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On the estimates of double exponential sums

by

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1. Introduction. In Theorem 1 of [K1], Kolesnik presented a very useful estimate for double exponential sums (known as the "AB-theorem", see [GK]). However, the proof of Theorem 1 of [K1] was not correct, for the use of the two-dimensional A-process (simultaneously for two variables) introduces a strong complication, and thus the derivatives of the resulting phase function are not of constant sizes, and, to carry out the necessary arguments, Kolesnik assumed on p. 164 that "each of the domains $D_1, D_2, \ldots$ is a bounded region in $\mathbb{R}^n$ such that any line parallel to any coordinate axis intersects it in $O(1)$ line segments, and the same is true for the intersection of D with the regions of the types $f_{\underline{x}_j}(\underline{x}) \le c$ or $f_{\underline{x}_j}(\underline{x}) \ge c$ for all considered functions $f(\underline{x})$", which was not verified in [K1]. In [GK] a corrected version of the AB-theorem was given for the special case of a monomial phase function, but the proof is also not completely satisfactory (see Remark 1 below). Our first aim is to give a rigorous proof of Kolesnik's AB-theorem by inventing some new techniques. We have the following theorem.

THEOREM 1. *Let* $X \ge 100$, $Y \ge 100$, $L = \log(XY)$, *and let real numbers* α *and* β *satisfy* $\alpha\beta \ne 0$, $\alpha + \beta \ne 1, 2$, $\alpha \ne 1, 2$, $\beta \ne 1, 2$. *Set*

$$D = \{(x, y) \mid x \in I,\ f_1(x) \le y \le f_2(x)\} \subseteq [X, 2X] \times [Y, 2Y],$$

where I *is a closed interval*, *the real function* $f_i(x)$ *is continuous and is either a linear function* ($f_i(x) = a_i x + b_i$) *or has continuous derivatives up to order two on* I, *and in the latter case it satisfies*

$$f_i^{(r)}(x) = \lambda_i (\varphi_i)_r x^{\varphi_i - r}(1 + O(\Phi)), \qquad i = 1, 2, \tag{1}$$

where $r = 0, 1, 2$, $\lambda_i > 0$, $\varphi_i \ne 0, 1$, $0 \le \Phi \le L^{-2}$, *and for real* ξ *and integral* s,

$$(\xi)_s = \xi(\xi - 1) \cdots (\xi - s + 1).$$

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Let $g(x,y)$ *be a real function on* $[X/2,4X]\times[Y/2,4Y]$; *consider its partial derivative* $g_{i,j}(x,y)$ *obtained by taking first the derivative of order* i *in* x, *and then the derivative of order* j *in* y, *for* $i\ge 0$ *and* $j\ge 0$. *Assume that* $g_{i,j}(x,y)$ *is continuous on* $[X/2,4X]\times[Y/2,4Y]$ *for* $i\ge 0$, $j\ge 0$ *and*

$$\begin{aligned} g_{i,j}(x,y) &= A(\alpha)_i(\beta)_j x^{\alpha-i}y^{\beta-j}(1+O(\Delta)) \text{ for } 1\le i+j\le 3, i\ge 0, j\ge 0,\\ g_{i,j}(x,y) &\ll FX^{-i}Y^{-j} \quad \text{for } 4\le i+j\le 5, i\ge 0, j\ge 0, \end{aligned} \tag{2}$$

where $F=|A|X^\alpha Y^\beta\gg 1$, $0\le\Delta\le L^{-2}$. *Then*, *for* $\tau=\Phi+\Delta$ *we have*

$$S_g(D)=\sum_{(a,b)\in D} e(g(a,b))$$
$$\ll (\sqrt[6]{F^2(XY)^3}+\sqrt[6]{F^{-2}(XY)^7}+\sqrt[10]{F^2X^5Y^9\tau^4}+\sqrt[10]{F^2X^9Y^5\tau^4})L^3,$$

where $e(\xi)=\exp(2\pi i\xi)$ *for a real* ξ. *In particular*, *for* $\Delta=\Phi=0$ *and* $F\gg XY$ *we have*

$$S_g(D)\ll(\sqrt[6]{F^2(XY)^3})L^3.$$

By taking into account an estimate for exponential sums having one variable with a general phase function similarly to that obtained by using the exponent pair $(11/30,16/30)$ $(=BA^3B(0,1))$, we can improve our Theorem 1 slightly for certain cases and get the following Theorem 2.

THEOREM 2. *Let* $X\ge 100$, $Y\ge 100$, $F=|A|X^\alpha Y^\beta\gg\max(X,Y)$, *and let real numbers* α *and* β *satisfy* $\alpha\beta\ne 0$, $\alpha<1$, $\beta<1$. *Set*

$$D=\{(x,y)\mid x\in I,\ f_1(x)\le y\le f_2(x)\}\subseteq[X,2X]\times[Y,2Y],$$

where I *is a closed interval*, $f_i(x)=\lambda_i x^{\varphi_i}$, $\lambda_i>0$. *Let* $g(x,y)=Ax^\alpha y^\beta$. *Then*, *for* $Z=X+Y$, $L=\log(F+2)$, *we have*

$$S_g(D)=\sum_{(a,b)\in D} e(g(a,b))$$
$$\ll L^{28}(\sqrt[86]{F^{26}(XY)^{43}Z^5}+\sqrt[56]{F^{18}(XY)^{28}Z}+\sqrt[76]{F^{-1}(XY)^{64}}$$
$$+\sqrt[164]{(XY)^{134}Z^5}+\sqrt[8]{F^{-1}(XY)^7Z}+\sqrt[14]{F^2(XY)^7Z^4}+\sqrt[42]{F^{17}Z^{30}}).$$

In addition, *if* $F\gg XY$, *then the terms* $\sqrt[76]{F^{-1}(XY)^{64}}$ *and* $\sqrt[14]{F^2(XY)^7Z^4}$ *can be neglected*, *for in* $(90')$ *and* $(91')$ *below we have* $B_4\ll B_8$ *and* $B_6\ll B_1$.

REMARK 1. Theorem 6.12 of [GK] is a special case of Theorem 1 of [K1]. To prove it, on p. 79 and p. 80 of [GK] the conditions (Ω_2) and (Ω_3) on the summation range E were introduced. By assuming (Ω_2) and (Ω_3) for the function $f(x,y)=Ax^\alpha y^\beta$ and the summation range D, Graham and Kolesnik [GK] asserted that it suffices to deduce their Theorem 6.12. However, as the proof of their Theorem 6.12 depends on their Lemmas 6.8 and 6.10, we find that they should also assume (Ω_3) for each function

$$f_1(m,n,q,r)=f(m+q,n+r)-f(m,n),$$

where $|q| \le Q$, $|r| \le R$, and Q and R are given on p. 83 of [GK]. The reason is that in their Lemmas 6.8 and 6.10 what they need to show in practice concerns the exponential sum $S_1(q,r)$ and not the original sum S. In addition, they need to show that the domains D_0, D_1 and D_2 on p. 84 are all of the type (Ω_2), for otherwise their argument cannot be carried out. The authors of [GK] did not explain how to verify the assumptions in applications, and even for the simplest case with $f(x,y) = Ax^{\alpha}y^{\beta}$ $(A\alpha\beta \ne 0,$ $\alpha, \beta \ne 1)$, it seems impossible to verify the condition (Ω_3) for each function $f_1(m,n,q,r)$ directly.

REMARK 2. It is plain that Kolesnik's more complicated arguments (using at least twice our Lemma 2) of [K1] to [K3] cannot be remedied by our method.

REMARK 3. Recently, we have found mistakes in some works on the distribution of 4-full numbers. In particular, our Theorem 2 cannot be used to yield a better result for the 4-full problem, but it can be used to get a result which is slightly better than 15/92 for the order of $\zeta(1/2+it)$.

2. Lemmas for the proof of Theorem 1. We need several lemmas. Lemma 1 is used to change the order of variables in a summation.

LEMMA 1. *Let D be an arbitrary summation range of the shape*

$$D = \{(x,y) \mid x \in I,\ f_1(x) \le y \le f_2(x)\},$$

where $D \subseteq [X,2X] \times [Y,2Y]$, $X \ge 10$, $Y \ge 10$, I is a closed interval, the real function $f_i(x)$ is either a linear function on I $(f_i(x) = a_i x + b_i)$, or has continuous derivatives up to order two on I, and in the latter case it satisfies

$$f_i^{(r)}(x) = \lambda_i (\varphi_i)_r x^{\varphi_i - r}(1 + O(\Phi)), \qquad i = 1,2, \tag{3}$$

where $r = 0,1,2$, $\lambda_i > 0$, $\varphi_i \ne 0,1$, $0 \le \Phi \le L^{-2}$, $L = \log(XY)$. Let g be an arbitrary real function on D. Then for an absolute constant C we have

$$S_g(D) = \sum_{1\le i\le C} S_g(D_i') + O(Z), \qquad Z = X + Y,$$

where $D_i' \subseteq D$, D_i' takes the form

$$\{(x,y) \mid c \le y \le d,\ g_1(y) \le x \le g_2(y)\}$$

and $g_i(y)$ is either a linear function or has continuous derivatives of order two on $[c,d]$, and in the latter case it satisfies, similarly to (3),

$$g_i^{(r)}(y) = \lambda_i^{-\mu_i}(\mu_i)_r y^{\mu_i - r}(1 + O(\Phi)), \qquad i = 1,2,\ r = 0,1,2,\ \mu_i = \varphi_i^{-1}.$$

Proof. We can assume the complicated case that both $f_1(x)$ and $f_2(x)$ are not constant on I. Let

$$\min_{x\in I} f_1(x) = y_1, \quad \max_{x\in I} f_1(x) = Y_1, \quad \min_{x\in I} f_2(x) = y_2, \quad \max_{x\in I} f_2(x) = Y_2.$$

If $f_1(x)$ is a linear function, then $f_1'(x) \neq 0$, and otherwise it follows from (3) that $f_1'(x) \neq 0$ on I (we can assume that X is greater than a suitable constant, for otherwise Lemma 1 follows trivially). The same can be said of $f_2'(x)$. Thus $f_1'(x)$ and $f_2'(x)$ do not change signs on I. Assume that $f_1'(x)$ and $f_2'(x)$ are positive on I (the other cases can be treated similarly). Then both $f_1(x)$ and $f_2(x)$ are strictly increasing on I. Therefore, for $y_i \leq y \leq Y_i$, $f_i(x) = y$ has a unique solution $x = F_i(y)$, and $F_i''(y)$ is continuous and $\neq 0$ on $[y_i, Y_i]$. In particular, if $f_i(x)$ satisfies (3), then by taking derivatives in y on both sides of $f_i(F_i(y)) = y$, we can verify that $F_i(y)$ satisfies

$$F_i^{(r)}(y) = \lambda_i^{-\mu_i}(\mu_i)_r y^{\mu_i - r}(1 + O(\Phi)), \qquad \mu_i = \varphi_i^{-1},\ r = 0, 1, 2.$$

Exchanging the roles of x and y we have $S_g(D) = S_g(D_1)$, where

$$D_1 = \{(x,y) \mid y_1 \leq y \leq Y_2,\ a \leq x \leq b,\ f_1(x) \leq y \leq f_2(x)\}, \qquad [a,b] = I.$$

If $y_1 \geq Y_2$, then $S_g(D_1) = O(1)$. Let $y_1 < Y_2$, and assume that $Y_1 \leq Y_2$. Then

$$S_g(D_1) = S_g(D_2) + S_g(D_3) + O(X),$$
$$D_2 = \{(x,y) \mid y_1 \leq y \leq Y_1,\ a \leq x \leq b,\ f_1(x) \leq y \leq f_2(x)\},$$
$$D_3 = \{(x,y) \mid Y_1 \leq y \leq Y_2,\ a \leq x \leq b,\ f_1(x) \leq y \leq f_2(x)\}.$$

Assume that $y_1 \geq y_2$. Because both $f_1(x)$ and $f_2(x)$ are strictly increasing, we have

$$D_2 = \{(x,y) \mid y_1 \leq y \leq Y_1,\ F_2(y) \leq x \leq F_1(y)\},$$
$$D_3 = \{(x,y) \mid Y_1 \leq y \leq Y_2,\ F_2(y) \leq x \leq b\}.$$

If $y_1 < y_2$, then for $y_2 \leq Y_1$ we have

$$S_g(D_2) = S_g(D_4) + S_g(D_5) + O(X),$$

where

$$\begin{aligned} D_4 &= \{(x,y) \mid y_1 \leq y \leq y_2,\ a \leq x \leq b,\ f_1(x) \leq y \leq f_2(x)\} \\ &= \{(x,y) \mid y_1 \leq y \leq y_2,\ a \leq x \leq F_1(y)\}, \\ D_5 &= \{(x,y) \mid y_2 \leq y \leq Y_1,\ a \leq x \leq b,\ f_1(x) \leq y \leq f_2(x)\} \\ &= \{(x,y) \mid y_2 \leq y \leq Y_1,\ F_2(y) \leq x \leq F_1(y)\}. \end{aligned}$$

Similarly, we can treat the summation on D_3. In case $Y_1 > Y_2$, we have

$$S_g(D_1) = S_g(D_6) + S_g(D_7) + O(X),$$

$$D_6 = \{(x,y) \mid y_1 \le y \le \max(y_1,y_2),\, a \le x \le b,\, f_1(x) \le y \le f_2(x)\},$$
$$D_7 = \{(x,y) \mid \max(y_1,y_2) \le y \le Y_2,\, a \le x \le b,\, f_1(x) \le y \le f_2(x)\}$$
$$= \{(x,y) \mid \max(y_1,y_2) \le y \le Y_2,\, F_2(y) \le x \le F_1(y)\}.$$

Evidently, for $y_1 \ge y_2$ we have $S_g(D_6) = O(X)$, and if $y_1 < y_2$, then

$$D_6 = \{(x,y) \mid y_1 \le y \le y_2,\, a \le x \le F_1(y)\}.$$

Thus Lemma 1 is proved in case both $f_1(x)$ and $f_2(x)$ are not constant on I. For the other cases, the argument is similar and easier. The proof is finished.

The next result is Weyl's inequality for two variables, which can be proved similarly to Lemma 5 of [HB].

LEMMA 2. *Let X and Y be positive numbers with $X \ge 100$, $Y \ge 100$, and M and N be positive integers with $M \le X$, $N \le Y$. Let the summation range D satisfy*

$$D = \{(a,b) \mid a \in I,\, b \in J_a\} \subseteq [X,X'] \times [Y,Y'],$$

where $X \le X' = O(X)$, $Y \le Y' = O(Y)$, I is an interval, J_a is an interval depending on a, and $Z(m,n)$ is a complex number for $X \le m \le X'$ and $Y \le n \le Y'$. Then

$$\Big|\sum_{(a,b)\in D} Z(a,b)\Big|^2 \le (1 + (X'-X)M^{-1})(1 + (Y'-Y)N^{-1})\widetilde{S},$$

where

$$\widetilde{S} = \sum_{|q|\le M}\sum_{|r|\le N} (1 - |q|M^{-1})(1 - |r|N^{-1}) S_{q,r},$$
$$S_{q,r} = \sum_{(a,b)\in D(q,r)} \overline{Z}(a,b) Z(a+q, b+r),$$
$$D(q,r) = \{(a,b) \mid (a,b) \in D,\, (a+q,b+r) \in D\}.$$

Proof. It is a particular case of Lemma 6.1 of [GK].

LEMMA 3. *Suppose $f(x)$ is a real function, $f''(x)$ is continuous and*

$$|f''(x)| \approx r, \qquad r > 0,$$

on the interval $[a,b]$, $b > a > 0$. Then

$$\sum_{a\le x\le b} e(f(x)) \ll (b-a)r^{1/2} + r^{-1/2}.$$

Proof. It is Lemma 2.2 of [GK].

LEMMA 4. *Let $f(x)$ be a real function such that $f^{(4)}(x)$ is continuous on the interval $[a,b]$ (for $k\ge 3$, $f^{(k)}(x)$ denotes the derivative of order k), and*

$$|f''(x)|\approx R^{-1},\qquad \beta_k(x)=f^{(k)}(x)/f''(x)=O(U^{2-k}),$$

where $R>0$, $U\ge 1$, $3\le k\le 4$, $1\le a<b\le 2a$. Then

$$\sum_{a\le n\le b} e(f(n))=\lambda\sum_{\alpha<\nu<\beta}|f''(x_\nu)|^{-1/2}e(f(x_\nu)-\nu x_\nu+1/8)+O(R^{1/2})$$
$$+O(\log(2+aR^{-1}))+O((a+R)U^{-1}).$$

Proof. It is a particular case of Lemma 3.6 of [GK].

LEMMA 5. *Let $a\approx X$, $b\approx Y$, $X\ge 1$, $Y\ge 1$, $qr\ne 0$, $\lambda=X/|r|+Y/|q|$, and*

$$|bq-\gamma_1 ar|\le \eta_1\lambda|qr|,\qquad |bq-\gamma_2 ar|\le \eta_2\lambda|qr|,$$

where γ_1 and γ_2 are absolute constants, $\gamma_1\ne\gamma_2$, and $\eta_1>0$, $\eta_2>0$. Then there is an absolute constant $\delta>0$ such that $\eta_1+\eta_2\ge 2\delta$.

Proof. From

$$|bq-\gamma_1 ar|\le \eta_1\lambda|qr|,\qquad |bq-\gamma_2 ar|\le \eta_2\lambda|qr|,$$

we get

$$b|q|\le\lambda\,\frac{\eta_1|\gamma_2|+\eta_2|\gamma_1|}{|\gamma_1-\gamma_2|}\,|qr|,\qquad a|r|\le\lambda\,\frac{\eta_1+\eta_2}{|\gamma_1-\gamma_2|}\,|qr|,$$

and thus

$$|qr|\lambda\ll a|r|+b|q|\le\lambda\,\frac{\eta_1(1+|\gamma_2|)+\eta_2(1+|\gamma_1|)}{|\gamma_1-\gamma_2|}\,|qr|$$
$$\le\frac{\lambda(\eta_1+\eta_2)(1+|\gamma_1|+|\gamma_2|)}{|\gamma_1-\gamma_2|}\,|qr|,$$

and the conclusion of Lemma 5 follows.

Our Lemma 6 improves the coefficient 2^{M+N} of the corresponding result of [C].

LEMMA 6. *Let $M,N\ge 1$, A_m, B_n, u_m and ν_n $(1\le m\le M,\ 1\le n\le N)$ be positive numbers, and $0\le Q_1<Q_2$. Then there is a number $q\in[Q_1,Q_2]$ such that*

$$\sum_{1\le m\le M}A_m q^{u_m}+\sum_{1\le n\le N}B_n q^{-\nu_n}\le (M+N)\Big(\sum_{1\le m\le M}A_m Q_1^{u_m}+\sum_{1\le n\le N}B_n Q_2^{-\nu_n}+\sum_{1\le m\le M}\sum_{1\le n\le N}(A_m^{\nu_n}B_n^{u_m})^{1/(u_m+\nu_n)}\Big).\tag{4}$$

Proof. Let $q = \max(Q_1, \min_{1\le i\le M} C_i)$, where

$$C_i = \min(Q_2, \min_{1\le j\le N} \lambda_{i,j}), \qquad \lambda_{i,j} = (B_j A_i^{-1})^{1/(u_i+\nu_j)}.$$

Obviously, $q \in [Q_1, Q_2]$. For a fixed integer m, $1 \le m \le M$, there exists an integer r, $1 \le r \le N$, such that

$$\max_{1\le j\le N} \lambda_{m,j} = \lambda_{m,r}.$$

Thus, if we denote the right side of (4) as $(M+N)S$, we have

$$A_m q^{u_m} \le A_m Q_1^{u_m} + A_m C_m^{u_m} \le A_m Q_1^{u_m} + A_m \lambda_{m,r}^{u_m} \le S. \tag{5}$$

For an integer n, $1 \le n \le N$, we have, for some integer k, $1 \le k \le M$,

$$\min_{1\le i\le M} \min(Q_2, \lambda_{i,n}) = \min(Q_2, \lambda_{k,n}).$$

Thus, from

$$C_i \ge \min(Q_2, \lambda_{i,n}),$$

we get

$$\begin{aligned} B_n q^{-\nu_n} &\le B_n(\min_{1\le i\le M} C_i)^{-\nu_n} \le B_n(\min_{1\le i\le M} \min(Q_2, \lambda_{i,n}))^{-\nu_n} \\ &= B_n(\min(Q_2, \lambda_{k,n}))^{-\nu_n} \le B_n Q_2^{-\nu_n} + B_n \lambda_{k,n}^{-\nu_n} \le S. \end{aligned} \tag{6}$$

Summing over m and n in (5) and (6) respectively yields the inequality (4).

