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Inductivity of the global root number

by

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The invariance of the Artin root number under induction can be proved without special effort. In fact if one develops the properties of Artin L-functions in the usual way, then the inductivity of the root number is obvious. In barest outline the argument is as follows. First one proves the inductivity of Artin L-functions themselves, and then one combines Brauer's induction theorem with the analytic continuation and functional equation of Hecke L-functions, thereby deriving the analytic continuation and functional equation of Artin L-functions. The inductivity of the gamma and exponential factors follows from the duplication formula and the properties of the Artin conductor respectively, so the "normalized L-function" (the product of the L-function and its exponential and gamma factors) is also inductive. Since the root number is the normalized L-function at s divided by the normalized dual L-function at $1-s$, the inductivity of the root number follows.

Artin L-functions can be viewed as the simplest examples of motivic L-functions, but in the general setting much less is known. Certainly the analytic continuation and functional equation are only conjectural, and even the basic objects—the L-function, conductor, and root number—are conjectural in the sense that their definition depends on compatibilities which are not yet known to hold in complete generality. Paradoxically, these very limitations on our knowledge make it easy to prove that motivic root numbers are inductive: Given that the root number itself is only conjectural, one is happy to use the conjectural functional equation, and then an argument along the lines of the proof sketched above goes through as before.

The present note deals with an intermediate case. We consider a setting in which the analytic continuation and functional equation of the L-function are in general still conjectural, while the L-function, conductor, and root number are well defined. The example to keep in mind is an elliptic curve over an arbitrary number field, where we know the required compatibili-

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ties but not the analytic continuation and functional equation. Under these circumstances a proof of inductivity that uses the functional equation is unsatisfactory, but as pointed out to me by Ralph Greenberg, an unconditional proof does not seem to have been explicitly recorded in the literature (cf. [3, p. 163]). The gap is easily filled, and the purpose of this note is to fill it. The key point is a local commutation relation (Proposition 3).

1. Pseudomotives. Let K be a number field, $\overline{K}$ a fixed algebraic closure of K, and $\mathbb{E}$ a *coefficient field*, by which we mean a number field contained in $\mathbb{C}$. Of course $\mathbb{E}$ can also be regarded as a subfield of its completions $\mathbb{E}_\lambda$ at the finite places λ of $\mathbb{E}$. A *family of λ-adic representations of* $\mathrm{Gal}(\overline{K}/K)$ *with coefficient field* $\mathbb{E}$ is a collection $\{\rho_\lambda\}$ of representations (i.e. continuous homomorphisms) $\rho_\lambda : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}(V_\lambda)$, where λ runs over the finite places of $\mathbb{E}$ and V_λ is a vector space over $\mathbb{E}_\lambda$ of some fixed finite dimension d independent of λ. We shall always assume that $\{\rho_\lambda\}$ satisfies the following condition:

(S) *There is a finite set S of finite places of K, independent of λ, such that ρ_λ is unramified (i.e. trivial on inertia) at all finite places $v \notin S \cup S_\ell$, where ℓ is the residue characteristic of λ and S_ℓ consists of the places of K with residue characteristic ℓ.*

The minimal set S with this property is the *exceptional set* of the family.

In practice, $\{\rho_\lambda\}$ is of interest only if the representations ρ_λ satisfy some sort of compatibility beyond (S). The usual condition is *strict compatibility* [8], or perhaps the slightly stronger condition denoted C_5 in [9] and referred to in [7] as *full compatibility*. But for present purposes full compatibility does not suffice even if supplemented by condition C_8 of [9] and even if $\{\rho_\lambda\}$ is assumed to come from a motive, because what we really need is compatibility at the level of Weil–Deligne groups (cf. [1, p. 571, Définition 8.8]), and the latter type of compatibility is not known to follow from the others. In order to define "compatibility at the level of Weil–Deligne groups"—the condition labeled (WD) below—we need to change notation temporarily from global to local.

