes the poles of $\widehat{\varphi}$, preserving their multiplicities. In this case $\widehat{\varphi}$ has a simple pole at ∞ and a pole of order $p-1$ at 0, so γ fixes 0 and ∞. Likewise, γ permutes the fixed points of $\widehat{\varphi}$, preserving their multiplicities. In this case, $\widehat{\varphi}$ has a fixed point of order p at ∞, and a simple fixed point at $1/p^2$, so γ fixes $1/p^2$ as well. Hence γ must be the identity map.

When $p = 2$, in Example 6.2 for $\varphi(z) = (z^2-1)/(2z)$, $K = \mathbb{C}_p$, and $H = \mathbb{Q}_2$, we have $\mathrm{MinResLoc}(\varphi) = \{Q\}$ where $Q = \zeta_{i,1/2}$. Here $D(i,1/2) = D(\sqrt{3},1/2)$ since $|i-\sqrt{3}| = 1/2$, so $Q = \gamma_1(\zeta_G) = \gamma_2(\zeta_G)$ where $\gamma_1 = \left[\begin{smallmatrix} 2 & i \\ 0 & 1 \end{smallmatrix}\right]$ and $\gamma_2 = \left[\begin{smallmatrix} 2 & \sqrt{3} \\ 0 & 1 \end{smallmatrix}\right]$. Since $\mathbb{Q}_2(i) \cap \mathbb{Q}_2(\sqrt{3}) = \mathbb{Q}_2$, we have $H_\varphi = \mathbb{Q}_2$. However, $D(i,1/2) \cap \mathbb{Q}_2$ is empty. Hence there can be no $\gamma \in \mathrm{GL}_2(\mathbb{Q}_2)$ with $\gamma(\zeta_G) = Q$. ∎

5. An algorithm. In this section we give an algorithm which determines $\mathrm{MinResLoc}(\varphi)$, using the fact that $\mathrm{MinResLoc}(\varphi)$ is contained in an explicit, computable tree.

Given $\varphi(z) \in K(z)$ with $d = \deg(\varphi) \geq 2$, put $a = \varphi(\infty) \in K \cup \{\infty\}$. The following algorithm finds the minimal value of $\mathrm{ordRes}_\varphi(\cdot)$ and determines $\mathrm{MinResLoc}(\varphi)$, by working in the tree $\Gamma_{\mathrm{Fix},\varphi^{-1}(a)}$. This tree is spanned by ∞, the finite fixed points, and the finite solutions to $\varphi(z) = a$, so it is convenient for computational purposes.

Algorithm A: *Minimize* $\mathrm{ordRes}_\varphi(\cdot)$*, find* $\mathrm{MinResLoc}(\varphi)$*, and find a* $\gamma \in \mathrm{GL}_2(K)$ *for which* $\mathrm{ordRes}(\varphi^\gamma)$ *is minimal.*

Given a complete nonarchimedean valued field K with absolute value $|x| = q^{-\mathrm{ord}(x)}$, and a function $\varphi(z) \in K(z)$ with $d = \deg(\varphi) \geq 2$:

(1) (Find the endpoints of $\Gamma_{\mathrm{Fix},\varphi^{-1}(a)}$.)
  - (a) Write $\varphi(z) = f(z)/g(z)$ with $f(z), g(z) \in K[z]$; set $a = \varphi(\infty)$.
  - (b) Find the roots of $f(z) - zg(z) = 0$ (the finite fixed points).
  - (c) If $a = \infty$, find the roots of $g(z) = 0$ (the finite poles). If $a \neq \infty$, find the roots of $f(z) - a \cdot g(z) = 0$ (the finite solutions to $\varphi(z) = a$).
  - (d) List the distinct roots from (b) and (c) as $\{\alpha_1, \ldots, \alpha_k\}$.

(2) (Minimize $\mathrm{ordRes}_\varphi(\cdot)$ on each path $[\alpha_i, \infty]$.) For $i = 1, \ldots, k$, do the following:
  - (a) Set $\gamma_i(z) = z + \alpha_i$, and compute $\varphi^{\gamma_i}(z) = \varphi(z+\alpha_i) - \alpha_i$.
  - (b) Find a normalized representation (F_i, G_i) for $\varphi^{\gamma_i}(z)$.
  - (c) Compute $R_i = \mathrm{ord}(\mathrm{Res}(F_i, G_i))$.

(d) If $F_i(X,Y) = a_d X^d + \cdots + a_0 Y^d$, $G_i(X,Y) = b_d X^d + \cdots + b_0 Y^d$, set $C_\ell = R_i - 2d\,\mathrm{ord}(a_\ell)$, $D_\ell = R_i - 2d\,\mathrm{ord}(b_\ell)$ for $\ell = 0, \ldots, d$.

(e) Minimize the piecewise affine function

$$\chi_i(t) = \max\Big(\max_{0\le\ell\le d}(C_\ell + (d^2+d-2d\ell)t),\\ \max_{0\le\ell\le d}(D_\ell + (d^2+d-2d(\ell+1)t)\Big).$$

(f) Record the minimum value of $\chi_i(t)$ as M_i, and record the set of points where it is achieved as a singleton $\{\zeta_{\alpha_i,r_i}\}$ or a segment $[\zeta_{\alpha_i,r_{i,1}}, \zeta_{\alpha_i,r_{i,2}}]$, where $r = q^{-t}$ for a given t.

(3) (Find the minimum.) Let $M = \min_{1\le i\le k} M_i$, and output

$$\text{“}\min(\mathrm{ordRes}_\varphi(\cdot)) = M\text{”}.$$

(4) (Find the minimal resultant locus.) Consider the indices i for which $M = M_i$:

(a) If for each such i, $\chi_i(t)$ achieved M at a single point, output

$$\text{“}\mathrm{MinResLoc}(\varphi) = \{\zeta_{\alpha_i,r_i}\}\text{”}$$

for any such i, and go to (5).

(b) If for some such i, $\chi_i(t)$ achieved M on a segment, then:

(i) Find the relevant nodes of the tree $\Gamma_{\mathrm{Fix},\varphi^{-1}(a)}$: for all (i,j) with $1 \le i < j \le k$ such that $M = M_i = M_j$, find $r_{ij} = |\,\alpha_i - \alpha_j|$, then record $\zeta_{\alpha_i,r_{ij}} = \zeta_{\alpha_j,r_{ij}}$ as a node.

(ii) Using the nodes, collate the segments $[\zeta_{\alpha_i,r_{i,1}}, \zeta_{\alpha_i,r_{i,2}}]$ into a single segment $[\zeta_{a,r_a}, \zeta_{b,r_b}]$, and output

$$\text{“}\mathrm{MinResLoc}(\varphi) = [\zeta_{a,r_a}, \zeta_{b,r_b}]\text{”}.$$

(5) (Find $\gamma \in \mathrm{GL}_2(K)$ for which $\mathrm{ordRes}(\varphi^\gamma) = M$.)

