

Khinchin's theorem in k dimensions with prime numerator and denominator

by

HUW JONES (Cardiff)

1. Introduction. This paper is devoted to the proof of the following result.

THEOREM. *Let $\psi_1(n), \dots, \psi_k(n)$ be non-increasing, positive functions of a positive integer variable n . Then the simultaneous inequalities given by:*

$$(1.1) \quad \left| \alpha_i - \frac{q_i}{p} \right| < \frac{\psi_i(p)}{p}$$

for $1 \leq i \leq k$ have infinitely many or finitely many solutions in primes p, q_i for almost all $\alpha = (\alpha_1, \dots, \alpha_k)$ in \mathbb{R}^k , according to whether the sum

$$(1.2) \quad \sum_{n=2}^{\infty} \frac{\psi_1(n) \dots \psi_k(n)}{(\log n)^{k+1}}$$

diverges or converges respectively.

In the English translation of "Continued Fractions" [7] by A. Ya. Khinchin the theorem to which the title refers is Theorem 32 on page 69. It can be stated as follows:

Suppose that $\psi(x)$ is a positive continuous function of a positive variable x and that $x\psi(x)$ is a non-increasing function. Then the inequality

$$\left| \alpha - \frac{m}{n} \right| < \frac{\psi(n)}{n}$$

has, for almost all α , an infinite number of solutions in integers m and n (with $n > 0$) if $\sum_{n=1}^{\infty} \psi(n)$ diverges. If on the other hand the sum converges then for almost all α there are only a finite number of solutions to the inequality.

This elegant theorem was proved by Khinchin in the context of continued fractions and using results of Borel and Bernstein. It has been the subject

of important work by Duffin and Schaffer and in their 1941 paper [2] the condition on $\psi(x)$ is considerably weaker in that merely $\psi(x)$ itself need be non-increasing. The question of the necessary and sufficient condition on $\psi(x)$ was investigated further by Duffin and Schaffer and is the exact subject of the Duffin–Schaffer conjecture. See Chapter 2 of [5] for a treatment of this question.

Glyn Harman has considered the question of restricting the integers m and n to sets of number theoretic interest and these results can be found in Chapter 6 of [5]. In [4] both m and n are restricted to primes and it is in that paper that our present theorem is proved in its one-dimensional form.

We have two things to show in the proof of our theorem; firstly that if the sum (1.2) converges then there are finitely many solutions to the inequalities for almost all $\alpha \in \mathbb{R}^k$ and secondly that if the sum diverges then for almost all $\alpha \in \mathbb{R}^k$ there are infinitely many solutions. The first implication is proved quite easily using the Borel–Cantelli Lemma and it is the second implication that takes by far the most work.

DEFINITIONS. Throughout the paper, p, q, r and s , with or without subscripts, will be primes (either positive or negative) unless stated otherwise.

For $\mathbf{x} \in \mathbb{R}^k$ let

$$|\mathbf{x}| = \max_{1 \leq i \leq k} (|x_i|).$$

Define $D_p \subseteq \mathbb{R}^k$ by

$$D_p = \bigcup_{\substack{s_i \neq p \\ 1 \leq i \leq k}} \left(\frac{s_1 - \psi_1(p)}{p}, \frac{s_1 + \psi_1(p)}{p} \right) \times \dots \times \left(\frac{s_k - \psi_k(p)}{p}, \frac{s_k + \psi_k(p)}{p} \right),$$

and

$$V(N) = \sum_{2 \leq p \leq N} \frac{\psi_1(p) \dots \psi_k(p)}{(\log p)^k}.$$

Also, let

$$\mathcal{D} = \bigcap_{q=2}^{\infty} \bigcup_{p=q}^{\infty} D_p$$

that is, the set of points $\alpha \in \mathbb{R}^k$ belonging to infinitely many of the D_p .

Throughout the proof we assume the Prime Number Theorem, the result that the divergence of (1.2) is equivalent to the divergence of

$$\sum_{p=2}^{\infty} \frac{\psi_1(p) \dots \psi_k(p)}{(\log p)^k}$$

and that $\psi_i(n) \leq 1/2$ for each i and for all n . Furthermore we will require the following results:

RESULT 1 (First Borel–Cantelli Lemma). *If $\{A_n\}$ is a sequence of measurable sets in a measure space whose measures have a finite sum, then the set of points that are in an infinite number of the given sets has measure zero.*

RESULT 2. *Let V be an open, Lebesgue-measurable subset of \mathbb{R}^k . Let (D_n) be a sequence of measurable subsets of V . Also, let $\mathcal{D} = \bigcap_{m=1}^\infty \bigcup_{n=m}^\infty D_n$, i.e. the set of points of V belonging to infinitely many of the D_n . Suppose that for every open cube $C \subseteq V$, we have*

$$(1.3) \quad \sum_{n=1}^\infty \lambda^k(B_n) = \infty,$$

$$(1.4) \quad \limsup_{N \rightarrow \infty} \left(\sum_{n=1}^N \lambda^k(B_n) \right)^2 \left(\sum_{1 \leq m, n \leq N} \lambda^k(B_m \cap B_n) \right)^{-1} \geq \delta \lambda^k(C),$$

where $B_n = C \cap D_n$ and $\delta > 0$ is a constant independent of C . Then $\lambda^k(V \setminus \mathcal{D}) = 0$.

For proof of this result see Section 4.

RESULT 3. *Let (\mathbf{x}_n) be a sequence of N points in \mathbb{R}^k . Let B be a box in $U_k \subseteq \mathbb{R}^k$ with side lengths β_1, \dots, β_k . Then for $|\theta_i| \leq 1$ ($1 \leq i \leq k$) and positive integer L ,*

$$\begin{aligned} & \sum_{\substack{n=1 \\ \mathbf{x}_n \in B + \mathbb{Z}^k}}^N 1 \\ &= \sum_{n=1}^N \left(\lambda(B) + 1 \cdot \frac{\theta_1 \mu_1(B)}{L+1} + 3 \cdot \frac{\theta_2 \mu_2(B)}{(L+1)^2} + \dots + (2k-1) \frac{\theta_k \mu_k(B)}{(L+1)^k} \right) \\ &+ \sum_{0 < |\mathbf{m}| \leq L} c(\mathbf{m}) \left| \sum_{n=1}^N e(\mathbf{m} \cdot \mathbf{x}_n) \right|, \end{aligned}$$

where

$$\mu_j(B) = \sum_{\substack{1 \leq i_a \leq k \\ i_b \neq i_c}} \beta_{i_1} \dots \beta_{i_{k-j}},$$

so e.g. $\mu_1(B) = \beta_1 \beta_2 \dots \beta_{k-1} + \dots + \beta_2 \beta_3 \dots \beta_k$, i.e. $\mu_j(B)$ is the $(k-j)$ -dimensional perimeter of B , and with

$$\begin{aligned} \mathbf{m} &= (m_1, \dots, m_k), \quad m_i \in \mathbb{Z}, \\ c(\mathbf{m}) &= k \prod_{j=1}^k \left(\frac{1}{L+1} + \min \left(\frac{1}{\pi |m_j|}, \beta_j, 1 - \beta_j \right) \right). \end{aligned}$$

Proof. This is a k -dimensional generalisation of Lemma 5.1 of [4].