Thus let K be a finite extension of $\mathbb{Q}_p$ for some prime $p < \infty$, and let $\overline{K}$ be a fixed algebraic closure of K. Write $\mathrm{W}(\overline{K}/K)$ and $\mathrm{WD}(\overline{K}/K)$ respectively for the Weil group and the Weil–Deligne group of K. We recall that as an abstract group, $\mathrm{W}(\overline{K}/K)$ is the subgroup of $\mathrm{Gal}(\overline{K}/K)$ generated by the inertia group I together with any Frobenius element, and as a topological group, $\mathrm{W}(\overline{K}/K)$ is characterized by the fact that the subgroups of I which are open in the Krull topology on I remain open in $\mathrm{W}(\overline{K}/K)$ and form a neighborhood basis at the identity. A representation of $\mathrm{W}(\overline{K}/K)$ is *unramified* if it is trivial on I. As for $\mathrm{WD}(\overline{K}/K)$, there is no need to recall the definition: It suffices to know that a finite-dimensional representation of $\mathrm{WD}(\overline{K}/K)$ over a topological field of characteristic zero can be identified with a pair $\boldsymbol{\rho} = (\rho, N)$, where ρ is a representation of $\mathrm{W}(\overline{K}/K)$ and N is a nilpotent endomorphism of the space of ρ satisfying $\rho(g)N\rho(g)^{-1} = \omega(g)N$ for all $g \in \mathrm{W}(\overline{K}/K)$. Here ω is the unramified character of $\mathrm{W}(\overline{K}/K)$ which on an arithmetic Frobenius element takes the value q, the order of the residue class field of K. We mention that if $N = 0$ then it is customary to identify $\boldsymbol{\rho}$ and ρ. The key point is this: If λ is a finite place of $\mathbb{E}$ of residue characteristic different from the residue characteristic of K, then a construction of Grothendieck and Deligne gives a map $\beta \mapsto \mathrm{GD}(\beta)$ from d-dimensional representations of $\mathrm{Gal}(\overline{K}/K)$ over $\mathbb{E}_\lambda$ to d-dimensional representations of $\mathrm{WD}(\overline{K}/K)$ over $\mathbb{E}_\lambda$ (see [10, pp. 515–516] and [1, pp. 566–571]). Strictly speaking, the map $\beta \mapsto \mathrm{GD}(\beta)$ is defined at the level of *isomorphism classes* of representations, so β and $\mathrm{GD}(\beta)$ are to be taken up to isomorphism.

Now let K denote a number field again, and write K_v for the completion of K at a place v of K and $\overline{K_v}$ for a fixed algebraic closure of K_v containing $\overline{K}$. If v is a finite place then the construction of Grothendieck and Deligne gives rise to a map

$$\{\rho_\lambda\} \mapsto \boldsymbol{\rho}_v \tag{1}$$

from families of λ-adic representations of $\mathrm{Gal}(\overline{K}/K)$ with coefficient field $\mathbb{E}$ to representations of $\mathrm{WD}(\overline{K_v}/K_v)$ over $\mathbb{C}$. The definition of (1) depends on two choices: First we choose a place λ of $\mathbb{E}$ of residue characteristic different from the residue characteristic of v, and we put $\beta = (\rho_\lambda)_v$, where $(\rho_\lambda)_v$ denotes the restriction of ρ_λ to the decomposition subgroup $\mathrm{Gal}(\overline{K_v}/K_v)$ of $\mathrm{Gal}(\overline{K}/K)$. Then we choose an embedding of $\iota : \mathbb{E}_\lambda \hookrightarrow \mathbb{C}$ extending the identity embedding $\mathbb{E} \subset \mathbb{C}$, and we let $\boldsymbol{\rho}_v$ be the complex representation obtained from $\mathrm{GD}(\beta)$ via extension of scalars under ι. We shall assume that the choices made are inconsequential:

(WD) *For every finite place v of K, the isomorphism class of the complex representation $\boldsymbol{\rho}_v$ of* $\mathrm{WD}(\overline{K_v}/K_v)$ *obtained from $\{\rho_\lambda\}$ by applying the construction of Grothendieck and Deligne is independent of the choices inherent in the construction.*

According to Deligne ([1, p. 571, Exemple 8.10]), this condition is satisfied if $\mathbb{E} = \mathbb{Q}$ and $\{\rho_\lambda\}$ is the family of ℓ-adic representations $\{\rho_{A,\ell}\}$ associated to H^1 of an abelian variety A over K. The special case of elliptic curves is much easier, of course; see for example [6, pp. 147–150].

Conditions (S) and (WD) do not yet provide an adequate framework for the present discussion. Using (S) and (WD) one can associate an L-function and a conductor to $\{\rho_\lambda\}$, but we are interested in the root number associated to $\{\rho_\lambda\}$, and to define it one needs an archimedean contribution.