LEMMA 7. *Let $f(x)$ be a real function which has continuous derivative $f'''(x)$ on $[a,b]$, where $[a,b] \subseteq [N, N']$, $N \ge 1$, and $N' = O(N)$. Moreover, assume that*

$$|f^{(r)}(x)| \approx \lambda N^{1-r} \quad \textit{for } x \in [a,b],$$

for all $1 \le r \le 4$, and some $\lambda > 0$. Then

$$\sum_{a\le n\le b} e(f(n)) \ll \min(\lambda^{1/6} N^{4/6}, \lambda^{1/14} N^{11/14}) + \lambda^{-1}.$$

Proof. This follows by the familiar arguments showing that $(1/6, 4/6)$ and $(1/14, 11/14)$ $(= A(1/6, 4/6))$ are exponent pairs.

3. Proof of Theorem 1. To estimate the exponential sum $S_g(D)$, we assume that

$$F \le \min(X^3, Y^3), \qquad \min(X, Y) \ge L^6; \tag{7}$$

the other cases will be treated easily later. Let

$$\max(XL/Y, YL/X) \le t \le XYL^{-4},\ M = (tX/Y)^{1/2},\ N = (tY/X)^{1/2}, \tag{8}$$

where t is a parameter. Obviously,

$$1 \le M \le XL^{-2}, \qquad 1 \le N \le YL^{-2}.$$

By Lemma 2 we get

(9) $$|S_g(D)|^2 = O((XY)^2(MN)^{-1} + XY(MN)^{-1}(S_1 + S_2 + S_3)),$$

where, for $D_1 = \{(a,b) \mid (a,b) \in D,\ (a+q, b+r) \in D\}$,

$$S_1 = \sum_{1\le|q|\le M}\ \sum_{1\le|r|\le N} |S_G(D_1)|, \qquad G(a,b) = g(a+q, b+r) - g(a,b),$$

$$S_2 = \sum_{1\le|r|\le N} \Big| \sum_{\substack{(a,b)\in D\\(a,b+r)\in D}} e(g(a,b+r) - g(a,b))\Big|,$$

$$S_3 = \sum_{1\le|q|\le M} \Big| \sum_{\substack{(a,b)\in D\\(a+q,b)\in D}} e(g(a+q,b) - g(a,b))\Big|.$$

To estimate S_1, it suffices to estimate $S_G(D_1)$ for $q > 0$. From

(10) $$\begin{gathered} X \le a \le 2X, \qquad Y \le b \le 2Y, \\ 1 \le |q| \le M \le XL^{-2}, \qquad 1 \le |r| \le N \le YL^{-2}, \end{gathered}$$

we get

$$\tfrac{1}{2}X \le a + q \le 4X, \qquad \tfrac{1}{2}Y \le b + r \le 4Y.$$

Let

(11) $$\varrho = \frac{|q|}{X} + \frac{|r|}{Y}, \qquad \lambda = \frac{\varrho XY}{|qr|} = \frac{Y}{|r|} + \frac{X}{|q|}.$$

By the assumption (2), making Taylor expansion, for real variables a, b, q and r satisfying (10), we get, for suitable numbers a', b' ($|a'-a| \le q$ and $|b'-b| \le r$),

(12) $$\begin{aligned} G_{n,s}(a,b) &= g_{n,s}(a+q,b+r) - g_{n,s}(a,b) \\ &= qg_{n+1,s}(a,b) + rg_{n,s+1}(a,b) + O(q^2|g_{n+2,s}(a',b')| \\ &\quad + |qrg_{n+1,s+1}(a',b')| + r^2|g_{n,s+2}(a',b')|) \\ &= A(\alpha)_n(\beta)_s a^{\alpha-n-1} b^{\beta-s-1}(\Phi_{n,s}(qb,ra) + O(XY\varrho(\varrho+\Delta))), \end{aligned}$$

where $n \ge 0$, $s \ge 0$, $n + s \le 2$, and $\Phi_{n,s}(\xi,\eta) = (\alpha-n)\xi + (\beta-s)\eta$. Let $G_{i,j} = G_{i,j}(a,b)$. From (12) we get

(13) $$G_{2,0}G_{0,2} - G_{1,1}^2 = A^2 a^{2\alpha-4} b^{2\beta-4}(\Phi(qb,ra) + O((XY\varrho)^2(\varrho+\Delta))),$$

where $\Phi(\xi,\eta)$ is the homogeneous polynomial

$$\begin{aligned} \Phi(\xi,\eta) &= (\alpha)_2(\beta)_2\Phi_{2,0}(\xi,\eta)\Phi_{0,2}(\xi,\eta) - (\alpha)_1^2(\beta)_1^2\Phi_{1,1}^2(\xi,\eta) \\ &= \alpha\beta(2-\alpha-\beta)(\alpha(\alpha-1)\xi^2 + 2(\alpha-1)(\beta-1)\xi\eta + \beta(\beta-1)\eta^2). \end{aligned}$$

Because $\alpha+\beta \ne 1$, we can show that the equation $\Phi(t,1) = 0$ has no double roots. Let the roots be θ_1 and θ_2, $|\theta_1| \le |\theta_2|$. We can assume that θ_1 and θ_2 are real. In case θ_1 and θ_2 are conjugate complex numbers, the polynomial

$\Phi(t,1)$ is positive definite or negative definite, and the treatment would be simpler. Let

$$\theta_3 = \frac{2-\beta}{\alpha}, \qquad \theta_4 = \frac{\beta}{\alpha-2},$$
$$D_2 = \{(a,b) \mid |bq - \theta_i ar| < \varrho \text{ for some } i\},$$
$$D_3 = \{(a,b) \mid (a,b) \in D_1,\ (a,b) \notin D_2\}.$$

For a fixed integer b, since $\theta_i \neq 0$, the number of integers a satisfying $|bq - \theta_i ar| < \varrho$ is $\ll 1$. Thus

$$|S_G(D_2)| \le \sum_{(a,b)\in D_2} 1 \ll Y, \qquad |S_G(D_1)| \ll |S_G(D_3)| + Y. \tag{14}$$

For $(a,b) \in D_3$ and each i, we have

$$\varrho \le |bq - \theta_i ar| \le 2(1+|\theta_i|)\lambda|qr|,$$

thus we can divide D_3 into $\ll L^4$ small ranges of the shape (note that $\lambda|qr| = \varrho XY$, cf. (11))

$$D_4 = \{(a,b) \mid \varepsilon_i\lambda|qr|\Delta_i \le bq - \theta_i ar \le \delta_i\lambda|qr|\Delta_i \text{ for } 1 \le i \le 4\} \cap D_1,$$

where $1/(XY) \le \Delta_i \le 2(1+\sum_{1\le i\le 4}|\theta_i|)$, and $(\varepsilon_i,\delta_i) = (-2,-1)$ or $(1,2)$. For example we can take $\Delta_i = (XY)^{-1}2^{k_i}$, $k_i \ge 0$ an integer. Consequently, for some particular range D_4 of the above shape, we have

$$|S_G(D_3)| \ll L^4|S_G(D_4)| + L^4X, \tag{15}$$

where the error term $O(L^4X)$ of (15) comes from counting the number of lattice points (a,b) satisfying

$$bq - \theta_i ar = \lambda\widetilde{\Delta}_i|qr|,\ \text{or } 2\lambda\widetilde{\Delta}_i|qr|,\ \text{or } -2\lambda\widetilde{\Delta}_i|qr|,\ \text{or } -\lambda\widetilde{\Delta}_i|qr|$$

for some i and $O(L)$ values of $\widetilde{\Delta}_i$. Let

$$R_1 = \max_{1\le i\le 4}(q^{-1}(\varepsilon_i\lambda|qr|\Delta_i + \theta_i ar)), \qquad R_2 = \min_{1\le i\le 4}(q^{-1}(\delta_i\lambda|qr|\Delta_i + \theta_i ar)).$$

Then from

$$D = \{(a,b) \mid a \in I,\ f_1(a) \le b \le f_2(a)\}$$

we have

$$D_4 = \{(a,b) \mid a \in I_1,\ B_1(a) \le b \le B_2(a)\},$$

where $I_1 = \{a \in I \mid (a+q) \in I\}$, and

$$B_1(a) = \max(f_1(a+q)-r, f_1(a), R_1), \qquad B_2(a) = \min(f_2(a+q)-r, f_2(a), R_2).$$

From now on, we assume that each interval appearing has length ≥ 10 if it is not empty, for the other cases can be treated easily. By discussing the monotonicity we can show that I can be divided into $O(1)$ disjoint intervals, on each of which both $B_1(a)$ and $B_2(a)$ have explicit forms. For instance, to compare $f_1(a+q)-r$ and $f_1(a)$ we let

$$G_1(a)=f_1(a+q)-f_1(a)-r.$$

If f_1 is a linear function on I_1, then G_1 is a constant which may depend on q and r. In case f_1 has continuous derivatives up to order 2 and satisfies (1) for $x\in I$, we find that

$$G_1'(a)=f_1'(a+q)-f_1'(a)$$

does not change its sign. It follows that $G_1(a)$ is either a constant on I_1, or strictly monotonic on I_1. In the latter case there is at most one number $c\in I$ such that $G_1(c)=0$, and $G_1(a)$ does not change its sign on $[a_1,c)$ and $(c,b_1]$ respectively, where $I_1=[a_1,b_1]$, $b_1-a_1\ge 10$. Assume for example $G_1(a)\le 0$ for $a\in[a_1,c]$. Then for $a\in[a_1,c]$ we have

$$\max(f_1(a+q)-r,f_1(a))=f_1(a).$$

Using the similar method, we can continue our monotonicity arguments by taking derivatives to get

$$|S_G(D_4)|\ll\sum|S_G(D_5)|+Y, \tag{16}$$

where the summation is taken over $O(1)$ disjoint ranges of the shape

$$D_5=\{(a,b)\mid a\in I',\ B_1(a)\le b\le B_2(a)\}\subseteq D_4,$$

I' is a suitable interval contained in I_1, $B_1(a)$ has an explicit expression, one of the three forms

$$f_1(a+q)-r,\qquad f_1(a),\qquad k_1a+k_2,\qquad (k_1,k_2)=(r\theta_iq^{-1},\varepsilon_i\lambda|r|\Delta_i),$$

for some i, $1\le i\le 4$, and similarly, $B_2(a)$ has one of the following three forms on I':

$$f_2(a+q)-r,\qquad f_2(a),\qquad k_1'a+k_2',\qquad (k_1',k_2')=(r\theta_jq^{-1},\delta_j\lambda|r|\Delta_j),$$

for some j, $1\le j\le 4$. We need to estimate each $S_G(D_5)$. As $\alpha+\beta\ne 2$, $\alpha\beta\ne 0$, by a calculation we find that θ_3 and θ_4 are not roots of $\Phi(t,1)=0$, thus $\theta_3,\theta_4\ne\theta_1,\theta_2$. From $\alpha+\beta\ne 2$ we also get $\theta_3\ne\theta_4$. Thus by Lemma 5 there exists an absolute constant $\delta>0$ (we consider α and β also as absolute constants, which is indeed the case in applications) such that

$$\Delta_i+\Delta_j\ge 2\delta\quad\text{ for } i\ne j,\ \{i,j\}\subseteq\{1,2,3,4\}. \tag{17}$$

Let $\widetilde{\Delta}=\min_{1\le i\le 4}\Delta_i$. We distinguish several cases to estimate $S_G(D_5)$.

Case (i): *$\widetilde{\Delta}\le L(\Delta+\varrho)$ and $\Delta_3>\delta$.* As $\Delta_3>\delta$, $\varrho\le 2L^{-1}$ and $\varrho\le L^{-2}$, for $(a,b)\in D_5$, from (12) we get

$$|G_{0,2}(a,b)|\approx F\varrho Y^{-2}. \tag{18}$$

For $(a,b)\in D_5$ and a fixed $a\in I'$, $G_{0,2}(a,b)$ is continuous on $[B_1(a),B_2(a)]$ (we can assume $B_1(a)<B_2(a)$), and

$$|bq-\theta_i ar|\le 2\varDelta_i\lambda|qr|\le 2L(\varDelta+\varrho)\lambda|qr| \tag{19}$$

for some i and $\varDelta_i$. Thus (using $\varrho\lambda\gg 1$, cf. (11))

$$\sum_{B_1(a)\le b\le B_2(a)} 1 = O(1+L(\varDelta+\varrho)\lambda|r|)=O(L(\varDelta+\varrho)\lambda|r|),$$

and by (18) and Lemma 3 we get

$$\Big|\sum_{B_1(a)\le b\le B_2(a)} e(G(a,b))\Big| \ll \Big(\sum_{B_1(a)\le b\le B_2(a)} 1\Big)(F\varrho Y^{-2})^{1/2}+(F\varrho Y^{-2})^{-1/2}$$
$$\ll (L(\varDelta+\varrho)\lambda|r|)(F\varrho Y^{-2})^{1/2}+(F\varrho Y^{-2})^{-1/2}.$$

Consequently,

$$S_G(D_5)=O((F\varrho^3)^{1/2}XL(\varDelta+\varrho)\lambda+XY(F\varrho)^{-1/2}). \tag{20}$$

Case (ii): *$\widetilde{\varDelta}\le L(\varDelta+\varrho)$ and $\varDelta_3\le\delta$.* In this case, we argue similarly to (i), but with the roles of a and b exchanged. From (17) we get $\varDelta_4>\delta$, and thus from (12), for $(a,b)\in D_5$ we get

$$|G_{2,0}(a,b)|\approx F\varrho X^{-2}. \tag{21}$$

By Lemma 1 we have

$$S_G(D_5)=\sum_{1\le i\le C} S_G(D_i')+O(X+Y), \tag{22}$$

where C is an absolute constant, and

$$D_i'=\{(a,b)\mid b\in I_i',\ a\in I(b)\}\subseteq D_5,$$

where I_i' and $I(b)$ are suitable intervals. For $(a,b)\in D_5$ and $b\in I_i'$, $G_{2,0}(a,b)$ is continuous on $I(b)$, and

$$|bq-\theta_i ar|\le 2\varDelta_i\lambda|qr|\le 2L(\varDelta+\varrho)\lambda|qr| \tag{23}$$

for some i and $\varDelta_i$. Thus (note that all $\theta_i\ne 0$, and $\varrho\lambda\gg 1$)

$$\sum_{a\in I(b)} 1 = O(1+L(\varDelta+\varrho)\lambda|q|)=O(L(\varDelta+\varrho)\lambda|q|),$$

and by (21) and Lemma 3 we get

$$\Big|\sum_{a\in I(b)} e(G(a,b))\Big| \ll \Big(\sum_{a\in I(b)} 1\Big)(F\varrho X^{-2})^{1/2}+(F\varrho X^{-2})^{-1/2}$$
$$\ll (L(\varDelta+\varrho)\lambda|q|)(F\varrho X^{-2})^{1/2}+(F\varrho X^{-2})^{-1/2}.$$

Consequently,

$$S_G(D_i')=O(YX^{-1}(F\varrho)^{1/2}L(\varDelta+\varrho)\lambda|q|+XY(F\varrho)^{-1/2}),$$

and by (22) we get

$$S_G(D_5) = O(Y(F\varrho^3)^{1/2}L(\Delta+\varrho)\lambda + XY(F\varrho)^{-1/2} + X + Y). \tag{24}$$

Case (iii): $\widetilde{\Delta} > L(\Delta+\varrho)$ *and* $\Delta_3 > \delta$. By (12) and (13) we find that for $(a,b) \in D_5$, the functions

$$G_{0,2}(a,b), \quad G_{2,0}(a,b), \quad G_{2,0}(a,b)G_{0,2}(a,b) - G_{1,1}^2(a,b)$$

do not change their signs, and satisfy

$$|G_{0,2}(a,b)| \approx F\varrho Y^{-2}, \quad |G_{2,0}(a,b)| \approx \Delta_4 F\varrho X^{-2}, \tag{25}$$

$$|G_{2,0}(a,b)G_{0,2}(a,b) - G_{1,1}^2(a,b)| \approx \min(\Delta_1,\Delta_2)(F\varrho X^{-1}Y^{-1})^2, \tag{26}$$

where to deduce (26) we have used (cf. (13))

$$\left|\Phi\left(\frac{qb}{ra},1\right)\right| \approx \left|\left(\frac{qb}{ra}-\theta_1\right)\left(\frac{qb}{ra}-\theta_2\right)\right| \approx \left(\frac{q}{X}\right)^2\Delta_1\Delta_2\lambda^2 \approx \left(\frac{\varrho Y}{|r|}\right)^2\Delta_1\Delta_2,$$

and $\max(\Delta_1,\Delta_2) \ge \delta$ (which follows from (26)). From (12) and (25) we have

$$|G_{0,2}| \approx F\varrho Y^{-2}, \quad G_{0,3} \ll F\varrho Y^{-3}, \quad G_{0,4} \ll F\varrho Y^{-4} \tag{27}$$

for $(a,b) \in D_5$, and $G_{i,j} = G_{i,j}(a,b)$. We can assume that $G_{0,2} > 0$ on D_5; the case of $G_{0,2} < 0$ can be treated similarly. For a fixed $a \in I'$ such that $B_1(a) < B_2(a) - 10$, by Lemma 5 and (27) we obtain

$$\begin{aligned}\sum_{B_1(a)\le b\le B_2(a)} & e(G(a,b)) \\ &= \sum_{\alpha_1\le u\le\alpha_2} Ke(K_1) + O(Y(F\varrho)^{-1/2}) + O(Y(F\varrho)^{-1}) + O(L),\end{aligned} \tag{28}$$

where

$$K = K(a,u) = (G_{0,2}(a,b(a,u)))^{-1/2},$$
$$K_1 = K_1(a,u) = G(a,b(a,u)) - ub(a,u) + 1/8,$$

and $b(a,u)$ is the solution of $G_{0,1}(a,b) = u$ for given numbers $a \in I'$ and $\alpha_1 \le u \le \alpha_2$, $\alpha_i = \alpha_i(a) = G_{0,1}(a,B_i(a))$. As $G_{0,2} = G_{0,2}(a,b) > 0$ for $(a,b) \in D_5$, for a fixed $a \in I'$ the function $G_{0,1}(a,b)$ is strictly increasing with respect to $b \in [B_1(a),B_2(a)]$. By (2), taking Taylor expansion we have

$$\begin{aligned}\alpha_i &= G_{0,1}(a,B_i(a)) = F_i(a) + O(FY^{-1}\varrho(\varrho+\Delta)),\\ F_i(a) &= Aa^{\alpha-1}B_i^{\beta-2}(q\alpha\beta B_i + ar(\beta)_2).\end{aligned} \tag{29}$$

For $a \in I'$, there are many choices of $(B_1(a),B_2(a))$; we assume the difficult case that

$$(B_1(a),B_2(a)) = (f_1(a+q)-r, f_2(a+q)-r),$$

and neither f_1 nor f_2 is a linear function. Then from (1) we deduce for $i = 1, 2$ that

$$
\begin{aligned}
&B_i(a) = \lambda_i a^{\varphi_i}(1 + O(\Phi_1)),\\
&B_i'(a) = \lambda_i \varphi_i a^{\varphi_i - 1}(1 + O(\Phi_1)) = \varphi_i B_i(a) a^{-1}(1 + O(\Phi_1)),
\end{aligned}
\tag{30}
$$

where $\Phi_1 = \Phi + \varrho$. From (29) and (30) we get

$$
\alpha_i = g_i(a) + O(\Delta_1), \quad g_i(a) = A a^{\alpha-1} b_i^{\beta-2}(q\alpha\beta b_i + ar(\beta)_2), \tag{31}
$$

where $\Delta_1 = F\varrho(\varrho + \Delta + \Phi)Y^{-1}$ and $b_i = b_i(a) = \lambda_i a^{\varphi_i}$. Let C_1 be a large absolute constant such that

$$
X_1(a) = g_1(a) + C_1\Delta_1 > \alpha_1, \quad X_2(a) = g_2(a) - C_1\Delta_1 < \alpha_2,
$$

which is possible in view of (31). As

$$
K = K(a, u) = O(Y(F\varrho)^{-1/2}), \quad X_i(a) - \alpha_i = O(\Delta_1), \quad i = 1, 2,
$$

it follows that

$$
\sum_{\alpha_1 \le u \le \alpha_2} Ke(K_1) = \sum_{X_1(a) \le u \le X_2(a)} Ke(K_1) + O(Y(F\varrho)^{-1/2}(1 + \Delta_1)), \tag{32}
$$

whether $X_1(a) \ge X_2(a)$ or not. We consider the set of real numbers a with $a \in I'$ and $B_2(a) \ge B_1(a) + 10$. Let

$$
F(a) = B_2(a) - B_1(a) - 10, \quad a \in I'.
$$

By (30) we have

$$
\begin{aligned}
F'(a) &= B_2'(a) - B_1'(a)\\
&= \lambda_2\varphi_2 a^{\varphi_2 - 1}(1 + O(\Phi_1)) - \lambda_1\varphi_1 a^{\varphi_1 - 1}(1 + O(\Phi_1)), \quad a \in I'.
\end{aligned}
\tag{33}
$$

