Nonvanishing of Siegel–Poincaré series II

by

SOUMYA DAS (Mumbai), WINFRIED KOHNEN (Heidelberg)
and JYOTI SENGUPTA (Mumbai)

1. Introduction. Fourier coefficients of cusp forms in general are mysterious objects and there are many open questions. One basic problem is to decide whether a given Fourier coefficient is zero or not. For example, there is a well-known conjecture due to Lehmer that the Ramanujan τ-function giving the Fourier coefficients of the discriminant function \( \Delta \) of weight 12 on \( \Gamma_1 := \text{SL}_2(\mathbb{Z}) \) never vanishes. Let \( P_{k,m} \) be the \( m \)th Poincaré series of weight \( k \) on \( \Gamma_1 \), by definition up to a nonzero scalar the dual cusp form of weight \( k \) of the functional giving the \( m \)th Fourier coefficient with respect to the Petersson scalar product. Then Lehmer’s conjecture can be reformulated by saying that \( P_{12,m} \neq 0 \) for all \( m \geq 1 \).

More generally one can ask if \( P_{k,m} \) is nonzero whenever the corresponding space of cusp forms is nonzero. Rankin [10] proved that \( P_{k,m} \neq 0 \) for \( m \leq k^2 - \epsilon \), for any \( \epsilon > 0 \) and \( k \) large (depending on \( \epsilon \)). Rankin’s idea was to analyze the \( m \)th Fourier coefficient of \( P_{k,m} \) which can be explicitly expressed in terms of Kloosterman sums and Bessel functions and to show that it is not zero for the \( m \) and \( k \) in question.

The above method was successfully extended to the case of Jacobi forms in [2]. One could try to use the same method in the case of Siegel cusp forms of arbitrary degree \( g \geq 2 \), with the aim to prove a nonvanishing result for the \( T \)th Siegel–Poincaré series, defined in a similar way to the case \( g = 1 \), but \( T > 0 \) now being a positive definite half-integral matrix of size \( g \). Unfortunately, however, this seems to be extremely cumbersome since no neat and simple formulas for its Fourier coefficients are known. In [3] the authors instead used the result of [2] together with explicit formulas for the Maass lift to prove nonvanishing results for these series in the case of degree 2 and even weight. These results essentially are in the same spirit as those for degree 1.

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In this paper, we use the idea from [3] along with the results of [1], [4] to prove (roughly speaking) that for each even \( k \geq 10 \) there exists a positive proportion of fundamental discriminants arising as discriminants of matrices \( T > 0 \) of size 2 such that the Siegel–Poincaré series of degree 2 and index \( nT \) with \( n \in \mathbb{N} \) are not zero, under some mild conditions on \( n \). Unlike [3], we use the algebraic description of the Fourier coefficients of Hecke eigenforms of degree 1 and Deligne’s estimate for the latter.

2. Main result. We will denote by \( T > 0 \) a positive definite half-integral matrix of size 2. We let \( D_T := -4 \det T < 0 \), the discriminant of \( T \).

Recall that a fundamental discriminant is either 1 or the discriminant of a quadratic field. Thus a fundamental discriminant is a nonzero integer \( D \) such that either \( D \) is squarefree and \( D \equiv 1 \) (mod 4), or \( D \equiv 0 \) (mod 4), \( D/4 \) squarefree and \( D/4 \equiv 2, 3 \) (mod 4).

We let \( S_k(\Gamma_2) \) be the space of Siegel cusp forms of integral weight \( k \) and degree 2 on \( \Gamma_2 := \text{Sp}_2(\mathbb{Z}) \). It is well-known that \( S_k(\Gamma_2) \neq \{0\} \) if and only if \( k \geq 10 \).

We denote by \( \Delta_2 \) the subgroup of \( \Gamma_2 \) consisting of all matrices of the form \( \left( \begin{array}{cc} E & S' \\ 0 & E \end{array} \right) \), with \( S = S' \in \mathbb{Z}^{2,2} \) and \( E \) the unit matrix of size 2. If \( T > 0 \) and \( k \geq 6 \) we define the \( T \)th Siegel–Poincaré series of weight \( k \) on \( \Gamma_2 \) by

\[
P_{k,T}(Z) = \sum_{\gamma = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in \Delta_2 \setminus \Gamma_2} \det(CZ + D)^{-k} e^{2\pi i \text{Tr}(T \gamma Z)}
\]

where \( Z \) is a variable in the Siegel upper half-space \( \mathcal{H}_2 \) of degree 2, i.e. \( Z \in \mathbb{C}^{2,2} \), \( Z = Z' \) and \( \Im(Z) > 0 \). It is well-known that the series converges absolutely and locally uniformly on \( \mathcal{H}_2 \) and is in \( S_k(\Gamma_2) \). If \( \langle , \rangle \) denotes the usual Petersson scalar product on \( S_k(\Gamma_2) \), then we have

\[
\langle F, P_{k,T} \rangle = c_k (\det T)^{-k+3/2} A(T) \quad (\forall F \in S_k(\Gamma_2))
\]

where \( A(T) \) (\( T > 0 \)) is the \( T \)th Fourier coefficient of \( F \) and

\[
c_k = 2\sqrt{\pi}(4\pi)^{3/2} \Gamma\left(k - \frac{3}{2}\right) \Gamma(k - 2).
\]