(a) Choose an endpoint of $\mathrm{MinResLoc}(\varphi)$, and write it as $\zeta_{B,|A|}$ with $A \in K^\times$, $B \in K$.

(b) Output “$\gamma = \left[\begin{smallmatrix} A & B \\ 0 & 1 \end{smallmatrix}\right]$”, then halt.

The correctness of Algorithm A follows from Theorem 1.1. Since φ has potential good reduction if and only if $M = 0$, Algorithm A determines whether or not φ has potential good reduction, and if so, it finds a $\gamma \in \mathrm{GL}_2(K)$ such that φ^γ has good reduction.

6. Examples. The minimal resultant locus is an equivariant canonically attached to φ. In this section we compute $\mathrm{MinResLoc}(\varphi)$ for several examples and begin a study of its geometrical, dynamical, and arithmetical properties. The examples justify some of the assertions made in §2, §3, and §4. In subsequent work ([13], [14]), the author considers the geometric and dynamical significance of MinResLoc(φ) from a theoretical standpoint, placing the examples below in a coherent framework.

Examples 6.1, 6.2, and 6.6 show that MinResLoc(φ) need not be contained in the tree spanned by the fixed points alone, or the poles alone. Examples 6.3, 6.5, and 6.7 show that when d is odd, MinResLoc(φ) can be either a point or a segment.

When φ has potential good reduction, MinResLoc(φ) consists of a single point, which is necessarily fixed by φ. When φ does not have potential good reduction, MinResLoc(φ) may or may not contain fixed points: Example 6.6 shows it can be a single point, which is fixed. Example 6.5 shows it can be a segment, which is pointwise fixed. Examples 6.3 and 6.4 show it can be single point which is not fixed; Example 6.7 shows it can be a segment of which no point is fixed.

If H_v is a local field and MinResLoc(φ) contains no H_v-rational type II points, there are exactly two H_v-rational type II points adjacent to it in the tree spanned by $\mathbb{P}^1(H_v)$. (This follows from Proposition 3.5: the path connecting the tree spanned by $\mathbb{P}^1(H_v)$ to MinResLoc(φ) cannot branch off the tree at an H_v-rational type II point, because at each such point MinResLoc(φ) lies in a tangent direction containing points of $\mathbb{P}^1(H_v)$, which is thus a direction corresponding to an edge of the tree.) The function $\mathrm{ordRes}_\varphi(\cdot)$ may take the same or different values at those points; its value is strictly larger at all other H_v-rational type II points. Example 6.2 gives a case where the minimum is taken on at one of the two adjacent H-rational type II points, and Example 6.4 gives a case where it is taken on at both points.

If $\deg(\varphi)$ is odd and $\varphi(z) \in H(z)$, MinResLoc(φ) can contain arbitrarily many H-rational type II points. Example 6.5, with $\varphi(z) = \frac{p^n z^3+z^2-p^n z}{-p^n z^2+z+p^n}$ and $H = \mathbb{Q}_p$, shows this: for that function MinResLoc(φ) is a segment of length $2n$, with $\mathbb{Q}_p$-rational endpoints, contained in the path $[0, \infty]$.

In all the examples, MinResLoc(φ) lies well inside the ball

$$\left\{ z \in \mathbf{H}^1_K : \rho(\zeta_G, z) \le \frac{2}{d-1}\,\mathrm{ordRes}(\varphi) \right\}$$

given by Theorem 1.1. Probably the radius $\frac{2}{d-1}\,\mathrm{ordRes}(\varphi)$ is not sharp. One can also ask where MinResLoc(φ) lies relative to the Berkovich Julia set of φ. Example 6.6 shows it can lie inside; Example 6.3 shows it can lie outside.

For the remainder of this section, we take $K = \mathbb{C}_p$, the completion of the algebraic closure of $\mathbb{Q}_p$. The valuation $\mathrm{ord}(\cdot)$ on $\mathbb{C}_p$ will be normalized so that $\mathrm{ord}(p) = 1$, and $|\cdot|_p = p^{-\mathrm{ord}(\cdot)}$ will be the usual absolute value on $\mathbb{C}_p$. We write $\mathrm{Res}(\varphi)$ for $\mathrm{Res}(F, G)$, where (F, G) is the homogenization of the pair of polynomials defining φ.

We first consider two examples where $\varphi(z)$ has potential good reduction.

EXAMPLE 6.1: The function

$$\varphi(z) = \frac{z^d - p}{z^{d-1}},$$

with p arbitrary and $K = \mathbb{C}_p$. Here $\mathrm{Res}(\varphi) = (-1)^{d(d-1)/2}p^{d-1}$, so $\mathrm{ordRes}(\varphi) = d-1$. The poles of φ are 0 and ∞, and there is a $(d+1)$-fold fixed point at ∞. The tree Γ spanned by the fixed points and poles is just the path $[0, \infty]$.

Consider $\mathrm{ordRes}_\varphi(\cdot)$ on $[0, \infty]$. Let $Q_A \in \mathbf{P}^1_K$ correspond to $D(0, |A|)$; by (2.8),

$$\begin{aligned}\mathrm{ordRes}_\varphi(Q_A) = (d-1) + (d^2+d)\,\mathrm{ord}(A) \\ -2d\min\big(\mathrm{ord}(A^d), \mathrm{ord}(p), \mathrm{ord}(A \cdot A^{d-1})\big) \\ = \max\big((d-1) + (d-d^2)\,\mathrm{ord}(A), (-d-1) + (d^2+d)\,\mathrm{ord}(A)\big).\end{aligned}$$

This achieves its minimum when $\mathrm{ord}(A) = 1/d$, and $\mathrm{ordRes}_\varphi(\zeta_{0,p^{1/d}}) = 0$.

Thus $\varphi(z)$ has potential good reduction at the point $\zeta_{0,p^{1/d}}$, and conjugation by $\gamma = \left[\begin{smallmatrix} p^{1/d} & 0 \\ 0 & 1 \end{smallmatrix}\right]$ achieves the necessary change of coordinates: indeed, $\varphi^\gamma(z) = (z^d - 1)/z^{d-1}$. Here $\rho(\zeta_G, \zeta_{0,p^{1/d}}) = 1/d < \frac{2}{d-1}\,\mathrm{ordRes}(\varphi) = 2$. Note also that $\mathrm{ordRes}_\varphi(\zeta_G) = d-1$ and $\mathrm{ordRes}_\varphi(\zeta_{0,p}) = d^2 - 1$.