In other words, at each infinite place v of K one needs a representation $\boldsymbol{\rho}_v$ of the Weil–Deligne group $\mathrm{WD}(\overline{K_v}/K_v)$, or equivalently a representation ρ_v of the Weil group $\mathrm{W}(\overline{K_v}/K_v)$, there being no distinction in the archimedean case between the Weil and Weil–Deligne groups. (For the sake of a uniform notation, we write $\boldsymbol{\rho}_v = (\rho_v, N_v)$ with $N_v = 0$.) To define these groups we again let K be a local field, this time archimedean. If $K \cong \mathbb{C}$ then

$$\mathrm{WD}(\overline{K}/K) = \mathrm{WD}(K/K) = K^\times, \tag{2}$$

and if $K = \mathbb{R}$ then

$$\mathrm{WD}(\overline{K}/K) = \overline{K}^\times \cup J\overline{K}^\times \tag{3}$$

with $J^2 = -1 \in \overline{K}^\times$ and $JzJ^{-1} = \overline{z}$ for $z \in \overline{K}$. Here $z \mapsto \overline{z}$ is complex conjugation on K, which is independent of the identification $\overline{K} \cong \mathbb{C}^\times$.

Now let K denote a number field again. We want to augment our families $\{\rho_\lambda\}$ by appending representations $\boldsymbol{\rho}_v$ of $\mathrm{WD}(\overline{K_v}/K_v)$ at the infinite places of K. If $\{\rho_\lambda\}$ comes from a motive then there is a natural candidate for $\boldsymbol{\rho}_v$, namely the representation of $\mathrm{WD}(\overline{K_v}/K_v)$ on the Hodge structure at v. But since it is not known that a family satisfying (S) and (WD) comes from a motive we make the following definition: A *pseudomotive* over K is an ordered pair $M = (\{\rho_\lambda\}, \{\boldsymbol{\rho}_v\})$, where $\{\rho_\lambda\}$ is a family satisfying (S) and (WD) and $\{\boldsymbol{\rho}_v\}$ is simply an assignment of a representation $\boldsymbol{\rho}_v$ of $\mathrm{WD}(\overline{K_v}/K_v)$ to each infinite place v of K. We require the dimension of $\boldsymbol{\rho}_v$ to be the same as the dimension d of the representations ρ_λ, and we refer to d as the *rank* of M.

Now let v be any place of K, finite or infinite. In effect we have defined a map

$$M \mapsto \boldsymbol{\rho}_v(M) \tag{4}$$

from isomorphism classes of pseudomotives over K to isomorphism classes of complex representations of $\mathrm{WD}(\overline{K_v}/K_v)$: If v is finite then (4) is simply the composition

$$M \mapsto \{\rho_\lambda\} \mapsto \boldsymbol{\rho}_v$$

where $M \mapsto \{\rho_\lambda\}$ is projection on the first coordinate and $\{\rho_\lambda\} \mapsto \boldsymbol{\rho}_v$ is (1). If v is infinite then (4) is the projection of M onto the v-component of its second coordinate.

2. The root number. Let K denote a finite extension of $\mathbb{Q}_p$, where $p \leqslant \infty$, and let $\boldsymbol{\rho}$ denote a finite-dimensional complex representation of $\mathrm{WD}(\overline{K}/K)$. As our definition of the *local root number* $W(\boldsymbol{\rho})$ we take

$$W(\boldsymbol{\rho}) = \frac{\varepsilon(\boldsymbol{\rho}, \psi^{\mathrm{can}}, dx)}{|\varepsilon(\boldsymbol{\rho}, \psi^{\mathrm{can}}, dx)|}, \tag{5}$$

where $\varepsilon(*)$ is the local epsilon factor of [1], ψ^{can} is the canonical additive character of K, and dx is any Haar measure on K. By the "canonical additive character of K" we mean $x \mapsto e^{-2\pi i \,\mathrm{tr}_{K/\mathbb{R}}(x)}$ if $p = \infty$ and $x \mapsto e^{2\pi i\{\mathrm{tr}_{K/\mathbb{Q}_p}(x)\}_p}$ if $p < \infty$, where $\{z\}_p$ denotes the sum of the nonintegral terms in the p-adic expansion of a number $z \in \mathbb{Q}_p$. The quantity $\varepsilon(\boldsymbol{\rho}, \psi^{\mathrm{can}}, dx)$ is actually a product of two factors,

$$\varepsilon(\boldsymbol{\rho}, \psi^{\mathrm{can}}, dx) = \varepsilon(\rho, \psi^{\mathrm{can}}, dx)\delta(\boldsymbol{\rho}), \tag{6}$$

where $\varepsilon(\rho, \psi^{\mathrm{can}}, dx)$ depends only on the representation ρ of $\mathrm{W}(\overline{K}/K)$, and $\delta(\boldsymbol{\rho})$ depends only on $\boldsymbol{\rho}$, not on the choice of Haar measure dx or the choice of an additive character (a choice which we have eliminated anyway by insisting on the canonical one). Thus

$$W(\boldsymbol{\rho}) = W(\rho)\Delta(\boldsymbol{\rho}) \tag{7}$$

with $W(\rho) = \varepsilon(\rho, \psi^{\mathrm{can}}, dx)/|\varepsilon(\rho, \psi^{\mathrm{can}}, dx)|$ and $\Delta(\boldsymbol{\rho}) = \delta(\boldsymbol{\rho})/|\delta(\boldsymbol{\rho})|$.

