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Algebraic independence results for reciprocal sums of Fibonacci numbers

by

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1. Introduction. Let $\{F_n\}_{n\geq 0}$ and $\{L_n\}_{n\geq 0}$ be Fibonacci numbers and Lucas numbers defined by

$$\begin{aligned} &F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad (n \geq 0), \\ &L_0 = 2, \quad L_1 = 1, \quad L_{n+2} = L_{n+1} + L_n \quad (n \geq 0). \end{aligned}$$

Duverney, Ke. Nishioka, Ku. Nishioka, and the last named author [3] (see also [2]) proved the transcendence of the numbers

$$\sum_{n=1}^{\infty} \frac{1}{F_n^{2s}}, \quad \sum_{n=1}^{\infty} \frac{1}{L_n^{2s}}, \quad \sum_{n=1}^{\infty} \frac{1}{F_{2n-1}^{s}}, \quad \sum_{n=1}^{\infty} \frac{1}{L_{2n}^{s}} \quad (s = 1, 2, \ldots)$$

by using Nesterenko's theorem on the Ramanujan functions $P(q)$, $Q(q)$, and $R(q)$ (see Section 2).

In [4] we proved that the numbers

$$\sum_{n=1}^{\infty} \frac{1}{F_n^{2}}, \quad \sum_{n=1}^{\infty} \frac{1}{F_n^{4}}, \quad \sum_{n=1}^{\infty} \frac{1}{F_n^{6}} \quad \left(\text{respectively, } \sum_{n=1}^{\infty} \frac{1}{L_n^{2}}, \quad \sum_{n=1}^{\infty} \frac{1}{L_n^{4}}, \quad \sum_{n=1}^{\infty} \frac{1}{L_n^{6}}\right)$$

are algebraically independent, and that each

$$\sum_{n=1}^{\infty} \frac{1}{F_n^{2s}} \quad \left(\text{respectively, } \sum_{n=1}^{\infty} \frac{1}{L_n^{2s}}\right) \quad (s = 4, 5, 6, \ldots)$$

is written as a rational (respectively, algebraic) function of these three numbers over $\mathbb{Q}$. For the reciprocal sum

$$\zeta_{\mathrm{F}}(s) = \sum_{n=1}^{\infty} \frac{1}{F_n^{s}}$$

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of Fibonacci numbers, this result reads: the numbers $\zeta_F(2)$, $\zeta_F(4)$, $\zeta_F(6)$ are algebraically independent, and for any integer $s \geq 4$,

$$\zeta_F(2s) - r_s\zeta_F(4) \in \mathbb{Q}(u,v), \quad u := \zeta_F(2), \quad v := \zeta_F(6), \tag{1.1}$$

with some $r_s \in \mathbb{Q}$ ($r_s = 0$ if and only if s is odd), where the rational function of u and v is explicit (see [4, Theorem 1, Example 1]); for example,

$$\zeta_F(8) - \frac{15}{14}\zeta_F(4) = \frac{P(u,v)}{378(4u+5)^2},$$
$$P(u,v) = 256u^6 - 3456u^5 + 2880u^4 + 1792u^3v - 11100u^3 + 20160u^2v - 10125u^2 + 7560uv + 3136v^2 - 1050v.$$

The formula (1.1) for the values $\zeta_F(2s)$ ($s = 4, 5, 6, \ldots$) can be regarded as an analogue of Euler's formula

$$\zeta(2s) = \frac{(-1)^{s-1}24^s B_{2s}}{2(2s)!}\zeta^s(2) \quad (s = 1, 2, \ldots)$$

for the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, where B_n are Bernoulli numbers, from which in this case the algebraic dependence of the values $\zeta(2s)$ follows immediately. By (1.1), any four values $\zeta_F(2s_1)$, $\zeta_F(2s_2)$, $\zeta_F(2s_3)$, $\zeta_F(2s_4)$ with positive integers s_i are algebraically dependent. It remains to establish whether any given three values $\zeta_F(2s_1), \zeta_F(2s_2)$, $\zeta_F(2s_3)$ with distinct positive integers s_i, or even two of them, are algebraically independent or not, and the purpose of the present paper is to give a complete answer to this question.

In this paper, we treat more general reciprocal sums including $\zeta_F(s)$ as a special case. Let $\alpha, \beta \in \mathbb{C}$ satisfy $|\beta| < 1$ and $\alpha\beta = -1$. We put

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad (n \geq 0). \tag{1.2}$$

In particular, if $\beta = (1-\sqrt{5})/2$ and $\beta = 1 - \sqrt{2}$, we have the Fibonacci numbers $U_n = F_n$ and the Pell numbers $U_n = P_n$, respectively.

Our main result is stated as follows:

THEOREM 1.1. *Let $\beta \in \overline{\mathbb{Q}}$ with $|\beta| < 1$ and $\alpha\beta = -1$, and set*

$$\Phi_{2s} := (\alpha - \beta)^{-2s} \sum_{n=1}^{\infty} \frac{1}{U_n^{2s}} \quad (s \geq 1),$$

where $\{U_n\}_{n\geq 1}$ is defined by (1.2). Let s_1, s_2, s_3 be distinct positive integers. Then the numbers $\Phi_{2s_1}, \Phi_{2s_2}, \Phi_{2s_3}$ are algebraically independent if and only if at least one of s_1, s_2, s_3 is even.

COROLLARY 1.2. *For any distinct positive integers s_1 and s_2, the numbers $\Phi(2s_1)$ and $\Phi(2s_2)$ are algebraically independent.*

EXAMPLE. It follows from the theorem that the numbers $\zeta_{\mathrm{F}}(2s_1)$, $\zeta_{\mathrm{F}}(2s_2)$, $\zeta_{\mathrm{F}}(2s_3)$ are algebraically dependent if and only if all s_i are odd. We give here an explicit relation for one of the dependent cases, $(s_1, s_2, s_3) = (1, 3, 5)$:

$$\begin{aligned}297(4\zeta_{\mathrm{F}}(2)+5)^2\zeta_{\mathrm{F}}(10) &- (4760\zeta_{\mathrm{F}}(2)+3500)\zeta_{\mathrm{F}}(6)^2\\ &+ (1600\zeta_{\mathrm{F}}(2)^4 - 12800\zeta_{\mathrm{F}}(2)^3 - 11250\zeta_{\mathrm{F}}(2)^2 - 9375\zeta_{\mathrm{F}}(2) - 9375)\zeta_{\mathrm{F}}(6)\\ &- 512\zeta_{\mathrm{F}}(2)^7 + 3520\zeta_{\mathrm{F}}(2)^6 - 4050\zeta_{\mathrm{F}}(2)^5\\ &+ 3750\zeta_{\mathrm{F}}(2)^4 + 9375\zeta_{\mathrm{F}}(2)^3 = 0.\end{aligned}$$

Our results for Φ_{2s} stated above in the special case of $\Phi_{2s} = \zeta_{\mathrm{F}}(2s)$ are based on the expressions of Φ_{2s} as polynomials of K/π, E/π, and k over $\mathbb{Q}$, where K and E are the complete elliptic integrals of the first and second kind with a suitably chosen modulus k (see Section 2). Such expressions of Φ_{2s} are obtained from the expressions of series of hyperbolic cosecants and secants in terms of K/π, E/π, k given by Zucker [9]. Additionally we need recursive relations for the coefficients of the power series expansions of Jacobian elliptic functions $\mathrm{ns}^2 z$ and $\mathrm{nd}^2 z$ (see Lemmas 2.3 and 2.4). The algebraic independence of Φ_2, Φ_4, Φ_6 can be proved by applying Nesterenko's theorem, which implies that the quantities $K/\pi, E/\pi, k$ expressing Φ_{2s} are algebraically independent (see Corollary 2.2), and the rational functions indicated in (1.1) in the case of $\Phi_{2s} = 5^{-s}\zeta_{\mathrm{F}}(2s)$ are obtained by eliminating $K/\pi, E/\pi, k$ from the expression of Φ_{2s} using Φ_2, Φ_4, Φ_6.