EXAMPLE 6.2: The function

$$\varphi(z) = \frac{z^2 - 1}{2z},$$

with $K = \mathbb{C}_2$. Here $\mathrm{Res}(\varphi) = -4$, so $\mathrm{ordRes}(\varphi) = 2$. The poles of φ are 0 and ∞, and the fixed points are ∞ and $\pm i$, where $i = \sqrt{-1}$. The tree Γ spanned by $\{0, \infty, i, -i\}$ has branch points at $\zeta_G = \zeta_{0,1}$ and $\zeta_{i,1/2}$.

First consider $\mathrm{ordRes}_\varphi(\cdot)$ on the path $[0, \infty]$. Let $Q_A = \zeta_{0,|A|} \in \mathbf{P}^1_K$; then

$$\mathrm{ordRes}_\varphi(Q_A) = \max\big(2 - 2\,\mathrm{ord}(A), 2 + 6\,\mathrm{ord}(A)\big).$$

This takes on its minimum when $\mathrm{ord}(A) = 0$, where $\mathrm{ordRes}_\varphi(Q_A) = 2$ and $Q_A = \zeta_G$. Next consider $\mathrm{ordRes}_\varphi(\cdot)$ on the path $[i, \infty]$. Let $\gamma = \left[\begin{smallmatrix} 1 & i \\ 0 & 1 \end{smallmatrix}\right]$, so $\gamma(0) = i$ and $\gamma(\infty) = \infty$. Then

$$\varphi^\gamma(z) = \frac{(z+i)^2 - 1}{2(z+i)} - i = \frac{z^2 - 4iz}{2z + 2i}.$$

Let Q_A be the point corresponding to the disc $D(i, |A|)$; then

$$\mathrm{ordRes}_\varphi(Q_A) = \max\big(2 - 2\,\mathrm{ord}(A), -2 + 2\,\mathrm{ord}(A)\big).$$

This achieves its minimum when $\mathrm{ord}(A) = 1$, and $\mathrm{ordRes}_\varphi(Q_A) = 0$. The corresponding point Q_A is $\zeta_{i,1/2}$; note that

$$\rho(\zeta_G, \zeta_{i,1/2}) = 1 < \frac{2}{d-1}\,\mathrm{ordRes}(\varphi) = 4.$$

Thus $\varphi(z)$ has potential good reduction at $\zeta_{i,1/2}$, and $\eta = \left[\begin{smallmatrix} 2 & i \\ 0 & 1 \end{smallmatrix}\right]$ achieves the necessary change of coordinates. One sees that

$$\varphi^{\eta}(z) = \frac{z^2 - 2iz}{2z + i}$$

indeed has good reduction. The nearest point to $\zeta_{i,1/2}$ in the tree spanned by $\mathbb{P}^1(\mathbb{Q}_2)$ is $\zeta_{1,1/\sqrt{2}}$; it is weakly $\mathbb{Q}_2$-rational but not $\mathbb{Q}_2$-rational, and one has $\operatorname{ordRes}_{\varphi}(\zeta_{1,1/\sqrt{2}}) = 1$. The nearest $\mathbb{Q}_2$-rational points in the tree are ζ_G and $\zeta_{1,1/2}$; one has $\operatorname{ordRes}_{\varphi}(\zeta_G) = 2$ and $\operatorname{ordRes}_{\varphi}(\zeta_{1,1/2}) = 4$.

In the remaining examples, φ does not have potential good reduction.

Example 6.3: The function

$$\varphi(z) = \frac{z^p - z}{p},$$

with $K = \mathbb{C}_p$ for an arbitrary prime p. It is known (see [1, Example 10.120]) that the Berkovich Julia set of $\varphi(z)$ is contained in $\mathbb{P}^1(\mathbb{C}_p)$ (indeed, it is precisely $\mathbb{Z}_p$), and its invariant measure μ_{φ} is the additive Haar measure on $\mathbb{Z}_p$. Thus, $\varphi(z)$ cannot have potential good reduction; if it did, its Berkovich Julia set would be the unique point $Q \in \mathbf{H}^1_K$ where it attained good reduction. Below we will give a direct proof that φ does not have potential good reduction.

Here $d = p$, and $\operatorname{Res}(\varphi) = p^p$, so $\operatorname{ordRes}(\varphi) = p$. The fixed points of $\varphi(z)$ are ∞ and the solutions $u_0, \dots, u_{p-1}$ to $z^p - (1+p)z = 0$. Since $z^p - (1+p)z \equiv z^p - z \equiv z(z-1)\cdots(z-(p-1)) \pmod{p}$, Hensel's lemma shows that each u_i belongs to $\mathbb{Z}_p$, and we can label the u_i so that $u_0 = 0$ and $u_i \equiv i \pmod{p}$ for $i = 1, \dots, p-1$. The poles of $\varphi(z)$ are all at ∞. The tree Γ spanned by $\{\infty, 0, u_1, \dots, u_{p-1}\}$ has ζ_G as its only branch point.

First consider $\operatorname{ordRes}_{\varphi}(\cdot)$ on the path $[0, \infty]$. As before, write Q_A for $\zeta_{0,|A|}$; by (2.8),

$$\begin{aligned} \operatorname{ordRes}_{\varphi}(Q_A) &= p + (p^2 + p)\operatorname{ord}(A) - 2p\min\bigl(\operatorname{ord}(A^p), \operatorname{ord}(A), \operatorname{ord}(pA)\bigr) \\ &= \max\bigl(p + (p - p^2)\operatorname{ord}(A), p + (p^2 - p)\operatorname{ord}(A)\bigr). \end{aligned}$$

The minimum is achieved at ζ_G (when $\operatorname{ord}(A) = 0$), and $\operatorname{ordRes}_{\varphi}(\zeta_G) = p$.

Next fix i with $1 \le i \le p-1$, and consider $\operatorname{ordRes}_{\varphi}(\cdot)$ on the path $[u_i, \infty]$. Taking $\gamma = \left[\begin{smallmatrix} 1 & u_i \\ 0 & 1 \end{smallmatrix}\right]$, we see that

$$\varphi^{\gamma}(z) = \frac{(z + u_i)^p - (z + u_i) - pu_i}{p} = \frac{a_p z^p + a_{p-1}z^{p-1} + \cdots + a_1 z}{p}$$

where $a_p = 1$, $a_j = \binom{p}{j} u_i^{p-j}$ for $j = 2, \dots, p-1$, and $a_1 = pu_i^{p-1} - 1$. In particular $\operatorname{ord}(a_p) = \operatorname{ord}(a_1) = 0$, and $\operatorname{ord}(a_j) = 1$ for $j = 2, \dots, p-1$. By formula (2.8) we have

$$\begin{aligned}\operatorname{ordRes}_\varphi(\gamma(Q_A)) &= \operatorname{ordRes}_{\varphi^\gamma}(Q_A)\\ &= p + (p^2+p)\operatorname{ord}(A) - 2p\min\bigl(\operatorname{ord}(a_pA^p),\dots,\operatorname{ord}(a_1A),\operatorname{ord}(pA)\bigr)\\ &= \max\bigl(p+(p-p^2)\operatorname{ord}(A), p+(p^2-p)\operatorname{ord}(A)\bigr).\end{aligned}$$

Again the minimum is achieved at ζ_G (when $\operatorname{ord}(A)=0$), and $\operatorname{ordRes}_\varphi(\zeta_G) = p$. Thus $\operatorname{MinResLoc}(\varphi) = \{\zeta_G\}$, and $\varphi(z)$ does not have potential good reduction. Here $\rho(\zeta_G,\zeta_G) = 0 < \frac{2}{d-1}\operatorname{ordRes}(\varphi) = 2p/(p-1)$. Note that $\varphi(\zeta_G) = \zeta_{0,p}$, so $\operatorname{MinResLoc}(\varphi)$ is not fixed by φ.

