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On the success and failure of Gram's Law and the Rosser Rule

by

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1. Introduction. Table 1 summarises the locations of the results which pertain to Gram's Law (GL), the Weak Gram Law (WGL), and Rosser's Rule (RR). The columns represent whether the phenomenon is true or false for infinitely many intervals, and for a positive proportion of intervals. Question marks denote a lack of knowledge about a particular statement; asterisks denote new results. Throughout this paper, the letter A will denote a positive quantity, not necessarily the same at each occurrence.

Table 1. Summary of results

	true		false	
	i.m. intervals	p.p. of intervals	i.m. intervals	p.p. of intervals
GL	?	?	§4.3	§5.2*
WGL	§3	§6*	§4.3	§5.3*
RR	?	?	§7.3	§7.6*

1.1. Definition and properties of the zeta-function. The *Riemann zeta-function*, defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \tag{1}$$

when $\sigma > 1$, can be shown by analytic continuation to be a meromorphic function, with a simple pole at $s = 1$, at which the residue is equal to unity. It is known that $\zeta(s)$ satisfies the following functional equation:

$$\zeta(s) = \chi(s)\zeta(1-s), \tag{2}$$

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where

$$\chi(s) = 2^s \pi^{s-1} \sin \tfrac{1}{2}s\pi \cdot \Gamma(1-s) = \pi^{s-1/2} \frac{\Gamma\left(\frac{1}{2}-\frac{1}{2}s\right)}{\Gamma\left(\frac{1}{2}s\right)}. \tag{3}$$

The product taken over the primes in (1) is absolutely convergent for $\sigma > 1$, whence it follows that $\zeta(s)$ has no zeroes in this region. Moreover, equations (2) and (3) show that the only zeroes of $\zeta(s)$ for $\sigma < 0$ are at the points $s = -2, -4, -6, \ldots$, since at these points $\sin \frac{1}{2}s\pi = 0$. These zeroes are called the *trivial zeroes*, and hereafter the term "zeroes" when applied to the zeta-function refers to those zeroes of $\zeta(s)$ with $0 \le \sigma \le 1$. The work of Hadamard and de la Vallée Poussin which lead to the proof of the prime number theorem (see e.g. [37, Ch. III]) shows that the zeroes of $\zeta(s)$ are confined to the *critical strip* defined as the region $0 < \sigma < 1$. It follows from (2) that the zeroes of $\zeta(s)$ are symmetric about the lines $t = 0$ and $\sigma = \frac{1}{2}$, the latter of which is hereafter referred to as the *critical line*. The conjecture that *all* the zeroes lie on the critical line is the Riemann hypothesis.

1.2. Location of zeroes. Following the work of Riemann, it has become customary to work with an entire function $\xi(s)$, to avoid difficulties that are encountered by the pole of $\zeta(s)$ at $s = 1$. One writes

$$\xi(s) = \tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\tfrac{1}{2}s\right)\zeta(s), \tag{4}$$

whence it is seen that $\xi(s)$ is entire and that the zeroes of $\zeta(s)$ coincide with the zeroes of $\xi(s)$. By (2) and (3) one can show that

$$\xi\left(\tfrac{1}{2}+it\right) = \xi\left(\tfrac{1}{2}-it\right),$$

whence, applying the reflection principle, it is seen that $\xi\left(\frac{1}{2}+it\right)$ is real. This is useful information indeed since, in a search for a zero of $\xi\left(\frac{1}{2}+it\right)$, one can now search for an interval (t_1, t_2) in which $\xi\left(\frac{1}{2}+it\right)$ changes sign. Since it is a real-valued function, the change of sign guarantees the presence of an odd ([1]) zero of $\xi\left(\frac{1}{2}+it\right)$ and hence of $\zeta\left(\frac{1}{2}+it\right)$ in the interval. This is the guiding principle behind all investigations regarding the whereabouts of zeroes of $\zeta\left(\frac{1}{2}+it\right)$. For convenience in calculation, it is the evaluation of a scaled multiple of $\xi\left(\frac{1}{2}+it\right)$ that is used, and this is introduced in the following section.

1.2.1. *The functions $Z(t)$ and $\theta(t)$.* By (3) it is clear that

$$\chi(s)\chi(1-s) = 1,$$

whence, by the reflection principle, it follows that

$$\left|\chi\left(\tfrac{1}{2}+it\right)\right| = 1.$$

([1]) Henceforth an *odd zero* refers to either an odd number of simple zeroes or a zero of odd multiplicity.