If $F'(a) \ne 0$ on I', then $F(a)$ is strictly monotonic on I'. We consider the difficult case that there is a number $a_1 \in I'$ with $F'(a_1) = 0$. It $\varphi_1 = \varphi_2$, from (33) we also get $\lambda_1 = \lambda_2$, and thus

$$
B_2(a) - B_1(a) = O(Y\Phi_1) = O(Y(\Phi + \varrho))
$$

for $a \in I'$, and by (27) and Lemma 3 we get (as $Y\varrho > |r| \ge 1$)

$$
\begin{aligned}
\sum_{B_1(a) \le b \le B_2(a)} e(G(a, b)) &\ll (1 + Y(\Phi + \varrho))(F\varrho Y^{-2})^{1/2} + (F\varrho Y^{-2})^{-1/2}\\
&\ll (\Phi + \varrho)(F\varrho)^{1/2} + Y(F\varrho)^{-1/2},
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
S_G(D_5) &= \sum_{a \in I'} \sum_{B_1(a) \le b \le B_2(a)} e(G(a, b))\\
&\ll X(F\varrho)^{1/2}(\Phi + \varrho) + XY(F\varrho)^{-1/2}.
\end{aligned}
\tag{35}
$$

Suppose $\varphi_1 \neq \varphi_2$. Then for any other $a_2 \in I'$ such that $F'(a_2) = 0$, from (33) we get

$$\frac{\lambda_1\varphi_1}{\lambda_2\varphi_2}(a_1^{\varphi_1-\varphi_2} - a_2^{\varphi_1-\varphi_2}) = O(\Phi_1),$$

which implies that $|a_2 - a_1| \le C_2 X\Phi_1$ for a suitable absolute constant C_2. Let

$$I_1' = \{a \in I' \mid |a - a_1| > C_2X\Phi_1\}, \qquad I_2' = \{a \in I' \mid |a - a_1| \le C_2X\Phi_1\}.$$

We have $F'(a) \neq 0$ on I_1', and I_1' can be divided into at most two disjoint intervals on each of which $F(a)$ is strictly monotonic. If $F\varrho \le Y$, by Lemma 3 and (27) we get

$$|S_G(D_5)| \ll X((F\varrho)^{1/2} + Y(F\varrho)^{-1/2}) \ll XY(F\varrho)^{-1/2}. \tag{36}$$

Suppose $F\varrho > Y$. By Lemma 3 and (27) we get (as $X\varrho \ge |q| \ge 1$)

$$\Big|\sum_{a\in I_2'}\sum_{B_1(a)\le b\le B_2(a)} e(G(a,b))\Big| \ll (1 + X(\Phi+\varrho))((F\varrho)^{1/2} + (F\varrho Y^{-2})^{-1/2}) \ll X(\Phi+\varrho)(F\varrho)^{1/2}. \tag{37}$$

From the observations on I_1' and (37), there is an interval $I_3' \subseteq I'$ such that

$$|S_G(D_5)| \ll \Big|\sum_{a\in I_3'}\sum_{B_1(a)\le b\le B_2(a)} e(G(a,b))\Big| + X(\Phi+\varrho)(F\varrho)^{1/2}, \tag{38}$$

and $F(a)$ is strictly monotonic on I_3' if $I_3' \neq \emptyset$. It follows that there are intervals $I_4', I_5' \subseteq I_3'$ such that $I_3' = I_4' \cup I_5'$, and

$$I_4' = \{a \in I_3' \mid F(a) \le 0\}, \qquad I_5' = \{a \in I_3' \mid F(a) > 0\}.$$

By Lemma 3 and (27) we have

$$\Big|\sum_{a\in I_4'}\sum_{B_1(a)\le b\le B_2(a)} e(G(a,b))\Big| \ll X((F\varrho Y^{-2})^{1/2} + (F\varrho Y^{-2})^{-1/2}) \ll XY^{-1}(F\varrho)^{1/2} + XY(F\varrho)^{-1/2}.$$

Thus (38) gives

$$|S_G(D_5)| \ll \Big|\sum_{a\in I_5'}\sum_{B_1(a)\le b\le B_2(a)} e(G(a,b))\Big| + X(\Phi+\varrho)(F\varrho)^{1/2} + XY(F\varrho)^{-1/2}. \tag{39}$$

We suppose $I_5' \neq \emptyset$. By (28), (32), (35) and (39), we have

$$|S_G(D_5)| \ll \Big|\sum_{a\in I_5'}\sum_{X_1(a)\le u\le X_2(a)} Ke(K_1)\Big| + XY(F\varrho)^{-1/2} + X(\Delta+\Phi+\varrho)(F\varrho)^{1/2} + XL, \tag{40}$$

and, by exchanging the order of summation

$$\sum_{a\in I_5'}\sum_{X_1(a)\le u\le X_2(a)} Ke(K_1)=\sum_{u_1\le u\le u_2}\sum_{a\in S(u)} Ke(K_1),$$

where $u_1=\min_{a\in I_5'} X_1(a)$, $u_2=\max_{a\in I_5'} X_2(a)$, and

$$S(u)=\{a\in I_5' \mid X_1(a)\le u\le X_2(a)\}.$$

The function $X_1(a)-u$ is monotonic on each of the $O(1)$ disjoint intervals contained in I_5', because from $\alpha+\beta\ne 2$ (which implies $|\alpha-1+\varphi_1(\beta-1)|+|\alpha+\varphi_1(\beta-2)|\ne 0$) we find that (cf. (31))

$$\begin{aligned}X_1'(a)&=g_1'(a)\\&=\beta A a^{\alpha-2}b_1^{\beta-2}[\alpha q(\alpha-1+\varphi_1(\beta-1))+(\beta-1)r(\alpha+\varphi_1(\beta-2))a/b_1]\\&=0\end{aligned}$$

has at most one solution on I_5'. A similar conclusion holds for $X_2(a)-u$. As the intersection of two closed intervals is either empty or a closed interval (maybe consisting of only one point), it follows that $S(u)$ consists of $O(1)$ disjoint intervals. Therefore for a suitable interval $I(u)\subseteq S(u)\subseteq I_5'$ we get

$$\Big|\sum_{a\in I_5'}\sum_{X_1(a)\le u\le X_2(a)} Ke(K_1)\Big|\ll\sum_{u_1\le u\le u_2}\Big|\sum_{a\in I(u)} Ke(K_1)\Big|. \tag{41}$$

For a u with $u_1\le u\le u_2$, suppose $I(u)\ne\emptyset$, and $I(u)$ is an interval of length >0. Then, for all $a\in I(u)$, $(a,b(a,u))\in D_5$, where $b(a,u)$ is the solution of

$$G_{0,1}(a,b(a,u))=u.$$

Taking derivatives in a of both sides of this equality, we get

$$\frac{\partial b}{\partial a}(a,u)=b_a'(a,u)=-G_{1,1}(a,b(a,u))/G_{0,2}(a,b(a,u)).$$

Thus (by (27))

$$\begin{aligned}K_a'(a,u)&=-\tfrac12 G_{0,2}^{-3/2}(G_{0,3}\cdot b_a'+G_{1,2})\\&=-\tfrac12 G_{0,2}^{-5/2}(G_{1,2}G_{0,2}-G_{1,1}G_{0,3})\ll YX^{-1}(F\varrho)^{-1/2},\end{aligned}$$

and by partial summation,

$$\Big|\sum_{a\in I(u)} Ke(K_1)\Big|\ll Y(F\varrho)^{-1/2}\Big|\sum_{a\in I_1(u)} e(K_1(a,u))\Big|, \tag{42}$$

where $I_1(u)\subseteq I(u)$, $I_1(u)$ is an interval. Let $I_1(u)\ne\emptyset$. For $a\in I(u)$, by (25) and (26) we have

$$\begin{gathered}(K_1)_a'=G_{1,0},\\ |(K_1)_a''|=|G_{2,0}+G_{1,1}b_a'|=|(G_{2,0}G_{0,2}-G_{1,1}^2)/G_{0,2}|\approx\varDelta(1)F\varrho X^{-2},\end{gathered}$$

where we let $\varDelta(1) = \min(\varDelta_1, \varDelta_2)$. By (27) and Lemma 3 we get

$$\Big|\sum_{a\in I_1(u)} e(K_1)\Big| \ll (F\varrho)^{1/2} + X(\varDelta(1))^{-1/2}(F\varrho)^{-1/2}. \tag{43}$$

Consequently, as $F\varrho > Y$ and $u_i = O(F\varrho Y^{-1})$, from (41)–(43) we deduce that

$$\Big|\sum_{a\in I_5'}\sum_{X_1(a)\le u\le X_2(a)} Ke(K_1)\Big| \ll F\varrho + X(\varDelta(1))^{-1/2}.$$

Therefore from (40) we obtain

$$\begin{aligned}(44)\quad & |S_G(D_5)| \\ & \ll F\varrho + XY(F\varrho)^{-1/2} + X(\varDelta+\varPhi+\varrho)(F\varrho)^{1/2} + XL + X(\varDelta(1))^{-1/2}.\end{aligned}$$

To eliminate the term $X(\varDelta(1))^{-1/2}$ of (44), similarly to the estimate (24) of (ii) we can derive that

$$\begin{aligned} S_G(D_5) &= O(YX^{-1}(F\varrho)^{1/2}\varDelta(1)\lambda|q| + XY(F\varrho)^{-1/2} + X + Y) \\ &= O((F\varrho)^{1/2}\varrho Y^2|r|^{-1}\varDelta(1) + XY(F\varrho)^{-1/2} + X + Y).\end{aligned}$$

Thus from (44) we get for $F\varrho > Y$ the estimate

$$\begin{aligned}(45)\qquad S_G(D_5) &= O(F\varrho + XY(F\varrho)^{-1/2} \\ &\qquad + X(\varDelta+\varPhi+\varrho)(F\varrho)^{1/2} + XL + Y + R), \\ R &= \min(X\varDelta(1)^{-1/2}, F^{1/2}\varrho^{3/2}Y^2|r|^{-1}\varDelta(1)) \\ &\le \sqrt[6]{F\varrho^3X^4Y^4|r|^{-2}}.\end{aligned}$$

In view of (36), we find that (45) also holds for $F\varrho \le Y$. For other choices of $(B_1(a), B_2(a))$, we can deduce similarly to prove (45).

CASE (iv): *$\widetilde{\varDelta} > L(\varDelta+\varrho)$ and $\varDelta_3 \le \delta$.* By (17) we have $\varDelta_1, \varDelta_2, \varDelta_4 \ge \delta$. Using the decomposition (22) and noting that D_i' has a similar form to D_5, but with the roles of a and b exchanged, we can treat $S_G(D_i')$ as we treated $S_G(D_5)$ in (iii), but with the roles of a and b exchanged, and we get for $F\varrho \ge X$, similarly to (44), the estimate

$$S_G(D_i') = O(F\varrho + XY(F\varrho)^{-1/2} + Y(\varDelta+\varPhi+\varrho)(F\varrho)^{1/2} + YL).$$

For $F\varrho < X$, similarly to (36) we get

$$S_G(D_i') = O(XY(F\varrho)^{-1/2}).$$

Thus the estimate

$$S_G(D_5) = O(F\varrho + XY(F\varrho)^{-1/2} + Y(\varDelta+\varPhi+\varrho)(F\varrho)^{1/2} + YL + X) \tag{46}$$

always holds.

CASE (v): *Final estimate.* From (20), (24), (45), and (46), we have (note that $\varrho\lambda \gg 1$, cf. (11))

$$S_G(D_5) = O(F\varrho + XY(F\varrho)^{-1/2} + LZ + Z(\Delta + \Phi + \varrho)(F\varrho)^{1/2}\varrho\lambda + \sqrt[6]{F\varrho^3X^4Y^4|r|^{-2}}). \tag{47}$$

where $Z = X + Y$. From (14)–(16) and (47) we obtain

$$L^{-5}|S_G(D_1)| \ll F\varrho + XY(F\varrho)^{-1/2} + Z + (F\varrho)^{1/2}Z(\Delta + \Phi + \varrho) + \sqrt[6]{F\varrho^3X^4Y^4|r|^{-2}}. \tag{48}$$

In case $\Phi(t,1) = 0$ has conjugate complex roots, we can argue similarly and relatively simply to obtain (48), for in this case

$$|G_{2,0}G_{0,2} - G_{1,1}^2| \approx (F\varrho X^{-1}Y^{-1})^2$$

always holds, and we only need to divide D_3 into $\ll L^2$ small ranges of the shape

$$\{(a,b) \mid \varepsilon_i\lambda|qr|\Delta_i \le bq - \theta_i ar \le \delta_i\lambda|qr|\Delta_i \text{ for } 3 \le i \le 4\} \cap D_1.$$

From (8) and (48) we get

$$L^{-6}\,\frac{XY}{t}\,S_1 \ll \sqrt{F^2XYt} + \sqrt[4]{F^{-2}(XY)^9t^{-1}} + XYZ + \sqrt[4]{F^2XYZ^4t^3} + \sqrt[4]{F^2X^3Y^3Z^4\tau^4t} + \sqrt[12]{F^2X^{19}Y^{15}t}, \tag{49}$$

where $\tau = \Delta + \Phi$. For $1 \le r \le N$, the condition "$(a,b) \in D$ and $(a, b+r) \in D$" is equivalent to

$$(a,b) \in D_1' = \{(a,b) \mid a \in I,\ f_1(a) \le b \le f_2(a) - r\}.$$

Let $G(a,b) = g(a,b+r) - g(a,b)$. We can estimate directly $S_G(D_1')$ as we estimated $S_G(D_5)$ in (iii), for now

$$|G_{2,0}G_{0,2} - G_{1,1}^2| \approx (F\varrho X^{-1}Y^{-1})^2, \qquad \varrho = |r|/Y,$$

always holds. Thus we can deduce, similarly to (45) (the additional "R" term will not emerge here, cf. (36) and (44)) that

$$S_G(D_1') = O(F|r|Y^{-1} + XY(F|r|Y^{-1})^{-1/2} + ZL + X(F|r|Y^{-1})^{1/2}(\Delta + \Phi + |r|/Y)). \tag{50}$$

A similar estimate with $r \in [-N, -1]$, and D_1' replaced by

$$\{(a,b) \mid a \in I,\ f_1(a) - r \le b \le f_2(a)\},$$

also holds. For $1 \le |q| \le M$, the condition "$(a,b) \in D$ and $(a+q, b) \in D$" is equivalent to

$$(a,b) \in D_1'' = \{(a,b) \mid a \in I_1,\ f_1(a) \le b \le f_2(a)\},$$

where I_1 is the interval determined by $a \in I$ and $(a+q) \in I$. Suppose $I_1 \neq \emptyset$. Let

$$G(a,b) = g(a+q,b) - g(a,b).$$

We can deduce similarly to (47) and (50) to get ($\varrho = |q|/X$)

$$S_G(D_1'') = O(F|q|X^{-1} + XY(F|q|X^{-1})^{-1/2} + ZL + X(F|q|X^{-1})^{1/2}(\varDelta + \varPhi + |q|/X)). \tag{51}$$

From (8), (50) and (51) we have

$$L^{-1}\frac{XY}{t}(S_2+S_3) \ll FZ + \sqrt[4]{F^{-2}t^{-3}X^7Y^7Z^4} + \sqrt{t^{-1}XYZ^4} + \sqrt[4]{t^{-1}F^2X^5YZ^4\tau^4} + \sqrt[4]{tF^2X^3Y^{-1}Z^4}. \tag{52}$$

As $t \geq \max(XL/Y, YL/X) \gg Z^2(XY)^{-1}$, we can compare similar terms of (49) and (52); for instance, we have

$$FZ \ll \sqrt{F^2XYt^2}, \quad \sqrt[4]{F^{-2}t^{-3}X^7Y^7Z^4} \ll \sqrt[4]{F^{-2}(XY)^9t^{-1}},$$
$$\sqrt{t^{-1}XYZ^4} \ll XYZ, \quad \sqrt[4]{t^{-1}F^2X^5YZ^4\tau^4} \ll \sqrt[4]{F^2X^3Y^3Z^4\tau^4t},$$
$$\sqrt[4]{tF^2X^3Y^{-1}Z^4} \ll \sqrt[4]{F^2XYZ^4t^3}.$$

Thus, from (9), (49) and (52) we get

$$L^{-6}|S_g(D)|^2 \ll (XY)^2t^{-1} + \sqrt{F^2XYt} + \sqrt[4]{F^{-2}(XY)^9t^{-1}} + XYZ + \sqrt[4]{F^2XYZ^4t^3} + \sqrt[4]{F^2X^3Y^3Z^4\tau^4t} + \sqrt[12]{F^2X^{19}Y^{15}t}. \tag{53}$$

The term $\sqrt[12]{F^2X^{19}Y^{15}t}$ of (53) can be neglected. Indeed, let the seven terms of (53) be $A_1, \ldots, A_7$. For $F \leq XY$, by Hölder's inequality we get

$$A_3 + A_5 \gg A_3^{2/3}A_5^{1/3} \gg A_7.$$

Let $F > XY$. Then for $X \geq Y^2$,

$$A_1 + A_5 \gg A_1^{8/21}A_5^{13/21} \gg A_7,$$

and for $X < Y^2$,

$$A_2 + A_3 \gg A_2^{4/9}A_3^{5/9} \gg A_7.$$

Thus we get from (53) the estimate

$$L^{-6}|S_g(D)|^2 \ll (XY)^2t^{-1} + \sqrt{F^2XYt} + \sqrt[4]{F^{-2}(XY)^9t^{-1}} + XYZ + \sqrt[4]{F^2XYZ^4t^3} + \sqrt[4]{F^2X^3Y^3Z^4\tau^4t}. \tag{54}$$

Suppose $X \geq Y$ in (54). Then we have

$$L^{-6}|S_g(D)|^2 \ll (XY)^2t^{-1} + \sqrt{F^2XYt} + \sqrt[4]{F^{-2}(XY)^9t^{-1}} + X^2Y + \sqrt[4]{F^2X^5Yt^3} + \sqrt[4]{F^2X^7Y^3\tau^4t}. \tag{55}$$

By (1) and Lemma 1 we have

$$S_g(D) = \sum_{1\le i\le C} S_g(D_i') + O(X),$$

where C is an absolute constant, and for each i,

$$D_i' = \{(a,b) \mid b \in I_i',\ g_1(b) \le a \le g_2(b)\} \subseteq D,$$

I_i' is an interval, g_1 and g_2 are suitable functions. Thus we can use Lemma 7 to estimate the sum over a in $S_g(D_i')$, and get

$$S_g(D_i') = O(Y(\sqrt[6]{FX^3} + X/F)) = O(\sqrt[6]{FX^3Y^6} + \sqrt[6]{F^{-2}(XY)^7}),$$

and thus

$$S_g(D) = O(\sqrt[6]{FX^3Y^6} + \sqrt[6]{F^{-2}(XY)^7} + X).$$

If $F \ge Y^3$ or $Y \le L^6$, then $\sqrt[6]{FX^3Y^6} \le L^3(\sqrt[6]{F^2(XY)^3})$, and we have

$$S_g(D) = O(L^3(\sqrt[6]{F^2(XY)^3} + \sqrt[6]{F^{-2}(XY)^7} + X)). \tag{56}$$

Suppose (7). The estimate (55) is derived for $t \ge XL/Y$ (cf. (8)). For $0 \le t < XL/Y$ and $F < Y^3$, we have

$$(XY)^2t^{-1} \gg XY^3L^{-1} \gg (\sqrt[3]{FX^3Y^6})L^{-1},$$

and thus from (55) and (56) we get for all t satisfying $0 \le t \le XYL^{-4}$ the estimate

$$\begin{aligned} L^{-6}|S_g(D)|^2 \ll{} & (XY)^2t^{-1} + \sqrt{F^2XYt} + \sqrt[4]{F^{-2}(XY)^9t^{-1}} \\ & + X^2Y + \sqrt[4]{F^2X^5Yt^3} + \sqrt[4]{F^2X^7Y^3\tau^4t} \\ & + \sqrt[3]{F^2(XY)^3} + \sqrt[3]{F^{-2}(XY)^7}. \end{aligned} \tag{57}$$

Suppose also $t \le \sqrt[3]{F^2(XY)^{-1}}$. Then $\sqrt[4]{F^{-2}(XY)^9t^{-1}} \ll (XY)^2t^{-1}$, and from (66) we get

$$\begin{aligned} L^{-6}|S_g(D)|^2 \ll{} & (XY)^2t^{-1} + \sqrt{F^2XYt} + X^2Y + \sqrt[4]{F^2X^5Yt^3} \\ & + \sqrt[4]{F^2X^7Y^3\tau^4t} + \sqrt[3]{F^2(XY)^3} + \sqrt[3]{F^{-2}(XY)^7} \end{aligned} \tag{58}$$

for $0 \le t \le \min(\sqrt[3]{F^2(XY)^{-1}}, XYL^{-4})$. By Lemma 6 we can choose t in this range in (58) to get

$$\begin{aligned} L^{-6}|S_g(D)|^2 \ll{} & \sqrt[3]{F^2(XY)^3} + \sqrt[3]{F^{-2}(XY)^7} \\ & + \sqrt[7]{F^2X^{11}Y^7} + \sqrt[5]{F^2X^9Y^5\tau^4} + X^2Y \\ ={} & B_1 + \cdots + B_5, \quad \text{say.} \end{aligned} \tag{59}$$

By Hölder's inequality we have

$$B_3 = B_1^{3/7}B_5^{4/7} \ll B_1 + B_5,$$

and

$$B_5 \ll B_1^{1/2}B_2^{1/2} \ll B_1 + B_2 \text{ for } X \le Y^2, \qquad B_5 \ll \sqrt[3]{X^7Y} \ll B_2 \text{ for } X > Y^2$$

(using (7)). Hence Theorem 1 follows from (59) in case $X \geq Y$. As X and Y are symmetric in (54), for the case $X < Y$ we can argue similarly. The proof of Theorem 1 is finished.