Example 6.4: The function

$$\varphi(z) = \frac{p^4z^3+pz+1}{p^6z^3},$$

where $K = \mathbb{C}_p$ and p is odd. Here $\operatorname{Res}(\varphi) = p^{18}$. The function $\varphi(z)$ has a triple pole at 0, and its fixed points are the roots of $-p^6z^4+p^4z^3+pz+1$. By the theory of Newton polygons, if the fixed points $u_1,\dots,u_4$ are ordered by increasing size, then $|u_1| = p$, $|u_2| = |u_3| = p^{3/2}$, and $|u_4| = p^2$. The tree Γ spanned by $\{0,u_1,u_2,u_3,u_4\}$ has branch points at $\zeta_{0,p}$ and $\zeta_{0,p^{3/2}}$.

Consider $\operatorname{ordRes}_\varphi(\cdot)$ on the path $[0,\infty]$; note that only the subsegment $[0,\zeta_{0,p^2}]$ is contained in Γ. Let $Q_A \in \mathbf{P}^1_K$ be the point corresponding to $D(0,|A|)$. Then $\operatorname{ordRes}_\varphi(Q_A)$ is given by

$$\max\bigl(-18-12\operatorname{ord}(A), -6-6\operatorname{ord}(A), 12+6\operatorname{ord}(A), 18+12\operatorname{ord}(A)\bigr).$$

This function achieves its minimum value of 3 when $\operatorname{ord}(A) = -3/2$; it has breaks when $\operatorname{ord}(A) = -1$, $\operatorname{ord}(A) = -3/2$, and $\operatorname{ord}(A) = -2$.

The initial segments of $[\zeta_{0,p^{3/2}},u_1]$ and $[\zeta_{0,p^{3/2}},u_4]$ belong to $[0,\infty]$, so $\operatorname{ordRes}_\varphi(\cdot)$ is increasing along them. To show that $\operatorname{ordRes}_\varphi(\cdot)$ achieves its minimum on $\mathbf{P}^1_K$ at $\zeta_{0,p^{3/2}}$, it suffices to check that it is increasing along $[\zeta_{0,p^{3/2}},u_2]$ and $[\zeta_{0,p^{3/2}},u_3]$.

Take $\gamma = \left[\begin{smallmatrix} p^{3/2} & 0\\ 0 & 1\end{smallmatrix}\right]$; conjugating φ by γ brings $\zeta_{0,p^{3/2}}$ to ζ_G. One finds

$$\varphi^\gamma(z) = \frac{z^3+z+p^{1/2}}{p^{1/2}z}.$$

The fixed points of φ^γ lie in the directions of 0, $\pm i$, and ∞ at ζ_G, where $i=\sqrt{-1}$; these correspond to the directions of u_1, u_2, u_3, and u_4 at $\zeta_{0,p^{3/2}}$, respectively. Since p is odd, the directions of $\pm i$ at ζ_G are distinct. Conjugating φ^γ by $\nu = \left[\begin{smallmatrix} 1 & i\\ 0 & 1\end{smallmatrix}\right]$ yields

$$\varphi^{\gamma\nu}(z) = \frac{(1-ip^{1/2})z^3+(3+i)z^2+(-2+3ip^{1/2})z}{p^{1/2}(z+i)^3}.$$

Since $\operatorname{ord}(-2+3ip^{1/2}) = 0$ when p is odd, it follows from Lemma 2.4 (or directly from formula (2.8)) that $\operatorname{ordRes}_\varphi(\cdot)$ is increasing in the direction of u_2 at $\zeta_{0,p^{3/2}}$. A similar argument applies for u_3.

Thus $\varphi(z)$ does not have potential good reduction: $\mathrm{MinResLoc}(\varphi) = \{\zeta_{0,p^{3/2}}\}$ with $\mathrm{ordRes}_\varphi(\zeta_{0,p^{3/2}}) = 3$. Here $\varphi(\zeta_{0,p^{3/2}}) = \zeta_{0,p}$, so $\mathrm{MinResLoc}(\varphi)$ is not fixed by φ. Note that $\rho(\zeta_G, \zeta_{0,p^{3/2}}) = 3/2 < \frac{2}{d-1}\,\mathrm{ordRes}(\varphi) = 18$ and that $\mathrm{ordRes}_\varphi(\zeta_{0,p}) = \mathrm{ordRes}_\varphi(\zeta_{0,p^2}) = 6$.

Example 6.5: The function

$$\varphi(z) = \frac{p^n z^3 + z^2 - p^n z}{-p^n z^2 + z + p^n},$$

where $n \geq 2$ and $K = \mathbb{C}_p$ for arbitrary p. Here $\mathrm{Res}(\varphi) = -4p^{4n}$, so $\mathrm{ordRes}(\varphi) = 4n + 2\,\mathrm{ord}(2)$. The fixed points of φ are 0, ± 1, and ∞, and the poles are $\alpha_\pm = (1 \pm \sqrt{1+4p^{2n}})/(-2p^n)$, where

$$\alpha_- = -p^n + p^{3n} + \cdots, \qquad \alpha_+ = 1/p^n + p^n - p^{3n} + \cdots .$$

The tree Γ spanned by $\{0, \pm 1, \infty, \alpha_-, \alpha_+\}$ has branch points at $\zeta_{0,1/p^n}$, $\zeta_{0,p^{-\mathrm{ord}(2)}}$, and ζ_{0,p^n}.

First consider $\mathrm{ordRes}_\varphi(\cdot)$ on the path $[0,\infty]$. Let $Q_A \in \mathbf{P}^1_K$ be the point corresponding to $D(0,|A|)$; then

$$\mathrm{ordRes}_\varphi(Q_A) = 2\,\mathrm{ord}(2) + \max\bigl(-2n - 6\,\mathrm{ord}(A), 4n, -2n + 6\,\mathrm{ord}(A)\bigr).$$

This takes its minimum value of $4n+2\,\mathrm{ord}(2)$ for all A with $\mathrm{ord}(A) \in [-n,n]$. Thus $\mathrm{MinResLoc}(\varphi)$ is a segment with $[\zeta_{0,1/p^n}, \zeta_{0,p^n}] \subseteq \mathrm{MinResLoc}(\varphi) \subset \Gamma$.