To prove the theorem, we have to examine whether given three numbers $\Phi_{2s_1}, \Phi_{2s_2}, \Phi_{2s_3}$, which are polynomials over $\mathbb{Q}$ of K/π, E/π, k, are algebraically independent or not. For this we give an algebraic independence criterion for such numbers (see Lemma 3.1). It seems difficult to apply the criterion directly to the rational functions in question, since they are given by rather involved recursive relations. Then we deduce from the criterion some sufficient conditions for the algebraic independence of $\Phi_{2s_1}, \Phi_{2s_2}, \Phi_{2s_3}$ with even s_i and prove their algebraic independence in Section 5. The remaining cases are treated similarly in the final section.

2. Preliminaries. In what follows, s and s_1, s_2, s_3 are always positive integers. The reciprocal sum Φ_{2s} in our theorem is written as a series of hyperbolic functions. In [9] Zucker gave a method of summing such series. He wrote them as q-series, and then expressed these q-series in closed form in terms of K, E, and k, where K and E are the complete elliptic integrals of the first and second kind with modulus $k \neq 0, \pm 1$ defined by

$$K = K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \qquad E = E(k) = \int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}}\, dt$$

for $k^2 \in \mathbb{C} \setminus (\{0\} \cup [1, \infty))$, and $K' = K(k')$ with $k^2 + k'^2 = 1$. Here the branch of each integrand is chosen so that it tends to 1 as $t \to 0$. The relation among q and these quantities is given by

$$q = e^{-\pi c}, \qquad c = K'/K.$$

By [9, Tables 1(i), 1(iv)], we have

$$\Sigma_1 := 2^{-2s} \sum_{\nu=1}^{\infty} \operatorname{cosech}^{2s}(\nu\pi c) = \frac{1}{(2s-1)!} \sum_{j=0}^{s-1} \sigma_{s-j-1}(s) A_{2j+1}(q),$$

$$\Sigma_2 := 2^{-2s} \sum_{\nu=1}^{\infty} \operatorname{sech}^{2s} \frac{(2\nu-1)\pi c}{2} = \frac{(-1)^{s-1}}{(2s-1)!} \sum_{j=0}^{s-1} \sigma_{s-j-1}(s) D_{2j+1}(q),$$

where

$$A_{2j+1}(q) = \sum_{n=1}^{\infty} \frac{n^{2j+1} q^{2n}}{1 - q^{2n}}, \qquad D_{2j+1}(q) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^{2j+1} q^n}{1 - q^{2n}},$$

and $\sigma_1(s), \ldots, \sigma_{s-1}(s)$ are the elementary symmetric functions of $-1, -2^2, \ldots, -(s-1)^2$ defined by

$$\sigma_i(s) = (-1)^i \sum_{1 \le r_1 < \cdots < r_i \le s-1} r_1^2 \cdots r_i^2 \quad (1 \le i \le s-1), \qquad \sigma_0(s) = 1 \quad (s \ge 1).$$

Now specializing $c = c(\beta)$ (or $q = q(\beta)$) as

$$q = e^{-\pi c} = \beta^2, \quad \beta = -e^{-\pi c/2},$$

where $\beta \in \overline{\mathbb{Q}}$ is given in the theorem, and decomposing our reciprocal sum into two parts, we have

$$\Phi_{2s} = (\alpha - \beta)^{-2s} \sum_{\nu=1}^{\infty} \frac{1}{U_{2\nu-1}^{2s}} + (\alpha - \beta)^{-2s} \sum_{\nu=1}^{\infty} \frac{1}{U_{2\nu}^{2s}} = \Sigma_1 + \Sigma_2,$$

an expression of Φ_{2s} by finite sums of q-series A_{2j+1} and D_{2j+1}. These q-series A_{2j+1} and D_{2j+1} are generated from Fourier expansions of the squares of Jacobian elliptic functions $\operatorname{ns}^2 z$ and $(1-k^2)\operatorname{nd}^2 z$:

$$\tag{2.1} \begin{cases} \left(\dfrac{2K}{\pi}\right)^2 \operatorname{ns}^2\left(\dfrac{2Kx}{\pi}\right) \\ \qquad = \dfrac{4K(K-E)}{\pi^2} + \operatorname{cosec}^2 x - 8 \displaystyle\sum_{j=0}^{\infty} (-1)^j A_{2j+1} \dfrac{(2x)^{2j}}{(2j)!}, \\ \left(\dfrac{2K}{\pi}\right)^2 (1-k^2)\operatorname{nd}^2\left(\dfrac{2Kx}{\pi}\right) \\ \qquad = \dfrac{4KE}{\pi^2} - 8 \displaystyle\sum_{j=0}^{\infty} (-1)^j D_{2j+1} \dfrac{(2x)^{2j}}{(2j)!}, \end{cases}$$

where

$$\operatorname{ns} z = 1/\operatorname{sn} z, \qquad \operatorname{dn} z = \sqrt{1 - k^2 \operatorname{sn}^2 z}, \qquad \operatorname{nd} z = 1/\operatorname{dn} z,$$

with $w = \operatorname{sn} z$ defined by

$$z = \int_0^w \frac{dw}{\sqrt{(1-w^2)(1-k^2w^2)}},$$

and the power series expansions of these elliptic functions give the expressions of the corresponding q-series in terms of K/π, E/π, k (cf. [5]). For example, we find in [7]

$$(2.2) \quad \begin{cases} P(q^2) := 1 - 24A_1(q) = \left(\dfrac{2K}{\pi}\right)^2 \left(\dfrac{3E}{K} - 2 + k^2\right), \\ Q(q^2) := 1 + 240A_3(q) = \left(\dfrac{2K}{\pi}\right)^4 (1 - k^2 + k^4), \\ R(q^2) := 1 - 504A_5(q) = \left(\dfrac{2K}{\pi}\right)^6 \dfrac{1}{2}(1+k^2)(1-2k^2)(2-k^2). \end{cases}$$

Here we state the theorem of Nesterenko and its corollary [6]. We denote by $\operatorname{tr.d.}(L : K)$ the transcendence degree of a field extension $L : K$.

Theorem 2.1 (Nesterenko's Theorem). *If $\rho \in \mathbb{C}$ with $0 < |\rho| < 1$, then*

$$\operatorname{tr.d.}(\mathbb{Q}(\rho, P(\rho), Q(\rho), R(\rho)) : \mathbb{Q}) \geq 3.$$

This combined with (2.2) implies the following:

Corollary 2.2. *If $q = e^{-\pi c} \in \overline{\mathbb{Q}}$ with $0 < |q| < 1$, then K/π, E/π, k are algebraically independent.*

Zucker's Tables 1(i) and 1(iv) in [9] exhibit expressions of A_{2j+1} and D_{2j+1} for $j = 0, 1, 2, 3$. We need these expressions for all $j \geq 0$, which can be deduced using Lemmas 2.3 and 2.4 below. In this way we obtain the expressions (4.4) and (6.3) necessary for the proof of Theorem 1.1.