For \( x > 0 \) we let \( \mathcal{N}_k(x) \) be the number of fundamental discriminants \( D_0 < 0 \) with \( |D_0| < x \) and with the property that \( P_{k,nT} \neq 0 \) for every \( T > 0 \) with \( D_T = D_0 \) and every \( n \in \mathbb{N} \) not divisible by 2, 3, 5, 7, 11, 13.

**Theorem 2.1.** Suppose that \( k \geq 10 \) is even. Then for every \( \epsilon > 0 \) we have

\[
\mathcal{N}_k(x) \geq \left(\frac{9}{16\pi^2} - \epsilon\right) x \quad (x \gg \epsilon 1).
\]

The proof will be given in Section 3. In Section 4 we deal with the case when some of the first six primes may divide \( n \). In this case, using certain
congruences for Hecke eigenvalues we can prove a result similar to that of
the theorem, but with some technical conditions involving \( n \) and \( D_T \), with
\( D_T \) fundamental. In Section 5 we briefly indicate a nonvanishing result for
Siegel–Poincaré series in higher degree \( g \), using the result in degree 2 and
the Ikeda lift.

3. Proof of Theorem 2.1. For \( l \) a positive integer we denote by \( S_{l+1/2}^+ \)
the space of cusp forms of weight \( l + 1/2 \) on the Hecke congruence subgroup
\( \Gamma_0(4) \) of level 4, having a Fourier expansion of the form
\[
\sum_{m \geq 1, (-1)^l m \equiv 0, 1 \pmod{4}} c(m)q^m
\]
where \( q = e^{2\pi i z} \) for \( z \in \mathcal{H} \), the complex upper half-plane. For details we
refer to [6], [11].

Let \( l \geq 6 \) if \( l \) is even and \( l \geq 9 \) if \( l \) is odd. Then one knows that the
number of fundamental discriminants \( D_0 \) with \( |D_0| < x \) and such that there
exists a cusp form \( h \in S_{l+1/2}^+ \) whose \(|D_0|\)th Fourier coefficient does not
vanish is
\[
\geq \left( \frac{9}{16\pi^2} - \epsilon \right) x \quad (x \gg \epsilon^1)
\]
for every \( \epsilon > 0 \). This was proved in [7] for \( l \) even and later in [1] for \( l \) odd.
Since \( S_{l+1/2}^+ \) has a basis of eigenforms for all Hecke operators, we can assume
that \( h \) is a Hecke eigenform.

We now let \( l = k - 1 \) with \( k \geq 10 \) even. In view of (1) and the above,
and \( S_{l+1/2}^+ \) has a basis of eigenforms for all Hecke operators, we can assume
that \( h \) is a Hecke eigenform.

We now let \( l = k - 1 \) with \( k \geq 10 \) even. In view of (1) and the above,
to prove the theorem it is sufficient to prove the existence of a form in
\( S_k(\Gamma_2) \) whose \( nT \)th Fourier coefficient does not vanish, for all \( T > 0 \) with
\( D_T = D_0 \) and all \( n \) not divisible by any of the first six primes, where
\( D_0 \) is a fundamental discriminant such that there exists a Hecke eigenform
\( h \in S_{k-1/2}^+ \) with \(|D_0|\)th coefficient nonzero.

Recall that one has a linear lifting map (Maass lift) from \( S_{k-1/2}^+ \) to
\( S_k(\Gamma_2) \) given explicitly on the level of Fourier coefficients by
\[
h = \sum_{m \geq 1, (-1)^l m \equiv 0, 3 \pmod{4}} c(m)q^m \mapsto \sum_{T > 0} A_h(T)e^{2\pi i \text{tr}(TZ)} \quad (Z \in \mathcal{H}_2)
\]
where
\[
A_h(T) = \sum_{d|c_T} d^{k-1}c(|D_T|/d^2)
\]
and \( c_T \) denotes the content of \( T \), i.e. \( c_T := \gcd(a, b, c) \) if \( T = \left( \begin{array}{cc} a & b/2 \\ b/2 & c \end{array} \right) \) with
\( a, b, c \) integers.
If \( D_T = D_0 \) is a fundamental discriminant and \( n \in \mathbb{N} \), then clearly

\[
A_h(nT) = \sum_{d|n} d^{k-1} c \left( \frac{|D_0|n^2}{d^2} \right).
\]

Now let \( h \) be a Hecke eigenform. We let

\[
f(z) = \sum_{n \geq 1} a(n) q^n \quad (z \in \mathcal{H})
\]

be the unique normalized Hecke eigenform of weight \( 2k - 2 \) on \( \Gamma_1 \) corresponding to \( h \) under the Shimura correspondence \([6], [11]\).