We next show that $\mathrm{MinResLoc}(\varphi) = [\zeta_{0,1/p^n}, \zeta_{0,p^n}]$. Since $n \geq 2 > \mathrm{ord}(2)$, the point $\zeta_{0,p^{-\mathrm{ord}(2)}}$ belongs to the interior of $[\zeta_{0,1/p^n}, \zeta_{0,p^n}]$. Since $\mathrm{MinResLoc}(\varphi)$ is a segment, $\mathrm{ordRes}_\varphi(\zeta)$ is necessarily increasing along the paths $[\zeta_{0,p^{-\mathrm{ord}(2)}}, \pm 1]$.

Note that the paths $[\zeta_{0,1/p^n}, \alpha_+]$ and $[\zeta_{0,1/p^n}, -p^n]$ share an initial segment. Take $\gamma = \left[\begin{smallmatrix} p^n & 0 \\ 0 & 1 \end{smallmatrix}\right]$, $\nu = \left[\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right]$ and set $\eta = \gamma \circ \nu = \left[\begin{smallmatrix} p^n & -p^n \\ 0 & 1 \end{smallmatrix}\right]$. Then η takes $[0,\infty]$ to the path $[-p^n, \infty]$, with $\eta(\zeta_G) = \zeta_{0,1/p^n}$. One computes

$$\varphi^\eta(z) = (\varphi^\gamma)^\nu(z) = \frac{p^{2n}(z-1)^3 - p^{2n}(z-1)^2 + z^2 - 2z + 2}{-p^{2n}(z-1)^2 + z}.$$

If we write the numerator of φ^η as $a_3 z^3 + a_2 z^2 + a_1 z + a_0$, then $\mathrm{ord}(a_2) = 0$, and it follows from (2.8) that $\mathrm{ordRes}_{\varphi^\eta}(\cdot)$ is increasing in the direction $\vec{v}_0$ at ζ_G. This means $\mathrm{ordRes}_\varphi(\cdot)$ is increasing in the direction $\vec{v}_{\alpha_-}$ at $\zeta_{0,1/p^n}$. By a similar argument, one sees that $\mathrm{ordRes}_\varphi(\cdot)$ is increasing in the direction $\vec{v}_{\alpha_+}$ at ζ_{0,p^n}.

Thus $\mathrm{MinResLoc}(\varphi)$ is the segment $[\zeta_{0,1/p^n}, \zeta_{0,p^n}]$, and the minimal value of $\mathrm{ordRes}_\varphi(\cdot)$ is $4n+2\,\mathrm{ord}(2)$; in particular $\varphi(z)$ does not have potential good reduction. Each point of $[\zeta_{0,1/p^n}, \zeta_{0,p^n}]$ is fixed by φ: ζ_{0,p^α} is an indifferent fixed point for $-n < \alpha < n$, and $\zeta_{0,1/p^n}$ and ζ_{0,p^n} are repelling fixed points of degree 2. In this case $\mathrm{MinResLoc}(\varphi)$ is contained in $\{z \in \mathbf{H}^1_K : \rho(\zeta_G, z) \leq n\}$, while $\frac{2}{d-1}\,\mathrm{ordRes}(\varphi) = 4n + 2\,\mathrm{ord}(2)$.

For rationality considerations involving $\mathrm{MinResLoc}(\varphi)$, it will be useful to note the relation between $\mathrm{ordRes}_{\varphi^\gamma}(\cdot)$ and $\mathrm{ordRes}_\varphi(\cdot)$ for conjugates φ^γ. For each $\gamma \in \mathrm{GL}_2(K)$, for all $Q \in \mathbf{P}^1_K$ one has

$$\mathrm{ordRes}_{\varphi^\gamma}(Q) = \mathrm{ordRes}_\varphi(\gamma(Q)). \tag{6.1}$$

To see this, by continuity it is enough to check it for type II points. Suppose $Q = \tau(\zeta_G)$ for some $\tau \in \mathrm{GL}_2(K)$. Then by the definitions,

$$\mathrm{ordRes}_{\varphi^\gamma}(Q) = \mathrm{ordRes}_{\varphi^\gamma}(\tau(\zeta_G)) = \mathrm{ordRes}_\varphi(\gamma(\tau(\zeta_G))) = \mathrm{ordRes}_\varphi(\gamma(Q)).$$

It follows that $\mathrm{MinResLoc}(\varphi^\gamma) = \gamma^{-1}(\mathrm{MinResLoc}(\varphi))$.

Take $u \in \mathbb{C}_p$ with $|u| = 1$, and let $\gamma_1 = \left[\begin{smallmatrix} 1 & u \\ -1 & u \end{smallmatrix}\right]$. One easily sees that

$$\varphi_1(z) := \varphi^{\gamma_1}(z) = \frac{-z^3 + (4p^n + 1)u^2 z}{(4p^n - 1)z^2 + u^2}, \tag{6.2}$$

and that $\gamma_1(\zeta_{u,1/p^n}) = \zeta_{0,p^n}$ and $\gamma_1(\zeta_{-u,1/p^n}) = \zeta_{0,1/p^n}$. Thus $\mathrm{MinResLoc}(\varphi_1)$ is the segment $[\zeta_{-u,1/p^n}, \zeta_{u,1/p^n}]$. When p is odd, the midpoint of this segment is $\zeta_G = \zeta_{0,1}$. When $p = 2$, its midpoint is $\zeta_{u,1/2}$.

Next conjugate $\varphi_1(z)$ by $\gamma_2 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & p^{1/2} \end{smallmatrix}\right]$. Then

$$\varphi_2(z) := (\varphi_1)^{\gamma_2}(z) = \frac{-z^3 + (4p^n + 1)u^2 pz}{(4p^n - 1)z^2 + pu^2} \tag{6.3}$$

and $\mathrm{MinResLoc}(\varphi_2) = [\zeta_{-up^{1/2},1/p^{n+1/2}}, \zeta_{up^{1/2},1/p^{n+1/2}}]$. When p is odd, the midpoint of this segment is $\zeta_{0,p^{-1/2}}$. When $p = 2$, its midpoint is $\zeta_{u2^{1/2},2^{-3/2}}$.