For the first implication we assume that the sum

$$\sum_{p=2}^{\infty} \frac{\psi_1(p) \dots \psi_k(p)}{(\log p)^k}$$

converges. Without loss of generality, we will show that the set of α in $(\mathbb{R}^+)^k$ for which there are infinitely many solutions to (1.1) has measure zero. Let $C_{\mathbf{M}} = [M_1, M_1 + 1] \times \dots \times [M_k, M_k + 1]$ for some vector $\mathbf{M} = (M_1, \dots, M_k) \in \mathbb{Z}^k$ and let $B_p = C_{\mathbf{M}} \cap D_p$, where D_p is as in the definitions. We have

$$B_p = C_{\mathbf{M}} \cap \left(\bigcup_{\substack{s_i \neq p \\ 1 \leq i \leq k}} \left(\frac{s_1 - \psi_1(p)}{p}, \frac{s_1 + \psi_1(p)}{p} \right) \times \dots \times \left(\frac{s_k - \psi_k(p)}{p}, \frac{s_k + \psi_k(p)}{p} \right) \right),$$

and we can say that

$$\sum_{p=2}^{\infty} \lambda^k(B_p) \leq \sum_{p=2}^{\infty} \frac{2^k \psi_1(p) \dots \psi_k(p)}{p^k} \sum_{s_1 \sim p} \dots \sum_{s_k \sim p} 1,$$

where $s_i \sim p$ denotes $pM_i < s_i < p(M_i + 1)$. Hence using the Brun–Titchmarsh inequality for the number of primes in an interval of length p we can say that the sum in question is

$$\leq C(k) \sum_{p=2}^{\infty} \frac{\psi_1(p) \dots \psi_k(p)}{(\log p)^k}.$$

Since the sum above converges the sum of the measures of the B_p is finite and so by the First Borel–Cantelli Lemma (Result 1) we can conclude that the set of α contained in infinitely many B_p has measure zero. Now the set of α contained in infinitely many D_p is just the union over all \mathbf{M} of those contained in infinitely many $C_{\mathbf{M}} \cap D_p$ and since a countable union of null sets is itself null we can conclude that this too is a null set. From this the result follows.

2. Construction. We now assume that

$$\sum_{p=2}^{\infty} \frac{\psi_1(p) \dots \psi_k(p)}{(\log p)^k}$$

diverges. We will show that the sequence (D_p) of measurable sets satisfy the conditions of Result 2 with $V = (\mathbb{R}^+)^k$. We will then be able to conclude that the set of α contained in infinitely many of the D_p has full measure, which is equivalent to the assertion that for almost all α there are infinitely

many solutions to the simultaneous inequalities (1.1). This then proves the theorem. It follows from the Prime Number Theorem that if (1.2) diverges then

$$\sum_{p=2}^{\infty} \lambda^k(D_p \cap C) = \infty,$$

where C is any cube in $(\mathbb{R}^+)^k$. Thus (1.3) holds, and this is the first condition of Result 2.

We now need to show that the second condition of Result 2 holds. Thus we show that given any $C \subseteq (\mathbb{R}^+)^k$, we have

$$(2.1) \quad \limsup_{N \rightarrow \infty} \left(\sum_{p=2}^N \lambda^k(D_p \cap C) \right)^2 \left(\sum_{2 \leq p, q \leq N} \lambda^k(D_p \cap D_q \cap C) \right)^{-1} \geq \delta \lambda^k(C),$$

where $\delta > 0$ is a constant independent of the choice of C . To show this we prove the following lemma.

LEMMA 1. *For $C \subseteq (\mathbb{R}^+)^k$ we have*

$$(2.2) \quad \lim_{N \rightarrow \infty} \frac{1}{V(N)} \sum_{p=2}^N \lambda^k(D_p \cap C) = 2^k \lambda^k(C)$$

and

$$(2.3) \quad \limsup_{N \rightarrow \infty} \frac{1}{(V(N))^2} \sum_{2 \leq p, q \leq N} \lambda^k(D_p \cap D_q \cap C) \leq \Delta \lambda^k(C),$$

where $\Delta > 0$ is independent of the choice of C .

Proof. For ease of notation we prove the lemma for cubes C of the form $(1, 1 + \theta)^k$ with $\theta \leq 1/2$ rather than for cubes with a corner at some general point $\boldsymbol{\eta}$ of $(\mathbb{R}^+)^k$. Also, note that Δ will be used to denote a positive real, independent of the cube C , and will not necessarily be the same at every occurrence. To show (2.2) note that

$$\sum_{p=2}^N \lambda^k(D_p \cap C) = \sum_{p=2}^N 2^k \frac{\psi_1(p) \cdots \psi_k(p)}{p^k} \sum_{s_1 \sim p} \cdots \sum_{s_k \sim p} 1$$

where $s_i \sim p$ denotes $p < s_i < p(1 + \theta)$. By the Prime Number Theorem, this is

$$\sum_{p=2}^N 2^k \frac{\psi_1(p) \cdots \psi_k(p)}{p^k} \cdot \frac{\theta^k p^k}{(\log p)^k} (1 + o(1))$$

and this is just $2^k \theta^k V(N) + o(V(N))$ as required.

For (2.3) note that since

$$\sum_{2 \leq p=q \leq N} \lambda^k(D_p \cap D_q \cap C) = 2^k \theta^k V(N) + o(V(N))$$

and the right-hand side above is $o(V(N))^2$, we need only show that

$$(2.4) \quad \sum_{2 \leq p < q \leq N} \lambda^k(D_p \cap D_q \cap C) \leq \Delta \lambda^k(C) (V(N))^2 + o(V(N))^2.$$

We will denote the sum on the left-hand side above by $S(N)$. Now D_q is a union of cubes each having a measure of

$$2^k \frac{\psi_1(q) \cdots \psi_k(q)}{q^k}.$$

Since this is also the maximum measure of the intersection of a cube in D_p and a cube in D_q , where $p < q$, we may write an upper bound on $\lambda^k(D_p \cap D_q \cap C)$ by bounding the number of such intersections and counting them with this weight. Suppose that

$$\left(\frac{s_1 - \psi_1(p)}{p}, \frac{s_1 + \psi_1(p)}{p} \right) \times \cdots \times \left(\frac{s_k - \psi_k(p)}{p}, \frac{s_k + \psi_k(p)}{p} \right)$$

intersects with

$$\left(\frac{s_1 - \psi_1(q)}{q}, \frac{s_1 + \psi_1(q)}{q} \right) \times \cdots \times \left(\frac{s_k - \psi_k(q)}{q}, \frac{s_k + \psi_k(q)}{q} \right) :$$

