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Nonvanishing of Siegel–Poincaré series II

by

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1. Introduction. Fourier coefficients of cusp forms in general are mysterious objects and there are many open questions. One basic problem is to decide whether a given Fourier coefficient is zero or not. For example, there is a well-known conjecture due to Lehmer that the Ramanujan τ-function giving the Fourier coefficients of the discriminant function Δ of weight 12 on $\Gamma_1 := \mathrm{SL}_2(\mathbb{Z})$ never vanishes. Let $P_{k,m}$ be the mth Poincaré series of weight k on Γ_1, by definition up to a nonzero scalar the dual cusp form of weight k of the functional giving the mth Fourier coefficient with respect to the Petersson scalar product. Then Lehmer's conjecture can be reformulated by saying that $P_{12,m} \neq 0$ for all $m \geq 1$.

More generally one can ask if $P_{k,m}$ is nonzero whenever the corresponding space of cusp forms is nonzero. Rankin [10] proved that $P_{k,m} \neq 0$ for $m \leq k^{2-\epsilon}$, for any $\epsilon > 0$ and k large (depending on ϵ). Rankin's idea was to analyze the mth Fourier coefficient of $P_{k,m}$ which can be explicitly expressed in terms of Kloosterman sums and Bessel functions and to show that it is not zero for the m and k in question.

The above method was successfully extended to the case of Jacobi forms in [2]. One could try to use the same method in the case of Siegel cusp forms of arbitrary degree $g \geq 2$, with the aim to prove a nonvanishing result for the Tth Siegel–Poincaré series, defined in a similar way to the case $g = 1$, but $T > 0$ now being a positive definite half-integral matrix of size g. Unfortunately, however, this seems to be extremely cumbersome since no neat and simple formulas for its Fourier coefficients are known. In [3] the authors instead used the result of [2] together with explicit formulas for the Maass lift to prove nonvanishing results for these series in the case of degree 2 and even weight. These results essentially are in the same spirit as those for degree 1.

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In this paper, we use the idea from [3] along with the results of [1], [4] to prove (roughly speaking) that for each even $k \geq 10$ there exists a positive proportion of fundamental discriminants arising as discriminants of matrices $T > 0$ of size 2 such that the Siegel–Poincaré series of degree 2 and index nT with $n \in \mathbb{N}$ are not zero, under some mild conditions on n. Unlike [3], we use the algebraic description of the Fourier coefficients of Hecke eigenforms of degree 1 and Deligne's estimate for the latter.

2. Main result. We will denote by $T > 0$ a positive definite half-integral matrix of size 2. We let $D_T := -4 \det T < 0$, the discriminant of T.

Recall that a *fundamental discriminant* is either 1 or the discriminant of a quadratic field. Thus a fundamental discriminant is a nonzero integer D such that either D is squarefree and $D \equiv 1 \pmod 4$, or $D \equiv 0 \pmod 4$, $D/4$ squarefree and $D/4 \equiv 2, 3 \pmod 4$.

We let $S_k(\Gamma_2)$ be the space of Siegel cusp forms of integral weight k and degree 2 on $\Gamma_2 := \mathrm{Sp}_2(\mathbb{Z})$. It is well-known that $S_k(\Gamma_2) \neq \{0\}$ if and only if $k \geq 10$.

We denote by Δ_2 the subgroup of Γ_2 consisting of all matrices of the form $\left(\begin{smallmatrix} E & S \\ 0 & E \end{smallmatrix}\right)$, with $S = S' \in \mathbb{Z}^{2,2}$ and E the unit matrix of size 2. If $T > 0$ and $k \geq 6$ we define the Tth Siegel–Poincaré series of weight k on Γ_2 by

$$P_{k,T}(Z) = \sum_{\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Delta_2 \backslash \Gamma_2} \det(CZ + D)^{-k} e^{2\pi i \ \mathrm{Tr}(T \cdot \gamma \circ Z)}$$

where Z is a variable in the Siegel upper half-space $\mathcal{H}_2$ of degree 2, i.e. $Z \in \mathbb{C}^{2,2}$, $Z = Z'$ and $\Im(Z) > 0$. It is well-known that the series converges absolutely and locally uniformly on $\mathcal{H}_2$ and is in $S_k(\Gamma_2)$. If $\langle \, , \, \rangle$ denotes the usual Petersson scalar product on $S_k(\Gamma_2)$, then we have

$$\langle F, P_{k,T} \rangle = c_k (\det T)^{-k+3/2} A(T) \qquad (\forall F \in S_k(\Gamma_2)) \tag{1}$$

where $A(T)$ ($T > 0$) is the Tth Fourier coefficient of F and

$$c_k = 2\sqrt{\pi}(4\pi)^{3-2k} \Gamma\left(k - \frac{3}{2}\right) \Gamma(k-2).$$