Lemma 2.3 ([4]). *The coefficients of the expansion*

$$\operatorname{ns}^2 z = \frac{1}{z^2} + \sum_{j=0}^{\infty} c_j z^{2j}$$

are given by

$$(2.3) \quad \begin{cases} c_0 = \dfrac{1}{3}(1+k^2), \quad c_1 = \dfrac{1}{15}(1 - k^2 + k^4), \\ c_2 = \dfrac{1}{189}(1+k^2)(1-2k^2)(2-k^2), \end{cases}$$

$$(2.4) \quad (j-2)(2j+3)c_j = 3 \sum_{i=1}^{j-2} c_i c_{j-i-1} \quad (j \geq 3).$$

Lemma 2.4 ([4]). *The coefficients of the expansion*

$$(1-k^2)\,\mathrm{nd}^2\, z = 1 - k^2 + \sum_{j=1}^{\infty} d_j z^{2j}$$

are given by

$$d_1 = k^2(1-k^2), \qquad d_2 = -\frac{1}{3}k^2(1-k^2)(1-2k^2), \tag{2.5}$$

$$j(2j-1)d_j = -2(1-2k^2)d_{j-1} - 3\sum_{i=1}^{j-2} d_i d_{j-i-1} \qquad (j \ge 3). \tag{2.6}$$

3. An algebraic independence criterion

Lemma 3.1 (Algebraic independence criterion). *Let $x_1, \dots, x_n \in \mathbb{C}$ be algebraically independent and let $y_j := U_j(x_1, \dots, x_n)$, where $U_j(X_1, \dots, X_n) \in \mathbb{Q}[X_1, \dots, X_n]$ $(1 \le j \le n)$. Assume that*

$$\det\left(\frac{\partial U_j}{\partial X_i}(x_1, \dots, x_n)\right) \ne 0. \tag{3.1}$$

Then the numbers $y_1, \dots, y_n$ are algebraically independent.

The main tool in proving Lemma 3.1 is the following lemma.

Lemma 3.2. *Let L be a field such that $\mathbb{Q} \subset L \subset \mathbb{C}$, and let $P_j(X_1, \dots, X_n) \in L[X_1, \dots, X_n]$ $(1 \le j \le n)$. Assume that $(x_1, \dots, x_n) \in \mathbb{C}^n$ satisfies the conditions*

$$P_j(x_1, \dots, x_n) = 0 \quad (1 \le j \le n) \quad \text{and} \quad \det\left(\frac{\partial P_j}{\partial X_i}(x_1, \dots, x_n)\right) \ne 0. \tag{3.2}$$

Then $L(x_1, \dots, x_n)$ is algebraic over L. In particular, all numbers $x_1, \dots, x_n$ are algebraic over L.

This lemma follows directly from the Corollary to Theorem 40 (page 126) in [8].

Proof of Lemma 3.1. For $U_j(X_1, \dots, X_n) \in \mathbb{Q}[X_1, \dots, X_n]$ $(1 \le j \le n)$ set

$$P_j(X_1, \dots, X_n) := U_j(X_1, \dots, X_n) - y_j \in \mathbb{Q}(y_1, \dots, y_n)[X_1, \dots, X_n] \qquad (1 \le j \le n).$$

Then the numbers $x_1, \dots, x_n$ satisfy the system

$$P_j(x_1, \dots, x_n) = 0 \quad (1 \le j \le n),$$

whereas the assumption (3.1) implies that the second condition in (3.2) is also fulfilled. Therefore, Lemma 3.2 with $L = \mathbb{Q}(y_1, \dots, y_n)$ is applicable, and we conclude that

$$\mathrm{tr.d.}(\mathbb{Q}(x_1, \dots, x_n) : \mathbb{Q}(y_1, \dots, y_n)) = 0.$$

By the assumption, we have $\mathrm{tr.d.}(\mathbb{Q}(x_1,\dots,x_n):\mathbb{Q})=n$. Applying the chain rule of transcendence degrees to the field extensions $\mathbb{Q}\subseteq\mathbb{Q}(y_1,\dots,y_n)\subseteq\mathbb{Q}(x_1,\dots,x_n)$, we get

$$\mathrm{tr.d.}(\mathbb{Q}(y_1,\dots,y_n):\mathbb{Q})=n,$$

as desired. ■

4. Sufficient conditions for algebraic independence. In this and all the subsequent sections, we assume that the condition on β in Theorem 1.1 is fulfilled, which means that k, K/π, E/π are algebraically independent (cf. Corollary 2.2). The Jacobian elliptic function $\mathrm{ns}^2 z+(k^2-1)\,\mathrm{nd}^2 z$ has the series expansion

$$\mathrm{ns}^2 z+(k^2-1)\,\mathrm{nd}^2 z=\frac{1}{z^2}+\frac{4k^2-2}{3}+\sum_{j=1}^{\infty}C_j z^{2j}\quad (j\ge 1),\tag{4.1}$$

where $C_j=c_j-d_j$, and $c_j,d_j\in\mathbb{Q}[k]$ are given recursively in Lemmas 2.3 and 2.4 (cf. [4]). We denote $C_j'=dC_j/dk$ as usual.

LEMMA 4.1. *Let $1<s_1<s_2<s_3$ be even integers. Assume that*

$$s_3C_{s_3-1}C'_{s_2-1}-s_2C_{s_2-1}C'_{s_3-1}\neq 0\tag{4.2}$$

as a polynomial in k. Then the numbers $\Phi_{2s_1},\Phi_{2s_2},\Phi_{2s_3}$ are algebraically independent.

REMARK. The condition (4.2) is equivalent to

$$C_{s_2-1}^{s_3}/C_{s_3-1}^{s_2}\notin\mathbb{Q},\tag{4.3}$$

which can be seen by integration and logarithmic derivation. We note that the condition (4.3), and so (4.2), does not hold for $(s_2,s_3)=(2,4)$. Indeed,

$$C_{s_2-1}=C_1=\frac{1}{15}-\frac{16}{15}k^2+\frac{16}{15}k^4,$$
$$C_{s_3-1}=C_3=\frac{1}{675}-\frac{32}{675}k^2+\frac{32}{75}k^4-\frac{512}{675}k^6+\frac{256}{675}k^8,$$

which satisfy $C_1^4=9C_3^2$.