The key properties of the local root number are: first, additivity—in other words, given two representations $\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$ of $\mathrm{WD}(\overline{K}/K)$, we have

$$W(\boldsymbol{\rho} \oplus \boldsymbol{\rho}') = W(\boldsymbol{\rho})W(\boldsymbol{\rho}') \tag{8}$$

—and second, inductivity in degree zero. The latter property can be stated as follows. Let $\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$ be representations of $\mathrm{WD}(\overline{K}/K)$ *of the same dimension.* Given a subfield F of K containing $\mathbb{Q}_p$, we have

$$\frac{W(\mathrm{ind}_{K/F}\,\boldsymbol{\rho})}{W(\mathrm{ind}_{K/F}\,\boldsymbol{\rho}')} = \frac{W(\boldsymbol{\rho})}{W(\boldsymbol{\rho}')} \qquad (\dim \boldsymbol{\rho} = \dim \boldsymbol{\rho}') \tag{9}$$

where $\mathrm{ind}_{K/F}$ denotes induction from $\mathrm{WD}(\overline{K}/K)$ to $\mathrm{WD}(\overline{K}/F)$. The notion of "induction from $\mathrm{WD}(\overline{K}/K)$ to $\mathrm{WD}(\overline{K}/F)$" requires no explanation if $p = \infty$, because $\mathrm{WD}(\overline{K}/K)$ is a subgroup of finite index in $\mathrm{WD}(\overline{K}/F)$, and $\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$ are representations of $\mathrm{WD}(\overline{K}/K)$ in the usual sense. However if $p < \infty$ then $\boldsymbol{\rho}$ is a pair (ρ, N), and a definition is in order. Let ω_F denote the unramified character of $\mathrm{W}(\overline{K}/F)$ sending an arithmetic Frobenius element to q_F, the order of the residue class field of F. We put

$$\mathrm{ind}_{K/F}\,\boldsymbol{\rho} = (\mathrm{ind}_{K/F}\,\rho, \omega_F^{-1} \cdot (1 \otimes N)), \tag{10}$$

where just as in the case $p = \infty$, the first coordinate on the right-hand side has the obvious interpretation. As for the second coordinate, let V be the space of ρ, and put $G = \mathrm{W}(\overline{K}/F)$ and $H = \mathrm{W}(\overline{K}/K)$. We may take the space of $\mathrm{ind}_{K/F}\,\rho$ to be $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$. Then $\omega_F^{-1} \cdot (1 \otimes N)$ is the nilpotent endomorphism of $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$ which sends a primitive tensor $g \otimes v$ to $\omega_F(g)^{-1}(g \otimes Nv)$ for all $g \in G$ and $v \in V$.

For the proof of (8) and (9) see [1]. We merely remark that in view of the decomposition (7), two identities must be verified in each case: for (8) one shows that

$$W(\rho \oplus \rho') = W(\rho)W(\rho'), \tag{11}$$

$$\Delta(\rho \oplus \rho') = \Delta(\rho)\Delta(\rho'), \tag{12}$$

and for (9) one shows that

$$\frac{W(\operatorname{ind}_{K/F} \rho)}{W(\operatorname{ind}_{K/F} \rho')} = \frac{W(\rho)}{W(\rho')} \quad (\dim \rho = \dim \rho'), \tag{13}$$

$$\Delta(\operatorname{ind}_{K/F} \boldsymbol{\rho}) = \Delta(\boldsymbol{\rho}). \tag{14}$$

Properties (11) and (13) are inherent in the very existence of local epsilon factors ([1, p. 535, Théorème 4.1]), and properties (12) and (14) are elementary (see for example [6, p. 142]).

We also mention a third property of local root numbers. Suppose that $p < \infty$. We say that the representation $\boldsymbol{\rho} = (\rho, N)$ is unramified if ρ is *unramified* and $N = 0$. Given our definition of $W(\boldsymbol{\rho})$ (in particular, the choice of ψ^{can} in (5)), it follows from paragraph 5.9 on p. 550 of [1] that if $\boldsymbol{\rho}$ is unramified and if K is unramified over $\mathbb{Q}_p$ then $W(\boldsymbol{\rho}) = 1$.

