ACTA ARITHMETICA
156.1 (2012)

The Ramsey-type version of a problem of Pomerance and Schinzel

by

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1. Introduction. A set H is called *product free* if $a, b \in H$ implies $ab \notin H$. Hajdu, Schinzel and Skałba [4] have shown that a product free subset of the positive integers can have upper density arbitrarily close to 1. Sárközy has suggested to investigate the Ramsey-type variation of the problem: is it true that for any r-colouring of $\mathbb{N}$ the equation $ab = c$ has a monochromatic solution different from the trivial solution $1 \cdot 1 = 1$? In particular he asked the question for squarefree numbers:

Problem 1. *Is it true that for any r-colouring of the squarefree numbers greater than 1 the equation $ab = c$ has a monochromatic solution?*

There are several other questions about density theorems, where the Ramsey-type version was answered positively; see for example [1], [5]. It is a consequence of Schur's theorem [9] that Sárközy's original problem always has a solution among the powers of 2.

Proposition 1. *For every r-colouring of the 2-powers the equation $ab = c$ has a nontrivial solution.*

Proof. Let us colour the 2-powers by r colours. We define a colouring of $\mathbb{N}$ by r colours in the following way. Let the colour of $x \in \mathbb{N}$ be the colour of 2^x. By Schur's theorem the equation $x + y = z$ has a monochromatic solution in $\mathbb{N}$. Then the equation $ab = c$ also has a monochromatic solution (for the original colouring) among the 2-powers, namely $a = 2^x$, $b = 2^y$, $c = 2^z$. ∎

Pomerance and Schinzel [7] have proved that for Problem 1 the answer is affirmative if $r = 2$. In this paper we settle the problem for arbitrary r, and extend the results to more general equations. We show that the equation

2010 *Mathematics Subject Classification*: Primary 11B75.
Key words and phrases: algebraic equation, Ramsey type, solution set.

$a_1 \cdots a_k = b_1 \cdots b_l$ has a nontrivial monochromatic solution for every r-colouring of the squarefree numbers.

2. Squarefree numbers. The result of Hajdu, Schinzel and Skałba implies that there is no density theorem for the equation $ab = c$. The following example shows that if $k \neq l$, then there is no density theorem for the equation $a_1 \cdots a_k = b_1 \cdots b_l$ either.

EXAMPLE 1. Let $A_n = \{4i + 2 : 0 \le i,\ 4i + 2 \le n\}$. If $a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_l \in A_n$, then the exponent of 2 is k in the canonical form of $a_1 \cdots a_k$ and l in $b_1 \cdots b_l$. Thus the equation $a_1 \cdots a_k = b_1 \cdots b_l$ does not have a solution in A_n if $k \neq l$. The size of A_n is $\frac{1}{4}n + O(1)$.

If $k = l$, then $a_1 = \cdots = a_k = b_1 = \cdots = b_k$ is a solution. We say that $a_1, \dots, a_k, b_1, \dots, b_l$ is a *primitive solution* of the equation $a_1 \cdots a_k = b_1 \cdots b_l$ if $a_1, \dots, a_k, b_1, \dots, b_l$ are pairwise distinct. From now on we look only for primitive solutions of equations.

In case $k = l$ there is a density theorem for primitive solutions if k and l are even.

PROPOSITION 2. *Let $k \in \mathbb{N}$ be even. For arbitrary $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that for every $n \ge N$ and $A \subseteq \{1, \dots, n\}$ with size $|A| \ge \varepsilon n$ the equation $a_1 a_2 \dots a_k = b_1 b_2 \dots b_k$ has a primitive solution in A.*

Proof. The proof is by induction on k. First let $k = 2$ and $\varepsilon > 0$ be arbitrary. The bound N is chosen later. Let $A \subseteq \{1, \dots, n\}$, where $n \ge N$ and $|A| > \varepsilon n$. In [2] it is proved that only $o(n^2)$ numbers can be found in the "multiplication table" of the integers up to n. As $A \subseteq \{1, \dots, n\}$, the set $A \cdot A = \{c_1 c_2 : c_1, c_2 \in A\}$ has at most $o(n^2)$ elements. There are $\binom{|A|}{2} = (\varepsilon^2/2)n^2 + o(n^2)$ pairs c_1, c_2 with $c_1, c_2 \in A$ and $c_1 \neq c_2$. Now, choose N such that $\binom{|A|}{2}$ is larger than the size of $A \cdot A$. Thus there exists an element in $A \cdot A$ which can be written as a product of two different elements of A in at least two different ways: $a_1 a_2 = b_1 b_2$. This way we have obtained a primitive solution.

Now, assume that $4 \le k \in 2\mathbb{N}$ and the statement holds for $k - 2$. Let $\varepsilon > 0$ be arbitrary. By the induction hypothesis there exists some N such that for any set $B \subseteq \{1, \dots, n\}$ with at least $(\varepsilon/3)n$ elements, the equations $a_1 \cdots a_{k-2} = b_1 \cdots b_{k-2}$ and $a_{k-1} a_k = b_{k-1} b_k$ have a primitive solution in B if $n \ge N$. Let $A \subseteq \{1, \dots, n\}$ have at least εn elements. If $n \ge 3/\varepsilon$, then A can be partitioned into two disjoint parts A_1 and A_2 both of size at least $(\varepsilon/3)n$. If $n \ge N$, then $a_1 \cdots a_{k-2} = b_1 \cdots b_{k-2}$ has a primitive solution in A_1 and $a_{k-1} a_k = b_{k-1} b_k$ has a primitive solution in A_2. Therefore, $a_1, \dots, a_k, b_1, \dots, b_k$ is a primitive solution of $a_1 \cdots a_k = b_1 \cdots b_k$ in A. ∎

The case when $k = l$ is odd is still open.