Example 6.6: The function

$$\varphi(z) = \frac{z^2}{(1 + pz)^4},$$

where $K = \mathbb{C}_p$ and $p \geq 5$. This function was studied by Favre and Rivera-Letelier ([9]; or see [1, Example 10.124]), who showed that its Berkovich Julia set is the segment $[\zeta_G, \zeta_{0,p^2}]$, and that its invariant measure μ_φ is the uniform measure of mass 1 on that segment (relative to the path distance). Here $\mathrm{Res}(\varphi) = p^8$. The poles of φ are all at $z = -1/p$, and the fixed points of φ are $z = 0$ and the roots of $1 + (4p-1)z + 6p^2z^2 + 4p^3z^3 + p^4z^4 = 0$. By the theory of Newton polygons, these roots can be labeled so that $|u_1| = 1$ and $|u_2| = |u_3| = |u_4| = p^{4/3}$. The tree Γ spanned by $\{0, -1/p, u_1, u_2, u_3, u_4\}$ has branch points at ζ_G, $\zeta_{0,p}$, and $\zeta_{0,p^{4/3}}$.

On the path $[0, \infty]$, we have

$$\mathrm{ordRes}_\varphi(\zeta_{0,|A|}) = \max\bigl(-24 - 20\,\mathrm{ord}(A), 8 + 4\,\mathrm{ord}(A)\bigr),$$

which takes its minimum value of $8/3$ at $\mathrm{ord}(A) = 4/3$. Conjugating by $\gamma = \left[\begin{smallmatrix} p^{-4/3} & 0 \\ 0 & 1 \end{smallmatrix}\right]$ gives

$$\varphi^\gamma(z) = \frac{z^2}{z^4 + 4p^{1/3}z^3 + 6p^{2/3}z^2 + 4pz + p^{4/3}}. \tag{6.4}$$

The fixed points u_2, u_3, u_4 lie in the directions $\vec{v}_{p^{4/3}}, \vec{v}_{\zeta_3 p^{4/3}}, \vec{v}_{\zeta_3^2 p^{4/3}}$ at $\zeta_{0,p^{4/3}}$, where ζ_3 is a primitive cube root of unity, and it is easily checked that $\mathrm{ordRes}_\varphi(\cdot)$ is increasing in each of those directions. Thus $\mathrm{MinResLoc}(\varphi) = \{\zeta_{0,p^{4/3}}\}$. Note that $\zeta_{0,p^{4/3}}$ is fixed by φ; indeed, by (6.4), $\zeta_{0,p^{4/3}}$ is a repelling fixed point of φ of degree 2. Also note that $\rho(\zeta_G, \zeta_{0,p^{4,3}}) = 4/3 < \frac{2}{d-1}\,\mathrm{ordRes}(\varphi) = 16/3$.

Example 6.7: The function

$$\varphi(z) = \frac{pz^3 + z^2}{p},$$

with $K = \mathbb{C}_p$ for an arbitrary prime p. Here $\mathrm{Res}(\varphi) = p^6$. The fixed points of $\varphi(z)$ are 0, ∞, and the solutions u_1, u_2 to $pz^2 + z - p = 0$:

$$u_1 = p + p^3 + \cdots, \qquad u_2 = -p^{-1} - p - p^3 + \cdots,$$

so that $|u_1| = 1/p$, $|u_2| = p$. The poles of $\varphi(z)$ are all at ∞. The tree Γ spanned by $\{0, \infty, u_1, u_2\}$ has branch points at $\zeta_{0,1/p}$ and $\zeta_{0,p}$.

First consider $\mathrm{ordRes}_\varphi(\cdot)$ on $[0,\infty]$. We have

$$\begin{aligned}\mathrm{ordRes}_\varphi(\zeta_{0,|A|}) &= 6 + 12\,\mathrm{ord}(A) - 6\min\bigl(\mathrm{ord}(pA^3), \mathrm{ord}(A^2), \mathrm{ord}(pA)\bigr)\\ &= \max\bigl(-6\,\mathrm{ord}(A), 6, 6 + \mathrm{ord}(A)\bigr).\end{aligned}$$

This takes the constant value 6 when $-1 \le \mathrm{ord}(A) \le 0$. By convexity, the minimum value of $\mathrm{ordRes}_\varphi(\cdot)$ on $\mathbf{P}^1_K$ is 6, and $\mathrm{MinResLoc}(\varphi)$ contains the segment $[\zeta_G, \zeta_{0,p}]$.

To see that $\mathrm{MinResLoc}(\varphi)$ contains no other points, note that the path $[\zeta_{0,p}, u_2]$ shares an initial segment with $[\zeta_{0,p}, p^{-1}]$. Conjugating φ by $\gamma = \left[\begin{smallmatrix} 1/p & 1/p \\ 0 & 1\end{smallmatrix}\right]$ yields $\varphi^\gamma(z) = (z^3 + 4z^2 + 5z + (2 - p^2))/p^2$; here $\gamma(0) = p^{-1}$ and $\gamma(\zeta_G) = \zeta_{0,1/p}$. One finds that $\mathrm{ordRes}_{\varphi^\gamma}(\zeta_{0,|A|})$ equals

$$\max\bigl(6 - 6\,\mathrm{ord}(A), 6 - 6\,\mathrm{ord}(5) + 6\,\mathrm{ord}(A), 6 - 6\,\mathrm{ord}(2) + 12\,\mathrm{ord}(A)\bigr).$$

Since either $\mathrm{ord}(5) = 0$ or $\mathrm{ord}(2) = 0$, the right side is increasing for small positive values of $\mathrm{ord}(A)$. Thus $\mathrm{ordRes}_\varphi(\cdot)$ is increasing along $[\zeta_{0,p}, u_2]$, and $\mathrm{MinResLoc}(\varphi) = [\zeta_G, \zeta_{0,p}]$. Note that $\mathrm{MinResLoc}(\varphi)$ is contained in

$$\{z \in \mathbf{H}^1_K : \rho(\zeta_G, z) \le 1\},$$

while $\frac{2}{d-1}\,\mathrm{ordRes}(\varphi) = 6$. For $0 \le \alpha \le 1$, we have $\varphi(\zeta_{0,p^\alpha}) = \zeta_{0,p^{2\alpha+1}}$, so no point of $\mathrm{MinResLoc}(\varphi)$ is fixed by φ.

7. The case $d = 1$. For completeness, in this section we consider $\mathrm{ordRes}_\varphi(\cdot)$ when $d = 1$, that is, when $\varphi(z) = \frac{f_1 z + f_0}{g_1 z + g_0} \in K(z)$ is such that $f_1 g_0 - f_0 g_1 \ne 0$. It is no longer true that $\mathrm{MinResLoc}(\varphi)$ is a point or a segment of finite path-length: the underlying reason for the difference is the simple fact that $1^2 - 1 = 0$, whereas $d^2 - d > 0$ when $d \ge 2$.