A necessary condition for this is given by

$$(2.5) \quad 0 < |r_i p - s_i q| < 2q\psi_i(p) \quad \text{for } i = 1, \dots, k;$$

i.e. that the centres of the two cubes are sufficiently close for intersection to occur. (Note that we need not consider equality with zero in the above expression since we are assuming here that $p \neq q$ and in our definition of D_p , $s_i \neq p$.) Since we are only counting intersections which lie in C the centres of both cubes must certainly lie in C and so we have the condition $\mathbf{r} \sim q$, meaning that $q < r_i < q(1 + \theta)$ for each i between 1 and k , and $\mathbf{r} \sim p$ similarly. This argument gives the following sum as an upper bound:

$$S(N) \leq \sum_{2 \leq p < q \leq N} 2^k \frac{\psi_1(q) \cdots \psi_k(q)}{q^k} \sum_{\mathbf{r} \sim q} \sum_{\mathbf{s} \sim p}^* 1,$$

the asterix denoting the summation condition given in (2.5). We now split up the range $2 \leq p < q \leq N$ into subranges by defining parameters P and Q as taking values $2(1 + \theta)^n$ in the range $(2, N(1 + \theta))$. Using the “ \sim ” notation introduced above we can rewrite the sum as

$$(2.6) \quad S(N) \leq \sum_{2 \leq P \leq Q \leq N} 2^k \frac{\psi_1(Q) \cdots \psi_k(Q)}{Q^k} \sum_{\substack{\mathbf{r}: Q \\ q \sim Q}} \sum_{\substack{\mathbf{s}: P \\ p \sim P}}^* 1.$$

Here, $\mathbf{r} : Q$ denotes $Q < r_i < Q(1 + \theta)^2$ for $i = 1, \dots, k$ and the asterisk denotes the following conditions:

$$0 < |r_i p - s_i q| < 3Q\psi_i(P) \quad \text{for } i = 1, \dots, k.$$

Now define $S(P, Q)$ to be the double sum part of (2.6). $S(P, Q)$ counts the number of solutions to a set of k inequalities involving integers alone and the proof of (2.3) and hence of the main theorem now rests on proving a suitable size upper bound for this.

3. Main lemma

LEMMA 2. *For $Q \geq P \geq \theta^{-2}$ there exists a small, positive real c (dependent only on the ψ functions), such that*

$$S(P, Q) \leq \frac{\Delta \theta^{k+2} P Q H_1 \dots H_k}{(\log Q)^{k+1} (\log P)^{k+1}} + O(H_1 \dots H_k P Q^{1-c}).$$

Here Δ is a positive constant dependent only on the dimension k and $H_i = 3Q\psi_i(P)$.

The criterion of Lemma 2 is that $Q \geq P \geq \theta^{-2}$; when this does not hold either (1) $Q < \theta^{-2}$ or (2) $P \leq \theta^{-2} < Q$. In either case we have, trivially,

$$S(P, Q) \leq \sum_{p \sim P} \sum_{q \sim Q} \sum_{\mathbf{r}: Q} \sum_{\mathbf{s}: P} 1.$$

In case (1) this is just $O(1)$ with respect to N while in case (2) it is

$$\ll \frac{Q^{k+1}}{(\log P)^{k+1} (\log Q)^{k+1}}.$$

Hence by (2.6), the contribution this gives to $S(N)$ has size

$$\ll \sum_{\substack{2 \leq P \leq Q \leq N \\ P \leq \theta^{-2}}} \frac{Q\psi_1(Q) \dots \psi_k(Q)}{(\log P)^{k+1} (\log Q)^{k+1}},$$

which is $O(V(N))$. By this and Lemma 2 then, $S(N)$ is

$$\leq \Delta \theta^k \sum_{2 \leq P \leq Q \leq N} \frac{P Q \theta^2 \psi_1(P) \dots \psi_k(P) \psi_1(Q) \dots \psi_k(Q)}{(\log P)^{k+1} (\log Q)^{k+1}} + o(V(N))^2$$

and this is

$$\leq \Delta \lambda^k (C)(V(N))^2 + o(V(N))^2.$$

Hence (2.4) holds, which directly implies (2.3), thus proving the theorem.

Let \mathcal{A} be a set of positive integers, Y, X positive reals and h an even positive integer. Put

$$R_d = \sum_{\substack{n \in \mathcal{A} \\ d|n}} 1 - \frac{\omega(d)}{d} X.$$

Then the number of members of \mathcal{A} having all their prime factors either dividing h or greater than Y is

$$\leq \frac{\Delta X}{(\log Y)^\kappa} \left(\frac{h}{\phi(h)} \right)^\kappa + O\left(\sum_{\substack{d \leq Y \\ (d,h)=1}} \mu^2(d) d^\varepsilon |R_d| \right).$$

Here $\mu(d)$ is the Möbius function, $\phi(h)$ is the Euler totient function, ε is an arbitrary positive real and Δ is a positive constant.

Proof. This is a special case of Theorem (5.2) of [3].

These sieve results will in fact give an upper bound on the number of solutions to the inequalities in integers s_i, r_i with prime factors larger than a given size so in this section s_i, r_i are not necessarily primes.

We require one more auxiliary lemma before beginning the proof of Lemma 2.

LEMMA 4. Let $U, V \geq 1$ be real and t, w integers with $(t, w) = 1$ and $V \leq 3w$. Then for any $\varepsilon > 0$, we have

$$\sum_{1 \leq u \leq U} \left| \sum_{\substack{1 \leq v \leq V \\ (v,w)=1}} e\left(\frac{\bar{v}ut}{w}\right) \right| \ll U w^{1/2+\varepsilon}.$$

Here \bar{v} denotes a solution of $xv \equiv 1 \pmod{w}$.

Proof. See Lemmas 6 and 7 in Chapter 2 of [6].

Proof of Lemma 2. CASE 1: $P < Q^{1/4}$. Fix s_1, p and q in the first equation, then re-arranging the equation shows that the number of solutions is

$$\pi\left(\frac{s_1q}{p} + \frac{H_1}{p}\right) - \pi\left(\frac{s_1q}{p} - \frac{H_1}{p}\right).$$

By the Brun–Titchmarsh inequality this is

$$\leq \frac{4H_1}{p \log(H_1/p)}.$$

Now since $P < Q^{1/4}$ and $H_1 \gg Q/P$ this is

$$\ll \frac{4H_1}{P \log Q}.$$

Since we get a similar bound by fixing s_2 in the second equation, the total number of solutions is

$$\ll \sum_{p \sim P} \sum_{q \sim Q} \sum_{s_1 : P} \sum_{s_2 : P} \frac{H_1}{P \log Q} \cdot \frac{H_2}{P \log Q},$$

which is

$$\leq \frac{\Delta \theta^4 P Q H_1 H_2}{(\log Q)^3 (\log P)^3},$$

using the fact that $P\theta \geq P^{1/2}$ and $Q\theta \geq Q^{1/2}$. This proves Case 1 of the lemma.

