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On the largest prime factor of the partition function of n

by

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For Andrzej Schinzel on his seventy-fifth birthday

1. Introduction. Let $p(n)$ be the partition function of n, which is the number of ways of writing $n = \lambda_1 + \cdots + \lambda_k$, where $k \geq 1$ and $1 \leq \lambda_1 \leq \cdots \cdots \leq \lambda_k$ are integers. There is a huge literature on this function, dealing with its size, congruence properties, recurrence relations, and so on. Let $P(m)$ be the largest prime factor of the positive integer m with the convention that $P(1) = 1$, and let $\omega(m)$ be the number of distinct prime factors of m. In response to a question of Erdős and Ivić, Schinzel showed that $\omega(\prod_{m=1}^{N} p(m))$ tends to infinity with N (Lemma 2 in [2]). His method used lower bounds for nonzero linear forms in logarithms of algebraic numbers. Later, Schinzel and Wirsing [6] proved the effective result

$$\omega\Big(\prod_{m=1}^{N} p(m)\Big) \geq (1-\varepsilon)\frac{\log N}{\log 2} \quad \text{if } N > N_0(\varepsilon), \tag{1.1}$$

valid for all $\varepsilon > 0$, without using linear forms in logarithms.

Here, we revisit Schinzel's original argument to prove the following result.

Theorem 1. *The set of n for which*

$$P(p(n)) > \log\log n$$

is of asymptotic density 1.

This improves a result of the second author from [3], where it is proved by a different method that $P(p(n)) > \log\log\log\log\log\log n$ for almost all positive integers n.

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Notation. We use $c_1, c_2, \dots$ for computable positive constants. We use the Landau symbols O and o and the Vinogradov symbols $\ll$, $\gg$ and $\asymp$ with their usual meanings. Recall that $A = O(B)$, $A \ll B$ and $B \gg A$ are all equivalent to the fact that the inequality $|A| \le cB$ holds with some constant c. The constants implied by these symbols in our arguments are absolute. Furthermore, $A \asymp B$ means that both $A \ll B$ and $B \ll A$ hold, and $A = o(B)$ and $A \sim B$ mean that A/B tends to 0 and to 1, respectively.

2. Preliminary results. We start with Rademacher's formula for $p(n)$ (Chapter 5 in [1]).

Lemma 1. *We have*

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k}\left[\frac{d}{dx}\left(\frac{\sinh((\pi/k)\sqrt{(2/3)(x-1/24)})}{\sqrt{x-1/24}}\right)\right]_{x=n}, \tag{2.1}$$

where

$$A_k(n) := \sum_{\substack{1\le h\le k\\ \gcd(h,k)=1}} \omega_{h,k} e^{-2\pi i n h/k}$$

with $\omega_{h,k}$ being the root of unity of order 24 given by

$$\omega_{h,k} := e^{\pi i s(h,k)},$$

and $s(h,k)$ is the Dedekind sum

$$s(h,k) := \sum_{\mu=1}^{k-1}\left(\frac{\mu}{k} - \left[\frac{\mu}{k}\right] - \frac{1}{2}\right)\left(\frac{h\mu}{k} - \left[\frac{h\mu}{k}\right] - \frac{1}{2}\right).$$

In practice, one may truncate the sum appearing in (2.1) at $k := \lfloor\sqrt{n}\rfloor$ and then the nearest integer to this partial sum is exactly the value of $p(n)$ when $n > n_0$ is sufficiently large. Since in the range $k \le \sqrt{n}$ the kth term of the expansion (2.1) is of order $O(\exp(c_1\sqrt{n}/k)$, where $c_1 := \pi\sqrt{2/3}$ and $A_1(n) = 1$, we get

$$p(n) = \frac{1}{\pi\sqrt{2}}\left[\frac{d}{dx}\left(\frac{\sinh(\pi\sqrt{(2/3)(x-1/24)})}{\sqrt{x-1/24}}\right)\right]_{x=n} + O(\exp(c_1\sqrt{n}/2)). \tag{2.2}$$