There are three cases to consider:

(1) the case where $\varphi(z) \equiv z$;
(2) the case where $\varphi(z)$ has exactly two distinct fixed points, which means there are a $\gamma \in \mathrm{GL}_2(K)$ and a $C \in K^\times$ with $|C| \le 1$ and $C \neq 1$ such that $\varphi^\gamma(z) = Cz$;
(3) the case where $\varphi(z)$ has a single fixed point of multiplicity 2, which means there is a $\gamma \in \mathrm{GL}_2(K)$ such that $\varphi^\gamma(z) = z + 1$.

It is easy to distinguish between these cases: the second case occurs when the Jordan normal form of the matrix corresponding to φ is $\left[\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right]$ with $\lambda \neq \mu$; the third case when it is $\left[\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right]$. In the second case $C = \lambda/\mu$ when the eigenvalues are ordered so that $|\lambda| \le |\mu|$. If φ and the eigenvalues are rational over a subfield $H \subset K$, then γ can be chosen to belong to $\mathrm{GL}_2(H)$.

We will need some terminology. Given points $x_0 \neq x_1 \in \mathbb{P}^1(K)$, the *strong tube of radius R around the path* $[x_0, x_1]$ is the set

$$T_{[x_0,x_1]}(R) = [x_0, x_1] \cup \{z \in \mathbf{H}^1_K : \rho(z, x) \le R \text{ for some } x \in [x_0, x_1]\}.$$

If $z \in \mathbf{P}^1_K$ corresponds to a sequence of nested discs $\{D(a_i, r_i)\}_{i \ge 1}$ by Berkovich's classification theorem (see [1, p. 5]), define $\mathrm{diam}_\infty(z) = \lim_{i\to\infty} r_i$; set $\mathrm{diam}_\infty(\infty) = \infty$. In particular, $\mathrm{diam}_\infty(\zeta_{a,r}) = r$. The *horodisc of codiameter R, tangent to the point* ∞, is the set

$$H_\infty(R) = \{z \in \mathbf{P}^1_K : \mathrm{diam}_\infty(z) \ge R\}.$$

The only type I point belonging to $H_\infty(R)$ is ∞; a point $\zeta_{a,r}$ of type II or III belongs to $H_\infty(R)$ if and only if $r \ge R$. For each $a \in K$, the intersection of the path $[a, \infty]$ with $H_\infty(R)$ is the ray $[\zeta_{a,R}, \infty]$. For each $S > R$, the point $\zeta_{0,S}$ belongs to $H_\infty(R)$; if $a \in K$ and $|a| \le S$, the intersection of $[a, \zeta_{0,S}]$ with $H_\infty(R)$ is

$$\{z \in [a, \zeta_{0,S}] : \rho(\zeta_{0,S}, z) \le \log(S/R)\}.$$

Thus $H_\infty(R)$ can be described informally as "the set of points in $\mathbf{P}^1_K$ accessible by moving the ray $[\zeta_{0,R}, \infty]$ without stretching, keeping it anchored at ∞". For an arbitrary $x_0 \in \mathbb{P}^1(K)$, a *horodisc tangent to x_0* is a set of the form $\gamma(H_\infty(R))$ for some R, where $\gamma \in \mathrm{GL}_2(K)$ is such that $\gamma(\infty) = x_0$.

THEOREM 7.1. *Suppose $\varphi(z) \in K(z)$ has degree $d = 1$. The function* $\mathrm{ordRes}_\varphi(\cdot)$ *on type* II *points extends to a function* $\mathrm{ordRes}_\varphi : \mathbf{P}^1_K \to [0, \infty]$ *which is piecewise affine and convex upwards on each path in $\mathbf{P}^1_K$, with respect to the logarithmic path distance. It is finite and continuous on $\mathbf{H}^1_K$ with respect to the strong topology, and achieves its minimum on a nonempty set* $\mathrm{MinResLoc}(\varphi) \subset \mathbf{P}^1_K$. *Furthermore:*

(1) *If $\varphi(z) = z$, then* $\mathrm{ordRes}_\varphi(\cdot) \equiv 0$ *and* $\mathrm{MinResLoc}(\varphi) = \mathbf{P}^1_K$.

(2) *If $\varphi(z)$ has exactly two fixed points x_0, x_1, let $\gamma \in \mathrm{GL}_2(K)$ and $C \in K^\times$ with $|C| \le 1$, $C \neq 1$, be such that $\varphi^\gamma(z) = Cz$. The minimal value*

of $\mathrm{ordRes}_\varphi(\cdot)$ *is* $\mathrm{ord}(C)$*, and* φ *has potential good reduction if and only if* $|C| = 1$*. When* $|C| < 1$*, or when* $|C| = 1$ *and* $|C-1| = 1$*, then* $\mathrm{MinResLoc}(\varphi)$ *is the path* $[x_0, x_1]$*. When* $|C| = 1$ *and* $|C-1| < 1$*, set* $R = \mathrm{ord}(C-1) > 0$*; then* $\mathrm{MinResLoc}(\varphi)$ *is the strong tube* $T_{[x_0,x_1]}(R)$*. The function* $\mathrm{ordRes}_\varphi(\cdot)$ *takes the value* ∞ *at each point of* $\mathbb{P}^1(K)\backslash\{x_0, x_1\}$*, and is continuous on* $\mathbf{P}^1_K\backslash\{x_0, x_1\}$ *relative to the strong topology.*

(3) *If* $\varphi(z)$ *has one fixed point* x_0*, let* $\gamma \in \mathrm{GL}_2(K)$ *be such that* $\varphi^\gamma(z) = z+1$*. Then the minimal value of* $\mathrm{ordRes}_\varphi(\cdot)$ *is* 0*,* φ *has potential good reduction, and* $\mathrm{MinResLoc}(\varphi)$ *is the horodisc tangent to* x_0 *given by* $\gamma(H_\infty(1))$*. The function* $\mathrm{ordRes}_\varphi(\cdot)$ *takes the value* ∞ *at each point of* $\mathbb{P}^1(K)\backslash\{x_0\}$*, and is continuous on* $\mathbf{P}^1_K\backslash\{x_0\}$ *relative to the strong topology.*

Proof. The fact that $\mathrm{ordRes}_\varphi(\cdot)$ extends from type II points to a function $\mathrm{ordRes}_\varphi : \mathbf{P}^1_K \to [0, \infty]$ which is piecewise affine and convex upwards on each path in $\mathbf{P}^1_K$ with respect to the logarithmic path distance, and is finite and continuous on $\mathbf{H}^1_K$ with respect to the strong topology, follows by the same argument as in the proof of Theorem 1.1. Indeed, $\mathrm{ordRes}_\varphi(\cdot)$ is Lipschitz continuous on $\mathbf{H}^1_K$, with Lipschitz constant $1^2 + 1 = 2$. To prove the remaining assertions, we will make explicit computations in each case.

When $\varphi(z) = z$, it is easy to see that $\varphi^\gamma(z) = z$ for each $\gamma \in \mathrm{GL}_2(K)$, and the assertions in part (1) of the theorem follow trivially.