CASE 2: $P \geq Q^{1/4}$ and there exists $\delta > 0$ such that $\psi_i(P) \gg P^{\delta-1}$, $i = 1, 2$. We fix q and apply Lemma 3a with $\kappa = 4$ and with

$$\mathcal{A} = \{r_1 r_2 s_1 s_2 \mid |r_i p - s_i q| < H_i, r_i : Q, s_i : P \text{ and } p \sim P\}.$$

Note that p is prime here. In order to estimate the size of R_d we need to look at the size of

$$\mathcal{A}_d = \{n \in \mathcal{A} \mid d|n\}.$$

For this we will obtain an asymptotic formula using the Fourier analysis Result 3. Writing a member of \mathcal{A}_d in the form

$$(r_1 x_1)(r_2 x_2)(s_1 y_1)(s_2 y_2)$$

where $x_1 x_2 y_1 y_2 = d$, we can obtain the number of elements in \mathcal{A}_d by counting solutions to

$$\left\| \frac{r_1 x_1 p}{q y_1} \right\| < \frac{H_1}{q y_1}, \quad \left\| \frac{r_2 x_2 p}{q y_2} \right\| < \frac{H_2}{q y_2},$$

where $\| \cdot \|$ denotes distance to the nearest integer. Using Result 3 with

$$B = \left(-\frac{H_1}{q y_1}, \frac{H_1}{q y_1} \right) \times \left(-\frac{H_2}{q y_2}, \frac{H_2}{q y_2} \right)$$

we get

$$\sum_{n \in \mathcal{A}_d} 1 = \sum_{x_1 x_2 y_1 y_2 = d} \left(\sum_{r_1, r_2, p} \left(\frac{4H_1 H_2}{y_1 y_2 q^2} + \frac{\theta_1}{L+1} \left(\frac{2H_1}{y_1 q} + \frac{2H_2}{y_2 q} \right) + \frac{3\theta_2}{(L+1)^2} \right) + \theta_3 E_3 \right).$$

In the above sum, r_1, r_2 and p lie in the following ranges:

$$r_1 : \left(\frac{Q}{x_1} \right), \quad r_2 : \left(\frac{Q}{x_2} \right), \quad p \sim P.$$

We let E_3 denote the exponential sum part of Result 3 and we will deal with this presently. Of the three summands within the r_1, r_2, p sum it is the first which will form the main term and the other terms we will label E_1 and E_2 respectively. The main term then is

$$\sum_{x_1 x_2 y_1 y_2 = d} \frac{4H_1 H_2}{y_1 y_2 q^2} \left(\frac{\theta(2 + \theta)Q}{x_1} + O(1) \right) \left(\frac{\theta(2 + \theta)Q}{x_2} + O(1) \right) \pi(P, \theta)$$

where $\pi(P, \theta)$ denotes the number of primes in the interval $(P, P(1 + \theta))$. The above is

$$\frac{4H_1 H_2 \theta^2 (2 + \theta)^2 Q^2 \pi(P, \theta)}{q^2} \sum_{x_1 x_2 y_1 y_2 = d} \frac{1}{x_1 x_2 y_1 y_2} + O\left(\frac{H_1 H_2 P Q}{q^2} \sum_{x_1 x_2 y_1 y_2 = d} \frac{1}{x_1 y_1 y_2} \right)$$

and can be written as

$$\frac{\omega(d)}{d} X + O\left(\frac{H_1 H_2 P d^\epsilon}{q} \right).$$

Hence we can say that

$$(3.2) \quad |R_d| = O\left(\frac{H_1 H_2 P d^\epsilon}{q} \right) + E_1 + E_2 + E_3.$$

Now letting $L = Q$, we have

$$E_1 = \sum_{x_1 x_2 y_1 y_2 = d} \sum_{r_1, r_2, p} \frac{\theta_1}{L + 1} \left(\frac{2H_1}{y_1 q} + \frac{2H_2}{y_2 q} \right).$$

This has order no larger than $(H_1 + H_2) P d^\epsilon$, which in turn is

$$(3.3) \quad \ll \frac{H_1 H_2 P d^\epsilon}{Q^\delta},$$

since we have assumed that there exists $\delta > 0$ such that $\psi_i(P) \gg P^{\delta-1}$.

The next error term, E_2 , is

$$\sum_{x_1 x_2 y_1 y_2 = d} \sum_{r_1, r_2, p} \frac{3\theta_2}{(L + 1)^2}.$$

This can easily be seen to have order no worse than

$$(3.4) \quad \frac{H_1 H_2 P d^\epsilon}{Q}.$$

The exponential sum term, E_3 , is

$$\sum_{x_1 x_2 y_1 y_2 = d} \sum_{0 < |\mathbf{m}| \leq L} c(\mathbf{m}) \left| \sum_{r_1, r_2, p} e\left(\frac{m_1 r_1 x_1 p}{y_1 q} + \frac{m_2 r_2 x_2 p}{y_2 q} \right) \right|.$$

Here $c(\mathbf{m})$ (see Result 3) is

$$2 \prod_{j=1}^2 \left(\frac{1}{L + 1} + \min \left(\frac{1}{\pi |m_j|}, \beta_j, 1 - \beta_j \right) \right).$$

We are taking

$$\beta_j = \frac{2H_j}{qy_j}$$

and the order of $c(\mathbf{m})$ will depend on the size of $|m_j|$ relative to this. Note that we have already set $L = Q$. Summing over r_1 and r_2 we can say that E_3 is

$$\leq \sum_{x_1 x_2 y_1 y_2 = d} \sum_{0 < |\mathbf{m}| \leq Q} \sum_{p \sim P} c(\mathbf{m}) \min\left(\frac{Q}{x_1}, \frac{1}{\left\| \frac{m_1 x_1 p}{qy_1} \right\|}\right) \min\left(\frac{Q}{x_2}, \frac{1}{\left\| \frac{m_2 x_2 p}{qy_2} \right\|}\right).$$

The largest term in the above expression arises when m_1 or m_2 are zero. Without loss of generality, let $m_2 = 0$; then we have a term which is

$$\ll \sum_{x_1 x_2 y_1 y_2 = d} \sum_{1 \leq |m_1| \leq Q} \sum_{p \sim P} \min\left(\frac{H_1}{q}, \frac{1}{|m_1|}\right) \frac{H_2}{qy_2} \cdot \frac{Q}{x_2} \cdot \frac{1}{\left\| \frac{m_1 x_1 p}{qy_1} \right\|}.$$

Splitting the range of m_1 into two sections shows that this is

$$\begin{aligned} &\ll H_2 \sum_{x_1 x_2 y_1 y_2 = d} \frac{1}{x_2 y_2} \sum_{\substack{1 \leq |m_1| \leq Q/H_1 \\ p \sim P}} \frac{H_1}{Q} \cdot \frac{1}{\left\| \frac{m_1 x_1 p}{qy_1} \right\|} \\ &+ H_2 \sum_{x_1 x_2 y_1 y_2 = d} \frac{1}{x_2 y_2} \sum_{\substack{Q/H_1 \leq |m_1| \leq Q \\ p \sim P}} \frac{1}{|m_1|} \cdot \frac{1}{\left\| \frac{m_1 x_1 p}{qy_1} \right\|}. \end{aligned}$$

Combining m_1 with p to give a single variable a we can say that this is

$$\begin{aligned} &\ll \frac{H_1 H_2}{Q} \sum_{x_1 x_2 y_1 y_2 = d} \frac{1}{x_2 y_2} \sum_{P \leq a < 2PQ/H_1} \frac{1}{\left\| \frac{ax_1}{qy_1} \right\|} \\ &+ \frac{H_1 H_2}{Q} \sum_{x_1 x_2 y_1 y_2 = d} \frac{1}{x_2 y_2} \sum_{k=0}^{2 \log Q} \frac{1}{2^k} \sum_{2^k PQ/H_1 \leq a < 2^{k+1} PQ/H_1} \frac{1}{\left\| \frac{ax_1}{qy_1} \right\|}. \end{aligned}$$