Proof of Lemma 4.1. By the method of Section 2, for any even integer s we have

$$\begin{aligned}\Phi_{2s}=\frac{1}{(2s-1)!}\biggl[&-\frac{(s-1)!^2}{24}\biggl(1-\Bigl(\frac{2K}{\pi}\Bigr)^2\Bigl(\frac{6E}{K}-5+4k^2\Bigr)\biggr)\\&+\sum_{j=1}^{s-1}\sigma_{s-j-1}(s)\frac{(-1)^j(2j)!}{2^{2j+3}}\biggl(a_j-\Bigl(\frac{2K}{\pi}\Bigr)^{2j+2}(c_j-d_j)\biggr)\biggr],\end{aligned}\tag{4.4}$$

where a_j is defined by the series

$$\operatorname{cosec}^2 z = \frac{1}{z^2} + \sum_{j=0}^{\infty} a_j z^{2j} \tag{4.5}$$

with

$$a_j = \frac{(-1)^j (2j+1) 2^{2j+2} B_{2j+2}}{(2j+2)!} \qquad (j \geq 0), \tag{4.6}$$

and c_j and d_j are even polynomials in k. It follows immediately from (2.3)–(2.6) that $\deg_k C_j = 2+2j$ $(j \geq 0)$. In Φ_{2s} replace k, K/π, E/π by independent variables X_1, X_2, X_3, respectively, and denote it by $\Phi_{2s}(X_1, X_2, X_3)$. Then

$$\begin{aligned}\frac{\partial \Phi_{2s}}{\partial X_1}(k, X_2, E/\pi) = \frac{1}{(2s-1)!}\Bigg[&\frac{(s-1)!^2}{3}(2X_2)^2 k \\ &+ \sum_{j=1}^{s-1} \sigma_{s-j-1}(s) \frac{(-1)^{j+1}(2j)!}{2^{2j+3}} (2X_2)^{2j+2} C_j'\Bigg],\end{aligned} \tag{4.7}$$

$$\begin{aligned}\frac{\partial \Phi_{2s}}{\partial X_2}(k, X_2, E/\pi) = \frac{1}{(2s-1)!}\Bigg[&\frac{(s-1)!^2}{6}(6E/\pi + 2X_2(4k^2-5)) \\ &+ \sum_{j=1}^{s-1} \sigma_{s-j-1}(s) \frac{(-1)^{j+1}(2j)!(j+1)}{2^{2j+1}} (2X_2)^{2j+1} C_j\Bigg],\end{aligned} \tag{4.8}$$

$$\frac{\partial \Phi_{2s}}{\partial X_3}(k, X_2, E/\pi) = \frac{(s-1)!^2}{2(2s-1)!}(2X_2). \tag{4.9}$$

Now, we apply Lemma 3.1 with

$$n = 3, \qquad x_1 = k, \qquad x_2 = K/\pi, \qquad x_3 = E/\pi,$$

and, with respect to (4.4),

$$U_j = \Phi_{2s_j}(X_1, X_2, X_3), \qquad y_j = \Phi_{2s_j}(k, K/\pi, E/\pi) \qquad (j = 1, 2, 3).$$

We put for brevity

$$\phi_i(j) = \phi_i(j)(X_1, X_2, X_3) := \frac{\partial \Phi_{2s_j}}{\partial X_i}(X_1, X_2, X_3) \qquad (i, j = 1, 2, 3).$$

Set

$$\begin{aligned}\Delta(X_1, X_2, X_3) :=& \det\begin{pmatrix} \phi_1(1) & \phi_1(2) & \phi_1(3) \\ \phi_2(1) & \phi_2(2) & \phi_2(3) \\ \phi_3(1) & \phi_3(2) & \phi_3(3) \end{pmatrix} \\ =& \big(\phi_1(1)\phi_2(2)\phi_3(3) + \phi_1(2)\phi_2(3)\phi_3(1) + \phi_1(3)\phi_2(1)\phi_3(2)\big) \\ &- \big(\phi_1(3)\phi_2(2)\phi_3(1) + \phi_1(1)\phi_2(3)\phi_3(2) + \phi_1(2)\phi_2(1)\phi_3(3)\big).\end{aligned} \tag{4.10}$$

We only have to prove the nonvanishing of the determinant $\Delta(k, K/\pi, E/\pi)$. In what follows, for a polynomial $f(X_1, X_2, X_3) \in \mathbb{Q}[X_1, X_2, X_3]$, let $\lambda(2X_2, f)$ denote the leading coefficient of $f(k, X_2, E/\pi)$ with respect to $2X_2$. We compute $\lambda(2X_2; \phi_i(j))$, the leading coefficient of $\phi_i(j)(k, X_2, E/\pi)$ with respect to $2X_2$. Noting $\sigma_0(s) = 1$, we get

$$\begin{cases} \lambda(2X_2; \phi_1(u)) = \dfrac{1}{(2s_u - 1)2^{2s_u+1}} C'_{s_u-1}, \\ \lambda(2X_2; \phi_2(v)) = \dfrac{s_v}{(2s_v - 1)2^{2s_v-1}} C_{s_v-1}, \\ \lambda(2X_2; \phi_3(w)) = \dfrac{(s_w - 1)!^2}{2(2s_w - 1)!}. \end{cases} \tag{4.11}$$

From $s_1 < s_2 < s_3$ we see that the maximum of

$$\deg_{X_2}(\phi_1(u)\phi_2(v)\phi_3(w)) = 2s_u + (2s_v - 1) + 1 = 2(s_u + s_v)$$

is attained when $(s_u, s_v) = (s_2, s_3)$ and $(s_u, s_v) = (s_3, s_2)$. This implies that the leading coefficient of $\Delta(k, X_2, E/\pi)$ satisfies

$$\begin{aligned} |\lambda(2X_2; \Delta)| &= |\lambda(2X_2; \phi_1(2)\phi_2(3)\phi_3(1) - \phi_1(3)\phi_2(2)\phi_3(1))| \\ &= \frac{(s_1 - 1)!^2 |s_3 C_{s_3-1} C'_{s_2-1} - s_2 C_{s_2-1} C'_{s_3-1}|}{2^{2(s_2+s_3)+1}(2s_2 - 1)(2s_3 - 1)(2s_1 - 1)!}, \end{aligned}$$

which does not vanish as a polynomial in k by the assumption (4.2). Since k, K/π, E/π are algebraically independent, we have $\Delta(k, K/\pi, E/\pi) \neq 0$, and therefore Lemma 4.1 follows from Lemma 3.1. ∎

In the next lemma, we replace the condition (4.2) by a simpler one, (4.13). We put

$$b_j := \frac{(-1)^j 2^{2j-1}}{(2j)!} - \frac{j+1}{2} a_j \quad (j \geq 1), \tag{4.12}$$

where the a_j are given by (4.6), in particular $b_j < 0$ if j is odd.

Lemma 4.2. *Let $1 < s_1 < s_2 < s_3$ be even integers. Assume that*

$$\frac{s_3}{s_2} \neq \frac{a_{s_2-1} b_{s_3-1}}{a_{s_3-1} b_{s_2-1}}. \tag{4.13}$$

Then the numbers $\Phi_{2s_1}, \Phi_{2s_2}, \Phi_{2s_3}$ are algebraically independent.

Proof. We put

$$C_{s-1} = \alpha_{s,0} + \alpha_{s,1} k^2 + \cdots + \alpha_{s,s} k^{2s} \quad (s \geq 2), \tag{4.14}$$

where $\alpha_{s,0}\alpha_{s,1} \neq 0$ will follow from (4.16) below. We assume that (4.3) does not hold, that is, for some rational number r,

$$\begin{aligned} &(\alpha_{s_2,0} + \alpha_{s_2,1} k^2 + \cdots + \alpha_{s_2,s_2} k^{2s_2})^{s_3} \\ &\qquad = r(\alpha_{s_3,0} + \alpha_{s_3,1} k^2 + \cdots + \alpha_{s_3,s_3} k^{2s_3})^{s_2}, \end{aligned}$$

or

$$\alpha_{s_2,0}^{s_3} + s_3\alpha_{s_2,1}\alpha_{s_2,0}^{s_3-1}k^2 + \cdots + \alpha_{s_2,s_2}^{s_3}k^{2s_2s_3}$$
$$= r(\alpha_{s_3,0}^{s_2} + s_2\alpha_{s_3,1}\alpha_{s_3,0}^{s_2-1}k^2 + \cdots + \alpha_{s_3,s_3}^{s_2}k^{2s_2s_3}).$$

In particular,

$$\alpha_{s_2,0}^{s_3} = r\alpha_{s_3,0}^{s_2} \quad \text{and} \quad s_3\alpha_{s_2,1}\alpha_{s_2,0}^{s_3-1} = rs_2\alpha_{s_3,1}\alpha_{s_3,0}^{s_2-1}.$$

From these equations, we get

$$\frac{s_3}{s_2} = \frac{\alpha_{s_2,0}\alpha_{s_3,1}}{\alpha_{s_3,0}\alpha_{s_2,1}}. \tag{4.15}$$

In what follows we shall prove that

$$\alpha_{j+1,0} = a_j \quad \text{and} \quad \alpha_{j+1,1} = b_j \qquad (j \geq 1). \tag{4.16}$$

Then (4.15) contradicts our hypothesis (4.13), and the lemma follows immediately from the remark to Lemma 4.1.