If one writes

$$\theta(t) = -\tfrac{1}{2}\arg\chi\big(\tfrac{1}{2}+it\big),$$

so that

$$\chi\big(\tfrac{1}{2}+it\big) = e^{-2i\theta(t)},$$

then one arrives at

$$Z(t) = e^{i\theta(t)}\zeta\big(\tfrac{1}{2}+it\big), \tag{5}$$

where $Z(t)$ is real-valued. To see that $Z(t)$ is real, note that (3) gives

$$\big\{\chi\big(\tfrac{1}{2}+it\big)\big\}^{-1/2} = \left\{\pi^{it}\frac{\Gamma\big(\frac{1}{4}-\frac{1}{2}it\big)}{\Gamma\big(\frac{1}{4}+\frac{1}{2}it\big)}\right\}^{-1/2} = \pi^{-it/2}\frac{\Gamma\big(\frac{1}{4}+\frac{1}{2}it\big)}{\big|\Gamma\big(\frac{1}{4}+\frac{1}{2}it\big)\big|},$$

whence, courtesy of (4),

$$Z(t) = -2\pi^{1/4}\frac{\xi\big(\frac{1}{2}+it\big)}{\big(t^2+\frac{1}{4}\big)\big|\Gamma\big(\frac{1}{4}+\frac{1}{2}it\big)\big|}, \tag{6}$$

wherein all terms appearing on the right side are real. For the function $\theta(t)$ one writes

$$\theta(t) = -\frac{1}{2}\left(t\log\pi + \arg\frac{\Gamma\big(\frac{1}{4}-\frac{1}{2}it\big)}{\Gamma\big(\frac{1}{4}+\frac{1}{2}it\big)}\right), \tag{7}$$

and appeals to Stirling's formula (see e.g. [33, Ch. IV, §42]) to show that

$$\theta(t) = \frac{1}{2}t\log\frac{t}{2\pi} - \frac{1}{2}t - \frac{\pi}{8} + O(t^{-1}), \tag{8}$$

$$\theta'(t) = \frac{1}{2}\log\frac{t}{2\pi} + O(t^{-2}), \qquad \theta''(t) \sim \frac{1}{2t}. \tag{9}$$

The function $Z(t)$ is sometimes called "Hardy's function" on account of its importance in Hardy's proof (see, e.g. [37, Ch. X]) that there are infinitely many zeroes on the critical line.

Recall that the purpose of the introduction of the functions $Z(t)$ and $\theta(t)$ is to find a zero of $\zeta\big(\frac{1}{2}+it\big)$ by finding an interval (t_1,t_2) in which $Z(t)$ changes sign. To this end, $Z(t_1)$ and $Z(t_2)$ are calculated, not directly (i.e. not from (6)) but from (5). The function $\theta(t)$ can be evaluated using (7), whence all that remains is to find a method to calculate $\zeta\big(\frac{1}{2}+it\big)$.

1.2.2. *The approximate functional equation.* Inside the critical strip one can approximate $\zeta(s)$ by the approximate functional equation of Hardy and Littlewood, given below as

THEOREM 1.1 (Hardy–Littlewood). *If h is a positive constant, $2\pi xy = t$, and $x, y > h$, then for $0 < \sigma < 1$ and $t > 0$,*

$$\zeta(s) = \sum_{n\le x} \frac{1}{n^s} + \chi(s)\sum_{n\le y}\frac{1}{n^{1-s}} + O(x^{-\sigma}) + O(t^{1/2-\sigma}y^{\sigma-1}). \tag{10}$$

Proof. See [37, Ch. IV, §§12–15]. ∎

Along the line $\sigma = \frac{1}{2}$ the error terms in the above equation are equal when $x \asymp y$. Thus one can take $\sigma = \frac{1}{2}$ and $x = y = \{t/2\pi\}^{1/2}$ in the approximate functional equation to show that

$$\zeta\big(\tfrac{1}{2} + it\big) = \sum_{n\le\sqrt{t/2\pi}} n^{-1/2-it} + \chi\big(\tfrac{1}{2}+it\big)\sum_{n\le\sqrt{t/2\pi}} n^{-1/2+it} + O(t^{-1/4}).$$

By multiplying both sides of this equation by $e^{i\theta(t)}$ one obtains

$$Z(t) = 2\sum_{n\le\sqrt{t/2\pi}} n^{-1/2}\cos\{\theta(t) - t\log n\} + O(t^{-1/4}). \tag{11}$$

The error term in (11) can be replaced, after the efforts of some careful analysis (see e.g. [37, Ch. IV, §16]), with an asymptotic series. Instead of the approximate functional equation, Gram, in [14], used Euler–Maclaurin summation and the following approximation (found in [37, Thm. 4.11]):

$$\zeta(s) = \sum_{n\le x}\frac{1}{n^s} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma}),$$

which is uniform in $\sigma \ge \sigma_0 > 0$, $|t| < 2\pi x/C$ where C is a given constant greater than 1. It was in the pursuance of values of $\zeta\big(\frac{1}{2}+it\big)$ via this method which led Gram to observe the phenomena underlying his eponymous principle.