4. Preliminaries to the proof of Theorem 2

(I) *The polynomials* $F_i(t)$, $3 \leq i \leq 7$. Let (cf. (22))

(1′) $F(t) = \Phi(t,1) = \alpha\beta(2-\alpha-\beta)[\alpha(\alpha-1)t^2+2(\alpha-1)(\beta-1)t+\beta(\beta-1)]$.

To introduce the polynomials $F_i(t)$ for $3 \leq i \leq 7$, which will be used in the proof of Theorem 2, we use deeper arguments in the setting of (iii) of §3, but with the choices (q and r are fixed)

$$g(a,b) = Aa^\alpha b^\beta, \qquad G(a,b) = A[(a+q)^\alpha(b+r)^\beta - a^\alpha b^\beta],$$

and correspondingly, we take "$\Delta = \Phi = 0$" in (iii) (§3). Thus using (42) we can differentiate the function (u is a fixed number)

$$K_1 = K_1(a,u) = G(a,b(a,u)) - ub(a,u) + 1/8,$$

using $b'_a = -G_{1,1}/G_{0,2}$ to get ($G_{i,j} = G_{i,j}(a,b(a,u))$)

(2′) $(K_1)'_a = G_{1,0}, \quad (K_1)''_a = (G_{2,0}G_{0,2}-G_{1,1}^2)G_{0,2}^{-1}, \quad (K_1)_a^{(i)} = P_iG_{0,2}^{1-c(i)}$,

where $3 \leq i \leq 5$, $c(3) = 4$, $c(4) = 6$, $c(5) = 8$, P_i takes the form

(3′) $P_i = P_i(a,b(a,u)) = \sum C(i;r_1,\ldots,r_k,i_1,j_1,\ldots,i_k,j_k)G_{i_1,j_1}^{r_1}\cdots G_{i_k,j_k}^{r_k}$,

$\sum$ means summation over lattice points $(r_1,\ldots,r_k,i_1,j_1,\ldots,i_k,j_k)$ satisfying the conditions

$$r_1+\cdots+r_k = c(i), \qquad i_1r_1+\cdots+i_kr_k = i,$$
$$j_1r_1+\cdots+j_kr_k = 2(c(i)-1), \qquad i_1+j_1,\ldots,i_k+j_k \leq i,$$

where $r_1,\ldots,r_k \geq 1$, $i_1,j_1,\ldots,i_k,j_k \geq 0$, $c(i) \geq k \geq 1$, $(i_1,j_1),\ldots,(i_k,j_k)$ are different from each other, and $C(i;r_1,\ldots,j_k)$ is a suitable integer. We note that for $3 \leq i \leq 5$, (2′) and (3′) can be proved by a direct computation (a procedure which may be described as "taking the formal derivatives") with unspecified coefficients $C(i;r_1,\ldots,j_k)$. Let $P = G_{2,0}G_{0,2} - G_{1,1}^2$, and

(4′) $$P_6 = P_4P - 3P_3^2, \qquad P_7 = P_5P^2 - 10PP_3P_4 + 15P_3^3.$$

Using (12) and (13), from (3′) and (4′) we get ($b = b(a,u)$)

(5′) $P_i(a,b) = A^{c(i)}a^{\varepsilon(\alpha,i)}b^{\delta(\beta,i)}(\Phi_i(qb,ra) + O((XY\varrho)^{c(i)}\varrho)), \quad 3 \leq i \leq 7$,

where $\varepsilon(\alpha,i) = (\alpha-1)c(i)-i$, $\delta(\beta,i) = (\beta-3)c(i)+2$ for $3 \leq i \leq 5$, $c(6) = 8$, $c(7) = 12$ and

$$\varepsilon(\alpha,6) = 8\alpha-14, \qquad \delta(\beta,6) = 8\beta-20,$$
$$\varepsilon(\alpha,7) = 12\alpha-21, \qquad \delta(\beta,7) = 12\beta-30,$$

$\Phi_i(\xi,\eta)$ is a homogeneous polynomial of degree $c(i)$,

$$(6') \qquad \Phi_i(\xi,\eta)=\lambda_i\xi^{c(i)}+\dots+\mu_i\eta^{c(i)}, \quad 3\le i\le 7,$$

λ_i and μ_i are constants. Let the polynomials $F_i(t)$ be defined by

$$(7') \qquad F_i(t)=\Phi_i(t,1)=\lambda_i t^{c(i)}+\dots+\mu_i, \quad 3\le i\le 7.$$

We will use the real roots of the equations $F_i(t)=0$ to define summation ranges. To estimate S_1 and S_3 of (9), we will use $\lambda_i\ne 0$ for $3\le i\le 7$, and to estimate S_2 of (9), we will also require $\mu_i\ne 0$ for $3\le i\le 7$ as a condition. Stimulated by the proof of Lemma 7 in [C], we manage to obtain the values of λ_i and μ_i for $3\le i\le 5$, and so for $i=6$ and 7, in terms of α and β. We have

LEMMA 8. *Let* $\alpha,\beta\ne 0,1,2$. *Then*

$$\begin{aligned}
\lambda_i&=(\alpha\beta(\beta-1))^{c(i)-1}\alpha(\alpha-1)\left(\frac{2-\alpha-\beta}{\beta-1}\right)_{i-1}, \quad 3\le i\le 5,\\
\lambda_6&=\alpha^8\beta^6(\alpha-1)^2(2-\alpha-\beta)^2(3-\alpha-2\beta)(2\alpha+3\beta-5),\\
\lambda_7&=\alpha^{12}\beta^9(2-\alpha-\beta)^3(\alpha-1)^3(\beta-1)^3(3-\alpha-2\beta)\\
&\quad\times(5-2\alpha-3\beta)(7-3\alpha-4\beta),\\
\mu_i&=\alpha\beta\left(\frac{2-\alpha-\beta}{\beta-2}\right)_{i-1}(\beta(\beta-1)(\beta-2))^{c(i)-1}, \quad 3\le i\le 5,\\
\mu_6&=\alpha^2\beta^8(\beta-1)^6(\beta-2)^2(2-\alpha-\beta)^2(4-\alpha-2\beta)(2\alpha+3\beta-6),\\
\mu_7&=\alpha^3\beta^{12}(2-\alpha-\beta)^3(\beta-1)^9(\beta-2)^3(4-\alpha-2\beta)\\
&\quad\times(6-2\alpha-3\beta)(8-3\alpha-4\beta),
\end{aligned}$$

where $(s)_{i-1}=s(s-1)\cdots(s-i+2)$ *for a real* s. *Thus for* $\alpha\beta\ne 0$, $\alpha,\beta<1$, *we have* $\lambda_i\mu_i\ne 0$ *for all* $3\le i\le 7$ (*note that* $m\alpha+n\beta\ne k$ *for any positive integers* m,n *and* k *with* $k\ge m+n$).

Proof. Obviously, λ_i and μ_i are independent of q and r. To calculate λ_i for $3\le i\le 5$, we choose the special values of q and r, $q=[N]=[\sqrt{Xt/Y}]$ and $r=1$, where t is the parameter given by (8). Thus for $L=\log(XY)$,

$$(8') \qquad q\gg L^{1/2}, \quad |qY|\gg XL^{1/2}, \quad \varrho\approx|q|/X,$$

and from (2′), (5′) and (6′) we find for $3\le i\le 5$ that

$$\begin{aligned}
(9') \qquad G_{0,2}^{c(i)-1}(K_1)_a^{(i)}&=A^{c(i)}a^{\varepsilon(\alpha,i)}b^{\delta(\beta,i)}(\lambda_i(qb)^{c(i)}+O((qY)^{c(i)-1}X)\\
&\quad+O((XY\varrho)^{c(i)}\varrho))\\
&=A^{c(i)}a^{\varepsilon(\alpha,i)}b^{\delta(\beta,i)}(\lambda_i(qb)^{c(i)}+O((qY)^{c(i)}L^{-1/2})).
\end{aligned}$$

Let

$$G(a,b)=A[(a+q)^\alpha(b+1)^\beta-a^\alpha b^\beta], \quad a\approx X,\ b\approx Y.$$

The assumption $G_{0,2} > 0$ is equivalent to $\alpha A\beta(\beta-1) > 0$, for by (12) and (8′) we have

$$G_{0,2}(a,b) = A\beta(\beta-1)\alpha a^{\alpha-1}b^{\beta-2}q(1+O(L^{-1/2})),$$

which implies that

$$G_{0,2}(a,b)(A\beta(\beta-1)\alpha q)^{-1} \approx X^{\alpha-1}Y^{\beta-2}.$$

Similarly, we have

$$G_{0,1}(a,b) = A\beta\alpha a^{\alpha-1}b^{\beta-1}q(1+O(L^{-1/2})), \tag{10′}$$

which implies that

$$G_{0,1}(a,b)(A\beta\alpha q)^{-1} \approx X^{\alpha-1}Y^{\beta-1}.$$

We compare the function ($a \approx X$, $b \approx Y$)

$$\widetilde{G}(a,b) = \alpha q A a^{\alpha-1}b^{\beta} \tag{11′}$$

with $G(a,b)$. For a fixed number u satisfying $u(A\beta\alpha q)^{-1} \approx X^{\alpha-1}Y^{\beta-1}$, and a real variable $a \in I(u)$ for a suitable interval $I(u)$ depending on u, $I(u) \subseteq I'_5$ (cf. (41)), let

$$\widetilde{K}_1 = \widetilde{K}(a,u) = \widetilde{G}(a,\widetilde{b}(a,u)) - u\widetilde{b}(a,u) + 1/8,$$

where $\widetilde{b}(a,u)$ is determined by $\widetilde{G}_{0,1}(a,\widetilde{b}(a,u)) = u$. Then (2′) is also valid with K_1, $G_{r,s}$ and P_i replaced by $\widetilde{K}_1$, $\widetilde{G}_{r,s}$ and $\widetilde{P}_i$, where

$$\widetilde{G}_{r,s} = \widetilde{G}_{r,s}(a,\widetilde{b}(a,u)),$$
$$\widetilde{P}_i = \widetilde{P}_i(a,\widetilde{b}(a,u)) = \sum C(i;r_1,\ldots,j_k)\widetilde{G}^{r_1}_{i_1,j_1}\cdots\widetilde{G}^{r_k}_{i_k,j_k}.$$

For $3 \le i \le 5$, we can show that

$$\begin{aligned} G_{0,2}^{c(i)-1}(K_1)_a^{(i)} \\ = \widetilde{G}_{0,2}^{c(i)-1}(\widetilde{K}_1)_a^{(i)} + O(L^{-1/2}(F|q|X^{-1})^{c(i)}X^{-i}Y^{-2(c(i)-1)}), \end{aligned} \tag{12′}$$

where $G_{i,j} = G_{i,j}(a,b(a,u))$, $\widetilde{G}_{i,j} = \widetilde{G}_{i,j}(a,\widetilde{b}(a,u))$, and $F = |A|X^{\alpha}Y^{\beta}$. In fact, from (10′) and (11′) we get

$$b(a,u) = \widetilde{b}(a,u)(1+O(L^{-1/2})), \qquad \widetilde{b}(a,u) = \left(\frac{u}{\alpha\beta qA}\,a^{1-\alpha}\right)^{1/(\beta-1)}, \tag{13′}$$

and thus by (12) ("$\Delta = 0$") and (8′) we have

$$\begin{aligned} G_{i,j} = G_{i,j}(a,b(a,u)) &= A(\alpha)_{i+1}(\beta)_j a^{\alpha-i-1}b^{\beta-j}q(1+O(L^{-1/2})) \\ &= A(\alpha)_{i+1}(\beta)_j a^{\alpha-i-1}(\widetilde{b}(a,u))^{\beta-j}q(1+O(L^{-1/2})) \\ &= \widetilde{G}_{i,j}(1+O(L^{-1/2})). \end{aligned}$$

Consequently, from (2′) and (3′) we get (using $\widetilde{G}_{i,j} \ll F(|q|/X)X^{-i}Y^{-j}$)

$$\begin{aligned}G_{0,2}^{c(i)-1}(K_1)_a^{(i)} &= P_i(a,b(a,u)) = \sum C(i;r_1,\ldots,j_k)G_{i_1,j_1}^{r_1}\cdots G_{i_k,j_k}^{r_k}\\ &= \sum C(i;r_1,\ldots,j_k)\widetilde{G}_{i_1,j_1}^{r_1}\cdots \widetilde{G}_{i_k,j_k}^{r_k}(1+O(L^{-1/2}))\\ &= \widetilde{P}_i(a,\widetilde{b}(a,u)) + O(L^{-1/2}(F|q|X^{-1})^{c(i)}X^{-i}Y^{-2(c(i)-1)})\\ &= \widetilde{G}_{0,2}^{c(i)-1}(\widetilde{K}_1)_a^{(i)} + O(L^{-1/2}(F|q|X^{-1})^{c(i)}X^{-i}Y^{-2(c(i)-1)}),\end{aligned}$$

and (12′) follows. We can compute the value of $\widetilde{G}_{0,2}^{c(i)-1}(\widetilde{K}_1)_a^{(i)}$ precisely in terms of the value of $\widetilde{b}(a,u)$ of (13′). We have

$$\begin{aligned}(\widetilde{K}_1)'_a &= \widetilde{G}_{1,0}(a,\widetilde{b}(a,u)) = A\alpha(\alpha-1)qa^{\alpha-2}\widetilde{b}^{\beta}\\ &= A\alpha(\alpha-1)q\left(\frac{u}{A\alpha\beta q}\right)^{\beta/(\beta-1)}a^{(2-\alpha-\beta)/(\beta-1)},\\ (\widetilde{K}_1)_a^{(i)} &= A\alpha(\alpha-1)q\left(\frac{u}{A\alpha\beta q}\right)^{\beta/(\beta-1)}\left(\frac{2-\alpha-\beta}{\beta-1}\right)_{i-1}a^{(2-\alpha-\beta)/(\beta-1)-i+1}\\ &= Aq\alpha(\alpha-1)\left(\frac{2-\alpha-\beta}{\beta-1}\right)_{i-1}a^{\alpha-1-i}\widetilde{b}^{\beta},\\ \widetilde{G}_{0,2}^{c(i)-1} &= (A\alpha\beta(\beta-1)qa^{\alpha-1}\widetilde{b}^{\beta-2})^{c(i)-1},\end{aligned}$$

and thus

$$(14')\quad \widetilde{G}_{0,2}^{c(i)-1}(\widetilde{K}_1)_a^{(i)} = (Aq)^{c(i)}a^{(\alpha-1)c(i)-i}\widetilde{b}^{(\beta-2)c(i)+2} \times (\alpha\beta(\beta-1))^{c(i)-1}\alpha(\alpha-1)\left(\frac{2-\alpha-\beta}{\beta-1}\right)_{i-1}.$$

From (12′), (13′) and (14′) we get

$$(15')\quad G_{0,2}^{c(i)-1}(K_1)_a^{(i)} = (Aq)^{c(i)}a^{(\alpha-1)c(i)-i}b^{(\beta-2)c(i)+2}(\alpha\beta(\beta-1))^{c(i)-1} \times \alpha(\alpha-1)\left(\frac{2-\alpha-\beta}{\beta-1}\right)_{i-1}(1+O(L^{-1/2})).$$

From (9′) and (15′) we get

$$(16')\qquad \lambda_i = (\alpha\beta(\beta-1))^{c(i)-1}\alpha(\alpha-1)\left(\frac{2-\alpha-\beta}{\beta-1}\right)_{i-1},\qquad 3\le i\le 5.$$

To calculate μ_i for $3\le i\le 5$, by taking $q=1$ and $r=[\sqrt{Yt/X}]$, similarly to (9′), from (2′), (5′) and (6′) we get

$$(17')\qquad G_{0,2}^{c(i)-1}(K)_a^{(i)} = A^{c(i)}a^{\varepsilon(\alpha,i)}b^{\delta(\beta,i)}(\mu_i(ra)^{c(i)} + O((ra)^{c(i)}L^{-1/2})),$$

where $G(a,b) = A[(a+1)^{\alpha}(b+r)^{\beta} - a^{\alpha}b^{\beta}]$, $a\approx X$, $b\approx Y$. Now we have

$$\begin{aligned}G_{0,1}(a,b) &= A\beta(\beta-1)a^{\alpha}b^{\beta-2}r(1+O(L^{-1/2})),\\ G_{0,2}(a,b) &= A\beta(\beta-1)(\beta-2)a^{\alpha}b^{\beta-3}r(1+O(L^{-1/2})),\end{aligned}$$

and the assumption $G_{0,2} > 0$ of (ii) of §3 is equivalent to $A\beta(\beta-1)(\beta-2) > 0$. In particular we have

$$G_{0,1}(a,b)(A\beta(\beta-1)r)^{-1} \approx X^{\alpha}Y^{\beta-2}.$$

We compare $G(a,b)$ with the function

$$\widetilde{G}(a,b) = \beta r A a^{\alpha} b^{\beta-1}.$$

For a fixed number u satisfying

$$u(A\beta(\beta-1)r)^{-1} \approx X^{\alpha}Y^{\beta-2},$$

and a real variable $a \in I(u)$ for a suitable interval $I(u)$ depending on u, $I(u) \subseteq I'_5$ (cf. (41)), let

$$\widetilde{K}_1 = \widetilde{K}_1(a,u) = \widetilde{G}(a,\widetilde{b}(a,u)) - u\widetilde{b}(a,u) - 1/8,$$

where $\widetilde{b}(a,u)$ is determined by $\widetilde{G}_{0,1}(a,\widetilde{b}(a,u)) = u$. For $3 \le i \le 5$, similarly to (12′) we can deduce that

$$(18') \quad G_{0,2}^{c(i)-1}(K_1)_a^{(i)} = \widetilde{G}_{0,2}^{c(i)-1}(\widetilde{K}_1)_a^{(i)} + O(L^{-1/2}(F|r|Y^{-1})^{c(i)}X^{-i}Y^{2-2c(i)}).$$