To appreciate the significance of this third property we return to a global setting. Let K be a number field again and let M be a pseudomotive over K. We define the *global root number* $W(M)$ of M by applying (4) at each place v of K:

$$W(M) = \prod_v W(\boldsymbol{\rho}_v(M)). \tag{15}$$

The product is meaningful because the map (1) sending $\{\rho_\lambda\}$ to $\boldsymbol{\rho}_v$ has the property that if $v \notin S$ (the exceptional set of $\{\rho_\lambda\}$) then $\boldsymbol{\rho}_v$ is unramified. Since S is finite and K is unramified at all but finitely many places we conclude that $W(\boldsymbol{\rho}_v(M)) = 1$ for all but finitely many v.

The point to be proved in this note is that the map $M \mapsto W(M)$ is invariant under induction. To formulate this assertion precisely, consider a pseudomotive $M = (\{\rho_\lambda\}, \{\boldsymbol{\rho}_v\})$ over K and a subfield F of K. We put

$$\operatorname{ind}_{K/F} M = (\{\operatorname{ind}_{K/F} \rho_\lambda\}, \{\bigoplus_{v|u} \operatorname{ind}_{K_v/F_u} \boldsymbol{\rho}_v\}), \tag{16}$$

where the second coordinate of $\operatorname{ind}_{K/F} M$ assigns to each infinite place u of F a direct sum over the places v of K lying above u, the direct summand at v being $\operatorname{ind}_{K_v/F_u} \boldsymbol{\rho}_v$.

Proposition 1. $W(\operatorname{ind}_{K/F} M) = W(M)$.

Implicit in Proposition 1 is the fact that $\operatorname{ind}_{K/F} M$ is a pseudomotive, in other words that (WD) holds with M replaced by $\operatorname{ind}_{K/F} M$. This will follow along with Proposition 1 itself from the fact that (4) commutes with induction. Let u denote a fixed place of F, finite or infinite. As before, the symbol $\bigoplus_{v|u}$ denotes a direct sum in which the summands are indexed by the places of K over u.

PROPOSITION 2. $\boldsymbol{\rho}_u(\operatorname{ind}_{K/F} M) \cong \bigoplus_{v|u} \operatorname{ind}_{K_v/F_u} \boldsymbol{\rho}_v(M)$.

If u is an infinite place of F then Proposition 2 is true by virtue of the definition (16), but if u is finite then there is something to check. First let us verify that Proposition 1 does indeed follow from Proposition 2.

3. Proof of Proposition 1 granting Proposition 2. As noted in the introduction, Proposition 1 is an easily proved classical fact in the case of Artin representations, and we shall use this known special case to deduce Proposition 1 from Proposition 2 in general. In fact all we need is the relation

$$W(\operatorname{ind}_{K/F} 1_K) = W(1_K), \tag{17}$$

where F is a subfield of K and 1_K is the one-dimensional trivial representation of $\operatorname{Gal}(\overline{K}/K)$. Now the local root numbers of the trivial representation are all equal to 1 and hence their product $W(1_K)$ is also 1, but it is more efficient not to make this simplification. Instead we write both sides of (17) as products over places and then we divide one side by the other. We obtain

$$1 = \frac{\prod_v W(1_{K_v})}{\prod_u \prod_{v|u} W(\operatorname{ind}_{K_v/F_u} 1_{K_v})} \tag{18}$$

by the standard formula for the restriction of an induced Artin representation to a decomposition group (see the last displayed formula on p. 12 of [11]).

Now consider an arbitrary pseudomotive $M = (\{\rho_\lambda\}, \{\boldsymbol{\rho}_v\})$ over K. Using the definition (15) in conjunction with Proposition 2, we have

$$W(\operatorname{ind}_{K/F} M) = \prod_u \prod_{v|u} W(\operatorname{ind}_{K_v/F_u} \boldsymbol{\rho}_v(M)). \tag{19}$$

Let d be the rank of M, or in other words the dimension of the representations $\boldsymbol{\rho}_v(M)$, and write $1_{K_v}^{\oplus d}$ for the direct sum of d copies of 1_{K_v}. Raising the two sides of (18) to the power d, multiplying by (19), and applying (8) and (9), we obtain

$$\begin{aligned} W(\operatorname{ind}_{K/F} M) &= \prod_u \prod_{v|u} \frac{W(\operatorname{ind}_{K_v/F_u} \boldsymbol{\rho}_v(M))}{W(\operatorname{ind}_{K_v/F_u} 1_{K_v}^{\oplus d})} \cdot \prod_v W(1_{K_v}^{\oplus d}) \\ &= \prod_v \frac{W(\boldsymbol{\rho}_v(M))}{W(1_{K_v}^{\oplus d})} \cdot \prod_v W(1_{K_v}^{\oplus d}). \end{aligned}$$

Making the obvious cancellation, we obtain Proposition 1.