2. Gram points. Gram calculated that $\Re\zeta\big(\frac{1}{2}+it\big)$ was very rarely negative, whereas $\Im\zeta\big(\frac{1}{2}+it\big)$ oscillated regularly between positive and negative values. Indeed he observed that the values of t such that $\Im\zeta\big(\frac{1}{2}+it\big) = 0$ interlaced with the zeroes of $\zeta\big(\frac{1}{2}+it\big)$.

Since $\theta(t)$ is ultimately increasing one can define points $\{g_\nu\}$ as those points which satisfy

$$\theta(g_\nu) = \nu\pi, \tag{12}$$

and, with a little care, one can show that the above equation has solutions for all $\nu \ge -1$. These, then, are the *Gram points*, at which (5) gives

$$\zeta\big(\tfrac{1}{2} + ig_\nu\big) = (-1)^\nu Z(g_\nu).$$

If $\zeta\big(\frac{1}{2}+it\big)$ is positive at successive Gram points g_ν and $g_{\nu+1}$, then the above equation shows that there must be a zero of $Z(t)$ for some $t \in (g_\nu, g_{\nu+1})$.

Gram calculated that $\zeta\left(\frac{1}{2}+ig_\nu\right)$ was positive for $-1 \le \nu \le 15$, and there is a good reason to suppose that this may hold for many values of ν. Consider (11) where $t = g_\nu$,

$$Z(g_\nu) = 2(-1)^\nu \sum_{n \le \sqrt{g_\nu/2\pi}} \frac{\cos(g_\nu \log n)}{n^{1/2}} + O(g_\nu^{-1/4}), \tag{13}$$

and note that the sum begins with $+1$, after which the terms are oscillatory and decreasing in magnitude. Provided there is not a conspiracy of a large quantity of negative terms, this initial $+1$ would dominate the sum. Thus one may expect that $(-1)^\nu Z(g_\nu) \sim 2$ "often": this is further explored in §3. One result which will be needed throughout this article is

LEMMA 2.1. *If $N_g(T)$ denotes the number of Gram points $g_\nu \in [0,T]$, then*

$$N_g(T) = \frac{T}{2\pi}\log T + O(T). \tag{14}$$

Also

$$g_\nu = O\left(\frac{\nu}{\log \nu}\right), \tag{15}$$

and if $g_\nu, g_\mu \in [T, 2T]$ then

$$g_\nu - g_\mu \sim \frac{2\pi(\nu-\mu)}{\log \nu} \sim \frac{2\pi(\nu-\mu)}{\log T}. \tag{16}$$

Proof. The first part follows from (12) and (8). For (15), note that (14) implies that $\nu\pi \sim \frac{1}{2} g_\nu \log(g_\nu/2\pi)$, so that $\log \nu \sim \log g_\nu$. Finally (16) is obtained by an application of the mean-value theorem to $\theta(t)$ using (9), and $\log \nu \sim \log g_\nu$, which has already been established. ∎

2.1. Gram's Law. It was Hutchinson [16] who proposed the notion of Gram's Law as given in

DEFINITION 2.2 (Gram's Law). Given Gram points g_ν and $g_{\nu+1}$, *Gram's Law* is said to hold if there is exactly one zero of $\zeta\left(\frac{1}{2}+it\right)$ for some t in the interval $(g_\nu, g_{\nu+1}]$.

Hutchinson's original definition (2) is couched in the zeroes of the function $Z(t)$. Adopting standard notation, Hutchinson's commentary runs thus:

> Gram calculated the first fifteen roots [of the function $Z(t)$] γ and called attention to the fact that the γ's and the g_ν's separate each other. I will refer to this property of the roots as *Gram's Law.*

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(2) In [16], rather confusingly, $Z(t)$ is labelled $\rho(t)$, and the Gram points are labelled γ_n.

It seems that zeroes of multiplicity greater than one were not considered in this definition: indeed the rest of [16] is concerned with finding a sign change in $Z(t)$. In verifying the Riemann hypothesis to a certain height, zeroes of multiplicity m are included as m simple zeroes. It is therefore natural to suppose that Hutchinson wrote "zero" for "simple zero". Furthermore, Definition 2.2 is not concerned with zeroes off the critical line, i.e. the presence of these zeroes does not contradict Gram's Law.