This in turn is

$$\begin{aligned} &\ll \frac{H_1 H_2}{Q} \sum_{x_1 x_2 y_1 y_2 = d} \frac{1}{x_2 y_2} \left(\frac{2P}{H_1} + 1\right) Q \log Q \\ &+ \frac{H_1 H_2}{Q} \sum_{x_1 x_2 y_1 y_2 = d} \frac{1}{x_2 y_2} \sum_{k=0}^{2 \log Q} \frac{1}{2^k} \left(\frac{2^k P}{H_1} + 1\right) Q \log Q, \end{aligned}$$

which is

$$\ll H_2 P d^\varepsilon \log^2 Q + H_1 H_2 d^\varepsilon \log Q.$$

Now since $\psi_1(P) \gg P^{\delta-1}$, this is

$$(3.5) \quad \ll \frac{H_1 H_2 P d^\varepsilon}{Q^\beta} (\log^2 Q),$$

where $\beta = \min(\delta, 1/4)$. The remaining part of E_3 has m_1 and m_2 both non-zero and for this part we start by noting that for any fixed p ,

$$\sum_{1 \leq |m_2| \leq Q} \frac{1}{\left\| \frac{m_2 x_2 p}{y_2 q} \right\|} \ll Q y_2 \log Q.$$

Using this bound for the m_2 part of the double exponential sum (having m_1 and m_2 both non-zero) shows that the sum is no more than

$$Q \log Q \sum_{x_1 x_2 y_1 y_2 = d} y_2 \sum_{1 \leq m_1 \leq Q} \frac{H_2}{q} \min \left(\frac{H_1}{q}, \frac{1}{|m_1|} \right) \sum_{p \sim P} \frac{1}{\left\| \frac{m_1 x_1 p}{y_1 q} \right\|}.$$

As before we can now combine the two variables m_1 and p to give a single variable a to get our bound. This is straightforward for $|m_1| \leq q/H_1$ and uses a splitting up argument for m_1 in the range $q/H_1 < |m_1| < Q$. Together these give a bound no more than

$$(3.6) \quad \frac{H_1 H_2 P d^{1+\varepsilon}}{Q^\beta} (\log^3 Q).$$

Looking at (3.2) and the bounds we have obtained for E_1 , E_2 and E_3 in (3.3)–(3.6) respectively, we can say that

$$(3.7) \quad |R_d| \ll \frac{H_1 H_2 P d^{1+\varepsilon}}{Q^\beta} (\log^2 Q).$$

This means that the error term of the sieve is

$$\sum_{d \leq Y} \mu^2(d) d^\varepsilon |R_d| \leq \frac{\Delta H_1 H_2 P Y^{2+\varepsilon}}{Q^{\beta-\eta}},$$

for any small $\eta > 0$, thus accounting for the $\log^2 Q$ term in (3.7). Let $Y = Q^\alpha$. Then taking $\alpha = \eta = \varepsilon = \beta/4$ makes this term

$$\leq \Delta \frac{H_1 H_2 P}{Q^c},$$

where $c = \beta/8$ say. By Lemma 3 the size of \mathcal{A} is

$$\leq \frac{\Delta X}{(\log Q)^4} + O\left(\frac{H_1 H_2 P}{Q^c}\right).$$

So summing over q we can say that the $S(P, Q)$ of Lemma 2 is

$$\leq \frac{\Delta \pi(P, \theta) H_1 H_2 \theta^2 (2 + \theta)^2 Q^2}{(\log Q)^4} \sum_{q \sim Q} \frac{1}{q^2} + O(H_1 H_2 P Q^{1-c}),$$

which is

$$\leq \frac{\Delta P \theta^3}{(\log P)} \cdot \frac{H_1 H_2}{(\log Q)^4} \cdot \frac{\theta Q}{(\log Q)} + O(H_1 H_2 P Q^{1-c}).$$

This completes Case 2 of the proof of Lemma 2.

CASE 3: $P \geq Q^{1/4}$ and there exists i such that $\psi_i(P) \ll P^{\delta-1}$ for all $\delta > 0$. Without loss of generality assume $\psi_1(P) \ll P^{\delta-1}$ for all $\delta > 0$ (and consequently that there exists $\delta' > 0$ such that $\psi_2(P) \gg P^{\delta'-1}$). The number of solutions to the equations for those $h_1 = |r_1p - s_1q|$ with $p | h_1$ is $\ll H_1H_2P \ll H_1H_2PQ^{3/4}$. For other values of h_1 we fix h_1 and p and apply the sieve result, Lemma 3b, with $\kappa = 5$ and with

$$\mathcal{A} = \{r_1r_2s_1s_2q \mid r_1p - s_1q = h_1, \ |r_2p - s_2q| < H_2, \ s_i : P \text{ and } q \sim Q\}.$$

Note that here we are including q in the sieve so we are not restricting q to being prime. We want to obtain a suitable bound on the size of

$$\mathcal{A}_d = \{n \in \mathcal{A} \mid d|n\}.$$

For squarefree d this is equivalent to the number of solutions to

$$\begin{aligned} x_1x_2y_1y_2z &= d, & x_1, \dots, z &\geq 1, \\ r_1x_1p - s_1y_1qz &= h_1 & \text{and} & \quad |r_2x_2p - s_2y_2qz| < H_2. \end{aligned}$$

Since we are considering d with $(h_1, d) = 1$ in this sieve, we also have $(h_1, x_1p) = 1$ and consequently $(s_1y_1z, x_1p) = 1$. Thus the above is equivalent to

$$\begin{aligned} x_1x_2y_1y_2z &= d, & x_1, \dots, z &\geq 1, \\ \overline{s_1y_1z}h_1 + q &\equiv 0 \pmod{x_1p} & \text{and} & \quad s_2y_2qz + h_2 \equiv 0 \pmod{x_2p}. \end{aligned}$$

Here the \bar{a} notation denotes the inverse of a modulo x_1p . We now convert the congruences to exponential sums and say that the number of solutions to the above is equal to

$$\sum_{x_1x_2y_1y_2z=d} \frac{1}{p^2x_1x_2} \sum_{\substack{0 \leq |m_1| \leq px_1/2 \\ 0 \leq |m_2| \leq px_2/2}} \sum_{q \sim Q/z} \sum_{\substack{s_1: P/y_1 \\ (s_1, px_1)=1}} \sum_{\substack{s_2: P/y_2 \\ 0 < |h_2| \leq H_2}} S$$

(with the convention that the value $m_i = -px_i/2$ is omitted from the summation if px_i is even), and where

$$S = e\left(\frac{m_1(\overline{s_1y_1z}h_1 + q)}{x_1p}\right) e\left(\frac{m_2(s_2y_2zq + h_2)}{x_2p}\right).$$