We have (cf. (13′))

$$\widetilde{b}(a,u) = \left(\frac{u}{\beta(\beta-1)rA}\,a^{-\alpha}\right)^{1/(\beta-2)},$$

thus

$$\begin{aligned}
(\widetilde{K}_1)'_a &= \widetilde{G}_{1,0}(a,\widetilde{b}(a,u)) = \alpha\beta r A a^{\alpha-1}\widetilde{b}^{\beta-1} \\
&= \alpha\beta r A\left(\frac{u}{\beta(\beta-1)rA}\right)^{(\beta-1)/(\beta-2)} a^{(2-\alpha-\beta)/(\beta-2)}, \\
(\widetilde{K}_1)_a^{(i)} &= \alpha\beta r A\left(\frac{u}{\beta(\beta-1)rA}\right)^{(\beta-1)/(\beta-2)} \\
&\quad\times\left(\frac{2-\alpha-\beta}{\beta-2}\right)_{i-1} a^{(2-\alpha-\beta)/(\beta-2)-i+1} \\
&= \alpha\beta r A\left(\frac{2-\alpha-\beta}{\beta-2}\right)_{i-1} a^{\alpha-i}\widetilde{b}^{\beta-1}, \\
\widetilde{G}_{0,2}^{c(i)-1} &= (\beta(\beta-1)(\beta-2)rAa^{\alpha}\widetilde{b}^{\beta-3})^{c(i)-1},
\end{aligned}$$

$$\begin{aligned}
(19') \quad \widetilde{G}_{0,2}^{c(i)-1}(\widetilde{K}_1)_a^{(i)} &= (Ar)^{c(i)}a^{\alpha(c(i)-1)+\alpha-i}\widetilde{b}^{(\beta-3)(c(i)-1)+\beta-1}\alpha\beta \\
&\quad\times\left(\frac{2-\alpha-\beta}{\beta-2}\right)_{i-1}(\beta(\beta-1)(\beta-2))^{c(i)-1} \\
&= (Ar)^{c(i)}a^{\alpha c(i)-i}b^{(\beta-3)c(i)+2}\alpha\beta\left(\frac{2-\alpha-\beta}{\beta-2}\right)_{i-1} \\
&\quad\times(\beta(\beta-1)(\beta-2))^{c(i)-1}(1+O(L^{-1/2})).
\end{aligned}$$

From (18′) and (19′) we have

$$(20')\qquad G_{0,2}^{c(i)-1}(K_1)_a^{(i)} = (Ar)^{c(i)} a^{\alpha c(i)-i} b^{(\beta-3)c(i)+2} \alpha\beta \binom{2-\alpha-\beta}{\beta-2}_{i-1} \times (\beta(\beta-1)(\beta-2))^{c(i)-1}(1+O(L^{-1/2})).$$

From (17′) and (20′) we obtain

$$(21')\qquad \mu_i = \alpha\beta \binom{2-\alpha-\beta}{\beta-2}_{i-1} (\beta(\beta-1)(\beta-2))^{c(i)-1} \quad \text{for } 3 \le i \le 5.$$

From (21′) we get

$$\begin{aligned} P(a,b) &= G_{2,0}(a,b)G_{0,2}(a,b) - G_{1,1}^2(a,b) \\ &= A^2 a^{2\alpha-4} b^{2\beta-4}(\Phi(qb,ra) + O((XY\varrho)^2(\varrho+\Delta))), \\ \Phi(qb,ra) &= \alpha^2(\alpha-1)\beta(2-\alpha-\beta)(qb)^2 + 2\alpha\beta(2-\alpha-\beta)(\alpha-1)(\beta-1)qrab \\ &\quad + \alpha\beta^2(\beta-1)(2-\alpha-\beta)(ra)^2, \end{aligned}$$

thus in view of (4′)–(7′), (16′), and (21′) we get

$$\begin{aligned} \lambda_6 &= \lambda\lambda_4 - 3\lambda_3^2 \\ &= \alpha^8\beta^6(\alpha-1)^2(\beta-1)^2(2-\alpha-\beta)^2(3-\alpha-2\beta)(2\alpha+3\beta-5), \\ \mu_6 &= \mu\mu_4 - 3\mu_3^2 \\ &= \alpha^2\beta^8(\beta-1)^6(\beta-2)^2(2-\alpha-\beta)^2(4-\alpha-2\beta)(2\alpha+3\beta-6), \\ \lambda_7 &= \lambda_5\lambda^2 - 10\lambda\lambda_3\lambda_4 + 15\lambda_3^3 \\ &= \alpha^{12}\beta^9(2-\alpha-\beta)^3(\alpha-1)^3(\beta-1)^3(3-\alpha-2\beta) \\ &\quad \times (5-2\alpha-3\beta)(7-3\alpha-4\beta), \\ \mu_7 &= \mu_5\mu^2 - 10\mu\mu_3\mu_4 + 15\mu_3^3 \\ &= \alpha^3\beta^{12}(2-\alpha-\beta)^3(\beta-1)^9(\beta-2)^3(4-\alpha-2\beta) \\ &\quad \times (6-2\alpha-3\beta)(8-3\alpha-4\beta), \end{aligned}$$

where

$$\lambda = \alpha^2(\alpha-1)\beta(2-\alpha-\beta), \qquad \mu = \alpha\beta^2(\beta-1)(2-\alpha-\beta).$$

The proof of Lemma 8 is finished.

(II) *The polynomials* $F_i(t)$, $8 \le i \le 12$. In (iv) of §3, to estimate each $S_G(D_i')$ of (22) similarly to the way we estimated $S_G(D_5)$ in (iii) of §3, we need to reverse the orders of a and b in the treatment. Consequently, similarly to (41) we arrive at the exponential sum

$$\sum_{\nu_1 \le \nu \le \nu_2} \Big| \sum_{b \in I(\nu)} K_2 e(K_3) \Big|,$$

where $\nu_i = O(F\varrho X^{-1})$, $F\varrho > X$, $I(\nu)$ is an interval depending on ν, $b \in I(\nu)$ implies that $b \approx Y$, and

$$K_2 = K_2(b,\nu) = |G_{2,0}(a(b,\nu),b)|^{-1/2},$$
$$K_3 = K_3(b,\nu) = G(a(b,\nu),b) - \nu a(b,\nu) + 1/8,$$

$a(b,\nu)$ is the solution of $G_{1,0}(a(b,\nu),b) = \nu$ for given b and ν, and

$$G(a,b) = A[(a+q)^\alpha (b+r)^\beta - a^\alpha b^\beta],$$

q and r satisfy (8). Then, for each fixed ν, by partial summation we need to estimate, similarly to (42) and (43), the sum

$$\Big|\sum_{b\in I_1(\nu)} e(K_3(b,\nu))\Big|,$$

for a suitable interval $I_1(\nu) \subseteq I(\nu)$. Our purpose here is to introduce a number of additional polynomials, which will be used to define summation ranges. Thus we can omit many detailed explanations, and we only need to consider (formal) derivatives. We have

$$(a(b,\nu))'_b = -G_{1,1}(a(b,\nu),b)/G_{2,0}(a(b,\nu),b).$$

Thus, for $G_{i,j} = G_{i,j}(a(b,\nu),b)$, we get, similarly to (2′),

(22′) $(K_3)'_b = G_{0,1},\ (K_3)''_b = (G_{2,0}G_{0,2} - G_{1,1}^2)/G_{2,0},\ (K_3)_b^{(i)} = P_{i+5}G_{2,0}^{1-c(i)}$,

where $3 \le i \le 5$, $c(3) = 4$, $c(4) = 6$, $c(5) = 8$, and

(23′) $$P_{i+5} = P_{i+5}(a(b,\nu),b) = \sum\nolimits_1 C(i+5; r_1,\ldots,r_k,i_1,j_1,\ldots,i_k,j_k)G_{i_1,j_1}^{r_1}\cdots G_{i_k,j_k}^{r_k},$$

$\sum_1$ means a suitable summation over lattice points $(r_1,\ldots,r_k,i_1,j_1,\ldots,i_k,j_k)$ satisfying

$$r_1+\cdots+r_k = c(i),\quad i_1r_1+\cdots+i_kr_k = 2(c(i)-1),$$
$$j_1r_1+\cdots+j_kr_k = i,\quad i_1+j_1,\ldots,i_k+j_k \le i,$$

$r_1,\ldots,r_k \ge 1$, $i_1,j_1,\ldots,i_k,j_k \ge 0$, $c(i) \ge k \ge 1$, $(i_1,j_1),\ldots,(i_k,j_k)$ are different from each other, and $C(i+5;r_1,\ldots,j_k)$ is a suitable integer. We can obtain the expression (23′) by a direct calculation. However, our argument will not need a precise value of each coefficient $C(i+5;r_1,\ldots,j_k)$. As in (I), let $P = G_{2,0}G_{0,2} - G_{1,1}^2$. Let

(24′) $$P_{11} = P_9P - 3P_8^2,\qquad P_{12} = P_{10}P^2 - 10PP_8P_9 + 15P_8^3.$$

Then, similarly to (5′), using (12) and (13), from (22′) and (24′) we obtain

(25′) $$P_{i+5} = P_{i+5}(a(b,\nu),b) = A^{c(i)}a^{\widetilde{\varepsilon}(\alpha,i)}b^{\widetilde{\delta}(\beta,i)}(\varPhi_{i+5}(qb,ra) + O((XY\varrho)^{c(i)}\varrho)),\quad 3 \le i \le 7,$$

where $\widetilde{\varepsilon}(\alpha,i)=c(i)(\alpha-3)+2$, $\widetilde{\delta}(\beta,i)=c(i)(\beta-1)-i$ for $3\le i\le 5$, $c(6)=8$, $c(7)=12$ and $\widetilde{\varepsilon}(\alpha,6)=8\alpha-20$, $\widetilde{\delta}(\beta,6)=8\beta-14$, $\widetilde{\varepsilon}(\alpha,7)=12\alpha-30$, $\widetilde{\delta}(\beta,7)=12\beta-12$, $\Phi_{i+5}(\xi,\eta)$ is a homogeneous polynomial of degree $c(i)$,

$$(26')\qquad \Phi_{i+5}(\xi,\eta)=\lambda_{i+5}\xi^{c(i)}+\cdots+\mu_{i+5}\eta^{c(i)},\quad 3\le i\le 7,$$

λ_{i+5} and μ_{i+5} are constants. Let the polynomials $F_{i+5}(t)$ be defined by

$$(27')\qquad F_{i+5}(t)=\Phi_{i+5}(t,1)=\lambda_{i+5}t^{c(i)}+\cdots+\mu_{i+5},\quad 3\le i\le 7.$$

To estimate S_1 of (9) the condition $\lambda_i\ne 0$ for $8\le i\le 12$ is necessary. We have

LEMMA 9. *Let* $\alpha,\beta\ne 0,1,2$. *Then*

$$\lambda_{i+5}=\alpha\beta\left(\frac{2-\alpha-\beta}{\alpha-2}\right)_{i-1}(\alpha(\alpha-1)(\alpha-2))^{c(i)-1},\quad 3\le i\le 5,$$
$$\lambda_{11}=\alpha^8\beta^2(\alpha-1)^6(\alpha-2)^2(2-\alpha-\beta)^2(4-2\alpha-\beta)(3\alpha+2\beta-6),$$
$$\lambda_{12}=\alpha^{12}\beta^3(\alpha-1)^9(\alpha-2)^3(2-\alpha-\beta)^3(4-2\alpha-\beta)(6-3\alpha-2\beta)\times(8-4\alpha-3\beta).$$

Thus "$\alpha\beta\ne 0$, $\alpha<1$ *and* $\beta<1$" *implies that* $\lambda_i\ne 0$ *for* $8\le i\le 12$.

Proof. To calculate λ_{i+5} for $3\le i\le 5$, we can choose special values of q and r as in (I) by letting $q=[\sqrt{Xt/Y}]$, $r=1$, where t is given by (8). Then from (22′), (25′) and (26′) we get for $a=a(b,\nu)$, similarly to (9′),

$$(28')\qquad G_{2,0}^{c(i)-1}(K_3)_b^{(i)}=P_{i+5}(a,b)$$
$$=A^{c(i)}a^{\widetilde{\varepsilon}(\alpha,i)}b^{\widetilde{\delta}(\beta,i)}(\lambda_{i+5}(qb)^{c(i)}+O((qY)^{c(i)}L^{-1/2})),\quad 3\le i\le 5.$$

As in (I), (11′)–(16′), we compare the two functions

$$G(a,b)=A[(a+q)^\alpha(b+1)^\beta-a^\alpha b^\beta],\qquad \widetilde{G}(a,b)=\alpha qAa^{\alpha-1}b^\beta.$$

Assume that $\alpha(\alpha-1)(\alpha-2)A>0$ (the contrary case can be treated similarly). For a suitable number ν, let

$$\widetilde{K}_3=\widetilde{K}_3(b,\nu)=\widetilde{G}(\widetilde{a}(b,\nu),b)-\nu\widetilde{a}(b,\nu)+1/8,$$

where $\widetilde{a}(b,\nu)$ is determined by $\widetilde{G}_{1,0}(\widetilde{a}(b,\nu),b)=\nu$ for real variables b belonging to a suitable interval. For $3\le i\le 5$ we can deduce similarly to (12′) that ($\widetilde{G}_{i,j}=\widetilde{G}_{i,j}(\widetilde{a}(b,\nu),b)$)

$$(29')\qquad G_{2,0}^{c(i)-1}(K_3)_b^{(i)}$$
$$=\widetilde{G}_{2,0}^{c(i)-1}(\widetilde{K}_3)_b^{(i)}+O(L^{-1/2}(F|q|X^{-1})^{c(i)}X^{2-2c(i)}Y^{-i}).$$

On the other hand, from

$$\widetilde{a}(b,\nu)=\left(\frac{\nu}{A\alpha(\alpha-1)q}\,b^{-\beta}\right)^{1/(\alpha-2)},$$

we get

$$
\begin{aligned}
(\widetilde{K}_3)'_b &= \widetilde{G}_{0,1}(\widetilde{a}(b,\nu),b) = \alpha\beta qA\widetilde{a}^{\alpha-1}b^{\beta-1}\\
&= \alpha\beta qA\left(\frac{\nu}{A\alpha(\alpha-1)q}\right)^{(\alpha-1)/(\alpha-2)} b^{(2-\alpha-\beta)/(\alpha-2)},\\
(\widetilde{K}_3)^{(i)}_b &= \alpha\beta qA\left(\frac{\nu}{A\alpha(\alpha-1)q}\right)^{(\alpha-1)/(\alpha-2)}\\
&\quad\times\left(\frac{2-\alpha-\beta}{\alpha-2}\right)_{i-1} b^{(2-\alpha-\beta)/(\alpha-2)-i+1}\\
&= \alpha\beta qA\left(\frac{2-\alpha-\beta}{\alpha-2}\right)_{i-1}\widetilde{a}^{\alpha-1}b^{\beta-i},\\
\widetilde{G}^{c(i)-1}_{2,0} &= (\alpha(\alpha-1)(\alpha-2)qA\widetilde{a}^{\alpha-3}b^{\beta})^{c(i)-1},
\end{aligned}
$$

$$
\begin{aligned}
(30')\quad \widetilde{G}^{c(i)-1}_{2,0}(\widetilde{K}_3)^{(i)}_b &= (qA)^{c(i)}\widetilde{a}^{(\alpha-3)c(i)+2}b^{\beta c(i)-i}\alpha\beta\\
&\quad\times(\alpha(\alpha-1)(\alpha-2))^{c(i)-1}\left(\frac{2-\alpha-\beta}{\alpha-2}\right)_{i-1}\\
&= (qA)^{c(i)}\widetilde{a}^{(\alpha-3)c(i)+2}b^{\beta c(i)-i}\alpha\beta(\alpha(\alpha-1)(\alpha-2))^{c(i)-1}\\
&\quad\times\left(\frac{2-\alpha-\beta}{\alpha-2}\right)_{i-1}(1+O(L^{-1/2})),
\end{aligned}
$$

because we have, similarly to (13′),

$$a(b,\nu)=\widetilde{a}(b,\nu)(1+O(L^{-1/2})).$$

From (29′) and (30′) we get

$$
\begin{aligned}
(31')\quad & G^{c(i)-1}_{2,0}(K_3)^{(i)}_b\\
&= (qA)^{c(i)}a^{(\alpha-3)c(i)+2}b^{\beta c(i)-i}\alpha\beta(\alpha(\alpha-1)(\alpha-2))^{c(i)-1}\\
&\quad\times\left(\frac{2-\alpha-\beta}{\alpha-2}\right)_{i-1}(1+O(L^{-1/2})).
\end{aligned}
$$

Thus it follows from (28′) and (31′) that

$$\lambda_{i+5}=\alpha\beta(\alpha(\alpha-1)(\alpha-2))^{c(i)-1}\left(\frac{2-\alpha-\beta}{\alpha-2}\right)_{i-1},\qquad 3\le i\le 5.$$

Consequently, from (22′), (24′), (25′) and similarly to the calculation of λ_6 and λ_7 of (I), we get

$$
\begin{aligned}
\lambda_{11} &= \lambda\lambda_9-3\lambda_8^2\\
&= \alpha^8\beta^2(\alpha-1)^6(\alpha-2)^2(2-\alpha-\beta)^2(4-2\alpha-\beta)(3\alpha+2\beta-6),
\end{aligned}
$$

$$\begin{aligned}\lambda_{12} &= \lambda^2\lambda_{10} - 10\lambda\lambda_8\lambda_9 + 15\lambda_8^3\\ &= \alpha^{12}\beta^3(\alpha-1)^9(\alpha-2)^3(2-\alpha-\beta)^3(4-2\alpha-\beta)\\ &\quad\times(6-3\alpha-2\beta)(8-4\alpha-3\beta),\end{aligned}$$

which proves Lemma 9.

Similarly, we can also calculate each μ_i for $8 \le i \le 12$ in terms of α and β. However, the condition "$\mu_i \ne 0$ for $8 \le i \le 12$", which is indeed the case in a lot of applications, is not imperative in the proof of Theorem 2 (cf. §5).

(III) *Additional lemmas for the proof of Theorem 2*

LEMMA 10. *Suppose the real function $f(x)$ has continuous derivative $f''(x)$ on an interval $[a,b]$, and it satisfies*

$$|f'(x)| \approx \lambda_1, \qquad |f''(x)| \approx \lambda_2,$$

where $\lambda_i > 0$. Then

$$\sum_{a\le x\le b} e(f(x)) \ll \lambda_1\lambda_2^{-1/2} + \lambda_1^{-1} + \log(2+\lambda_1).$$

Proof. Suppose that

$$\lambda_1 \ll |f'(x)| \le C\lambda_1$$

on $[a,b]$, where C is an absolute constant. If $C\lambda_1 \le 1/2$, by Lemma 4.19 of [T] we have

$$\sum_{a\le x\le b} e(f(x)) \ll \lambda_1^{-1},$$

and if $C\lambda_1 > 1/2$, then by Lemmas 4.7 and 4.4 of [T] we obtain

$$\sum_{a\le x\le b} e(f(x)) \ll (1+\lambda_1)\lambda_2^{-1/2} + \log(2+\lambda_1) \ll \lambda_1\lambda_2^{-1/2} + \log(2+\lambda_1).$$

Combining these two estimates gives the assertion of the lemma.

LEMMA 11. *Let $I = [X, X']$, $X' > X \ge 1$, let Q be a positive integer, and Z_n $(X \le n \le X')$ be complex numbers. Then*

$$\Big|\sum_{n\in I} Z_n\Big|^2 \le (1+(X'-X)Q^{-1}) \sum_{|q|\le Q}(1-|q|Q^{-1}) \sum_{n,n+q\in I} \overline{Z}_n Z_{n+q}.$$

Proof. This can be proved similarly to Lemma 5 of [HB]. If $Q \ge 1$ and Q is not an integer, to ensure the validity of the inequality we should replace Q^{-1} by $[Q]^{-1}$.