For $x > 0$ we let $\mathcal{N}_k(x)$ be the number of fundamental discriminants $D_0 < 0$ with $|D_0| < x$ and with the property that $P_{k,nT} \neq 0$ for every $T > 0$ with $D_T = D_0$ and every $n \in \mathbb{N}$ not divisible by $2, 3, 5, 7, 11, 13$.

THEOREM 2.1. *Suppose that $k \geq 10$ is even. Then for every $\epsilon > 0$ we have*

$$\mathcal{N}_k(x) \geq \left(\frac{9}{16\pi^2} - \epsilon\right) x \qquad (x \gg_\epsilon 1).$$

The proof will be given in Section 3. In Section 4 we deal with the case when some of the first six primes may divide n. In this case, using certain congruences for Hecke eigenvalues we can prove a result similar to that of the theorem, but with some technical conditions involving n and D_T, with D_T fundamental. In Section 5 we briefly indicate a nonvanishing result for Siegel–Poincaré series in higher degree g, using the result in degree 2 and the Ikeda lift.

3. Proof of Theorem 2.1. For l a positive integer we denote by $S^+_{l+1/2}$ the space of cusp forms of weight $l+1/2$ on the Hecke congruence subgroup $\Gamma_0(4)$ of level 4, having a Fourier expansion of the form

$$\sum_{m\geq 1,\,(-1)^l m\equiv 0,1\,(\mathrm{mod}\,4)} c(m)q^m$$

where $q = e^{2\pi iz}$ for $z \in \mathcal{H}$, the complex upper half-plane. For details we refer to [6], [11].

Let $l \geq 6$ if l is even and $l \geq 9$ if l is odd. Then one knows that the number of fundamental discriminants D_0 with $|D_0| < x$ and such that there exists a cusp form $h \in S^+_{l+1/2}$ whose $|D_0|$th Fourier coefficient does not vanish is

$$\geq \left(\frac{9}{16\pi^2} - \epsilon\right)x \quad (x \gg_\epsilon 1)$$

for every $\epsilon > 0$. This was proved in [7] for l even and later in [1] for l odd. Since $S^+_{l+1/2}$ has a basis of eigenforms for all Hecke operators, we can assume that h is a Hecke eigenform.

We now let $l = k-1$ with $k \geq 10$ even. In view of (1) and the above, to prove the theorem it is sufficient to prove the existence of a form in $S_k(\Gamma_2)$ whose nTth Fourier coefficient does not vanish, for all $T > 0$ with $D_T = D_0$ and all n not divisible by any of the first six primes, where D_0 is a fundamental discriminant such that there exists a Hecke eigenform $h \in S^+_{k-1/2}$ with $|D_0|$th coefficient nonzero.

Recall that one has a linear lifting map (*Maass lift*) from $S^+_{k-1/2}$ to $S_k(\Gamma_2)$ given explicitly on the level of Fourier coefficients by

$$h = \sum_{m\geq 1,\,(-1)^l m\equiv 0,3\,(\mathrm{mod}\,4)} c(m)q^m \mapsto \sum_{T>0} A_h(T)e^{2\pi i\,\mathrm{tr}(TZ)} \qquad (Z \in \mathcal{H}_2)$$

where

$$A_h(T) = \sum_{d\mid c_T} d^{k-1}c(|D_T|/d^2)$$

and c_T denotes the *content* of T, i.e. $c_T := \gcd(a,b,c)$ if $T = \left(\begin{smallmatrix} a & b/2 \\ b/2 & c \end{smallmatrix}\right)$ with a, b, c integers.