4. Proof of Proposition 2. As already noted, if u is an infinite place of F then there is nothing to prove, so suppose that u is finite, and choose a place λ of $\mathbb{E}$ of residue characteristic distinct from the residue characteristic of u. Appealing once again to the last displayed formula on p. 12 of [11], we have

$$(\mathrm{ind}_{K/F}\,\rho_\lambda)_u \cong \bigoplus_{v|u} \mathrm{ind}_{K_v/F_u}(\rho_\lambda)_v. \tag{20}$$

Now the map $\beta \mapsto \mathrm{GD}(\beta)$ commutes with formation of direct sums, and consequently when we apply it to both sides of (20), we obtain

$$\mathrm{GD}((\mathrm{ind}_{K/F}\,\rho_\lambda)_u) \cong \bigoplus_{v|u} \mathrm{GD}(\mathrm{ind}_{K_v/F_u}(\rho_\lambda)_v). \tag{21}$$

The induction functor on *complex* representations of the Weil–Deligne group was defined by (10), and the same definition is valid for representations over any topological field of characteristic zero, in particular $\mathbb{E}_\lambda$. Hence it is at least meaningful to write $\mathrm{ind}_{K_v/F_u}\,\mathrm{GD}((\rho_\lambda)_v)$. We claim that in fact

$$\mathrm{GD}(\mathrm{ind}_{K_v/F_u}(\rho_\lambda)_v) \cong \mathrm{ind}_{K_v/F_u}\,\mathrm{GD}((\rho_\lambda)_v). \tag{22}$$

Substituting (22) in (21) and extending scalars via $\iota : \mathbb{E}_\lambda \hookrightarrow \mathbb{C}$, we obtain the proposition.

Thus it suffices to prove the commutation relation (22). This is a purely local statement, and therefore we revert to a local notation. Let p be a prime number.

Proposition 3. *Let F be a finite extension of $\mathbb{Q}_p$, let K be a finite extension of F, and let β be a finite-dimensional representation of* $\mathrm{Gal}(\overline{K}/K)$ *over $\mathbb{E}_\lambda$, where λ is of residue characteristic $\ell \neq p$. Then* $\mathrm{GD}(\mathrm{ind}_{K/F}\,\beta) \cong \mathrm{ind}_{K/F}\,\mathrm{GD}(\beta)$.

Proof. Let V be the space of β. Recall also that I denotes the inertia subgroup of $\mathrm{W}(\overline{K}/K)$. Although the construction of Grothendieck and Deligne is most naturally thought of as a map on *isomorphism classes* of representations, if we fix an arithmetic Frobenius element $\sigma \in \mathrm{W}(\overline{K}/K)$ and a nonzero homomorphism $I \to \mathbb{Q}_\ell$ then the construction gives an explicit model for $\mathrm{GD}(\beta)$ as a representation of $\mathrm{WD}(\overline{K}/K)$ on the same vector space V. For our purposes σ can be chosen arbitrarily, but it is important to take $I \to \mathbb{Q}_\ell$ to be $t_\ell|I$, where $t_\ell : I_F \to \mathbb{Q}_\ell$ is any nonzero homomorphism and I_F denotes the inertia subgroup of $\mathrm{W}(\overline{K}/F)$. Such a map t_ℓ necessarily factors through the tame quotient of I_F, and its existence follows from the fact that the tame quotient is the product of the groups $\mathbb{Z}_q$ for $q \neq p$. Since I has finite index in I_F and $\mathbb{Q}_\ell$ is torsion-free as an additive group, $t_\ell|I$ is indeed nonzero. The formal properties of the Kummer pairing give

$$t_\ell(gig^{-1}) = \omega_F(g)t_\ell(i) \qquad (g \in \mathrm{W}(\overline{K}/F),\ i \in I_F), \tag{23}$$

where as before, ω_F is the unramified character of $\mathrm{W}(\overline{K}/F)$ sending an arithmetic Frobenius element to q_F.

So far we have not described the recipe for obtaining $\mathrm{GD}(\beta)$ from β. We do so now. Write $\mathrm{GD}(\beta) = \boldsymbol{\rho} = (\rho, N)$. Any nilpotent endomorphism of a finite-dimensional vector space over a field of characteristic zero can be exponentiated, and N is uniquely characterized by the property that

$$\beta(i) = \exp(t_\ell(i)N) \tag{24}$$

for all i in some open subgroup J of I. As for ρ, we write an arbitrary element $h \in \mathrm{W}(\overline{K}/K)$ in the form $h = i\sigma^n$ with $i \in I$ and $n \in \mathbb{Z}$, and we put

$$\rho(h) = \exp(-t_\ell(i)N)\beta(h). \tag{25}$$

It is a fact that ρ is a representation of $\mathrm{W}(\overline{K}/K)$ and that the pair $\boldsymbol{\rho} = (\rho, N)$ is a representation of $\mathrm{WD}(\overline{K}/K)$.