By Lemma 2.3, we may put

$$\mathrm{ns}^2(z,k) = u_0(z) + u_1(z)k^2 + O(k^4) \qquad (k \to 0, z \to 0)$$

with $u_0(z) = z^{-2} + O(1)$, $u_1(z) = O(1)$ $(z \to 0)$. Using the estimate

$$\mathrm{sn}(z,k) = \sin z - \frac{k^2}{4}(z - \sin z\cos z)\cos z + O(k^4)$$

(cf. [1, 16.13.1]), we obtain around $z = 0$

$$\mathrm{ns}^2(z,k) = \operatorname{cosec}^2 z + \left(\frac{1}{2} - \frac{1}{4z}(z^2\operatorname{cosec}^2 z)'\right)k^2 + O(k^4) \tag{4.17}$$

as $k \to 0$. We recall the definition of the polynomials $C_j = C_j(k) = c_j(k) - d_j(k)$ by the series expansion of $\mathrm{ns}^2(z,k) + (k^2-1)\,\mathrm{nd}^2(z,k)$ in (4.1). By (4.17) one has

$$\frac{1}{z^2} + \frac{1}{3} + \sum_{j=1}^{\infty} \alpha_{j+1,0}z^{2j} = \mathrm{ns}^2(z,0) = \operatorname{cosec}^2 z = \frac{1}{z^2} + \sum_{j=0}^{\infty} a_j z^{2j}. \tag{4.18}$$

This proves the first identity in (4.16). Next, note that $(k^2-1)\,\mathrm{nd}^2(z,k)+1 = k^2(1 - \mathrm{sn}^2(z,0)) + O(k^4)$ with

$$1 - \mathrm{sn}^2(z,0) = \cos^2 z = \frac{1}{2} + \frac{1}{2}\cos(2z) = \frac{1}{2} + \sum_{j=0}^{\infty} \frac{(-1)^j(2z)^{2j}}{2(2j)!}.$$

Thus, from $\mathrm{ns}^2(z,k) = z^{-2} + \sum_{j=0}^{\infty} c_j(k)z^{2j}$ we compute the following generating function for the numbers $\alpha_{j+1,1}$:

$$\sum_{j=1}^{\infty} \alpha_{j+1,1}z^{2j} = \left[\frac{1}{2}\frac{d^2}{dk^2}\Big(\sum_{j=1}^{\infty} c_j(k)z^{2j}\Big)\right]_{k=0} + \sum_{j=1}^{\infty} \frac{(-1)^j 2^{2j-1}}{(2j)!}z^{2j}$$
$$= \sum_{j=1}^{\infty} \frac{c_j''(0)}{2}z^{2j} + \sum_{j=1}^{\infty} \frac{(-1)^j 2^{2j-1}}{(2j)!}z^{2j}.$$

It follows that

$$\alpha_{j+1,1} = \frac{c_j''(0)}{2} + \frac{(-1)^j 2^{2j-1}}{(2j)!} \qquad (j \geq 1).$$

The second identity in (4.16) is verified by using $c_j''(0) = -(j+1)a_j$ $(j \geq 1)$, which follows immediately from (4.17), or by differentiating the recurrence formula from Lemma 2.3 twice with respect to k and using $a_j = c_j(0)$, $c_j'(0) = 0$ for $j \geq 1$. ∎

5. Algebraic independence of Φ_{2s} for even s. In this section we shall prove the following result.

PROPOSITION 5.1. *Let s_1, s_2, s_3 be distinct even positive integers. Then the numbers $\Phi_{2s_1}, \Phi_{2s_2}, \Phi_{2s_3}$ are algebraically independent.*

For this, we shall show that for even $s_1 < s_2 < s_3$ the condition (4.13) in Lemma 4.2 is satisfied.

We remark that all $a_0, a_1, \ldots$ defined by (4.6) are positive.

LEMMA 5.2. *Let $j \geq k+2 \geq 4$ be integers. Then*

$$\frac{a_j}{a_k} > 4^{j-k}\frac{(2k)!}{(2j)!}.$$

Moreover, for every $j \geq 1$,

$$\frac{a_j}{a_{j-1}} > \frac{j+1}{2\pi^2 j}.$$

Proof. By (4.6) and the following inequalities for Bernoulli numbers (cf. [1, 23.1.15]):

$$\frac{2(2n)!}{(2\pi)^{2n}} < |B_{2n}| < \frac{2(2n)!}{(2\pi)^{2n}(1-2^{1-2n})} \qquad (n \geq 1),$$

we have

$$\frac{(2j+1)2^{2j+3}}{(2\pi)^{2j+2}} < a_j < \frac{(2j+1)2^{2j+3}}{(2\pi)^{2j+2}(1-2^{-2j-1})} \quad (j \geq 0),$$

which yields, for any nonnegative integers j, k,

$$\frac{a_j}{a_k} > \frac{2j+1}{2k+1}4^{j-k}(2\pi)^{2k-2j}(1-2^{-2k-1}). \tag{5.1}$$

If $k = j-1$,

$$\frac{a_j}{a_{j-1}} > \frac{(2j+1)(1-2^{1-2j})}{(2j-1)\pi^2} \geq \frac{j+1}{2j\pi^2},$$

which is the second inequality. Suppose that $m := j - k \geq 2$, $k \geq 2$. Observing that $2\pi/(2k+3) \leq 2\pi/7 < 1 - 2^{-2k-1}$, we have

$$\begin{aligned}
\frac{(2k)!}{(2j)!} &= \frac{1}{(2k+1)\cdots(2j)} \leq \frac{(2k+3)^2}{(2k+1)(2k+2)} \cdot \frac{1}{(2k+3)^{2m}} \\
&= \frac{(2k+3)^2}{(2k+2)(2k+2m+1)} \cdot \frac{2j+1}{2k+1} \cdot (2\pi)^{2k-2j} \cdot \left(\frac{2\pi}{2k+3}\right)^{2m} \\
&\leq \frac{2j+1}{2k+1}(2\pi)^{2k-2j}\frac{2\pi}{7} < \frac{2j+1}{2k+1}(2\pi)^{2k-2j}(1-2^{-2k-1}).
\end{aligned}$$