The part of this sum with $m_1 = m_2 = 0$ is our main term and is equal to

$$(3.8) \quad X \frac{\omega(d)}{d} + O\left(\frac{H_2Qd^\varepsilon}{p}\right),$$

where $X = 2H_2P^2Q\theta^3(2 + \theta)^2/p^2$. The remainder term in the sieve, R_d , is thus

$$(3.9) \quad \sum_{x_1 x_2 y_1 y_2 z = d} \frac{1}{p^2 x_1 x_2} \sum_{\substack{0 \leq |m_1| \leq px_1/2 \\ 0 \leq |m_2| \leq px_2/2 \\ m_1, m_2 \text{ not both } 0}} \sum_{q \sim Q/z} \sum_{\substack{s_1: P/y_1 \\ (s_1, px_1) = 1}} \sum_{\substack{s_2: P/y_2 \\ 0 < |h_2| \leq H_2}} S + O(H_2 Q d^\epsilon).$$

We now consider the parts of this sum where either m_1 or m_2 are zero. Firstly, the part with $m_1 = 0$ is

$$\ll \frac{1}{p} \sum_{x_1 x_2 y_1 y_2 z = d} \frac{1}{x_1 x_2 y_1} \times \sum_{1 \leq |m_2| \leq px_2/2} \left| \sum_{\substack{q \sim Q/z \\ s_2: P/y_2}} \sum_{0 < |h_2| \leq H_2} e\left(\frac{m_2(s_2 y_2 z q + h_2)}{x_2 p}\right) \right|.$$

Now since

$$\left| \sum_{0 < |h_2| \leq H_2} e\left(\frac{m_2 h_2}{x_2 p}\right) \right| \leq \min\left(H_2, \frac{x_2 p}{|m_2|}\right),$$

the exponential sum above is

$$\ll \frac{1}{p} \sum_{x_1 x_2 y_1 y_2 z = d} \frac{1}{x_1 x_2 y_1} \times \sum_{1 \leq |m_2| \leq px_2/2} \min\left(H_2, \frac{x_2 p}{|m_2|}\right) \sum_{q \sim Q/z} \left| \sum_{s_2: P/y_2} e\left(\frac{m_2 s_2 y_2 z q}{x_2 p}\right) \right|.$$

We now sum over s_2 and split the sum into two parts distinguished by the size of m_2 to give the following bound:

$$\frac{H_2}{p} \sum_{x_1 x_2 y_1 y_2 z = d} \frac{1}{x_1 x_2 y_1} \sum_{1 \leq |m_2| \leq px_2/H_2} \sum_{q \sim Q/z} \frac{1}{\left\| \frac{m_2 y_2 z q}{x_2 p} \right\|} + \sum_{x_1 x_2 y_1 y_2 z = d} \frac{1}{x_1 y_1} \sum_{px_2/H_2 \leq |m_2| \leq px_2/2} \frac{1}{|m_2|} \sum_{q \sim Q/z} \frac{1}{\left\| \frac{m_2 y_2 z q}{x_2 p} \right\|}.$$

We then combine m_2 with q giving a single variable a running over a large range. This will produce a $\sum_{d|a} 1$ term which adds a factor of Q^ϵ to the bound. The bound we obtain by doing this is

$$\sum_{x_1 x_2 y_1 y_2 z = d} \frac{1}{x_1 y_1 z} Q^{1+\epsilon} \log P + \left(\sum_{x_1 x_2 y_1 y_2 z = d} \frac{1}{x_1 y_1 z} Q^{1+\epsilon} \log P \log H_2 + \sum_{x_1 x_2 y_1 y_2 z = d} \frac{1}{x_1 y_1} H_2 Q^\epsilon \log P \right).$$

This is no more than $QQ^\varepsilon d^\varepsilon$, and since $H_2 \gg QP^{-\delta'}$, the bound is

$$(3.10) \quad \ll H_2 P^{\delta'} Q^\varepsilon d^\varepsilon.$$

Next we need to bound the part of (3.9) that has $m_2 = 0$. This is

$$\begin{aligned} &\ll \frac{H_2}{p} \sum_{x_1 x_2 y_1 y_2 z = d} \frac{1}{x_1 x_2 y_2} \\ &\quad \times \sum_{1 \leq |m_1| \leq p x_1 / 2} \left| \sum_{q \sim Q/z} \sum_{\substack{s_1: P/y_1 \\ (s_1, p x_1) = 1}} e\left(\frac{m_1(\overline{s_1 y_1 z} h_1 + q)}{x_1 p}\right) \right|. \end{aligned}$$

Now for $|m_1| \leq p x_1 / 2$ we have

$$\left| \sum_{q \sim Q/z} e\left(\frac{m_1 q}{x_1 p}\right) \right| \leq \min\left(\frac{Q}{z}, \frac{x_1 p}{|m_1|}\right)$$

(note that q now runs over all integers in its range), so the exponential sum above is no more than

$$\begin{aligned} &\frac{H_2}{p} \sum_{x_1 x_2 y_1 y_2 z = d} \frac{1}{x_1 x_2 y_2} \\ &\quad \times \sum_{1 \leq |m_1| \leq p x_1 / 2} \min\left(\frac{Q}{z}, \frac{x_1 p}{|m_1|}\right) \left| \sum_{\substack{s_1: P/y_1 \\ (s_1, p x_1) = 1}} e\left(\frac{m_1 \overline{s_1 y_1 z} h_1}{x_1 p}\right) \right|. \end{aligned}$$

We may now apply Lemma 4 to the inner sum above giving a factor $(x_1 p)^{1/2+\varepsilon}$. We then sum over m_1 either directly, for $|m_1| \leq p x_1 z / q$, or in blocks for $|m_1| > p x_1 z / q$ thus obtaining a bound for the above of

$$H_2 \sum_{x_1 x_2 y_1 y_2 z = d} \frac{x_1^{1/2+\varepsilon}}{x_2 y_2 z} p^{1/2+\varepsilon} + H_2 \sum_{x_1 x_2 y_1 y_2 z = d} \frac{x_1^{1/2+\varepsilon}}{x_2 y_2} \log(P/z) p^{1/2+\varepsilon}.$$

Combined these are

$$(3.11) \quad \ll H_2 p^{1/2+\varepsilon} d^{1/2+\varepsilon}.$$

The remaining part of the multiple exponential sum in (3.9) is

$$\sum_{x_1 x_2 y_1 y_2 z = d} \frac{1}{p^2 x_1 x_2} \sum_{\substack{1 \leq |m_1| \leq p x_1 / 2 \\ 1 \leq |m_2| \leq p x_2 / 2}} \sum_{q \sim Q/z} \sum_{\substack{s_1: P/y_1 \\ (s_1, p x_1) = 1}} \sum_{\substack{s_2: P/y_2 \\ 0 < |h_2| \leq H_2}} S,$$

where

$$S = e\left(\frac{m_1(\overline{s_1 y_1 z} h_1 + q)}{x_1 p}\right) e\left(\frac{m_2(s_2 y_2 z q + h_2)}{x_2 p}\right).$$