**Lemma 3.1.** For any \( n \geq 1 \), one has

\[
A_h(nT) = c(|D_0|) \prod_{p^n \parallel n} \left( \sum_{\mu=0}^{\nu} p^{(\nu-\mu)(k-1)} \left( a(p^\mu) - p^{-2} \left( \frac{D_0}{p} \right) a(p^{\mu-1}) \right) \right).
\]

Here \( p^n \parallel n \) means that \( p^n \) exactly divides \( n \) and we set \( a(n) = 0 \) if \( n \) is not an integer.

**Proof.** From (2) we conclude that

\[
\sum_{n \geq 1} A_h(nT)n^{-s} = \zeta(s - k + 1) \sum_{n \geq 1} c(|D_0|n^2)n^{-s},
\]

and by \([6]\) we have

\[
\sum_{n \geq 1} c(|D_0|n^2)n^{-s} = c(|D_0|) \frac{L(f, s)}{L(s - k + 2, \chi_{D_0})},
\]

where \( L(f, s) \) is the Hecke \( L \)-function of \( f \) and \( \chi_{D_0} \) is the quadratic character attached to \( D_0 \).

Since the \( p \)-Euler factor of \( L(f, s) \) is equal to

\[
\sum_{\mu \geq 0} a(p^\mu)p^{-\mu s}
\]

and that of \( 1/L(s - k + 2, \chi_{D_0}) \) is

\[
1 - \left( \frac{D_0}{p} \right) p^{k-2-s}
\]

our claim easily follows. \( \blacksquare \)

By the lemma, we now see that it is enough to prove that

\[
S_p(\nu) := \sum_{\mu=0}^{\nu} p^{(\nu-\mu)(k-1)} \left( a(p^\mu) - p^{-2} \left( \frac{D_0}{p} \right) a(p^{\mu-1}) \right) \neq 0
\]

for all \( \nu \geq 0 \) and for each prime \( p > 13 \). Assume on the contrary that
$S_p(\nu) = 0$ for some $\nu$ and $p$ as above. Clearly then $\nu \geq 1$, and it follows that

$$-1 = \sum_{\mu=1}^{\nu} p^{-\mu(k-1)} \left( a(p^\mu) - p^{k-2} \left( \frac{D_0}{p} \right) a(p^{\mu-1}) \right).$$

We now invoke Deligne’s bound

$$|a(p^\mu)| \leq (\mu + 1)p^{\mu(k-3/2)}.$$  

Taking absolute values in (3) we obtain

$$1 \leq \sum_{\mu=1}^{\nu} \left( \frac{\mu}{p^{\mu/2+1/2}} + \frac{\mu + 1}{p^{\mu/2}} \right).$$

We put $\alpha := p^{-1/2}$ and then use the elementary estimates

$$\sum_{\mu=1}^{\nu} \mu\beta^\mu < \frac{\beta}{(1-\beta)^2}, \quad \sum_{\mu=1}^{\nu} \beta^\mu < \frac{\beta}{1-\beta}, \quad (0 < \beta < 1)$$

with $\beta = \alpha$. Then from (4) we deduce that

$$1 < \frac{\alpha(\alpha + 1)}{(1-\alpha)^2} + \frac{\alpha}{1-\alpha}, \quad \text{i.e.,} \quad \alpha^2 - 4\alpha + 1 < 0,$$

which is equivalent to saying that $\alpha_1 < \alpha < \alpha_2$ where $\alpha_1 = 2 - \sqrt{3}$ and $\alpha_2 = 2 + \sqrt{3}$ are the roots of the polynomial $x^2 - 4x + 1$. This is a contradiction if $p > 13$, completing the proof of the theorem.

4. The excluded primes. If in the definition of $N_k(x)$ we allowed that one of the primes 2, 3, 5, 7, 11 or 13 divides $n$, then our arguments leading to the assertion of the theorem would break down, since we cannot use Deligne’s bound to show that $S_p(\nu) \neq 0$ for all $\nu$.