The first term of the expansion (2.2) is, after some calculation,

$$\begin{aligned} &\frac{1}{\pi\sqrt{2}}\left[\frac{d}{dx}\left(\frac{\sinh(\pi\sqrt{(2/3)(x-1/24)})}{\sqrt{x-1/24}}\right)\right]_{x=n} \\ &\quad = \frac{e^{c_1\sqrt{n-1/24}}}{2\pi\sqrt{2}(n-1/24)}\left(\frac{\pi}{\sqrt{6}} - \frac{1}{2\sqrt{n-1/24}}\right) + O(\exp(-c_1\sqrt{n})). \end{aligned} \tag{2.3}$$

Putting together (2.2) and (2.3), we get our working formula

$$p(n) = e^{c_1\sqrt{n-1/24}} f(n) + O(e^{c_1\sqrt{n}/2}), \tag{2.4}$$

where

$$f(n) := \frac{1}{4\sqrt{3}(n-1/24)}\left[1 - \frac{c_2}{\sqrt{n-1/24}}\right] \tag{2.5}$$

with $c_2 = \sqrt{3/2}/\pi$.

We shall also need a result of Matveev [4] from transcendental number theory. But first, some notation. For an algebraic number η having

$$F(X) := a_0 \prod_{i=1}^{d} (X - \eta^{(i)}) \in \mathbb{Z}[X]$$

as minimal polynomial over the integers, the *logarithmic height* of η is defined as

$$h(\eta) := \frac{1}{d}\Big(\log|a_0| + \sum_{i=1}^{d} \log\max\{|\eta^{(i)}|, 1\}\Big).$$

With this notation, Matveev [4] proved the following deep theorem.

Lemma 2. *Let $\mathbb{K}$ be a field of degree D, $\eta_1, \ldots, \eta_k$ be nonzero elements of $\mathbb{K}$, and $b_1, \ldots, b_k$ be integers. Define $B := \max\{|b_1|, \ldots, |b_k|\}$ and $\Lambda := 1 - \prod_{i=1}^{k} \eta_i^{b_i}$. Let $A_1, \ldots, A_k$ be real numbers such that*

$$A_j \geq \max\{Dh(\eta_j), |\log \eta_j|, 0.16\}, \qquad j = 1, \ldots, k.$$

Then, assuming that $\Lambda \neq 0$, we have

$$\log|\Lambda| > -3 \cdot 30^{k+4}(k+1)^{5.5} D^2 (1 + \log D)(1 + \log(kB)) \prod_{i=1}^{k} A_i.$$

We shall use the above result only when $\eta_1, \ldots, \eta_k$ are rational. So, $\mathbb{K} := \mathbb{Q}$, $D = 1$, and the logarithmic height of $\eta := r/s$ with nonzero coprime integers r and s is just $\log(\max\{|r|, |s|\})$.

3. The proof of Theorem 1. We let x be a large positive real number, and $2 = p_1 < p_2 < \cdots$ be the increasing sequence of prime numbers. We write $r := r(x)$ for a function tending slowly to infinity and let

$$\mathcal{N}_r(x) := \{n \in [x, 2x) : P(p(n)) \leq p_r\}. \tag{3.1}$$

Our goal is to show that if $r(x)$ is chosen such that $p_r \leq \log\log x$ then $\#\mathcal{N}_r(x) = o(x)$ as $x \to \infty$, since once we have done that, Theorem 1 will follow by replacing x with $x/2$, then with $x/4$, and so on, and then summing up all these estimates.

So assume that $n \in \mathcal{N}_r(x)$ and write

$$p(n) =: p_1^{a_1} \cdots p_r^{a_r}. \tag{3.2}$$

Combining (3.2) with (2.4), we get

$$e^{c_1\sqrt{n-1/24}} f(n) - p_1^{a_1} \cdots p_r^{a_r} = O(e^{(c_1/2)\sqrt{n}}).$$

Dividing through by $e^{c_1\sqrt{n-1/24}} f(n)$, we get

$$1 - e^{-c_1\sqrt{n-1/24}} f(n)^{-1} p_1^{a_1} \cdots p_r^{a_r} = O(ne^{-(c_1/2)\sqrt{n}}) = O(e^{-c_3\sqrt{n}}),$$

where $c_3 := c_1/3$. Taking logarithms, we get

$$|c_1\sqrt{n-1/24} + \log f(n) - a_1 \log p_1 - \cdots - a_r \log p_r| = O(e^{-c_3\sqrt{n}}). \tag{3.3}$$

We let $z := \log x$, $K := \lfloor z^{1/2} \rfloor$, and assume that there exists an interval $[n, n+z) \subset [x, 2x)$ containing K numbers $n_1 < \cdots < n_K$ such that $P(p(n_i)) \le p_r$ for all $i = 1, \ldots, K$.