Combining this with (5.1), we obtain the first inequality. ∎

Proof of Proposition 5.1. We may assume that $s_1 < s_2 < s_3$. It follows from (4.12) and Lemma 5.2 that

$$\begin{aligned}
&\frac{s_3}{s_2} - \frac{a_{s_2-1}b_{s_3-1}}{a_{s_3-1}b_{s_2-1}} \\
&\quad = \frac{s_3}{s_2} - \frac{-\dfrac{s_3}{2} - \dfrac{2^{2s_3-3}}{a_{s_3-1}(2s_3-2)!}}{-\dfrac{s_2}{2} - \dfrac{2^{2s_2-3}}{a_{s_2-1}(2s_2-2)!}} = \frac{s_3}{s_2} - \frac{s_3 + \dfrac{2^{2s_3-2}}{a_{s_3-1}(2s_3-2)!}}{s_2 + \dfrac{2^{2s_2-2}}{a_{s_2-1}(2s_2-2)!}} \\
&\quad = \frac{s_3\dfrac{2^{2s_2-2}}{a_{s_2-1}(2s_2-2)!} - s_2\dfrac{2^{2s_3-2}}{a_{s_3-1}(2s_3-2)!}}{s_2\left(s_2 + \dfrac{2^{2s_2-2}}{a_{s_2-1}(2s_2-2)!}\right)} \\
&\quad = \frac{\dfrac{s_2}{a_{s_3-1}}\,\dfrac{2^{2s_2-2}}{(2s_2-2)!}}{s_2\left(s_2 + \dfrac{2^{2s_2-2}}{a_{s_2-1}(2s_2-2)!}\right)} \cdot \left(\frac{s_3}{s_2}\,\frac{a_{s_3-1}}{a_{s_2-1}} - 2^{2(s_3-s_2)}\frac{(2s_2-2)!}{(2s_3-2)!}\right) \\
&\quad > \frac{\dfrac{s_2}{a_{s_3-1}}\,\dfrac{2^{2s_2-2}}{(2s_2-2)!}}{s_2\left(s_2 + \dfrac{2^{2s_2-2}}{a_{s_2-1}(2s_2-2)!}\right)} \cdot \left(\frac{a_{s_3-1}}{a_{s_2-1}} - 4^{s_3-s_2}\frac{(2s_2-2)!}{(2s_3-2)!}\right).
\end{aligned}$$

The hypotheses of Lemma 5.2 are satisfied for $j = s_3 - 1$ and $k = s_2 - 1$, since $s_3 - s_2 \geq 2$, $s_2 - 1 \geq 4 - 1 = 3$, and $(s_2-1)(s_3-1) \equiv 1 \bmod 2$. Therefore, we conclude that

$$\frac{s_3}{s_2} - \frac{a_{s_2-1}b_{s_3-1}}{a_{s_3-1}b_{s_2-1}} > 0,$$

so that condition (4.13) is satisfied. Thus, Proposition 5.1 follows from Lemma 4.2. ∎

6. Results with odd indices. In the preceding sections 4 and 5 all the indices s_1, s_2, s_3 were assumed to be even. In this section we treat the remaining cases in which at least one index is odd. Thus, we complete the proof of the main theorem stated in Section 1. We need the expressions of Φ_{2s} for odd s. Apart from $C_j^- := C_j(k) = c_j(k) - d_j(k)$ and $b_j^+ := b_j$ from (4.12) we additionally need

(6.1)
$$C_j^+ := c_j(k) + d_j(k) \quad (j \geq 1), \qquad b_j^- := \frac{(-1)^j 2^{2j-1}}{(2j)!} + \frac{j+1}{2} a_j \quad (j \geq 1),$$

for which we know that $\deg_k C_j^+ \leq 2 + 2j$ and

$$b_j^+ < 0 \quad (j \text{ odd}), \qquad b_j^- > 0 \quad (j \text{ even}). \tag{6.2}$$

For any odd integer s we have the representation (cf. [4])

$$\begin{aligned}\Phi_{2s} = \frac{1}{(2s-1)!}\Bigg[& \frac{(s-1)!^2}{24}\bigg(1 - \Big(\frac{2K}{\pi}\Big)^2 (1-2k^2)\bigg) \\ & + \sum_{j=1}^{s-1} \sigma_{s-j-1}(s) \frac{(-1)^j (2j)!}{2^{2j+3}} \bigg(a_j - \Big(\frac{2K}{\pi}\Big)^{2j+2} C_j^+\bigg)\Bigg].\end{aligned} \tag{6.3}$$

If s_1, s_2, s_3 are odd, then it follows from (6.3) with Lemmas 2.3 and 2.4 that

$$\Phi_{2s_1}, \Phi_{2s_2}, \Phi_{2s_3} \in \mathbb{Q}(k, K/\pi),$$

so that these three numbers are algebraically dependent. We split the remaining cases into two parts:

(6.4) Case 1: Two indices s_i are odd. Case 2: Two indices s_i are even.

Recall the function $\Phi_{2s}(X_1, X_2, X_3)$ obtained from Φ_{2s} by the substitution $(k, K/\pi, E/\pi) \mapsto (X_1, X_2, X_3)$. We write $(\partial\Phi_{2s}/\partial X_i)(k, X_2, E/\pi)$ as in (4.7)–(4.9), but now assuming s to be odd:

$$\begin{aligned}\frac{\partial\Phi_{2s}}{\partial X_1}(k, X_2, E/\pi) = \frac{1}{(2s-1)!}\Bigg[& \frac{(s-1)!^2}{6}(2X_2)^2 k \\ & + \sum_{j=1}^{s-1} \sigma_{s-j-1}(s) \frac{(-1)^{j+1}(2j)!}{2^{2j+3}} (2X_2)^{2j+2} (C_j^+)'\Bigg],\end{aligned} \tag{6.5}$$

$$\begin{aligned}\frac{\partial\Phi_{2s}}{\partial X_2}(k, X_2, E/\pi) = \frac{1}{(2s-1)!}\Bigg[& -\frac{(s-1)!^2}{6} 2X_2(1-2k^2) \\ & + \sum_{j=1}^{s-1} \sigma_{s-j-1}(s) \frac{(-1)^{j+1}(2j)!(j+1)}{2^{2j+1}} (2X_2)^{2j+1} C_j^+\Bigg],\end{aligned} \tag{6.6}$$

$$\frac{\partial\Phi_{2s}}{\partial X_3}(k, X_2, E/\pi) = 0. \tag{6.7}$$

First we assume that $1 \notin \{s_1, s_2, s_3\}$. Then without loss of generality we have the following two cases:

$$\begin{array}{lll} \text{Case 1:} & 3 \le s_1 < s_2 \text{ odd}, & 2 \le s_3 \text{ even}; \\ \text{Case 2:} & 3 \le s_1 \text{ odd}, & 2 \le s_2 < s_3 \text{ even}. \end{array}$$

Case 1. The determinant $\Delta(X_1, X_2, X_3)$ defined by (4.10) is simplified to

$$\Delta(X_1, X_2, X_3) = \phi_1(1)\phi_2(2)\phi_3(3) - \phi_1(2)\phi_2(1)\phi_3(3),$$

since $\phi_3(1) = \phi_3(2) = 0$ by (6.7). For $i, j \in \{1, 2\}$ we get from (6.5) and (6.6) the leading coefficients of $\phi_i(j)(k, X_2, E/\pi)$ with respect to $2X_2$:

$$\begin{aligned} \lambda(2X_2; \phi_1(u)) &= -\frac{1}{(2s_u - 1)2^{2s_u+1}}(C^+_{s_u-1})', \\ \lambda(2X_2; \phi_2(v)) &= -\frac{s_v}{(2s_v - 1)2^{2s_v-1}}C^+_{s_v-1}, \end{aligned}$$

and $\lambda(2X_2; \phi_3(3))$ was already computed (see (4.11)). Hence we get

$$|\lambda(2X_2; \Delta)| = \frac{(s_3 - 1)!^2|s_2C^+_{s_2-1}(C^+_{s_1-1})' - s_1C^+_{s_1-1}(C^+_{s_2-1})'|}{2^{2(s_1+s_2)+1}(2s_1 - 1)(2s_2 - 1)(2s_3 - 1)!}.$$