However, it is sometimes possible to show that $S_p(\nu) \neq 0$, for many $\nu$, even for the excluded primes, if one invokes a different idea, namely congruence properties for the Hecke eigenvalues $a(n)$ of the normalized Hecke eigenform $f$, modulo small primes $l$, proved by Hatada [4]. Recall that the $a(n)$ are algebraic integers lying in an algebraic number field determined by $f$, and by a congruence modulo $l$ we mean the corresponding congruence modulo the ideal $(l)$ generated by $l$.

We want to illustrate this in the simplest case $l = 2$. According to [4] one has $a(p) \equiv 0 \pmod{2}$ whenever $p$ is an odd prime. Reducing the generating series

$$\sum_{\mu \geq 0} a(p^\mu)X^\mu = \frac{1}{1 - a(p)X + p^{k-2}X^2}$$
modulo (2), we deduce that
\[ a(p^\mu) \equiv \begin{cases} 0 \pmod{(2)} & \text{if } \mu \equiv 1 \pmod{2}, \\ 1 \pmod{(2)} & \text{otherwise}. \end{cases} \]

This easily implies that
\[ S_p(\nu) \equiv \begin{cases} \nu + 1 \pmod{(2)} & \text{if } (p, D_0) = 1, \\ \lceil \nu/2 \rceil + 1 \pmod{(2)} & \text{otherwise}. \end{cases} \]

Hence if \( n \) is odd and \( p^\nu \parallel n \) implies that \( \nu \) is even when \( (p, D_0) = 1 \) and that \( \lceil \nu/2 \rceil \) is even when \( p \mid D_0 \), then \( \prod_{p^\nu \parallel n} S_p(\nu) \equiv 1 \pmod{(2)} \) for all odd \( n = \prod_{p^\nu \parallel n} p^\nu \) and in particular this product is not zero.

Thus if one modifies the definition of \( N_k(x) \) appropriately, one can also formulate a corresponding assertion as in the theorem allowing any \( n \) as above (in particular divisible by 2, 3, 5, 7, 11 or 13, under the given conditions).

5. Some remarks in higher degrees. In [5] Ikeda gave a generalization of the Maass lift in the case of Hecke eigenforms (usually called the Saito–Kurokawa lift in this case) to higher degrees. More specifically, whenever \( l \equiv g \pmod{2} \), starting with a normalized Hecke eigenform \( f \) of weight 2\( l \) on \( \Gamma \) he constructs a Siegel–Hecke eigenform \( F \) of weight \( l + g \) and degree \( 2g \) whose standard zeta function (up to a Riemann zeta function) is the product of shifted Hecke \( L \)-functions of \( f \). Moreover, the Fourier coefficients of \( F \) are given by a complicated product expression, involving the Fourier coefficients of a Hecke eigenform \( h \in S_{l+1/2}^+ \) corresponding to \( f \) under the Shimura correspondence, and a finite product over primes \( p \) of modified local singular (Laurent) polynomials at \( p \), evaluated at the \( p \)-Satake parameters of \( f \).

Later on in [8], a linear version of the Ikeda lift was given, as a linear map from \( S_{l+1/2}^+ \) to \( S_{l+g}(\Gamma_{2g}) \), the space of Siegel cups forms of weight \( l + g \) on \( \Gamma_{2g} = \text{Sp}_{2g}(\mathbb{Z}) \). If \( g = 2 \), it was proved that the formulas given (after replacing \( l \) by \( k - 1 \)) coincide with those giving the Maass lift in Section 3. We will denote the Fourier coefficients of the lift of \( h \in S_{l+1/2}^+ \) by \( A_{h,2g}(T) \) \((T > 0 \text{ of size } 2g)\). In particular then \( A_{h,2}(T) = A_h(T) \) in previous notation.

Now suppose that \( g \equiv 1 \pmod{4} \). Fix a positive definite even integral unimodular matrix \( T_0 \) of size \( 2g - 2 \). (Note that \( 2g - 2 \equiv 0 \pmod{8} \) and therefore such a matrix exists, as is well-known.) Then it was proved in [9] that
\[ A_{h,2g}\left(T \oplus \frac{1}{2}T_0\right) = A_{h,2}(T) \]
for any \( T > 0 \text{ of size } 2 \).

From this equality one can therefore obtain nonvanishing results for Siegel–Poincaré series of index \( nT \oplus \frac{1}{2}T_0 \) in degree \( 2g \) (under the given conditions), using our previous results.
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References


Soumya Das, Jyoti Sengupta
School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road
Mumbai 400005, India
E-mail: somu@math.tifr.res.in
sengupta@math.tifr.res.in

Winfried Kohnen
Mathematisches Institut
Ruprecht-Karls-Universität Heidelberg
D-69120 Heidelberg, Germany
E-mail: winfried@mathi.uni-heidelberg.de

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