Indeed, otherwise we can split $[x, 2x)$ into $O(x/z)$ intervals of length z, each one containing at most $K-1$ elements of $\mathcal{N}_r(x)$, and then

$$\#\mathcal{N}_r(x) \ll \left(\frac{x}{z}\right) \cdot (K-1) = O\left(\frac{x}{(\log x)^{1/2}}\right) = o(x) \quad \text{as } x \to \infty, \tag{3.4}$$

which is what we want to prove.

For $i = 1, \ldots, K$, write

$$p(n_i) = \prod_{j=1}^{r} p_j^{\alpha_{i,j}}.$$

Put

$$g(x) := c_1\sqrt{x - 1/24} + \log f(x).$$

We let $y = \lfloor x^{1/4} \rfloor$, $y_i \in \{0, 1, \ldots, \lfloor y \rfloor\}$ and compute

$$\sum_{i=1}^{K} y_i g(n_i). \tag{3.5}$$

The absolute value of the vector in (3.5) is $O(Kyx^{1/2})$ and there are $(\lfloor y \rfloor + 1)^K$ such vectors. Consequently, by the Pigeonhole Principle, there is a nonzero vector $\mathbf{y} := (y_1, \ldots, y_K)$ with integer components $|y_i| \le y$ for all $i = 1, \ldots, K$, such that

$$\Big|\sum_{i=1}^{K} y_i g(n_i)\Big| \ll \frac{Ky\sqrt{x}}{(\lfloor y \rfloor + 1)^K - 1} \ll \frac{x}{y^K} = \frac{1}{x^{K/4-1}} \ll \frac{1}{x^{K/5}}. \tag{3.6}$$

Writing down (3.3) for $n := n_i$, $i = 1, \ldots, K$, we get

$$\Big|g(n_i) - \sum_{j=1}^{r} \alpha_{i,j} \log p_j\Big| = O(\exp(-c_3\sqrt{x})) \quad \text{for } i = 1, \ldots, K.$$

Then, taking linear combinations of the above relations with the coefficients $\mathbf{y} = (y_1, \ldots, y_K)$, we see that for large x,

$$\Big|\sum_{i=1}^{K} y_i g(n_i) - \sum_{j=1}^{r} \beta_j \log p_j\Big| \ll Ky \exp(-c_3\sqrt{x}) \le \exp(-c_4\sqrt{x}), \tag{3.7}$$

where we can take $c_4 := c_3/2$ and

$$\beta_j := \sum_{i=1}^{K} y_i \alpha_{i,j} \quad \text{for all } j = 1, \ldots, r. \tag{3.8}$$

Comparing the upper bounds from (3.6) and (3.7), we get

$$\begin{aligned} \Big|\sum_{j=1}^{r} \beta_j \log p_j\Big| &\le \Big|\sum_{i=1}^{K} y_i g(n_i)\Big| + O(\exp(-c_4\sqrt{x})) \\ &= O\bigg(\frac{1}{x^{K/5}} + \frac{1}{\exp(c_4\sqrt{x})}\bigg) = O\bigg(\frac{1}{x^{K/5}}\bigg). \end{aligned} \tag{3.9}$$

We distinguish two cases. In the first case, we assume that

$$\Gamma := \sum_{j=1}^{r} \beta_j \log p_j$$

is nonzero. Hence,

$$|\Gamma| \le \frac{1}{x^{K/6}} \tag{3.10}$$

for all large enough x. Now Γ is nonzero but $\Gamma = o(1)$, so that we have $\Gamma \sim e^{\Gamma} - 1 =: \Lambda \ne 0$ as $x \to \infty$, and we can use Matveev's result (Lemma 2) to find a lower bound on this last expression.

We take, in the notation of Lemma 2,

$$k := r, \quad \eta_j := p_j \quad \text{and} \quad b_j := \beta_j \quad \text{for } j = 1, \ldots, r.$$

Clearly, $\mathbb{K} := \mathbb{Q}$, so $D = 1$, and $A_j := \log p_j$ for $j = 1, \ldots, r$.