This has order of magnitude no greater than

$$\sum_{x_1 \dots z} \frac{1}{p^2 x_1 x_2} \sum_{\substack{m_1, m_2 \\ q \sim Q/z}} \min \left(H_2, \frac{x_2 p}{|m_2|} \right) \\ \times \left| \sum_{\substack{s_1: P/y_1 \\ (s_1, p x_1) = 1}} e \left(\frac{m_1 (\overline{s_1 y_1 z} h_1 + q)}{x_1 p} \right) \right| \left| \sum_{s_2: P/y_2} e \left(\frac{m_2 s_2 y_2 z q}{x_2 p} \right) \right|;$$

here we have summed over h_2 . Next we sum over s_2 to give a bound of

$$\sum_{x_1 \dots z} \frac{1}{p^2 x_1 x_2} \sum_{\substack{m_2 \\ q \sim Q/z}} \min \left(H_2, \frac{x_2 p}{|m_2|} \right) \min \left(\frac{P}{y_2}, \frac{1}{\left\| \frac{m_2 y_2 z q}{x_2 p} \right\|} \right) T,$$

where

$$T = \sum_{1 \leq |m_1| \leq p x_1 / 2} \left| \sum_{\substack{s_1: P/y_1 \\ (s_1, p x_1) = 1}} e \left(\frac{m_1 (\overline{s_1 y_1 z} h_1 + q)}{x_1 p} \right) \right|.$$

We can now apply Lemma 4 directly to T . For the sum over m_2 and q we consider the parts where $x_2 p$ does and does not divide $m_2 y_2 z q$ separately. In the first case we combine the variables q and m_2 to give a single variable running over a large range. This will also produce a Q^ϵ term as an upper bound on the number of divisors of the variable. This part of the sum then is

$$\ll \sum_{x_1 \dots z} \frac{1}{p^2 x_1 x_2} (x_2 p) \log(x_2 p) \left(\frac{Q}{z x_2 p} \right) x_2 p \log(x_2 p) Q^\epsilon (x_1 p) (x_1 p)^{1/2 + \epsilon/2},$$

which is

$$\ll P^{1/2 + \epsilon} Q^{1 + \epsilon} \sum_{x_1 x_2 y_1 y_2 z = d} x_1^{1/2 + \epsilon/2}.$$

This gives a bound of $P^{1/2 + \epsilon} Q^{1 + \epsilon} d^{1/2 + \epsilon}$, and since $H_2 \gg Q P^{-\delta}$, this is

$$(3.12) \quad \ll H_2 P^{1/2 + \epsilon} Q^\epsilon d^{1/2 + \epsilon}.$$

When $x_2 p$ divides $m_2 y_2 z q$ we are forced to take the trivial bound P/y_2 on the sum over s_2 but since this occurs for a small range of m_2 values the bound we get is less than the bound given above.

By (3.10)–(3.12) the bound for the remainder term R_d as expressed in (3.9) is

$$\ll \frac{H_2 Q^{1 + \epsilon} d^{1/2 + \epsilon}}{p^{1/2}}.$$

Hence

$$\sum_{\substack{d \leq Y \\ (d, h) = 1}} \mu^2(d) d^\epsilon |R_d| \ll H_2 Q^{1 + \epsilon} P^{-1/2} Y^{3/2 + 2\epsilon}.$$

By taking $Y = P^{1/4}$ and both occurrences of ε as $1/100$ we can apply Lemma 3b to state that for fixed h_1 and p , the number of solutions to the inequalities (3.1) is

$$(3.13) \quad \leq \frac{\Delta H_2 \theta^3 Q}{\log^5 P} \left(\frac{h_1}{\phi(h_1)} \right)^5 + O(H_2 Q^{63/64}).$$

Now

$$\sum_{1 \leq h_1 \leq H_1} \left(\frac{h_1}{\phi(h_1)} \right)^5 < \Delta' H_1,$$

so summing (3.13) over p and h_1 gives the bound

$$\frac{\Delta H_1 H_2 \theta^4 P Q}{(\log P)^3 (\log Q)^3} + O(H_1 H_2 P Q^{63/64}).$$

Here we used the fact that $4 \log P \geq \log Q$. This completes the third and final case of the proof of Lemma 2.

We should now mention some details regarding the proof of Lemma 2 for $k > 2$. The proof still falls into the three cases used when $k = 2$. If we look at these in turn, Case 1 generalises instantly and the difference is merely one of notation. Case 2 is where we have P close to Q and with none of the $\psi_i(P)$ functions being “small”. This case too generalises in a straightforward way. The sieve result used is again Lemma 3a, applied with $\kappa = 2k$. Result 3 is employed, giving us the equation

$$\sum_{n \in \mathcal{A}_d} 1 = \sum_{x_1 \dots x_k y_1 \dots y_k = d} \left(\sum_{r_1, \dots, r_k, p} \left(\frac{2^k H_1 \dots H_k}{y_1 \dots y_k q^k} + T_1 + \dots + T_k \right) + E \right),$$

where the T_j terms are of the form

$$T_j = (2j - 1) \frac{\theta_j}{(L + 1)^j} \sum_{\substack{1 \leq i_a \leq k \\ i_b \neq i_c}} \frac{2H_{i_1}}{qy_{i_1}} \times \dots \times \frac{2H_{i_{k-j}}}{qy_{i_{k-j}}}.$$

It should be noted that the parameter L can be set to Q as before. The exponential sum E looks like

$$\sum_{x_1 \dots x_k y_1 \dots y_k = d} \sum_{r_1, \dots, r_k, p} c(\mathbf{m}) \left| \sum_{r_1, \dots, r_k, p} e \left(\frac{m_1 r_1 x_1 p}{y_1 q} + \dots + \frac{m_k r_k x_k p}{y_k q} \right) \right|.$$

Since this is symmetrical in the k variables, the generalisation from $k = 2$ is immediate. In Case 3 of the proof one might expect complications to arise from the fact that we have more than two $\psi(p)$ functions of different sizes. However, for each k only one of the functions can be small in the sense that

$$\psi(p) \ll p^{\delta-1} \quad \text{for any } \delta > 0,$$

since if more than one of these functions had this property then our sum (1.2) would converge. It may be the case that the product of l of the functions is small in the sense given above thus forcing the product of the remaining $k - l$ functions to be large in the sense that

$$\psi_{l+1}(p) \dots \psi_k(p) \gg p^{-\delta} \quad \text{for any } \delta > 0,$$

but no problems arise from functions being large in this sense. For Case 3 then we employ a $(2k + 1)$ -dimensional sieve with \mathcal{A} equal to

$$\{r_1 \dots r_k s_1 \dots s_k q \mid r_1 p - s_1 q = h_1, |r_i p - s_i q| < H_i, s_i : P \text{ and } q \sim Q\}.$$

The remainder terms in the sieve are dealt with using exponential sums as for $k = 2$ but over k variables this time, $m_1 \dots m_k$. The part of the sum having all m_i zero except m_1 is dealt with using the Kloosterman sum estimate as before.

4. Additional proofs. The proof of Result 2 is as follows.

Firstly we need the following lemma which, given a measurable subset A of \mathbb{R}^k , provides a useful sufficient condition for $\mathbb{R}^k \setminus A$ to be a null set.

LEMMA 5. *Let A be as above and suppose that for every open cube $C \subseteq \mathbb{R}^k$ we have*

$$\lambda^k(A \cap C) \geq \delta \lambda^k(C)$$

where $\delta > 0$ is a constant independent of the choice of C . Then $\lambda^k(\mathbb{R}^k \setminus A) = 0$.