Similarly to the remark following Lemma 4.1, we see that this leading coefficient does not vanish if

$$(C^+_{s_1-1})^{s_2}/(C^+_{s_2-1})^{s_1} \notin \mathbb{Q}. \tag{6.8}$$

Putting

$$C^+_{s-1} = \beta_{s,0} + \beta_{s,1}k^2 + \cdots + \beta_{s,s}k^{2s} \quad (s \ge 3, s \text{ odd}), \tag{6.9}$$

it follows as in the proof of Lemma 4.2 that (6.8) results from the condition

$$\frac{s_2}{s_1} \ne \frac{\beta_{s_1,0}\beta_{s_2,1}}{\beta_{s_2,0}\beta_{s_1,1}}.$$

Here $\beta_{j+1,0} = a_j \ne 0$ $(j \ge 1)$, and

$$\beta_{j+1,1} = \frac{c_j''(0)}{2} - \frac{(-1)^j 2^{2j-1}}{(2j)!} \quad (j \ge 1),$$

so that $\beta_{j+1,1} = -b_j^- \ne 0$ $(j \ge 1)$ follows from (6.1) and (4.17). Note that we have $s_1 \ge 3$ by the assumptions of Case 1. Finally, applying Lemma 5.2 with even $j = s_2 - 1$, $k = s_1 - 1 \ge 2$, we deduce (6.8) as in the proof of Proposition 5.1 from

$$\frac{s_2}{s_1} > \frac{\beta_{s_1,0}\beta_{s_2,1}}{\beta_{s_2,0}\beta_{s_1,1}} = \frac{a_{s_1-1}b^-_{s_2-1}}{a_{s_2-1}b^-_{s_1-1}} = \frac{s_2 + \dfrac{2^{2s_2-2}}{a_{s_2-1}(2s_2-2)!}}{s_1 + \dfrac{2^{2s_1-2}}{a_{s_1-1}(2s_1-2)!}}.$$

Hence, we have proved that $\Phi_{2s_1}, \Phi_{2s_2}, \Phi_{2s_3}$ are algebraically independent over $\mathbb{Q}$.

Case 2. The determinant $\Delta(X_1, X_2, X_3)$ takes the form

$$\begin{aligned}\Delta(X_1, X_2, X_3) = &\big(\phi_1(1)\phi_2(2)\phi_3(3) - \phi_1(2)\phi_2(1)\phi_3(3)\big)\\ &+ \big(\phi_1(3)\phi_2(1)\phi_3(2) - \phi_1(1)\phi_2(3)\phi_3(2)\big).\end{aligned}$$

Here we have

$$\begin{aligned}\deg_{X_2}(\phi_1(1)\phi_2(2)\phi_3(3)) = \deg_{X_2}(\phi_1(2)\phi_2(1)\phi_3(3)) = 2(s_1 + s_2),\\ \deg_{X_2}(\phi_1(3)\phi_2(1)\phi_3(2)) = \deg_{X_2}(\phi_1(1)\phi_2(3)\phi_3(2)) = 2(s_1 + s_3),\end{aligned}$$

where, by the assumption of Case 2, $\deg_{X_2} \Delta = 2(s_1 + s_3)$. Hence we get

$$\begin{aligned}|\lambda(2X_2; \Delta)| &= |\lambda(2X_2, \phi_1(3)\phi_2(1)\phi_3(2)) - \lambda(2X_2, \phi_1(1)\phi_2(3)\phi_3(2))|\\ &= \frac{(s_2-1)!^2|s_1 C^+_{s_1-1}(C^-_{s_3-1})' - s_3 C^-_{s_3-1}(C^+_{s_1-1})'|}{2^{2(s_1+s_3)+1}(2s_1-1)(2s_3-1)(2s_2-1)!}.\end{aligned} \tag{6.10}$$

Assume that the right-hand side vanishes, namely

$$(C^-_{s_3-1})^{s_1}/(C^+_{s_1-1})^{s_3} \in \mathbb{Q}.$$

We express C^-_{s-1} and C^+_{s-1} as in (4.14) and (6.9), respectively. We then get

$$\frac{s_3}{s_1} = \frac{\alpha_{s_3,1}\beta_{s_1,0}}{\alpha_{s_3,0}\beta_{s_1,1}} = -\frac{a_{s_1-1}b^+_{s_3-1}}{a_{s_3-1}b^-_{s_1-1}}. \tag{6.11}$$

Here, we may have $s_1 < s_3$, or $s_1 > s_3$. To handle all possible situations, we distinguish four cases:

$$\begin{aligned}&\text{Case 2.1:}\quad s_1 \le s_3 - 3, &&\text{Case 2.2:}\quad s_1 \ge s_3 + 3,\\ &\text{Case 2.3:}\quad s_1 = s_3 - 1, &&\text{Case 2.4:}\quad s_1 = s_3 + 1.\end{aligned}$$

Case 2.1. We have $s_1 - 1 \equiv 0 \bmod 2$, $s_3 - 1 \equiv 1 \bmod 2$, and $s_3 \ge 6$, $s_3 - s_1 \ge 3$. As in the proof of Proposition 5.1 we get

(6.12)

$$\frac{s_3}{s_1} + \frac{a_{s_1-1}b^+_{s_3-1}}{a_{s_3-1}b^-_{s_1-1}} > \frac{\dfrac{s_1}{a_{s_3-1}}\,\dfrac{2^{2s_1-2}}{(2s_1-2)!}}{s_1\left(s_1 + \dfrac{2^{2s_1-2}}{a_{s_1-1}(2s_1-2)!}\right)} \cdot \left(\frac{a_{s_3-1}}{a_{s_1-1}} - 4^{s_3-s_1}\frac{(2s_1-2)!}{(2s_3-2)!}\right).$$

From Lemma 5.2 with $j = s_3 - 1 \ge 5$ and $k = s_1 - 1 \ge 2$ we conclude that the right-hand side of (6.12) is positive. Thus, (6.11) does not hold in Case 2.1.

Case 2.2. We have $s_1 - s_3 \ge 3$ with $s_3 \ge 4$. Using $s_3/s_1 < 1$, one gets instead of (6.12) the inequality

(6.13)

$$\frac{s_3}{s_1}+\frac{a_{s_1-1}b^+_{s_3-1}}{a_{s_3-1}b^-_{s_1-1}}<\frac{\dfrac{s_1}{a_{s_3-1}}\dfrac{2^{2s_1-2}}{(2s_1-2)!}}{s_1\left(s_1+\dfrac{2^{2s_1-2}}{a_{s_1-1}(2s_1-2)!}\right)}\cdot\left(\frac{a_{s_3-1}}{a_{s_1-1}}-4^{s_3-s_1}\frac{(2s_1-2)!}{(2s_3-2)!}\right).$$

Here, we apply Lemma 5.2 with $j=s_1-1\geq 6$ and $k=s_3-1\geq 3$. Finding the relation

$$\frac{a_{s_1-1}}{a_{s_3-1}}>4^{s_1-s_3}\frac{(2s_3-2)!}{(2s_1-2)!},$$

it follows that the right-hand side of (6.13) is negative, which contradicts (6.11).