PROBLEM 2. *Is it true that for every odd $k > 1$ and $\varepsilon > 0$ there exists some N such that for every $N \leq n$ and $A \subseteq \{1, \dots, n\}$ with size at least εn the equation $a_1 \cdots a_k = b_1 \cdots b_k$ has a primitive solution in A?*

For the main result of the paper the following form of Ramsey's theorem will be used ([3], [6]):

RAMSEY'S THEOREM. *Let r and t be positive integers. Let us colour the at most t-element subsets of a set S by r colours. Then for every positive integer n there exists a positive integer d such that if $|S| > d$, then S has a subset H with n elements such that any two subsets of the same size not greater than t have the same colour, that is, for any $H_1, H_2 \subseteq H$ with $|H_1| = |H_2| \leq t$ the colour of H_1 and H_2 is the same.*

By Ramsey's theorem, for every n there exists d such that, if $|S| > d$, then there exists a subset $H \subseteq S$ with $|H| = n$ such that every one-element subset of H has the same colour, every two-element subset of H has the same colour, and so on, every subset of H with t elements has the same colour. The bound for this integer d is called a *Ramsey number* and the best known bound is multiply exponential in r.

The following version of Rado's theorem is also needed ([6], [8]):

RADO'S THEOREM. *Let $v \geq 2$. Let $c_i \in \mathbb{Z} \setminus \{0\}$, $1 \leq i \leq v$, be constants such that there exists a nonempty $D \subseteq \{c_i : 1 \leq i \leq v\}$ such that $\sum_{d \in D} d = 0$. If there exist distinct integers (not necessarily positive) y_i such that $\sum c_i y_i = 0$, then for every natural number r there exists some t such that for every r-colouring of the set $\{1, \dots, t\}$ the equation*

$$c_1 x_1 + \cdots + c_v x_v = 0$$

has a monochromatic solution $b_1, \dots, b_v$ in $\{1, \dots, t\}$, where the b_i's are distinct.

Now we prove that for every r-colouring of the squarefree numbers the equation $a_1 \cdots a_k = b_1 \cdots b_l$ has a primitive monochromatic solution if $k \geq 2$.

THEOREM 1. *For every $k \geq 2$, any $l, r \in \mathbb{N}$ and every r-colouring of the squarefree numbers greater than 1 the equation*

$$a_1 \cdots a_k = b_1 \cdots b_l \tag{1}$$

has a primitive monochromatic solution.

Proof. The squarefree numbers are in a one-to-one correspondence with the finite subsets of primes. To each squarefree number we assign the set of its prime divisors. The product of two squarefree numbers is squarefree if and only if the two sets are disjoint. Moreover, in this case the product corresponds to the union of the two subsets.

For a given r-colouring of the squarefree numbers we define a colouring of the finite subsets of primes. Each subset is coloured by the colour of the product of its elements. If we find nonempty subsets of primes $A_1, \dots, A_k$, $B_1, \dots, B_l$ such that

(i) $\bigcup A_i = \bigcup B_j$,
(ii) $A_1, \dots, A_k, B_1, \dots, B_l$ are pairwise distinct,

then $a_i = \prod_{p \in A_i} p$ for $1 \le i \le k$ and $b_j = \prod_{p \in B_j} p$ for $1 \le j \le l$ is a primitive monochromatic solution of (1). Now we show that the sets A_i, B_j with the above conditions exist with the additional property

(iii) the sizes $|A_1| = \alpha_1, \dots, |A_k| = \alpha_k, |B_1| = \beta_1, \dots, |B_l| = \beta_l$ are distinct.

The equation

$$\alpha_1 + \cdots + \alpha_k = \beta_1 + \cdots + \beta_l \tag{2}$$

is equivalent to

$$\alpha_1 + \cdots + \alpha_k - \beta_1 - \cdots - \beta_l = 0,$$

hence Rado's theorem applies with $v = k + l$, and $c_i = 1$, $y_i = i$ if $1 \le i \le k$ and $c_i = -1$, $y_i = -i$ if $k < i < v$ and $c_v = -1$, $y_v = (v-1)v/2$. Let t be chosen such that for every r-colouring of $\{1, \dots, t\}$ equation (2) has a monochromatic solution. Now, apply Ramsey's theorem for this t and $n = t \max(k, l)$. There is a number d such that for every r-colouring of the subsets of the first d primes there is a subset H of primes such that $|H| = n$, and for every $j \le t$ the j-element subsets of H have the same colour. Let us colour the elements of $\{1, \dots, t\}$ by r colours in the following way: for $1 \le i \le t$ let the colour of i be the colour of the i-element subsets of H. By Rado's theorem there exists a monochromatic solution of (2). Let $m = \alpha_1 + \cdots + \alpha_k = \beta_1 + \cdots + \beta_l$, where $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_l$ are distinct positive integers not greater than t. Consider an arbitrary partition $A_1, \dots, A_k$ of type $\alpha_1, \dots, \alpha_k$ and an arbitrary partition $B_1, \dots, B_l$ of type $\beta_1, \dots, \beta_l$ of the first m primes in H. These sets satisfy conditions (i)–(iii), so the statement is proved. ■

Acknowledgments. The author is greateful to A. Sárközy for turning his attention to this topic and for the useful hints. This research was supported by the Hungarian National Foundation for Scientific Research, grants K67870 and K100246.

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Received on 31.3.2011
and in revised form on 20.6.2011 (6655)