Next assume φ has exactly two distinct fixed points x_0 and x_1, and let $\gamma \in \mathrm{GL}_2(K)$ be such that $\varphi^\gamma(z) = Cz$ with $|C| \le 1$, $C \ne 1$. After relabeling x_0, x_1 if necessary, we can assume that $\gamma(0) = x_0$ and $\gamma(\infty) = x_1$. Given $A \in K^\times$ and $B \in K$, put $\tau = \tau_{A,B} = \left[\begin{smallmatrix} A & B \\ 0 & 1 \end{smallmatrix}\right]$. As A and B vary, the points $\zeta_{B,|A|} = \tau_{A,B}(\zeta_G)$ range over all type II points in $\mathbf{H}^1_K$. Consider the representation $(F^\gamma(X,Y), G^\gamma(X,Y)) = (CX, Y)$ for φ^γ. One sees easily that $\mathrm{ordRes}(\varphi^\gamma) = \mathrm{ord}(C)$ and $(F^{\gamma\tau}, G^{\gamma\tau}) = (ACX + B(C-1)Y, AY)$, which gives

$$\begin{aligned} (7.1) \qquad & \mathrm{ordRes}_{\varphi^\gamma}(\zeta_{B,|A|}) \\ & \quad = \max\big(\mathrm{ord}(C), \mathrm{ord}(C) - 2\,\mathrm{ord}(B) - 2\,\mathrm{ord}(C-1) + 2\,\mathrm{ord}(A)\big). \end{aligned}$$

When $|C| < 1$, or when $|C| = |C-1| = 1$, formula (7.1) simplifies to

$$\mathrm{ordRes}_{\varphi^\gamma}(\zeta_{B,|A|}) = \max\big(\mathrm{ord}(C), \mathrm{ord}(C) + 2\,\mathrm{ord}(A) - 2\,\mathrm{ord}(B)\big).$$

When $B = 0$, then $\mathrm{ordRes}_{\varphi^\gamma}(\zeta_{0,|A|}) = \mathrm{ord}(C)$ for all A, so $\mathrm{ordRes}_{\varphi^\gamma}(\cdot) \equiv \mathrm{ord}(C)$ on the path $[0, \infty]$. Next suppose that $B \ne 0$. The path $[B, \infty]$ meets $[0, \infty]$ at $\zeta_{0,|B|}$, and for $|A| \le |B|$ we see that $\mathrm{ordRes}_{\varphi^\gamma}(\zeta_{B,|A|}) = \mathrm{ord}(C) - 2\,\mathrm{ord}(A/B) > \mathrm{ord}(C)$. Thus $\mathrm{ordRes}_\varphi(\cdot)$ increases as one moves away from $[0, \infty]$, and $\mathrm{ordRes}_{\varphi^\gamma}(B) = \infty$. It follows that $\mathrm{MinResLoc}(\varphi^\gamma) = [0, \infty]$ and that $\mathrm{ordRes}_{\varphi^\gamma}(x) = \infty$ for all $x \in \mathbb{P}^1(K)\backslash\{0, \infty\}$. By Proposition 2.3, $\mathrm{ordRes}_{\varphi^\gamma}(\cdot)$ is continuous on $\mathbf{P}^1_K\backslash\{0, \infty\}$ relative to the strong topology.

When $|C-1|<1$, formula (7.1) becomes

$$\mathrm{ordRes}_{\varphi^\gamma}(\zeta_{B,|A|}) = \max\bigl(0, -2\,\mathrm{ord}(C-1) + 2\,\mathrm{ord}(A) - 2\,\mathrm{ord}(B)\bigr).$$

When $B=0$, then $\mathrm{ordRes}_{\varphi^\gamma}(\zeta_{0,|A|}) = 0$ for all A, so $\mathrm{ordRes}_{\varphi^\gamma}(\cdot) = 0$ on $[0,\infty]$. When $B \neq 0$, for $|A| \le |B|$ we see that $\mathrm{ordRes}_{\varphi^\gamma}(\zeta_{B,|A|}) = 0$ if $\mathrm{ord}(A/B) \le \mathrm{ord}(C-1)$, while $\mathrm{ordRes}_{\varphi^\gamma}(\zeta_{B,|A|}) = -2\,\mathrm{ord}(C-1) + 2\,\mathrm{ord}(A/B) > 0$ if $\mathrm{ord}(A/B) > \mathrm{ord}(C-1)$. Setting $R = \mathrm{ord}(C-1)$, we see that $\mathrm{MinResLoc}(\varphi^\gamma)$ is the strong tube $T_{[0,\infty]}(R)$ and that $\mathrm{ordRes}_{\varphi^\gamma}(x) = \infty$ for all $x \in \mathbb{P}^1(K)\backslash\{0,\infty\}$. By Proposition 2.3, $\mathrm{ordRes}_{\varphi^\gamma}(\cdot)$ is continuous on $\mathbf{P}^1_K\backslash\{0,\infty\}$ relative to the strong topology. Transferring these assertions back to φ using formula (6.1), we obtain part (2) of the theorem.

Finally suppose φ has exactly one fixed point x_0. Let $\gamma \in \mathrm{GL}_2(K)$ be such that $\varphi^\gamma(z) = z+1$; then $\gamma(\infty) = x_0$. Given $A \in K^\times$ and $B \in K$, let $\tau = \tau_{A,B}$ be as above. Consider the representation $(F^\gamma(X,Y), G^\gamma(X,Y)) = (X+Y, Y)$ for φ^γ. Then $\mathrm{ordRes}(\varphi^\gamma) = 0$ and $(F^{\gamma\tau}, G^{\gamma\tau}) = (AX+Y, AY)$, which gives

$$\mathrm{ordRes}_{\varphi^\gamma}(\zeta_{B,|A|}) = \max(0, 2\,\mathrm{ord}(A)). \tag{7.2}$$

For each $B \in K$, formula (7.2) shows that $\mathrm{ordRes}_{\varphi^\gamma}(\zeta_{B,|A|}) = 0$ if $|A| \ge 1$, while $\mathrm{ordRes}_{\varphi^\gamma}(\zeta_{B,|A|}) = 2\,\mathrm{ord}(A) > 0$ if $|A| < 1$. Thus $\mathrm{MinResLoc}(\varphi^\gamma)$ is the horodisc $H_\infty(1)$, and $\mathrm{ordRes}_{\varphi^\gamma}(x) = \infty$ for all $x \in \mathbb{P}^1(K)\backslash\{\infty\}$. By Proposition 2.3, $\mathrm{ordRes}_{\varphi^\gamma}(\cdot)$ is continuous on $\mathbf{P}^1_K\backslash\{\infty\}$ relative to the strong topology. Transferring these assertions back to φ using formula (6.1), we obtain part (3) of the theorem. ∎

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