In the next lemma, we extend the action of the exponent pair $(11/30, 16/30) = BA^2(1/2, 1/2)$ to a general class of functions.

LEMMA 12. *Let $f(x)$ be a real function on $[a,b]$ whose derivatives of orders* 1 *to* 5 *are continuous and satisfy* (*for* $1 \le a < b \le 2a$)

$$|f''(x)| \approx \lambda_2, \quad |f'''(x)| \approx \lambda_3, \quad |f^{(4)}(x)| \ll \lambda_4,$$
$$\lambda_3 \approx \lambda_2 U^{-1},\ \lambda_4 \ll \lambda_2 U^{-2},\ 0 < U \ll a,$$
$$|(f^{(4)}(x)f''(x) - 3(f'''(x))^2)(f''(x))^{-5}| \approx r_1 > 0,$$
$$|(f^{(5)}(x)(f''(x))^2 - 10f''(x)f'''(x)f^{(4)}(x) + 15(f'''(x))^3)(f''(x))^{-7}| \approx r_2 > 0.$$

Then

$$\Big|\sum_{a\le x\le b} e(f(x))\Big| \ll \sqrt[30]{\eta^{28} r_1^2 r_2^{-1}\lambda_2^{13}} + \sqrt[8]{\eta^5\lambda_3^2\lambda_2^{-2}r_1^{-2}}$$
$$+ (\sqrt[4]{\eta\lambda_2^3\lambda_3^{-2}} + \sqrt[8]{\eta^7\lambda_2^3} + (\eta\lambda_2)^{1/2} + 1)L + \lambda_2^{-1/2} + \eta U^{-1},$$

where $\eta = b - a$ *and* $L = \log(2 + (b-a)\lambda_2)$.

Proof. Assume $\eta\lambda_2 \ge 1$. By Lemma 4 we have

$$(32') \qquad \sum_{a\le x\le b} e(f(x))$$
$$= \lambda \sum_{y_1\le y\le y_2} |f''(x_y)|^{-1/2} e(F(y)) + O(\lambda_2^{-1/2}) + O(L) + O((b-a+\lambda_2^{-1})U^{-1}),$$

where $F(y) = f(x_y) - yx_y$, x_y is the solution of $f'(x) = y$ for $y \in [y_1, y_2]$, and $y_1 = \min(f'(a), f'(b))$, $y_2 = \max(f'(a), f'(b))$, $\lambda = 1$ or $-i$ according as $f''(x) > 0$ or $f''(x) < 0$ on $[a,b]$. We can suppose $\lambda = 1$ without loss of generality in our treatment. Then $y_1 = f'(a)$, $y_2 = f'(b)$. Let $F_0(y) = (f''(x_y))^{-1/2}$. Then for $y \in [y_1, y_2]$ we have

$$F_0'(y) = -\tfrac{1}{2}(f''(x_y))^{3/2} f'''(x_y)(x_y)'_y = -\tfrac{1}{2}(f''(x_y))^{5/2} f'''(x_y),$$

and thus $F_0'(y)$ keeps a constant sign. By partial summation we get

$$(33') \qquad \sum_{y_1\le y\le y_2} F_0(y) e(F(y))$$
$$= -\int_{y_1}^{y_2} \Big(\sum_{y_1\le y\le t} e(F(y))\Big) F_0'(t)\,dt + F_0(y_2)\sum_{y_1\le y\le y_2} e(F(y))$$
$$\ll (|F_0(y_1)| + |F_0(y_2)|)\Big|\sum_{y_1\le y\le y_3} e(F(y))\Big| \ll \lambda_2^{-1/2}\Big|\sum_{y_1\le y\le y_3} e(F(y))\Big|,$$

where y_3 is some suitable number in $[y_1, y_2]$. We assume the difficult case that $y_3 \ge y_1 + 100$. Let $y_3 - y_1 = \delta$, and $I = [y_1, y_3]$. By Lemma 11 we get

$$(34') \quad |\Sigma|^2 = \Big|\sum_{y\in I} e(F(y))\Big|^2 \ll \delta^2 Q^{-1} + \delta Q^{-1}\sum_{1\le|q|\le Q} |\Sigma_1|, \quad \Sigma_1 = \sum_{y\in I_1} e(F_1(y)),$$

where $I_1 = \{y \mid y \in I,\, y+q \in I\}$, $F_1(y) = F(y+q) - F(y)$, $Q \in [1, \delta/\log\delta]$ is a parameter. Applying Lemma 11 repeatedly we get

$$(35') \qquad |\Sigma_1|^2 \ll \delta^2 Q_1^{-1} + \delta Q_1^{-1} \sum_{1\le|q_1|\le Q_1} |\Sigma_2|, \qquad \Sigma_2 = \sum_{y\in I_2} e(F_2(y)),$$

$$(36') \qquad |\Sigma_2|^2 \ll \delta^2 Q_2^{-1} + \delta Q_2^{-1} \sum_{1\le|q_2|\le Q_2} |\Sigma_3|, \qquad \Sigma_3 = \sum_{y\in I_3} e(F_3(y)),$$

where $I_2 = \{y \mid y \in I_1,\, y+q_1 \in I_1\}$, $I_3 = \{y \mid y \in I_2,\, y+q_2 \in I_2\}$, and

$$F_2(y) = F_1(y+q_1) - F_1(y), \qquad F_3(y) = F_2(y+q_2) - F_2(y),$$

Q_1 and Q_2 are parameters which belong to $[1, \delta/\log\delta]$. We suppose that $|I_3| \ge 10$, where $|I_3|$ is the length of I_3. For a real variable $y \in I_1$, we have

$$(37') \qquad F_1(y) = q\int_0^1 F'(y+tq)\,dt,$$

for $y \in I_2$ by (37′) we have

$$(38') \qquad F_2(y) = q_1\int_0^1 F_1'(y+t_1q_1)\,dt_1 = qq_1\int_0^1\int_0^1 F''(y+tq+t_1q_1)\,dt\,dt_1,$$

and for $y \in I_3$ by (38′) we get

$$(39') \qquad F_3(y) = q_2\int_0^1 F_2'(y+t_2q_2)\,dt_2$$

$$= qq_1q_2\int_0^1\int_0^1\int_0^1 F'''(y+tq+t_1q_1+t_2q_2)\,dt\,dt_1\,dt_2.$$

For all $y \in I$, from $f'(x_y) = y$ we get $(x_y)' = 1/f''(x_y)$. Thus, taking derivatives of $F(y)$ we get

$$(40') \qquad F'(y) = -x_y, \quad F''(y) = -1/f''(x_y), \quad F'''(y) = f'''(x_y)(f''(x_y))^{-3},$$

$$(41') \qquad F^{(4)}(y) = (f^{(4)}(x_y)f''(x_y) - 3(f'''(x_y))^2)(f''(x_y))^{-5},$$

$$(42') \qquad F^{(5)}(y) = \frac{f^{(5)}(x_y)(f''(x_y))^2 - 10f''(x_y)f'''(x_y)f^{(4)}(x_y) + 15(f'''(x_y))^3}{(f''(x_y))^7}.$$

For $y \in I_3$, by (39′), (41′), (42′) and our assumption we get

$$|F_3'(y)| = \left|qq_1q_2\int_0^1\int_0^1\int_0^1 F^{(4)}(y+tq+t_1q_1+t_2q_2)\,dt\,dt_1\,dt_2\right| \approx r_1|qq_1q_2|,$$

$$|F_3''(y)| = \left|qq_1q_2\int_0^1\int_0^1\int_0^1 F^{(5)}(y+tq+t_1q_1+t_2q_2)\,dt\,dt_1\,dt_2\right| \approx r_2|qq_1q_2|,$$

thus by Lemma 7 we have

$$|\Sigma_3| \ll r_1 r_2^{-1/2}|qq_1q_2|^{1/2} + (r_1|qq_1q_2|)^{-1} + 1.$$

Consequently, by (36′) we get

$$|\Sigma_2|^2 \ll \delta^2 Q_2^{-1} + r_1 r_2^{-1/2}|qq_1|^{1/2}\delta Q_2^{1/2} + \delta Q_2^{-1} r_1^{-1}|qq_1|^{-1}L. \tag{43′}$$

If $\delta^2 \ge \delta r_1^{-1}|qq_1|^{-1}L$, from (43′) we get

$$|\Sigma_2|^2 \ll \delta^2 Q_2^{-1} + r_1 r_2^{-1/2}|qq_1|^{1/2}\delta Q_2^{1/2}.$$

Let $Q_2 = \min(\delta/L, \sqrt[3]{\delta^2 r_1^{-2} r_2 |qq_1|^{-1}})$. We get (for $Q_2 < 1$, (43′) holds trivially)

$$|\Sigma_2|^2 \ll \delta L + \sqrt[3]{\delta^4 r_1^2 r_2^{-1}|qq_1|}. \tag{44′}$$

If $\delta^2 < \delta(r_1|qq_1|)^{-1}L$, from (38′), (40′), (41′) and our assumption we get (for $y \in I_2$)

$$|F_2'(y)| = \Big|qq_1\int_0^1\int_0^1 F'''(y+tq+t_1q_1)\,dt\,dt_1\Big| \approx |qq_1|\lambda_3\lambda_2^{-3},$$

$$|F_2''(y)| = \Big|qq_1\int_0^1\int_0^1 F^{(4)}(y+tq+t_1q_1)\,dt\,dt_1\Big| \approx |qq_1|r_1.$$

Hence by Lemma 10 we get the estimate

$$\begin{aligned}|\Sigma_2| &\ll |qq_1|^{1/2}\lambda_3\lambda_2^{-3}r_1^{-1/2} + (|qq_1|\lambda_3\lambda_2^{-3})^{-1} + 1\\ &\ll \delta^{-1/2}\lambda_3\lambda_2^{-3}r_1^{-1}L^{1/2} + (|qq_1|\lambda_3\lambda_2^{-3})^{-1} + 1.\end{aligned} \tag{45′}$$

From (44′) and (45′) we always have

$$\begin{aligned}|\Sigma_2| \ll{}& \sqrt[6]{\delta^4 r_1^2 r_2^{-1}|qq_1|} + \delta^{-1/2}\lambda_3\lambda_2^{-3}r_1^{-1}L^{1/2}\\ &+ (|qq_1|\lambda_3\lambda_2^{-3})^{-1} + \delta^{1/2}L^{1/2}.\end{aligned} \tag{46′}$$

From (35′) and (46′) we get

$$\begin{aligned}|\Sigma_1|^2 \ll{}& \delta^2 Q_1^{-1} + \sqrt[6]{\delta^{10} r_1^2 r_2^{-1} Q_1|q|} + \delta^{1/2}\lambda_3\lambda_2^{-3}r_1^{-1}L^{1/2}\\ &+ \delta\lambda_3^{-1}\lambda_2^3|q|^{-1}Q_1^{-1}L + \delta^{3/2}L^{1/2}.\end{aligned}$$

If $\delta^2 \ge \delta\lambda_3^{-1}\lambda_2^3|q|^{-1}L$, then

$$|\Sigma_1|^2 \ll \delta^2 Q_1^{-1} + \sqrt[6]{\delta^{10} r_1^2 r_2^{-1} Q_1|q|} + \delta^{1/2}\lambda_3\lambda_2^{-3}r_1^{-1}L^{1/2} + \delta^{3/2}L^{1/2}. \tag{47′}$$

Let $Q_1 = \min(\delta L^{-1/2}, \sqrt[7]{\delta^2 r_1^{-2} r_2 |q|^{-1}})$ in (47′). We get

$$|\Sigma_1|^2 \ll \sqrt[7]{\delta^{12} r_1^2 r_2^{-1}|q|} + \delta^{1/2}\lambda_3\lambda_2^{-3}r_1^{-1}L^{1/2} + \delta^{3/2}L^{1/2}. \tag{48′}$$

If $\delta^2 < \delta\lambda_3^{-1}\lambda_2^3|q|^{-1}L$, by (37′), (40′) and our assumption we get

$$|F_1'(y)| = \Big|q\int_0^1 F''(y+tq)\,dt\Big| \approx |q|\lambda_2^{-1},$$

$$|F_1''(y)| = \Big|q\int_0^1 F'''(y+tq)\,dt\Big| \approx |q|\lambda_3\lambda_2^{-3},$$

where $y \in I_1$. Hence by Lemma 10 we have

$$(49') \qquad |\Sigma_1| \ll (|q|\lambda_3^{-1}\lambda_2)^{1/2} + |q|^{-1}\lambda_2 + 1 \ll (\lambda_2^4\lambda_3^{-2}\delta^{-1}L)^{1/2} + \lambda_2|q|^{-1} + 1.$$

From (48′) and (49′) we always have the estimate

$$(50') \quad |\Sigma_1| \ll \sqrt[14]{\delta^{12}r_1^2r_2^{-1}|q|} + \Big(\sqrt[4]{\delta\lambda_3^2\lambda_2^{-6}r_1^{-2}} + \sqrt{\lambda_2^4\lambda_3^{-2}\delta^{-1}} + \delta^{3/4}\Big)L^{1/2} + \lambda_2|q|^{-1}.$$

It follows from (34′) and (50′) that

$$(51') \quad |\Sigma|^2 \ll \delta^2Q^{-1} + \Big(\sqrt[14]{\delta^{26}r_1^2r_2^{-1}Q} + \sqrt[4]{\delta^5\lambda_3^2\lambda_2^{-6}r_1^{-2}} + \sqrt{\delta\lambda_2^4\lambda_3^{-2}} + \delta^{7/4}\Big)L^{1/2} + \lambda_2\delta Q^{-1}L.$$

For $\delta^2 \geq \delta\lambda_2L$, $\delta^2Q^{-1} \geq \lambda_2\delta Q^{-1}L$, let $Q = \min(\delta L^{-1/2}, \sqrt[15]{\delta^2r_1^{-2}r_2})$ in (51′). We obtain

$$|\Sigma|^2 \ll \sqrt[15]{\delta^{28}r_1^2r_2^{-1}} + \Big(\sqrt[4]{\delta^5\lambda_3^2\lambda_2^{-6}r_1^{-2}} + \sqrt{\delta\lambda_2^4\lambda_3^{-2}} + \delta^{7/4}\Big)L^{1/2}.$$

For $\delta^2 < \delta\lambda_2L$, $|\Sigma| = O(\lambda_2L)$, and thus

$$(52') \quad |\Sigma| \ll \sqrt[30]{\delta^{28}r_1^2r_2^{-1}} + \Big(\sqrt[8]{\delta^5\lambda_3^2\lambda_2^{-6}r_1^{-2}} + \sqrt[4]{\delta\lambda_2^4\lambda_3^{-2}} + \delta^{7/8} + \lambda_2\Big)L.$$

As $\delta \ll 1 + \eta\lambda_2$, if $\eta\lambda_2 \geq 1$, the conclusion of Lemma 12 follows from (32′), (33′) and (52′). In case $\eta\lambda_2 < 1$, it follows from Lemma 3, for we have

$$\Big|\sum_{a\leq x\leq b} e(f(x))\Big| \ll (b-a)\lambda_2^{1/2} + \lambda_2^{-1/2} = \eta\lambda_2^{1/2} + \lambda_2^{-1/2} \ll \lambda_2^{-1/2}.$$

The proof of Lemma 12 is finished.

Remark. In §2.3 of A. Ivić's book *The Riemann Zeta-Function*, it is claimed that if $f(x)$ is a real function, having continuous derivatives of any order on the interval $[N, 2N]$, $N \geq 1$, and satisfying

$$|f^{(k)}(x)| \approx \lambda N^{1-k}, \qquad k \geq 1,\ \lambda > 0,$$

then for any exponent pair (p,q), we have the estimate

$$(*)\qquad \Big|\sum_{n\le x\le 2n} e(f(x))\Big| \ll \lambda^p N^q + \lambda^{-1}.$$

However, taking $f(x) = Cx^{3/2}$, $C = 2(27)^{-1/2}$, and using Lemma 4 of the present paper to calculate the exponential sum of $(*)$, it is easy to observe that the estimate of $(*)$ does not hold for $(p,q) = (2/7, 4/7)$ and $(11/53, 33/53)$. The mistake comes from relying heavily on a famous paper of E. Phillips (published in Quart. J. Math. in 1933), simplifying van der Corput's method, in which the proof of a key result of van der Corput (Lemma 7 of the reference [C] of the present paper) was not given. However, using induction and Lemmas 10 and 11 of our paper, we find that $(*)$ does hold for $(p,q) = A^r B(0,1)$, where r is any non-negative integer.

5. Proof of Theorem 2. A polynomial with real coefficients can be factorized as the product of positive definite quadratic polynomials with real coefficients and linear polynomials with real coefficients. Let the real roots of the equation (cf. §4, (I) and (II)) $F_i(t) = 0$ be $\theta_{i1}, \ldots, \theta_{ik_i}$, where $|\theta_{i1}| \le \cdots \le |\theta_{ik_i}|$, multiple roots are counted with multiplicity, $i = 3, 6, 7, 8, 11$, and 12 and $k_3 \le 4$, $k_6 \le 8$, $k_7 \le 12$, $k_8 \le 4$, $k_{11} \le 8$, $k_{12} \le 12$. Here we note that, by the condition of Theorem 2 and Lemmas 8 and 9, $F_i(t) \not\equiv 0$ for these i. In case $k_i = 0$ for some i, the argument below will be simpler. Thus we assume the difficult case that $k_i \ge 1$ for all these i. As at the beginning of §3, assume that θ_{11} and θ_{12} ($|\theta_{11}| \le |\theta_{12}|$) are the real roots of $F(t) = 0$ (cf. §4, (I)), and $\theta_{21} = (2-\beta)/\alpha$, $\theta_{22} = \beta/(\alpha-2)$. Assume (7) and (8). As at the beginning of §3, by (18), to estimate the exponential sum $S_g(D)$, we need to estimate the exponential sum $S_G(D_1)$. As at the beginning of §3 (of course, we consider the difficult case that θ_{11} and θ_{12} are real), let

$$D_2 = \{(a,b) \mid |bq - \theta_{ij} ar| < \varrho \text{ for some } (i,j)\},$$
$$D_3 = \{(a,b) \mid (a,b) \in D_1,\ (a,b) \notin D_2\}.$$

As at the beginning of §3 we get $S_G(D_2) = O(Y)$. Thus

$$(53')\qquad |S_G(D_1)| \ll |S_G(D_3)| + Y.$$

As is (15) we have, for some particular D_4 (note that $k_1 + \cdots + k_{12} \le 52$),

$$(54')\qquad |S_G(D_3)| \ll L^{52} |S_G(D_4)| + L^{52} X,$$

where (note that for $(i,j) \in \varGamma$, θ_{ij} is real, and vice versa)

$$D_4 = \{(a,b) \mid \varepsilon_{ij}\lambda |qr| \varDelta_{ij} \le bq - \theta_{ij} ar \le \delta_{ij}\lambda |qr| \varDelta_{ij},\ (i,j) \in \varGamma\} \cap D_1,$$
$$\varGamma = \{(i,j) \mid i = 1,2,3,6,7,8,11,12,\ 1 \le j \le k_i,\ k_1 = k_2 = 2\},$$

Δ_{ij}, ε_{ij} and δ_{ij} are numbers satisfying $1/(XY) \le \Delta_{ij} \le 2(1+\sum_{(i,j)\in\Gamma} |\theta_{ij}|)$, $(\varepsilon_{ij}, \delta_{ij}) = (-2,-1)$ or $(1,2)$. Suppose $q > 0$. Let

$$R_1 = \max_{(i,j)\in\Gamma}(q^{-1}(\varepsilon_{ij}\lambda|qr|\Delta_{ij}+\theta_{ij}ar)), \quad R_2 = \min_{(i,j)\in\Gamma}(q^{-1}(\delta_{ij}\lambda|qr|\Delta_{ij}+\theta_{ij}ar)).$$

Then as at the beginning of §3 we have

$$D_4 = \{(a,b) \mid a \in I_1,\ B_1(a) \le b \le B_2(a)\},$$

where $I_1 = \{a \in I \mid (a+q) \in I\}$, and

$$B_1(a) = \max(f_1(a+q)-r, f_1(a), R_1), \qquad B_2(a) = \min(f_2(a+q)-r, f_2(a), R_2).$$

Using the method of showing (16), by taking derivatives to discuss monotonicity we get

$$(55') \qquad |S_G(D_4)| \ll \sum_{D_5} |S_G(D_5)| + Y,$$

where the summation is taken over $O(1)$ disjoint ranges D_5 of the shape

$$D_5 = \{(a,b) \mid a \in I',\ B_1(a) \le b \le B_2(a)\} \subseteq D_4,$$

I' is a suitable interval contained in I_1, $B_1(a)$ has one of the three forms

$$f_1(a+q)-r, \quad f_1(a), \quad k_1a+k_2, \quad (k_1,k_2) = (r\theta_{ij}q^{-1}, \varepsilon_{ij}\lambda|r|\Delta_{ij}),$$

for some $(i,j) \in \Gamma$, and similarly, $B_2(a)$ has one of the following three forms on I':

$$f_2(a), \quad f_2(a+q)-r, \quad k'_1a+k'_2, \quad (k'_1,k'_2) = (r\theta_{i'j'}q^{-1}, \delta_{i'j'}\lambda|r|\Delta_{i'j'}),$$

for some $(i',j') \in \Gamma$. Note that Δ_{11} and Δ_{12} correspond to Δ_1 and Δ_2 in §3, and Δ_{21} and Δ_{22} correspond to Δ_3 and Δ_4 in §3 (see the beginning of §3). Thus, as in (17), we have

$$(56') \qquad \Delta_{ps} + \Delta_{p's'} \ge 2\delta \quad \text{for } p,p',s,s' = 1 \text{ or } 2,\ (p,s) \ne (p',s'),$$

where δ is a suitably small positive constant which depends only on α and β, which, as at the beginning of §3, enables us to carry out the argument below. Let $\widetilde{\Delta} = \min_{(i,j)\in\Gamma} \Delta_{ij}$. We distinguish several cases to estimate the exponential sum $S_G(D_5)$.