If $D_T = D_0$ is a fundamental discriminant and $n \in \mathbb{N}$, then clearly

$$A_h(nT) = \sum_{d|n} d^{k-1} c\left(|D_0|\frac{n^2}{d^2}\right). \tag{2}$$

Now let h be a Hecke eigenform. We let

$$f(z) = \sum_{n\geq 1} a(n) q^n \qquad (z \in \mathcal{H})$$

be the unique normalized Hecke eigenform of weight $2k-2$ on Γ_1 corresponding to h under the Shimura correspondence [6], [11].

Lemma 3.1. *For any $n \geq 1$, one has*

$$A_h(nT) = c(|D_0|) \prod_{p^\nu \| n} \left(\sum_{\mu=0}^{\nu} p^{(\nu-\mu)(k-1)} \left(a(p^\mu) - p^{k-2}\left(\frac{D_0}{p}\right) a(p^{\mu-1}) \right) \right).$$

Here $p^\nu \,\|\, n$ means that p^ν exactly divides n and we set $a(n) = 0$ if n is not an integer.

Proof. From (2) we conclude that

$$\sum_{n\geq 1} A_h(nT) n^{-s} = \zeta(s-k+1) \sum_{n\geq 1} c(|D_0|n^2) n^{-s},$$

and by [6] we have

$$\sum_{n\geq 1} c(|D_0|n^2) n^{-s} = c(|D_0|) \frac{L(f,s)}{L(s-k+2, \chi_{D_0})},$$

where $L(f,s)$ is the Hecke L-function of f and χ_{D_0} is the quadratic character attached to D_0.

Since the p-Euler factor of $L(f,s)$ is equal to

$$\sum_{\mu\geq 0} a(p^\mu) p^{-\mu s}$$

and that of $1/L(s-k+2, \chi_{D_0})$ is

$$1 - \left(\frac{D_0}{p}\right) p^{k-2-s}$$

our claim easily follows. ∎

By the lemma, we now see that it is enough to prove that

$$S_p(\nu) := \sum_{\mu=0}^{\nu} p^{(\nu-\mu)(k-1)} \left(a(p^\mu) - p^{k-2}\left(\frac{D_0}{p}\right) a(p^{\mu-1}) \right) \neq 0$$

for all $\nu \geq 0$ and for each prime $p > 13$. Assume on the contrary that

$S_p(\nu) = 0$ for some ν and p as above. Clearly then $\nu \geq 1$, and it follows that

$$-1 = \sum_{\mu=1}^{\nu} p^{-\mu(k-1)} \left(a(p^\mu) - p^{k-2} \left(\frac{D_0}{p} \right) a(p^{\mu-1}) \right). \tag{3}$$

We now invoke Deligne's bound

$$|a(p^\mu)| \leq (\mu+1) p^{\mu(k-3/2)}.$$

Taking absolute values in (3) we obtain

$$1 \leq \sum_{\mu=1}^{\nu} \left(\frac{\mu}{p^{\mu/2+1/2}} + \frac{\mu+1}{p^{\mu/2}} \right). \tag{4}$$

We put $\alpha := p^{-1/2}$ and then use the elementary estimates

$$\sum_{\mu=1}^{\nu} \mu\beta^\mu < \frac{\beta}{(1-\beta)^2}, \quad \sum_{\mu=1}^{\nu} \beta^\mu < \frac{\beta}{1-\beta} \qquad (0 < \beta < 1)$$

with $\beta = \alpha$. Then from (4) we deduce that

$$1 < \frac{\alpha(\alpha+1)}{(1-\alpha)^2} + \frac{\alpha}{1-\alpha}, \quad \text{i.e.,} \quad \alpha^2 - 4\alpha + 1 < 0,$$

which is equivalent to saying that $\alpha_1 < \alpha < \alpha_2$ where $\alpha_1 = 2 - \sqrt{3}$ and $\alpha_2 = 2+\sqrt{3}$ are the roots of the polynomial x^2-4x+1. This is a contradiction if $p > 13$, completing the proof of the theorem.

4. The excluded primes. If in the definition of $\mathcal{N}_k(x)$ we allowed that one of the primes $2, 3, 5, 7, 11$ or 13 divides n, then our arguments leading to the assertion of the theorem would break down, since we cannot use Deligne's bound to show that $S_p(\nu) \neq 0$ for all ν.