We also use the fact that $p_m \le (m+1)^2$ for all positive integers m (see, for example, (3.13) in [5]).

As for B, observe that

$$\alpha_{i,j} \le \log p(n)/\log p_j \ll x^{1/2} \quad \text{for all } j = 1, \ldots, r \text{ and } i = 1, \ldots, K.$$

Therefore, using (3.8), we deduce that

$$|\beta_j| \ll yKx^{1/2} = o(x) \quad \text{as } x \to \infty.$$

So, we can take $B := x$ for all sufficiently large x, and then indeed

$$B \ge \max\{|\beta_j| : j = 1, \ldots, r\}.$$

Lemma 2 shows that there exists some absolute constant c_5 such that

$$|\Lambda| > \exp\bigl(-c_5^r(\log x)(\log(r+1)^2)^r\bigr). \tag{3.11}$$

Comparing (3.11) with (3.10) and using the fact that $|\Lambda| \sim |\Gamma|$ as $x \to \infty$, we get

$$(2c_5 \log(r+1))^r \geq K/7$$

for large x. In turn, this implies

$$r \log\log(r+1) \geq \log K + O(1) \geq c_6 \log\log x$$

for large x, where we can take $c_6 := 1/3$. Hence,

$$r \gg \frac{\log\log x}{\log\log\log\log x}.$$

From the Prime Number Theorem (or the Chebyshev estimates), we get

$$p_r \gg r \log r \gg (\log\log x)\biggl(\frac{\log\log\log x}{\log\log\log\log x}\biggr).$$

Note that the right-hand side above is of order at least $\log\log x$, which for large x contradicts our assumption that $p_r \leq \log\log x$. Thus, we get a contradiction assuming that $\Gamma \neq 0$.

Now we deal with the harder case when $\Gamma = 0$. Then (3.7) becomes

$$\Bigl|\sum_{i=1}^{K} y_i g(n_i)\Bigr| = O(\exp(-c_4\sqrt{x})). \tag{3.12}$$

We write each $n_i := n + \lambda_i$ for $i = 1, \ldots, K$ (note that $\lambda_1 = 0$, although we will not use this information), and write the Taylor series

$$g(n_i) = \sum_{k=0}^{\infty} \frac{g^{(k)}(n)}{k!}\lambda_i^k \qquad \text{for all } i = 1, \ldots, K,$$

which, via estimate (3.12), yields

$$\Bigl|\sum_{k=0}^{\infty} \frac{g^{(k)}(n)}{k!} \sum_{i=1}^{K} y_i \lambda_i^k\Bigr| = O(\exp(-c_4\sqrt{x})). \tag{3.13}$$

We need the derivatives of $g(y)$. Observe that

$$g(t) = \sqrt{t - 1/24} - \log\biggl(t - \frac{1}{24}\biggr) - \log c_7 + \log\biggl(1 - \frac{c_2}{\sqrt{t - 1/24}}\biggr),$$

where $c_7 := 4\sqrt{3}$ and $c_2 := \sqrt{3/2}/\pi$. For $k \geq 1$, one checks easily, by induction, that

$$\frac{d^k}{dt^k}\sqrt{t-1/24} = (-1)^{k-1}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\cdots\left(\frac{2k-3}{2}\right)\frac{1}{(t-1/24)^{(2k-1)/2}} \tag{3.14}$$
$$= (-1)^{k-1}\frac{(2k-2)!}{(k-1)!2^{2k-1}(t-1/24)^{(2k-1)/2}},$$

and

$$\frac{d^k}{dt^k}\log\left(t-\frac{1}{24}\right) = (-1)^{k-1}\frac{(k-1)!}{(t-1/24)^k}. \tag{3.15}$$