Of course the recipe just described applies to $\mathrm{GD}(\mathrm{ind}_{K/F}\,\beta)$ as well as to $\mathrm{GD}(\beta)$. We put $G = \mathrm{W}(\overline{K}/F)$ and $H = \mathrm{W}(\overline{K}/K)$, and we take the space of $\mathrm{ind}_{K/F}\,\beta$ to be $\mathbb{E}_\lambda[G] \otimes_{\mathbb{E}_\lambda[H]} V$. We write this space as $\mathbb{E}_\lambda[G] \otimes_\beta V$ to emphasize that H acts on V through β. Let $\boldsymbol{\pi} = \mathrm{GD}(\mathrm{ind}_{K/F}\,\beta)$ and write $\boldsymbol{\pi} = (\pi, P)$. Then

$$(\mathrm{ind}_{K/F}\,\beta)(i) = \exp(t_\ell(i)P) \tag{26}$$

for all i in some open subgroup J_F of I_F. Furthermore, if σ_F is a fixed arithmetic Frobenius element of G, so that every $g \in G$ has the form $g = i\sigma_F^n$ with $i \in I_F$ and $n \in \mathbb{Z}$, then

$$\pi(g) = \exp(-t_\ell(i)P)(\mathrm{ind}_{K/F}\,\beta)(g) \tag{27}$$

as before. We shall prove that

$$\boldsymbol{\pi} \cong \mathrm{ind}_{K/F}\,\boldsymbol{\rho}, \tag{28}$$

as asserted by the proposition.

Before we can prove (28) we need to derive a more explicit formula for P, and to derive the formula we need to examine the domain of validity of (24). While (24) is assumed to hold on *some* open subgroup J of I, the set of *all* $i \in I$ satisfying (24) is a subgroup J' of I, necessarily open since it contains J. Now if $h \in H$ and $i \in J'$ then $\beta(hih^{-1}) = \beta(h)\exp(t_\ell(i)N)\beta(h)^{-1}$ and consequently

$$\beta(hih^{-1}) = \exp(t_\ell(i)\beta(h)N\beta(h)^{-1}). \tag{29}$$

But from (25) we have $\beta(h) = \exp(t_\ell(i)N)\rho(h)$, and ρ is the first coordinate of the representation $\boldsymbol{\rho} = (\rho, N)$ of $\mathrm{WD}(\overline{K}/K)$. Thus conjugation by $\rho(h)$ multiplies N by $\omega(h)$, while $\exp(t_\ell(i)N)$ commutes with N. We conclude that $\beta(h)N\beta(h)^{-1} = \omega(h)N$. Making this substitution in (29), and taking account of (23) and the fact that $\omega_F|H = \omega$, we obtain

$$\beta(hih^{-1}) = \exp(t_\ell(hih^{-1})N). \tag{30}$$

Thus J' is normal in $\mathrm{W}(\overline{K}/K)$. Now let $g_1, \ldots, g_n$ be left coset representatives for $\mathrm{W}(\overline{K}/K)$ in $\mathrm{W}(\overline{K}/F)$, and let J'' be the intersection of the conjugates $g_i J' g_i^{-1}$. After declaring J'' to be the new J, we may assert that (24) holds for all i in an open subgroup J of I which is normal in $\mathrm{W}(\overline{K}/F)$. Of course since K/F is finite it follows that J is open in I_F as well.

Now consider $(\mathrm{ind}_{K/F}\,\beta)|J$. Given $j \in J$, $g \in G$, and $v \in V$, we have

$$(\mathrm{ind}_{K/F}\,\beta)(j)(g \otimes_\beta v) = jg \otimes_\beta v = g \otimes_\beta \beta(g^{-1}jg)(v)$$

because $g^{-1}jg$ belongs to J and therefore to H. In fact since $g^{-1}jg \in J$, we have

$$(\mathrm{ind}_{K/F}\,\beta)(j)(g \otimes_\beta v) = g \otimes_\beta \exp(t_\ell(j)\omega_F(g)^{-1}N)(v) \tag{31}$$

by (24) and (23). Recalling that P is uniquely characterized by the fact that (26) holds for all i in some open subgroup of I_F, we conclude from (31) that P is the nilpotent endomorphism of $\mathbb{E}_\lambda[G] \otimes_\beta V$ which on primitive tensors $g \otimes v$ is given by

$$P(g \otimes_\beta v) = g \otimes_\beta (\omega_F(g)^{-1}Nv). \tag{32}$$

This is the desired formula for P.