Proof. Corollary 6.2.6 on page 184 of [1] states that, if S is a (Lebesgue) measurable subset of \mathbb{R}^k , then almost all points of S are density points of S . Applying this to the set A^c , the complement of A relative to \mathbb{R}^k , and assuming for a contradiction that $\lambda^k(A^c) \neq 0$, we get

$$\exists x \in A^c \forall \varepsilon > 0 \exists \delta' > 0$$

$$\left| \frac{\lambda^k(A^c \cap C)}{\lambda^k(C)} - 1 \right| < \varepsilon \quad \text{for all cubes } C \text{ with } e(C) < \delta'.$$

(Here $e(C)$ denotes the edge length of C .) Applying the above with $\varepsilon = \delta/2$ (where δ is as in the lemma) and letting C be an open cube with $e(C) < \delta'_\varepsilon$ we have:

$$\left| \frac{\lambda^k(A^c \cap C)}{\lambda^k(C)} - 1 \right| < \frac{\delta}{2},$$

hence $1 - \lambda^k(A^c \cap C)/\lambda^k(C) < \delta/2$ and so $\lambda^k(A^c \cap C) > (1 - \delta/2)\lambda^k(C)$.

But

$$\begin{aligned}\lambda^k(C) &= \lambda^k(A^c \cap C) + \lambda^k(A \cap C) \\ &> (1 - \delta/2)\lambda^k(C) + \delta\lambda^k(C) \quad (\text{by assumption in the lemma}) \\ &= (1 + \delta/2)\lambda^k(C) \quad (\text{contradiction, since } \delta > 0).\end{aligned}$$

Hence we must have $\lambda^k(A^c) = 0$, which proves the lemma.

The result will now follow from two lemmas which allow us to apply Lemma 5 with $A = V^c \cup \mathcal{D}$.

LEMMA 6. *Let C be an open cube with $C \subseteq V$ and let $B_n = C \cap D_n$. Then*

$$\lambda^k(\mathcal{D} \cap C) \geq \delta\lambda^k(C).$$

Proof. Firstly, we apply Lemma 5 on p. 17 of [8] to the sequence of measurable sets (B_n) and get

$$(4.1) \quad \lambda^k(\mathcal{B}) \geq \lim_{N \rightarrow \infty} \left(\sum_{n=1}^{\infty} \lambda^k(B_n) \right)^2 \left(\sum_{1 \leq m, n \leq N} \lambda^k(B_m \cap B_n) \right)^{-1},$$

where $\mathcal{B} = C \cap \mathcal{D}$. The lemma requires a finite measure space but since the sets (B_n) all lie in the finite cube C this is satisfied. The result then follows from (4.1) and (1.4) of Result 2.

LEMMA 7. *Let $A = V^c \cup \mathcal{D}$ and C be an open cube in \mathbb{R}^k . Then*

$$\lambda^k(A \cap C) \geq \delta\lambda^k(C)$$

where $\delta > 0$ is a constant independent of the choice of C .

Proof. Let $G = C \cap V$; then G is an open, measurable subset of \mathbb{R}^k . Hence Lemma 1.4.2 (p. 28) of [1] ensures we can say

$$G = \bigcup_{i=1}^{\infty} C_i \quad \text{where the } C_i \text{ are disjoint half-open cubes.}$$

We have

$$\begin{aligned} (*) \quad \lambda^k(\mathcal{D} \cap G) &= \lambda^k\left(\mathcal{D} \cap \bigcup_{i=1}^{\infty} C_i\right) = \lambda^k\left(\bigcup_{i=1}^{\infty} (\mathcal{D} \cap C_i)\right) \quad (C_i \text{ disjoint}) \\ &= \sum_{i=1}^{\infty} \lambda^k(\mathcal{D} \cap C_i) = \sum_{i=1}^{\infty} \lambda^k(\mathcal{D} \cap C_i^\circ) \\ &\geq \sum_{i=1}^{\infty} \delta\lambda^k(C_i^\circ) \quad (\text{by Lemma 6}) \\ &= \sum_{i=1}^{\infty} \delta\lambda^k(C_i) = \delta\lambda^k\left(\bigcup_{i=1}^{\infty} C_i\right) = \delta\lambda^k(G).\end{aligned}$$

Hence,

$$\begin{aligned}
 \lambda^k(A \cap C) &= \lambda^k((V^c \cup \mathcal{D}) \cap C) \\
 &= \lambda^k(V^c \cap C) + \lambda^k(\mathcal{D} \cap C) && \text{(since } \mathcal{D} \subseteq V) \\
 &= \lambda^k(C \setminus V) + \lambda^k(\mathcal{D} \cap (C \cap V)) && \text{(since } \mathcal{D} \subseteq V) \\
 &= \lambda^k(C \setminus V) + \lambda^k(\mathcal{D} \cap G) \\
 &\geq \lambda^k(C \setminus V) + \delta \lambda^k(G) && \text{(by (*))} \\
 &\geq \delta \lambda^k(C \setminus V) + \delta \lambda^k(C \cap V) && \text{(letting } \delta < 1) \\
 &= \delta \lambda^k(C) && \text{(as required).}
 \end{aligned}$$

We have shown that $A = V^c \cup \mathcal{D}$ fulfils the criteria of Lemma 5 so we have $\lambda^k(\mathbb{R}^k \setminus (V^c \cup \mathcal{D})) = 0$; but this implies that $\lambda^k(V \cap (\mathbb{R}^k \setminus \mathcal{D})) = 0$ because $\mathcal{D} \subseteq V \subseteq \mathbb{R}^k$ and so $\lambda^k(V \setminus \mathcal{D}) = 0$ as required.

Acknowledgements. This work was undertaken while the author was in receipt of a Cardiff University studentship. The author would like to thank Prof. Glyn Harman for his encouragement and good advice.

References

- [1] D. L. Cohn, *Measure Theory*, Birkhäuser, Boston, 1980.
- [2] R. J. Duffin and A. C. Schaffer, *Khinchine's problem in metric Diophantine approximation*, Duke Math. J. 8 (1941), 243–255.
- [3] H. Halberstam and H.-R. Richert, *Sieve Methods*, Academic Press, New York, 1974.
- [4] G. Harman, *Metric Number Theory*, Clarendon Press, Oxford, 1998.
- [5] —, *Metric Diophantine approximation with two restricted variables III. Two prime numbers*, J. Number Theory 29 (1988), 364–375.
- [6] C. Hooley, *Applications of Sieve Methods*, Cambridge, 1976.
- [7] A. Ya. Khinchin, *Continued Fractions*, English translation of third edition by Univ. of Chicago Press, 1964.
- [8] V. G. Sprindžuk, *Metric Theory of Diophantine Approximations*, Winston and Wiley, Washington, 1979.

School of Mathematics
 Cardiff University
 Senghennydd Road
 PO Box 926
 Cardiff CF2 4HY, Wales, U.K.
 E-mail: JonesHR@cardiff.ac.uk
 HuwJ@zuken.co.uk

*Received on 26.10.1998
 and in revised form on 27.12.1999*

(3495)