Case 2.3. Put $s:=s_1\geq 3$. By (4.12) and (6.1), equation (6.11) takes the form

$$\frac{s+1}{s}=\frac{\dfrac{2^{2s}}{(2s)!a_s}+(s+1)}{\dfrac{2^{2s-2}}{(2s-2)!a_{s-1}}+s},$$

or, equivalently,

$$\frac{a_s}{a_{s-1}}=\frac{2}{(s+1)(2s-1)}. \tag{6.14}$$

Then, from the second inequality in Lemma 5.2, it follows that

$$\frac{s+1}{2\pi^2 s}<\frac{2}{(s+1)(2s-1)},$$

which does not hold for $s\geq 4$. The equality (6.14) is also false for $s=3$, since $a_3/a_2=7/50$.

Case 2.4. Put $s:=s_1\geq 5$. Again, we have (6.14), which is impossible as shown in Case 2.3.

Now it remains to discuss the two cases (6.4) with $1\in\{s_1,s_2,s_3\}$. Then the arguments are restricted to the following two cases:

Case 1: $s_1=1<s_2$ odd, $2\leq s_3$ even;
Case 2: $s_1=1,$ $2\leq s_2<s_3$ even.

Case 1. By (6.3), Φ_{2s_1} has the simple form

$$\Phi_2=\frac{1}{24}\left(1-\left(\frac{2K}{\pi}\right)^2(1-2k^2)\right).$$

This implies that

$$\frac{\partial\Phi_2}{\partial X_1}(k,X_2,E/\pi)=\frac{k}{6}(2X_2)^2,$$

$$\frac{\partial \Phi_2}{\partial X_2}(k, X_2, E/\pi) = \frac{2k^2-1}{6} \cdot 2X_2,$$
$$\frac{\partial \Phi_2}{\partial X_3}(k, X_2, E/\pi) = 0.$$

We now have

$$\Delta(X_1, X_2, X_3) = \phi_1(1)\phi_2(2)\phi_3(3) - \phi_1(2)\phi_2(1)\phi_3(3),$$
$$\lambda(2X_2, \phi_1(1)\phi_2(2)\phi_3(3)) = -\frac{ks_2(s_3-1)!^2}{6(2s_2-1)2^{2s_2}(2s_3-1)!} \cdot C^+_{s_2-1},$$
$$\lambda(2X_2, \phi_1(2)\phi_2(1)\phi_3(3)) = -\frac{(2k^2-1)(s_3-1)!^2}{6(2s_2-1)2^{2s_2+2}(2s_3-1)!} \cdot (C^+_{s_2-1})',$$
$$\deg_{X_2}(\phi_1(1)\phi_2(2)\phi_3(3)) = \deg_{X_2}(\phi_1(2)\phi_2(1)\phi_3(3)) = 2 + 2s_2.$$

Hence, it follows that

$$|\lambda(2X_2, \Delta)| = \frac{(s_3-1)!^2}{6(2s_2-1)2^{2s_2}(2s_3-1)!}\left|ks_2C^+_{s_2-1} - \frac{2k^2-1}{4}(C^+_{s_2-1})'\right|.$$

We assume that the right-hand side vanishes, namely

$$C^+_{s_2-1}/(2k^2-1)^{s_2} \in \mathbb{Q}.$$

Then, writing C^+_{s-1} as (6.9) with $\beta_{s,0} = a_{s-1}$, $\beta_{s,1} = -b^-_{s-1}$, we have $\beta_{s_2,0} = -r$ and $\beta_{s_2,1} = 2s_2r$ for some nonvanishing $r \in \mathbb{Q}$. Hence

$$-2s_2 = \frac{\beta_{s_2,1}}{\beta_{s_2,0}} = -\frac{b^-_{s_2-1}}{a_{s_2-1}}. \tag{6.15}$$

It follows for $j = s_2 - 1 \equiv 0 \bmod 2$ from (6.1) and (6.15) that

$$2s_2 = \frac{b^-_{s_2-1}}{a_{s_2-1}} = \frac{s_2}{2} + \frac{2^{2s_2-3}}{(2s_2-2)!a_{s_2-1}},$$

or, equivalently,

$$s_2 = \frac{2^{2s_2-2}}{3(2s_2-2)!a_{s_2-1}}. \tag{6.16}$$

Since $s_2 > 1$ is odd, we have $s_2 \geq 3$. (6.16) does not hold for $s_2 = 3, 5, \ldots, 19$, since the right-hand side takes the values

$$63\ (s_2 = 3), \quad 33\ (s_2 = 5), \quad 3\ (s_2 = 7),$$
$$\text{less than } 1\ (s_2 = 9, 11, \ldots, 19).$$

Therefore, we may assume $s_2 \geq 21$. Next, we apply Lemma 5.2 with $k = 2$ and $j \geq 4$:

$$a_j > 4^{j-2}\frac{4!}{(2j)!}a_2 = 4^{j-2}\frac{4!}{(2j)!}\,\frac{2}{189} = \frac{1}{63}\,\frac{2^{2j}}{(2j)!}.$$

Thus we estimate the right-hand side of (6.16) by

$$s_2 < \frac{2^{2s_2-2}}{3(2s_2-2)!} \cdot \frac{63(2s_2-2)!}{2^{2s_2-2}} = 21,$$

which contradicts our assumption on s_2. As before, it follows that Φ_2, Φ_{2s_2}, Φ_{2s_3} are algebraically independent.

Case 2. Here we have a situation described by (6.10) with $s_1 = 1$, and $C^+_{s_1-1}$ replaced by $(1-2k^2)/4$. Therefore it remains to investigate

$$C^-_{s_3-1}/(2k^2-1)^{s_3} \in \mathbb{Q}.$$

Writing C^-_{s-1} as in (4.14), we know that $\alpha_{s,0} = a_{s-1}$ and $\alpha_{s,1} = b^+_{s-1}$. Since s_3 is even, it follows from (4.12) that

$$-2s_3 = \frac{b^+_{s_3-1}}{a_{s_3-1}} = -\frac{s_3}{2} - \frac{2^{2s_3-3}}{(2s_3-2)!a_{s_3-1}},$$

or, equivalently,

$$s_3 = \frac{2^{2s_3-2}}{3(2s_3-2)!a_{s_3-1}}. \tag{6.17}$$

By $s_3 > s_2 \geq 2$ we have $s_3 \geq 4$. (6.17) does not hold for $s_3 = 4, 6, 8, \ldots, 18$, since the right-hand side takes the values

$$20\ (s_3 = 4), \quad \frac{2730}{691}\ (s_3 = 6),$$
$$\text{less than } 1\ (s_3 = 8, 10, \ldots, 18).$$

Therefore, we may assume $s_3 \geq 20$. Applying Lemma 5.2 with $j = s_3 - 1$, $k = 3$, and $a_3 = 1/675$, we get

$$a_{s_3-1} > a_3 4^{s_3-4} \frac{720}{(2s_3-2)!} = \frac{2^{2s_3-4}}{15(2s_3-2)!},$$

which can be used to estimate the right-hand side of (6.17):

$$s_3 < \frac{2^{2s_3-2}}{3(2s_3-2)!} \cdot \frac{15(2s_3-2)!}{2^{2s_3-4}} = 20.$$

This contradiction completes the proof of the theorem. ■

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