We now return to the proof of (28). By (10), we have $\mathrm{ind}_{K/F}\,\boldsymbol{\rho} = (\mathrm{ind}_{K/F}\,\rho, \omega_F^{-1} \cdot (1 \otimes N))$. In principle, the space of $\mathrm{ind}_{K/F}\,\rho$ could be written as $\mathbb{E}_\lambda[G] \otimes_{\mathbb{E}_\lambda[H]} V$, but for clarity we write it as $\mathbb{E}_\lambda[G] \otimes_\rho V$. To prove (28) we must show that there is a linear isomorphism from $\mathbb{E}_\lambda[G] \otimes_\rho V$ to $\mathbb{E}_\lambda[G] \otimes_\beta V$ which intertwines $\mathrm{ind}_{K/F}\,\rho$ with π and $\omega_F^{-1} \cdot (1 \otimes N)$ with P.

Consider the embedding $\psi : V \to \mathbb{E}_\lambda[G] \otimes_\beta V$ given by $v \mapsto 1 \otimes_\beta v$. We claim that ψ intertwines ρ and $\pi|H$. Indeed take $h \in H$ and write $h = i\sigma^n$ with $i \in I$ and $n \in \mathbb{Z}$. By (25), we have

$$\psi(\rho(h)(v)) = 1 \otimes_\beta \exp(-t_\ell(i)N)(\beta(h)(v)) \tag{33}$$

On the other hand, by (27),

$$\pi(h)(1 \otimes_\beta v) = \exp(-t_\ell(i)P)(1 \otimes_\beta \beta(h)(v)) \tag{34}$$

From (32) we see that the right-hand side of (34) coincides with that of (33), verifying the claim.

Now view V as an H-submodule of $\mathbb{E}_\lambda[G] \otimes_\rho V$ via the embedding $v \mapsto 1 \otimes_\rho v$, so that ψ is the map $1 \otimes_\rho v \mapsto 1 \otimes_\beta v$. Applying the universal property of induction, we obtain a linear map $\Psi : \mathbb{E}_\lambda[G] \otimes_\rho V \to \mathbb{E}_\lambda[G] \otimes_\beta V$ which extends ψ and intertwines $\mathrm{ind}_{K/F}\,\rho$ and π. The map Ψ is surjective, because $1 \otimes_\beta V$ spans $\mathbb{E}_\lambda[G] \otimes_\beta V$ as a G-module. Since $\mathbb{E}_\lambda[G] \otimes_\rho V$ and $\mathbb{E}_\lambda[G] \otimes_\beta V$ are both of dimension $[K : F] \dim(V)$ over $\mathbb{E}_\lambda$, we conclude that Ψ is an isomorphism.

We must still check that $\Psi \circ (\omega_F^{-1} \cdot (1 \otimes N)) = P \circ \Psi$. For $g \in G$ and $v \in V$,

$$\Psi((\omega_F^{-1} \cdot (1 \otimes N))(g \otimes_\rho v)) = \omega_F(g)^{-1} \Psi((\mathrm{ind}_{K/F}\, \rho)(g)(1 \otimes_\rho Nv)).$$

Now the right-hand side is $\omega_F(g)^{-1}\pi(g)\Psi(1 \otimes_\rho Nv)$, because Ψ intertwines $\mathrm{ind}_{K/F}\, \rho$ and π. Furthermore, Ψ extends ψ, so we find

$$\Psi((\omega_F^{-1} \cdot (1 \otimes N))(g \otimes_\rho v)) = \omega_F(g)^{-1}\pi(g)(1 \otimes_\beta Nv). \tag{35}$$

On the other hand, since $g \otimes_\rho v = (\mathrm{ind}_{K/F}\, \rho)(g)(1 \otimes_\rho v)$, the fact that Ψ intertwines $\mathrm{ind}_{K/F}\, \rho$ and π also gives

$$P(\Psi(g \otimes_\rho v)) = P(\pi(g)\Psi(1 \otimes_\rho v)). \tag{36}$$

But $P\pi(g) = \omega_F(g)^{-1}\pi(g)P$, because $\boldsymbol{\pi} = (\pi, P)$ is a representation of $\mathrm{WD}(\overline{K}/F)$. Making this substitution in (36), rewriting $\Psi(1 \otimes_\rho v))$ as $1 \otimes_\beta v$, and appealing to (32), we see that the right-hand sides of (35) and (36) coincide, whence (28) follows. ■

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