Finally, using the Taylor series expansion for $\log(1-y)$, we easily get

$$\log\left(1-\frac{c_2}{\sqrt{t-1/24}}\right) = -\sum_{j\geq 1}\frac{c_2^j}{j(t-1/24)^{j/2}}, \tag{3.16}$$

and taking derivatives, we arrive at

$$\frac{d^k}{dt^k}\left(\log\left(1-\frac{c_2}{\sqrt{t-1/24}}\right)\right) = -\sum_{j\geq 1}\frac{c_2^j}{j}\frac{d^k}{dy^k}\left(\frac{1}{(t-1/24)^{j/2}}\right) \tag{3.17}$$
$$= (-1)^{k+1}\sum_{j\geq 1}\frac{c_2^j}{j}\left(\frac{j(j+2)\cdots(j+2(k-1))}{2^k(t-1/24)^{j/2+k}}\right).$$

To get a contradiction, we shall show that for large x, inequality (3.13) leads to the conclusion that

$$\sum_{i=1}^{K} y_i\lambda_i^k = 0 \qquad \text{for } k=0,1,\ldots,K-1. \tag{3.18}$$

Granted that, $\mathbf{y}$ is a zero of the linear map with nonzero (Vandermonde) determinant, so $\mathbf{y}=\mathbf{0}$, which is a contradiction.

Now we prove by induction on k that (3.18) must hold for large x.

Put

$$M_k(t) := \left|\frac{g^{(k)}(t)}{k!}\right| \qquad \text{for } t\in[n,n+z].$$

Relations (3.14), (3.15) and (3.17) show easily that

$$M_k(t) \asymp \frac{1}{k^{3/2}n^{k-1/2}} \tag{3.19}$$

uniformly in $k\leq K$, $t\in[n,n+z]$ and $n\in[x,2x]$. Indeed, applying Stirling's formula to (3.14), we have

$$\frac{1}{k!}\left|\frac{d^k}{dt^k}\sqrt{t-1/24}\right| = \frac{(2k-2)!}{(k-1)!k!2^{2k-1}(t-1/24)^{k-1/2}} \tag{3.20}$$
$$\asymp \frac{1}{k^{1/2}2^{2k-1}}\,\frac{((2k-2)/e)^{2k-2}}{(k/e)^k((k-1)/e)^{k-1}}\,\frac{1}{n^{k-1/2}}\left(1+O\left(\frac{z}{n}\right)\right)^{k-1/2}$$
$$\asymp \frac{1}{k^{3/2}}\left(1-\frac{1}{k}\right)^{k-1}\frac{1}{n^{k-1/2}}\left(1+O\left(\frac{zK}{x}\right)\right) \asymp \frac{1}{k^{3/2}n^{k-1/2}},$$

uniformly for $k \le K$ and $n \in [x, 2x]$. From (3.15), we have

$$\frac{1}{k!}\left|\frac{d^k}{dt^k}\left(\log\left(t-\frac{1}{24}\right)\right)\right| = \frac{1}{k(t-1/24)^k} \tag{3.21}$$
$$= \frac{1}{kn^k}\left(1+O\left(\frac{z}{n}\right)\right)^k = \frac{1}{kn^k}\left(1+O\left(\frac{Kz}{x}\right)\right) \asymp \frac{1}{kn^k}.$$

For (3.17), put

$$a_{j,k} := \frac{c_2^j}{j}\left(\frac{j(j+2)\cdots(j+2(k-1))}{2^k(t-1/24)^{j/2}}\right) \qquad \text{for } j \ge 1,$$

and observe that

$$\frac{a_{j+1,k}}{a_{j,k}} = c_2\left(\frac{j}{j+1}\right)\left(\frac{(j+1)\cdots(j+1+2(k-1))}{j\cdots(j+2(k-1))}\right)\frac{1}{(t-1/24)^{1/2}}$$
$$\ll \left(\frac{j+1+2(k-1)}{j}\right)\frac{1}{(t-1/24)^{1/2}} \ll \frac{K}{x^{1/2}} = o(1)$$

uniformly in $j \ge 1$, $k \le K$, and $n \in [x, 2x]$ as $x \to \infty$, which shows that in the series of (3.17), the first term dominates. Thus,

$$\frac{1}{k!}\left|\frac{d^k}{dt^k}\log\left(1-\frac{c_2}{\sqrt{t-1/24}}\right)\right| \asymp \frac{1\cdot 3\cdots(2k-1)}{k!2^k(t-1/24)^{k+1/2}} \tag{3.22}$$
$$= \frac{2k!}{2^{2k}k!^2n^{k+1/2}}\left(1+O\left(\frac{z}{n}\right)\right)^{k+1/2}$$
$$\asymp \frac{1}{k^{1/2}}\,\frac{(2k/e)^{2k}}{2^{2k}(k/e)^{2k}}\,\frac{1}{n^{k+1/2}}\left(1+O\left(\frac{Kz}{x}\right)\right) \asymp \frac{1}{k^{1/2}n^{k+1/2}}.$$

Since the terms arising from (3.20)–(3.22) are of different orders of magnitude, with the dominating term coming from (3.20), we get (3.19).