However, it is sometimes possible to show that $S_p(\nu) \neq 0$, for many ν, even for the excluded primes, if one invokes a different idea, namely congruence properties for the Hecke eigenvalues $a(n)$ of the normalized Hecke eigenform f, modulo small primes l, proved by Hatada [4]. Recall that the $a(n)$ are algebraic integers lying in an algebraic number field determined by f, and by a congruence modulo l we mean the corresponding congruence modulo the ideal (l) generated by l.

We want to illustrate this in the simplest case $l = 2$. According to [4] one has $a(p) \equiv 0 \pmod{(2)}$ whenever p is an odd prime. Reducing the generating series

$$\sum_{\mu \geq 0} a(p^\mu) X^\mu = \frac{1}{1 - a(p)X + p^{k-2}X^2}$$

modulo (2), we deduce that

$$a(p^{\mu}) \equiv \begin{cases} 0 \pmod{(2)} & \text{if } \mu \equiv 1 \pmod{2}, \\ 1 \pmod{(2)} & \text{otherwise.} \end{cases}$$

This easily implies that

$$S_p(\nu) \equiv \begin{cases} \nu + 1 \pmod{(2)} & \text{if } (p, D_0) = 1, \\ [\nu/2] + 1 \pmod{(2)} & \text{otherwise.} \end{cases}$$

Hence if n is odd and $p^{\nu} \parallel n$ implies that ν is even when $(p, D_0) = 1$ and that $[\nu/2]$ is even when $p \mid D_0$, then $\prod_{p^{\nu} \parallel n} S_p(\nu) \equiv 1 \pmod{(2)}$ for all odd $n = \prod_{p^{\nu} \parallel n} p^{\nu}$ and in particular this product is not zero.

Thus if one modifies the definition of $\mathcal{N}_k(x)$ appropriately, one can also formulate a corresponding assertion as in the theorem allowing any n as above (in particular divisible by $2, 3, 5, 7, 11$ or 13, under the given conditions).

5. Some remarks in higher degrees. In [5] Ikeda gave a generalization of the Maass lift in the case of Hecke eigenforms (usually called the Saito–Kurokawa lift in this case) to higher degrees. More specifically, whenever $l \equiv g \pmod 2$, starting with a normalized Hecke eigenform f of weight $2l$ on Γ_1 he constructs a Siegel–Hecke eigenform F of weight $l+g$ and degree $2g$ whose standard zeta function (up to a Riemann zeta function) is the product of shifted Hecke L-functions of f. Moreover, the Fourier coefficients of F are given by a complicated product expression, involving the Fourier coefficients of a Hecke eigenform $h \in S^{+}_{l+1/2}$ corresponding to f under the Shimura correspondence, and a finite product over primes p of modified local singular (Laurent) polynomials at p, evaluated at the p-Satake parameters of f.

Later on in [8], a linear version of the Ikeda lift was given, as a linear map from $S^{+}_{l+1/2}$ to $S_{l+g}(\Gamma_{2g})$, the space of Siegel cups forms of weight $l+g$ on $\Gamma_{2g} = \mathrm{Sp}_{2g}(\mathbb{Z})$. If $g = 2$, it was proved that the formulas given (after replacing l by $k-1$) coincide with those giving the Maass lift in Section 3. We will denote the Fourier coefficients of the lift of $h \in S^{+}_{l+1/2}$ by $A_{h,2g}(T)$ ($T > 0$ of size $2g$). In particular then $A_{h,2}(T) = A_h(T)$ in previous notation.

Now suppose that $g \equiv 1 \pmod 4$. Fix a positive definite even integral unimodular matrix T_0 of size $2g-2$. (Note that $2g-2 \equiv 0 \pmod 8$ and therefore such a matrix exists, as is well-known.) Then it was proved in [9] that

$$A_{h,2g}\left(\mathcal{T} \oplus \frac{1}{2} T_0\right) = A_{h,2}(\mathcal{T})$$

for any $\mathcal{T} > 0$ of size 2.

From this equality one can therefore obtain nonvanishing results for Siegel–Poincaré series of index $n\mathcal{T} \oplus \frac{1}{2} T_0$ in degree $2g$ (under the given conditions), using our previous results.

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