Case (i): *$\widetilde{\Delta} \le L\varrho$ and $\Delta_{21} > \delta$.* In this case, we reason as in (i) of §3. Thus, similarly to (20), we get

$$(57') \qquad S_G(D_5) = O(X(F\varrho^3)^{1/2}L\varrho\lambda + XY(F\varrho)^{-1/2}),$$

where, as in (11), $\varrho = |q|/X + |r|/Y$ and $\lambda = \varrho XY/|qr|$.

Case (ii): *$\widetilde{\Delta} \le L\varrho$ and $\Delta_{21} \le \delta$.* In this case, we argue similarly to (ii) of §3 (if $\widetilde{\Delta} = \Delta_{ij}$ and $\theta_{ij} = 0$ for some $(i,j) \in \Gamma$, the treatment is simpler). Thus, similarly to (24), we deduce the estimate

$$(58') \qquad S_G(D_5) = O(Y(F\varrho^3)^{1/2}L\varrho\lambda + XY(F\varrho)^{-1/2} + X + Y).$$

Case (iii): $\widetilde{\Delta} > L\varrho$ *and* $\Delta_{21} > \delta$. This case is similar to (iii) of §3. Thus, as in (40), (41) and (42), for $F\varrho > Y$ we get (with "$\Delta = \Phi = 0$")

$$(59') \quad |S_G(D_5)| \ll \Big|\sum_{a\in I_5'}\sum_{X_1(a)\le u\le X_2(a)} Ke(K_1)\Big| + XY(F\varrho)^{-1/2} + X(F\varrho^3)^{1/2} + XL,$$

$$(60') \quad \Big|\sum_{a\in I_5'}\sum_{X_1(a)\le u\le X_2(a)} Ke(K_1)\Big| \ll \sum_{u_1\le u\le u_2}\Big|\sum_{a\in I(u)} Ke(K_1)\Big|$$

and

$$(61') \quad \Big|\sum_{a\in I(u)} Ke(K_1)\Big| \ll Y(F\varrho)^{-1/2}\Big|\sum_{a\in I_1(u)} e(K_1(a,u))\Big|,$$

where I_5', $I(u)$ and $I_1(u)$ are suitable intervals,

$$I_1(u) \subseteq I(u) \subseteq S(u) = \{a \in I_5' \mid X_1(a) \le u \le X_2(a)\},$$

$a \in I_5'$ implies that $B_1(a) < B_2(a) - 10$, $X_1(a)$ and $X_2(a)$ are suitable functions, $[X_1(a), X_2(a)] \subseteq [\alpha_1(a), \alpha_2(a)]$ in case $X_1(a) \le X_2(a)$, $u_1, u_2 = O(F\varrho Y^{-1})$, and, without losing the generality, assuming that $G_{0,2} > 0$ on D_5,

$$K = K(a,u) = (G_{0,2}(a, b(a,u)))^{-1/2},$$
$$K_1 = K_1(a,u) = G(a, b(a,u)) - ub(a,u),$$

$b(a,u)$ is the solution of $G_{0,1}(a, b(a,u)) = u$ for given a and u, and for a number u, $u_1 \le u \le u_2$, if $I_1(u) \ne \emptyset$ and $I_1(u)$ is an interval of length > 0 then for all $a \in I_1(u)$, we have $(a, b(a,u)) \in D_5$. As in (iii) of §3 we have

$$(62') \quad (b(a,u))'_a = -G_{1,1}(a, b(a,u))/G_{0,2}(a, b(a,u)).$$

We take derivatives in $a \in I_1(u)$ based on (62′) and use the notations of (I) of §4 to get (cf. (2′), (4′))

$$(K_1)'_a = G_{1,0}, \quad (K_1)''_a = (G_{2,0}G_{0,2} - G_{1,1}^2)G_{0,2}^{-1} = PG_{0,2}^{-1},$$
$$(K_1)^{(i)}_a = P_i G_{0,2}^{1-c(i)}, \quad 3 \le i \le 5,$$
$$(K_1)''_a(K_1)^{(4)}_a - 3((K_1)'''_a)^2 = (P_4P - 3P_3^2)G_{0,2}^{-6} = P_6G_{0,2}^{-6},$$
$$(K_1)^{(5)}_a((K_1)''_a)^2 - 10(K_1)''_a(K_1)'''_a(K_1)^{(4)}_a + 15((K_1)'''_a)^3$$
$$= (P_5P^2 - 10PP_3P_4 + 15P_3^3)G_{0,2}^{-9} = P_7G_{0,2}^{-9},$$

where $G_{i,j} = G_{i,j}(a, b(a,u))$, $c(3) = 4$, $c(4) = 6$, $c(5) = 8$, $c(6) = 8$, $c(7) = 12$, and P_i satisfies (5′), that is,

$$P_i = A^{c(i)}a^{\varepsilon(\alpha,i)}b^{\delta(\beta,i)}(\Phi_i(qb, ra) + O((XY\varrho)^{c(i)}\varrho)), \quad 3 \le i \le 7,$$

where

$$\varepsilon(\alpha,i) = (\alpha-1)c(i) - i, \quad \delta(\beta,i) = (\beta-3)c(i) + 2, \quad 3 \le i \le 5,$$

$\varepsilon(\alpha,6)=8\alpha-14,\ \delta(\beta,6)=8\beta-20,\ \varepsilon(\alpha,7)=12\alpha-21,\ \delta(\beta,7)=12\beta-30,$

$\Phi_i(\xi,\eta)$ is a homogeneous polynomial for $3\le i\le 7$, which, by the assumptions of Theorem 2 on α and β and Lemma 8, has the form

$$\Phi_i(\xi,\eta)=\lambda_i\xi^{c(i)}+\dots+\mu_i\eta^{c(i)},\qquad \lambda_i\mu_i\ne 0.$$

For each $3\le i\le 7$, since the complex roots appear in conjugate pairs, we know that $2\,|\,(c(i)-k_i)$. Thus from $|qb|+|ra|\approx XY\varrho=|qr|\lambda$, for $(a,b)\in D_5$ we get

$$|\Phi_i(qb,ra)|\approx(XY\varrho)^{2(c(i)-k_i)/2}(\Delta_{i1}XY\varrho)\cdots(\Delta_{ik_i}XY\varrho)=\Delta_i(XY\varrho)^{c(i)},$$
$$\Delta_i=\Delta_{i1}\cdots\Delta_{ik_i}\ll 1.$$

Consequently, if $\widetilde{\Delta}^{12}>L\varrho$ and X is sufficiently large, then by the definition of the summation range D_5 in (III) of §4, we get (cf. the beginning of §3, (26), and (I) of §4)

$$|\Phi_i(qb,ra)|\gg\widetilde{\Delta}^{12}(XY\varrho)^{c(i)}>L\varrho(XY\varrho)^{c(i)},$$
$$|P_i|\approx A^{c(i)}X^{\varepsilon(\alpha,i)}Y^{\delta(\beta,i)}(XY\varrho)^{c(i)}\Delta_i,$$
$$|(K_1)''_a|\approx\lambda_2,\qquad|(K_1)'''_a|\approx\lambda_3,\qquad(K_1)^{(4)}_a\ll\lambda_4,$$
$$|((K_1)''_a(K_1)^{(4)}_a-3((K_1)'''_a)^2)((K_1)''_a)^{-5}|\approx r_1,$$
$$|((K_1)^{(5)}_a((K_1)''_a)^2-10(K_1)''_a(K_1)'''_a(K_1)^{(4)}_a+15((K_1)'''_a)^3)((K_1)''_a)^{-7}|\approx r_2,$$

where

$$\lambda_2=\Delta(1)F\varrho X^{-2},\qquad\Delta(1)=\min(\Delta_{11},\Delta_{12}),$$
$$\lambda_3=F\varrho X^{-3}\Delta_3,\qquad\lambda_4=F\varrho X^{-4},$$
$$r_1=(F\varrho)^{-3}X^4(\Delta(1))^{-5}\Delta_6,\qquad r_2=(F\varrho)^{-4}X^5(\Delta(1))^{-7}\Delta_7.$$

Note that Lemma 11 holds also for $0<U<1$. Let

$$U=\min(\lambda_2\lambda_3^{-1},(\lambda_2\lambda_4^{-1})^{1/2}),\quad\eta=|I_1(u)|>10,\quad L=\log(2+FXY).$$

For every real number $a\in I_1(u)$, we have $(a,b(a,u))\in D_5$. Thus by Lemma 11 we get (for some terms, using simply the estimate $\eta\ll X$)

$$(63')\qquad\Big|\sum_{a\in I_1(u)}e(K_1(a,u))\Big|$$
$$\ll\sqrt[30]{\eta^{28}(F\varrho)^{11}X^{-23}\Delta_7^{-1}}+\sqrt[8]{\eta^5(F\varrho)^3X^{-4}\Delta_6^{-2}}$$
$$+\,(\sqrt[4]{F\varrho X^{-1}\Delta_3^{-2}}+\sqrt[8]{(F\varrho)^3X}+\sqrt{F\varrho X^{-1}}+1)L$$
$$+\,\sqrt{X^2(F\varrho\Delta(1))^{-1}}+(\Delta(1))^{-1}.$$

Hence, from (59′)–(61′), (63′), $u_i=O(F\varrho Y^{-1})$ and $F\varrho\ge Y$ we get the estimate

$$(64') \quad L^{-1}S_G(D_5) \ll \sqrt[30]{(F\varrho)^{-4}X^{-23}Y^{30}\Delta_7^{-1}}\cdot \widetilde{S}(28/30) + \sqrt[8]{(F\varrho)^{-1}X^{-4}Y^{8}\Delta_6^{-2}}\cdot \widetilde{S}(5/8) + \sqrt[4]{(F\varrho)^{3}X^{-1}\Delta_3^{-2}} + \sqrt[8]{(F\varrho)^{7}X} + F\varrho X^{-1/2} + X(\Delta(1))^{-1/2} + (F\varrho)^{1/2}(\Delta(1))^{-1} + XY(F\varrho)^{-1/2} + XF^{1/2}\varrho^{3/2},$$

where, for $0 < c < 1$,

$$\widetilde{S}(c) = \sum_{u_1\le u\le u_2} |I(u)|^c,$$

and $|I(u)|$ is the length of the interval $I(u)$. By Hölder's inequality we get

$$(65') \quad \widetilde{S}(c) \ll \Big(\sum_{u_1\le u\le u_2} 1\Big)^{1-c}\Big(\sum_{u_1\le u\le u_2} |I(u)|\Big)^{c} \ll (F\varrho Y^{-1})^{1-c}\Big(\sum_{u_1\le u\le u_2}\sum_{a\in I(u)} 1 + F\varrho Y^{-1}\Big)^{c} \ll \Big(\sum_{u_1\le u\le u_2}\sum_{a\in I(u)} 1\Big)^{c}\cdot (F\varrho Y^{-1})^{1-c} + F\varrho Y^{-1}.$$

Similarly to the arguments between (40) and (41), we have

$$(66') \quad \sum_{u_1\le u\le u_2}\sum_{a\in I(u)} 1 \le \sum_{u_1\le u\le u_2}\sum_{a\in S(u)} 1 \le \sum_{a\in I_5'}\sum_{X_1(a)\le u\le X_2(a)} 1 \le \sum_{a\in I_5'}\sum_{\alpha_1(a)\le u\le \alpha_2(a)} 1,$$

where $\alpha_i(a) = G_{0,1}(a, B_i(a))$ for a given $(B_1(a), B_2(a))$. Without loss of generality, we have assumed that $G_{0,2} > 0$ on D_5. As $a\in I_5'$ implies that $B_1(a) < B_2(a)$, thus $\alpha_1(a) < \alpha_2(a)$. For every pair of real numbers (a, u), with $a \in I_5'$ and $u\in [\alpha_1(a), \alpha_2(a)]$, we have

$$(67') \quad |b(a,u)q - \theta_{ij}ar| \ll \lambda|qr|\widetilde{\Delta}$$

for some pair $(i,j)\in\Gamma$, because the condition on (a,u) implies that $(a, b(a,u)) \in D_5$. From

$$\frac{\partial b(a,u)}{\partial u} = (G_{0,2}(a, b(a,u)))^{-1},$$

(27) and (67′), we know that for each given real number $a\in I_5'$, the number of integers u satisfying $\alpha_1(a)\le u\le \alpha_2(a)$ is $\ll 1 + |r|\lambda F\varrho Y^{-2}\widetilde{\Delta} \ll 1 + \lambda\widetilde{\Delta}F\varrho^2Y^{-1}$ ($\varrho = |q|/X + |r|/Y$, cf. (11)), that is,

$$(68') \quad \sum_{\alpha_1(a)\le u\le\alpha_2(a)} 1 \ll 1 + \lambda F\varrho^2 Y^{-1}\widetilde{\Delta}.$$

Therefore, in view of (64′)–(66′), $\varDelta_7 \geq \widetilde{\varDelta}^{12}$, $\varDelta_6 \geq \widetilde{\varDelta}^{8}$ and $\varDelta_3 \geq \widetilde{\varDelta}^{4}$, we obtain (using $\varrho\lambda \gg 1$ and $X\widetilde{\varDelta} \gg L$)

$$\widetilde{S}(c) \ll F\varrho Y^{-1}[(XY(F\varrho)^{-1})^c + (\varrho\lambda\widetilde{\varDelta}X)^c],$$

$$\begin{aligned}(69') \qquad L^{-1}S_G(D_5) \ll{}& \sqrt[30]{(F\varrho)^{-2}X^5Y^{28}\widetilde{\varDelta}^{-12}} + \sqrt[30]{F^{26}\varrho^{54}\lambda^{28}X^5} \\ &+ \sqrt[8]{(F\varrho)^2XY^5\widetilde{\varDelta}^{-16}} + \sqrt[8]{(F\varrho)^7\varrho^5\lambda^5X\widetilde{\varDelta}^{-11}} \\ &+ \sqrt[4]{(F\varrho)^3X^{-1}\widetilde{\varDelta}^{-8}} + \sqrt[8]{(F\varrho)^7X} + F\varrho X^{-1/2} \\ &+ X(\varDelta(1))^{-1/2} + (F\varrho)^{1/2}(\varDelta(1))^{-1} \\ &+ XY(F\varrho)^{-1/2} + X(F\varrho^3)^{1/2}.\end{aligned}$$

Recall that $|(K_1)''_a| \approx \varDelta(1)F\varrho X^{-2}$, and $\widetilde{\varDelta} > L\varrho$. Thus similarly to (43), by using Lemma 3 we get

$$\Big|\sum_{a\in I_1(u)} e(K_1)\Big| \ll \Big(\sum_{a\in I_1(u)} 1\Big)(F\varrho X^{-2})^{1/2} + (\varDelta(1)F\varrho X^{-2})^{-1/2}.$$

Consequently, from (59′)–(61′), and $u_i = O(F\varrho Y^{-1})$, $F\varrho \geq Y$, we get

$$\begin{aligned}&L^{-1}S_G(D_5) \\ &\quad \ll X^{-1}Y\Big(\sum_{u_1\leq u\leq u_2}\sum_{a\in I_1(u)} 1\Big) + X(\varDelta(1))^{-1/2} + XY(F\varrho)^{-1/2} + X(F\varrho^3)^{1/2}.\end{aligned}$$

From (66′) and (68′) we have

$$\sum_{u_1\leq u\leq u_2}\sum_{a\in I_1(u)} 1 \leq \sum_{u_1\leq u\leq u_2}\sum_{a\in I(u)} 1 \ll X + \lambda F\varrho^2XY^{-1}\widetilde{\varDelta},$$

thus

$$\begin{aligned}(70') \qquad L^{-1}S_G(D_5) \ll{}& F\varrho^2\lambda\widetilde{\varDelta} + Y + X(\varDelta(1))^{-1/2} \\ &+ XY(F\varrho)^{-1/2} + X(F\varrho^3)^{1/2}.\end{aligned}$$

If $\widetilde{\varDelta}^{12} \leq L\varrho$, by (70′) we have

$$\begin{aligned}(71') \qquad L^{-2}S_G(D_5) \ll{}& F\lambda\varrho^{25/12} + Y + X(\varDelta(1))^{-1/2} \\ &+ XY(F\varrho)^{-1/2} + X(F\varrho^3)^{1/2}.\end{aligned}$$

Suppose $\widetilde{\varDelta}^{12} > L\varrho$. Then the estimate (69′) can be derived. From (69′) and (70′) we obtain

$$\begin{aligned}(72') \quad L^{-1}S_G(D_5) \ll{}& R_1 + R_2 + R_3 + R_4 + \sqrt[30]{F^{26}\varrho^{54}\lambda^{28}X^5} \\ &+ \sqrt[8]{(F\varrho)^7X} + F\varrho X^{-1/2} + X(\varDelta(1))^{-1/2} \\ &+ (F\varrho)^{1/2}(\varDelta(1))^{-1} + XY(F\varrho)^{-1/2} + X(F\varrho^3)^{1/2},\end{aligned}$$

where

$$R_1 = \min(\sqrt[30]{(F\varrho)^{-2}X^5Y^{28}\widetilde{\Delta}^{-12}}, F\varrho^2\lambda\widetilde{\Delta}) \le \sqrt[42]{F^{10}\varrho^{22}X^5Y^{28}\lambda^{12}},$$
$$R_2 = \min(\sqrt[8]{(F\varrho)^2XY^5\widetilde{\Delta}^{-16}}, F\varrho^2\lambda\widetilde{\Delta}) \le \sqrt[24]{F^{18}\varrho^{34}XY^5\lambda^{16}},$$
$$R_3 = \min(\sqrt[8]{(F\varrho)^7\varrho^5\lambda^5X\widetilde{\Delta}^{-11}}, F\varrho^2\lambda\widetilde{\Delta}) \le \sqrt[19]{F^{18}\varrho^{34}X\lambda^{16}},$$
$$R_4 = \min(\sqrt[4]{(F\varrho)^3X\widetilde{\Delta}^{-8}}, F\varrho^2\lambda\widetilde{\Delta}) \le \sqrt[12]{(F\varrho)^{11}(\varrho\lambda)^8X}.$$

From (71′) and (72′) we get the estimate

$$(73') \quad L^{-2}S_G(D_5) \ll \sqrt[42]{F^{10}\varrho^{22}X^5Y^{28}\lambda^{12}} + \sqrt[24]{F^{18}\varrho^{34}XY^5\lambda^{16}} + \sqrt[19]{F^{18}\varrho^{34}X\lambda^{16}} + \sqrt[30]{F^{26}\varrho^{54}\lambda^{28}X^5} + \sqrt[12]{(F\varrho)^{11}(\varrho\lambda)^8X} + F\lambda\varrho^{25/12} + \sqrt[8]{(F\varrho)^7X} + XY(F\varrho)^{-1/2} + X(F\varrho^3)^{1/2} + F\varrho X^{-1/2} + X(\Delta(1))^{-1/2} + Y + (F\varrho)^{1/2}(\Delta(1))^{-1}.$$

To diminish the terms involving $\Delta(1)$ in (73′), similarly to the estimate between (44) and (45) of (iii) of §3, we have

$$(74') \qquad S_G(D_5) = O((F\varrho)^{1/2}\varrho Y^2|r|^{-1}\Delta(1) + XY(F\varrho)^{-1/2} + X + Y).$$

The estimate (73′) is derived for $F\varrho > Y$. Assume also $F\varrho > X$. Then, as $\varrho\lambda \gg 1$ (cf. (11)), we have

$$\sqrt[19]{F^{18}\varrho^{34}X\lambda^{16}} = \sqrt[19]{(F\varrho)^{18}(\varrho\lambda)^{16}X} \gg \sqrt[12]{(F\varrho)^{11}(\varrho\lambda)^8X}.$$

Thus by (73′) and (74′), for $F\varrho > Y$ and $F\varrho > X$ we have the estimate

$$(75') \quad L^{-2}S_G(D_5) \ll \sqrt[42]{F^{10}\varrho^{22}X^5Y^{28}\lambda^{12}} + \sqrt[24]{F^{18}\varrho^{34}XY^5\lambda^{16}} + \sqrt[19]{F^{18}\varrho^{34}X\lambda^{16}} + \sqrt[30]{F^{26}\varrho^{54}\lambda^{28}X^5} + F\lambda\varrho^{25/12} + \sqrt[8]{(F\varrho)^7X} + XY(F\varrho)^{-1/2} + X(F\varrho^3)^{1/2} + F\varrho X^{-1/2} + X + Y + R_5 + R_6,$$

where

$$R_5 = \min(X(\Delta(1))^{-1/2}, (F\varrho^3)^{1/2}Y^2|r|^{-1}\Delta(1)) \le \sqrt[6]{F\varrho^3X^4Y^4|r|^{-2}},$$
$$R_6 = \min((F\varrho)^{1/2}(\Delta(1))^{-1}, (F\varrho^3)^{1/2}Y^2|r|^{-1}\Delta(1)) \le \sqrt{F\varrho^2Y^2|r|^{-1}}.$$

If $F\varrho \le Y$ or $F\varrho \le X$, then similarly to (36) or as in (iv) of §3 (using (22)) we deduce that

$$S_G(D_5) \ll XY(F\varrho)^{-1/2} + X + Y,$$

and thus (75′) holds as well.