Now we are ready to prove that (3.18) must hold. Take $k = 0$ and use the Taylor's formula with remainder at $k = 1$ in (3.12) to get

$$\begin{aligned}(3.23)\qquad \sqrt{n}\Big|\sum_{i=1}^{K} y_i\Big| &\ll \Big|g^{(0)}(n)\sum_{i=1}^{K} y_i\lambda_i^0\Big| \\ &\ll \max_{n\le t\le n+z}\left\{\left|\frac{g^{(1)}(t)}{1!}\right|\right\}\sum_{i=1}^{K}|y_i\lambda_i| + \exp(-c_4\sqrt{x}) \\ &\ll \frac{yKz}{n^{1/2}} + \exp(-c_4\sqrt{x}) \ll \frac{yKz}{n^{1/2}},\end{aligned}$$

giving

$$(3.24)\qquad \Big|\sum_{i=1}^{k} y_i\Big| \ll \frac{yKz}{n} \ll \frac{yKz}{x} = o(1) \quad \text{as } x\to\infty.$$

Since the left-hand side of (3.24) is an integer, we get

$$(3.25)\qquad \sum_{i=1}^{K} y_i = 0,$$

which is the desired relation (3.12) with $k = 0$. Assume now by induction that (3.12) holds for all exponents $0, 1, \ldots, k-1$, for some $k \le K-1$. Applying again the Taylor formula with remainder at k in (3.12) and the induction hypothesis, as well as calculation (3.19), we get

$$\begin{aligned}\frac{1}{k^{3/2}n^{k-1/2}}\Big|\sum_{i=1}^{K} y_i\lambda_i^k\Big| &\ll \Big|\frac{g^{(k)}(n)}{k!}\sum_{i=1}^{K} y_i\lambda_i^k\Big| = \Big|\sum_{\ell=0}^{k}\frac{g^{(\ell)}(n)}{\ell!}\sum_{i=1}^{K} y_i\lambda_i^\ell\Big| \\ &\ll \max_{n\le t\le n+z}\left\{\left|\frac{g^{(k+1)}(t)}{(k+1)!}\right|\right\}\sum_{i=1}^{K}|y_i\lambda_i|^{k+1} + \exp(-c_4\sqrt{x}) \\ &\ll \frac{yKz^K}{(k+1)^{3/2}n^{k+1-1/2}} + \exp(-c_4\sqrt{x}) \ll \frac{yKz^K}{(k+1)^{3/2}n^{k+1-1/2}},\end{aligned}$$

where the last inequality follows because the term $\exp(-c_4\sqrt{x})$ is of a smaller order than

$$\frac{1}{K^{3/2}n^{K+1}} > \exp\big(-(\log(2x))^{3/2} - \log\log x\big).$$

Thus, from the above calculation we get

$$(3.26)\qquad \Big|\sum_{i=1}^{K} y_i\lambda_i^k\Big| \ll \frac{yKz^K}{n} \ll \frac{yKz^K}{x} = o(1) \quad \text{as } x\to\infty,$$

because

$$yKz^K \le x^{1/4}(\log x)^{1/2}(\log x)^{(\log x)^{1/2}} = x^{1/4}\exp(O((\log x)^{1/2}\log\log x) = o(x)$$

as $x \to \infty$. Since yKz^h/x is an integer, we find that $\sum_{i=1}^K y_i\lambda_i^k = 0$, as desired. Thus, we obtained a contradiction, assuming that $\Gamma = 0$. Hence, both cases $\Gamma = 0$ and $\Gamma \neq 0$ yield contradictions, so the conclusion is that an interval $[n, n+z] \subset [x, 2x)$ cannot contain K members of $\mathcal{N}_r(x)$. Now the argument used previously to derive (3.4) yields the desired conclusion.

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