Case (iv): *$\widetilde{\Delta} > L\varrho$ and $\Delta_{21} \le \delta$.* In this case, from (56′) we have $\Delta_{11}, \Delta_{12}, \Delta_{22} > \delta$. Similarly to (22), we have

$$S_G(D_5) = \sum_{1\le i\le C} S_G(D_i') + O(X+Y),$$

where D_i' has a similar form to D_5, but with the roles of a and b exchanged. Hence, in view of (iv) of §3 and (ii) of §5, we can estimate $S_G(D_i')$ similarly to the estimation of $S_G(D_5)$ in (iii) of §5, but with the roles of a and b exchanged. In case $\widetilde{\Delta} = \Delta_{ij}$ and $\theta_{ij} = 0$, the treatment is simpler. Correspondingly, similarly to (75′) we obtain

$$\begin{aligned} L^{-2} S_G(D_i') \ll{}& \sqrt[42]{F^{10}\varrho^{22}X^{28}Y^5\lambda^{12}} + \sqrt[24]{F^{18}\varrho^{34}X^5Y\lambda^{16}} \\ &+ \sqrt[19]{F^{18}\varrho^{34}Y\lambda^{16}} + \sqrt[30]{F^{26}\varrho^{54}\lambda^{28}Y^5} + F\lambda\varrho^{25/12} \\ &+ \sqrt[8]{(F\varrho)^7Y} + XY(F\varrho)^{-1/2} + Y(F\varrho^3)^{1/2} \\ &+ F\varrho Y^{-1/2} + X + Y + \sqrt[6]{F^4\varrho^6X^4Y^{-1}|q|^{-2}} \\ &+ \sqrt[6]{F\varrho^3X^4Y^4|q|^{-2}} + \sqrt{F\varrho^2X^2|q|^{-1}}. \end{aligned}$$

Thus

$$\begin{aligned} (76') \quad L^{-2} S_G(D_5) \ll{}& \sqrt[42]{F^{10}\varrho^{22}X^{28}Y^5\lambda^{12}} + \sqrt[24]{F^{18}\varrho^{34}X^5Y\lambda^{16}} \\ &+ \sqrt[19]{F^{18}\varrho^{34}Y\lambda^{16}} + \sqrt[30]{F^{26}\varrho^{54}\lambda^{28}Y^5} + F\lambda\varrho^{25/12} \\ &+ \sqrt[8]{(F\varrho)^7Y} + XY(F\varrho)^{-1/2} + Y(F\varrho^3)^{1/2} \\ &+ F\varrho Y^{-1/2} + X + Y + \sqrt[6]{F^4\varrho^6X^4Y^{-1}|q|^{-2}} \\ &+ \sqrt[6]{F\varrho^3X^4Y^4|q|^{-2}} + \sqrt{F\varrho^2X^2|q|^{-1}}. \end{aligned}$$

Case (v): *Final estimate.* From (57′), (58′), (75′) and (76′), we get (note that $\varrho\lambda \gg 1$)

$$\begin{aligned} (77') \quad L^{-2} S_G(D_5) \ll{}& Z(F\varrho^3)^{1/2}\varrho\lambda + XY(F\varrho)^{-1/2} + \sqrt[42]{F^{10}\varrho^{22}X^5Y^5Z^{23}\lambda^{12}} \\ &+ Z + \sqrt[24]{F^{18}\varrho^{34}XYZ^4\lambda^{16}} + \sqrt[19]{F^{18}\varrho^{34}Z\lambda^{16}} \\ &+ \sqrt[30]{F^{26}\varrho^{54}\lambda^{28}Z^5} + F\lambda\varrho^{25/12} + \sqrt[8]{(F\varrho)^7Z} \\ &+ F\varrho(X^{-1/2} + Y^{-1/2}) + \sqrt[6]{F^4\varrho^6\lambda^2X^{-1}Y^{-1}Z^3} \\ &+ \sqrt[6]{F\varrho^3\lambda^2X^2Y^2Z^2} + \sqrt{F\varrho^2\lambda Z}, \end{aligned}$$

where $Z = X+Y$. As $\varrho\lambda \gg 1$, we have $F\varrho^{25/12}\lambda \gg F\varrho(X^{-1/2}+Y^{-1/2})$. Thus from (53′)–(55′) and (77′) we get the estimate

$$\begin{aligned} (78') \quad L^{-54}|S_G(D_1)| \ll{}& Z(F\varrho^3)^{1/2}\varrho\lambda + XY(F\varrho)^{-1/2} \\ &+ \sqrt[42]{F^{10}\varrho^{22}X^5Y^5Z^{23}\lambda^{12}} + \sqrt[24]{F^{18}\varrho^{34}XYZ^4\lambda^{16}} \\ &+ \sqrt[19]{F^{18}\varrho^{34}Z\lambda^{16}} + \sqrt[30]{F^{26}\varrho^{54}\lambda^{28}Z^5} + F\lambda\varrho^{25/12} \\ &+ \sqrt[8]{(F\varrho)^7Z} + \sqrt[6]{F\varrho^3\lambda^2(XYZ)^2} + \sqrt{F\varrho^2\lambda Z}. \end{aligned}$$

If $F(\xi)=0$ or one of the equations $F_i(\xi)=0$ $(i=3,6,7,8,11,12)$ does not have real roots, then $F(\xi)$ or $F_i(\xi)$ is positive or negative definite, and we can derive $(78')$ similarly and relatively simply. For $0<\alpha\le 1$, as (cf. (11))

$$\varrho=\frac{|q|}{X}+\frac{|r|}{Y}\ll\frac{M}{X}+\frac{N}{Y}\ll\left(\frac{t}{XY}\right)^{1/2},\qquad \varrho\lambda=\varrho^2XY|qr|^{-1}\ll\frac{|q|Y}{|r|X}+\frac{|r|X}{|q|Y|},$$

we have

$$\varrho^\alpha\ll\left(\frac{t}{XY}\right)^{\alpha/2},$$

$$(79')\qquad \sum_q\sum_r(\varrho\lambda)^\alpha\ll\left(\frac{Y}{X}\right)^\alpha\sum_q\sum_r\left|\frac{q}{r}\right|^\alpha+\left(\frac{X}{Y}\right)^\alpha\sum_q\sum_r\left|\frac{r}{q}\right|^\alpha \ll MNL=tL,$$

where $1\le|q|\le M$, $1\le|r|\le N$, and t is a suitable parameter. Distinguishing the cases $|r|<Y|q|X^{-1}$ or $|r|\ge Y|q|X^{-1}$, we obtain

$$\sum_q\sum_r XY(F\varrho)^{-1/2}\ll\sqrt[4]{F^{-2}(XY)^5t^3}.$$

From $(79')$ we also have

$$\sum_q\sum_r\sqrt[30]{F^{26}\varrho^{54}\lambda^{28}Z^5}\ll\sqrt[30]{F^{26}Z^5t^{43}(XY)^{-13}}L.$$

We can use a similar method to handle other terms of $(78')$ and obtain

$$(80')\qquad L^{-55}\frac{XY}{t}S_1\ll\sqrt[4]{F^2XYZ^4t^3}+\sqrt[4]{F^{-2}(XY)^9t^{-1}}$$
$$+\sqrt[42]{F^{10}t^5(XY)^{42}Z^{23}}$$
$$+\sqrt[24]{F^{18}t^9(XY)^{16}Z^4}+\sqrt[19]{F^{18}t^9(XY)^{10}Z}$$
$$+\sqrt[30]{F^{26}t^{13}(XY)^{17}Z^5}+\sqrt[24]{F^{24}t^{13}(XY)^{11}}$$
$$+\sqrt[16]{F^{14}t^7(XY)^9Z^2}+\sqrt[12]{F^2t(XY)^{15}Z^4}$$
$$+XYZ+\sqrt[4]{F^2t(XY)^3Z^2}$$
$$=\sum_{1\le i\le 11}A_i,\qquad\text{say.}$$

Then, as in the proof of Theorem 1, we need to estimate S_2 and S_3 of (9). For $1\le r\le N$ we estimate $S_G(D_1')$, where $G(a,b)=g(a,b+r)-g(a,b)$, and

$$D_1'=\{(a,b)\mid a\in I,\ f_1(a)\le b\le f_2(a)-r\}.$$

We can use directly the method of (iii) of §5 (cf. also (iv) of §3) for estimating $S_G(D_5)$ to deal with $S_G(D_1')$. Let $F|r|/Y>\max(X,Y)$. Then corresponding to (75′) we obtain the estimate (note that $\varrho = |r|/Y$)

$$(81') \quad L^{-1}S_G(D_1') \ll \sqrt[30]{\left(F\,\frac{|r|}{Y}\right)^{26} X^5} + \sqrt[8]{\left(F\,\frac{|r|}{Y}\right)^{7} X} + F\,\frac{|r|}{Y}\,X^{-1/2} + XY\left(F\,\frac{|r|}{Y}\right)^{-1/2} + XF^{1/2}\left(\frac{|r|}{Y}\right)^{3/2} + Z,$$

as the corresponding Δ_7, Δ_6, Δ_3 and $\Delta(1)$ are all ≈ 1 in (63′), by the assumptions on α and β (corresponding to the homogeneous polynomials $\Phi(\xi,\eta)$ and $\Phi_i(\xi,\eta)$ of (I) of §4 and (iii) of §5, now we have the polynomials $\lambda\xi^2$ and $\lambda_i\xi^{c(i)}$). As in (iii) of §5 (cf. (36)), we find that (81′) is also true if $F|r|/Y \le \max(X,Y)$. The estimate (81′) holds also if $-N \le r \le -1$ and D_1' is replaced by $\{(a,b) \mid a \in I_1,\ f_1(a) - r \le b \le f_2(a)\}$. Similarly to (81′), using the method of (I) of §4 and (iii) of §5 (but the details are simpler), we deduce that

$$(82') \quad L^{-1}S_G(D_2') \ll \sqrt[30]{\left(F\,\frac{|q|}{X}\right)^{26} Z^5} + \sqrt[8]{\left(F\,\frac{|q|}{X}\right)^{7} Z} + F\,\frac{|q|}{X}\,Z^{-1/2} + XY\left(F\,\frac{|q|}{X}\right)^{-1/2} + ZF^{1/2}\left(\frac{|q|}{X}\right)^{3/2} + Z.$$

From (81′) and (82′) we get respectively (cf. (9))

$$(83') \quad L^{-2}\,\frac{XY}{t}\,S_2 \ll \sqrt[30]{F^{26}t^{-2}X^7Y^{32}} + \sqrt[16]{F^{14}X^3Y^{17}t^{-1}} + FYX^{-1/2} + \sqrt[4]{F^{-2}X^7Y^{11}t^{-3}} + \sqrt[4]{F^2X^3Y^3t} + \sqrt{X^3Y^3t^{-1}}$$

and

$$(84') \quad L^{-2}\,\frac{XY}{t}\,S_3 \ll \sqrt[30]{F^{26}X^{37}Y^2t^{-2}} + \sqrt[16]{F^{14}X^{19}Y^2t^{-2}} + FX^{1/2} + \sqrt[4]{F^{-2}X^{11}Y^7t^{-3}} + \sqrt[4]{F^2X^7Y^{-1}t} + \sqrt{X^5Yt^{-1}}.$$

As $t \ge \max(XL/Y, YL/X)$ (cf. (8)), we can compare similar terms of (80′), (83′) and (84′). We have

$$\sqrt[30]{F^{26}t^{-2}X^7Y^{32}},\ \sqrt[30]{F^{26}t^{-2}X^{37}Y^2} \ll A_6,$$
$$\sqrt[16]{F^{14}X^3Y^{17}t^{-1}},\ \sqrt[16]{F^{14}X^{19}Yt^{-1}} \ll A_8,$$
$$FYX^{-1/2}, FX^{1/2} \ll A_7, \qquad \sqrt[4]{F^{-2}X^7Y^{11}t^{-3}}, \sqrt[4]{F^{-2}X^{11}Y^7t^{-3}} \ll A_2,$$
$$\sqrt[4]{F^2X^3Y^3t}, \sqrt[4]{F^2X^7Y^{-1}t} \ll A_1, \qquad \sqrt{X^3Y^3t^{-1}}, \sqrt{X^5Yt^{-1}} \ll A_{10}.$$

Consequently, by (9) we get the estimate

$$(85') \qquad L^{-55}|S_g(D)|^2 \ll (XY)^2t^{-1} + \sum_{1\le i\le 11} A_i,$$

provided that $t \geq \max(XL/Y, YL/X)$. For $1 \leq t \leq \max(XL/Y, YL/X)$ and $F < \min(X^3, Y^3)$, we have

$$(LXY)^2 t^{-1} > \min(XY^3, X^3Y) > \sqrt[3]{FX^3Y^3 \min(X^3, Y^3)}.$$

By the method of proving the estimate (56), we get

$$S_g(D) = O(\sqrt[6]{FX^3Y^6} + XY/F) = O(\sqrt[6]{FX^3Y^6}),$$

and, by exchanging the roles of a and b handling $S_g(D)$ directly as we did for $S_g(D_1')$, we obtain the estimate

$$S_g(D) = O(\sqrt[6]{FX^6Y^3}).$$

Thus we have

$$(86') \qquad |S_g(D)| = O(\sqrt[6]{FX^3Y^3 \min(X^3, Y^3)}),$$

which implies, for $1 \leq t \leq \max(XL/Y, YL/X)$ and $F < \min(X^3, Y^3)$, the estimate

$$(87') \qquad |S_g(D)|^2 = O((LXY)^2 t^{-1}).$$

By (85′) and (87′), assuming $0 \leq t \leq XYL^{-4}$ and (7), we always have the estimate

$$(88') \qquad L^{-55}|S_g(D)|^2 \ll (XY)^2 t^{-1} + \sum_{1 \leq i \leq 11} A_i.$$

Suppose $X^2 \geq Y$ and $Y^2 \geq X$. Then

$$A_3 \ll A_2^{132/287} A_6^{155/287} \ll A_2 + A_6, \quad A_4 \ll A_2^{7/82} A_6^{75/82} \ll A_2 + A_6,$$
$$A_8 \ll A_2^{1/20} A_6^{19/20} \ll A_2 + A_6, \quad A_{10} \ll A_1^{1/2} A_2^{1/2} \ll A_1 + A_2,$$
$$A_9 \ll A_2^{1/3} A_{11}^{2/3} \ll A_2 + A_{11}, \quad A_{11} \ll A_2^{11/41} A_6^{30/41} \ll A_2 + A_6,$$

and it follows from (88′) that

$$(89') \quad L^{-55}|S_g(D)|^2 \ll (XY)^2 t^{-1} + \sqrt[4]{F^2 XYZ^4 t^3} + \sqrt[4]{F^{-2}(XY)^9 t^{-1}} + \sqrt[19]{F^{18} t^9 (XY)^{10} Z} + \sqrt[30]{F^{26} t^{13} (XY)^{17} Z^5} + \sqrt[24]{F^{24} t^{13} (XY)^{11}}$$

for all $0 \leq t \leq XYL^{-4}$. By Lemma 3, we can choose t in the range $0 \leq t \leq XYL^{-4}$ in (89′) to get

$$(90') \; L^{-56}|S_g(D)|^2 \ll \sqrt[28]{F^{18}(XY)^{28} Z} + \sqrt[43]{F^{26}(XY)^{43} Z^5} + \sqrt[37]{F^{24}(XY)^{37}} + \sqrt[38]{F^{-1}(XY)^{64}} + \sqrt[4]{F^{-1}(XY)^7 Z} + \sqrt[7]{F^2(XY)^7 Z^4} + \sqrt[55]{(XY)^{91} Z} + \sqrt[82]{(XY)^{134} Z^5} + (XY)^2 F^{-1/2} + XYL^3 = \sum_{1 \leq i \leq 10} B_i, \quad \text{say}.$$

By (7), $F \gg \max(X, Y)$ and $Z \gg (XY)^{1/2}$, we deduce that

$$B_3 \ll B_1, \quad B_7 \ll B_8, \quad B_{10} \ll B_1, \quad B_9 \ll B_5.$$

Consequently, by assuming $X^2 \ge Y$, $Y^2 \ge X$, and (7) we obtain

$$(91') \qquad L^{-28}|S_g(D)| \ll \sqrt[56]{F^{18}(XY)^{28}Z} + \sqrt[86]{F^{26}(XY)^{43}Z^5} + \sqrt[76]{F^{-1}(XY)^{64}} + \sqrt[8]{F^{-1}(XY)^7Z} + \sqrt[14]{F^2(XY)^7Z^4} + \sqrt[164]{(XY)^{134}Z^5}.$$

If (7) holds, but $X > Y^2$ or $Y > X^2$, then from

$$\sqrt[6]{FX^3Y^6} \ll \sqrt[6]{FX^6} \ll \sqrt{B_5}, \quad \text{or} \quad \sqrt[6]{FX^6Y^3} \ll \sqrt[6]{FY^6} \ll \sqrt{B_5}$$

and (86′) (note that (86′) is derived without assuming $X \le Y^2$ or $Y \le X^2$), we find that (91′) still holds. Assume that (7) is not true, that is, $F \ge \min(X^3, Y^3)$ or $\min(X, Y) \le L^6$. If $F \ge Y^3$, then by Lemma 7 we get

$$(92') \qquad |S_g(D)| \ll Y\sqrt[14]{FX^{10}} \ll \sqrt[42]{F^{17}X^{30}} \ll \sqrt[42]{F^{17}Z^{30}}.$$

If $L^6 \ge Y$, then similarly we have (cf. (90′))

$$(93') \qquad |S_g(D)| \ll Y\sqrt[14]{FX^{10}} \ll L^6\sqrt{B_6}.$$

The cases of $F \ge X^3$ or $X \le L^6$ can be treated similarly. Theorem 2 follows from (91′), (92′) and (93′).

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