Calculation of the first 15 sign changes of $Z(t)$ and thus of the first 15 zeroes of $\zeta\left(\frac{1}{2}+it\right)$ was published by Gram in 1903. By considering the Euler-product formula, it was shown by Gram that these first 15 zeroes were the *only* non-trivial zeroes of $\zeta(s)$ up to height $t = 50$. Each of these zeroes was found to lie between successive Gram points. Gram's work was continued by Backlund [2] in 1914, who showed that $N_0(200) = N(200) = 79$, and that Gram's Law is true up to this height. Here, and elsewhere, the function $N_0(T)$ denotes the number of zeroes of $\zeta\left(\frac{1}{2}+it\right)$ for $0 < t < T$. This result was extended by Hutchinson [op. cit.], who in 1925 found the first 138 roots of $\zeta\left(\frac{1}{2}+it\right)$. Precisely one zero was contained in each Gram interval with four exceptions, occurring in two "pairs". The interval $(g_{125}, g_{126}]$ does not contain a zero of $\zeta\left(\frac{1}{2}+it\right)$, whereas the interval $(g_{126}, g_{127}]$ contains two zeroes. Likewise the interval $(g_{133}, g_{134}]$ contains two zeroes while the adjacent interval $(g_{134}, g_{135}]$ contains none. Moreover it was shown that these are the only complex zeroes of $\zeta(s)$ at this height. Table 2 shows the progress made in calculating zeroes of $\zeta\left(\frac{1}{2}+it\right)$: each method used Gram's Law or a variant thereof.

Table 2. Calculation of the first n zeroes of $\zeta(\frac{1}{2}+it)$ up to height $T(n)$

Year	Author	n	$T(n)$
1903	Gram [14]	15	66
1914	Backlund [2]	79	200
1925	Hutchinson [16]	138	300
1936	Titchmarsh [36]	1 041	1 467
1953	Turing [40]	1 104	1 540
1956	Lehmer [23]	15 000	14 041
1970	Lehman [22]	25 000	170 571
1969	Rosser, Yohe and Schoenfeld [30]	$3.5 \cdot 10^6$	$1.9 \cdot 10^6$
1979	Brent [4]	$4 \cdot 10^7$	$1.8 \cdot 10^7$
1983	van de Lune and te Riele [41]	$3 \cdot 10^8$	$1.2 \cdot 10^8$
1986	van de Lune, te Riele and Winter [42]	$1.5 \cdot 10^9$	$5.5 \cdot 10^8$
2004	Gourdon [13]	$1 \cdot 10^{13}$	$2.5 \cdot 10^{12}$

2.2. The Weak Gram Law. Some sources cite Gram's Law slightly differently to Definition 2.2. Sometimes the statement that

$$(-1)^{\nu} Z(g_{\nu}), (-1)^{\nu+1} Z(g_{\nu+1}) > 0, \tag{17}$$

is given as an equivalent definition to Gram's Law. Clearly this is implied by Definition 2.2, and the remarks after (13). However all that this alternative definition guarantees is the presence of an odd zero in the interval $(g_{\nu}, g_{\nu+1}]$, which quite possibly is actually 3, or 5, etc. zeroes.

It is for this reason that the property in (17) is called the "Weak Gram Law". The distinction appears, at first glance, to be one wrought from an over-zealous insistence on detail. However, Table 1 shows that the state of knowledge about Gram's Law is far less complete than that about the Weak Gram Law. Throughout this article various results will be proved about Gram's Law and the Weak Gram Law, the latter as defined in (17).

3. The Weak Gram Law is true infinitely often. The argument in this section is due to Titchmarsh [34], and it may also be found in [37, Ch. X, §6]. It is shown that $(-1)^{\nu} Z(g_{\nu})$ is positive on the average. A by-product of this theorem is a proof that there must be an infinity of zeroes of $\zeta\left(\frac{1}{2} + it\right)$.

Consider the sum

$$Z_1(g_{\nu}) = \sum_{n \le \sqrt{g_{\nu}/2\pi}} \frac{\cos(g_{\nu} \log n)}{\sqrt{n}} = 1 + \frac{\cos(g_{\nu} \log 2)}{\sqrt{2}} + \cdots,$$

that is, (13) but without the factor $2(-1)^{\nu}$ or the error term. One hopes to show that after the first term there is a fair amount of cancellation in this sum. What is needed is the following simple result regarding exponential sums and integrals.

LEMMA 3.1. *Let $f(x)$ be a continuous function which is differentiable over the interval $[a, b]$. If $f'(x)$ is monotonic and $|f'(x)| \le \theta < 1$, then*

$$\sum_{a < n \le b} e^{2\pi i f(n)} = \int_a^b e^{2\pi i f(x)}\, dx + O(1).$$

Proof. This is Lemma 4.18 in [37]. ∎

With a fixed M, examine the sum of $Z_1(g_{2\nu})$:

$$\sum_{\nu=M+1}^{N} Z_1(g_{2\nu}) = \sum_{\nu=M+1}^{N} \sum_{n \le \sqrt{g_{2\nu}/2\pi}} \frac{\cos(g_{2\nu} \log n)}{\sqrt{n}}. \tag{18}$$

This is a sum first over n, then over the Gram points indexed by ν. Note first that the $n = 1$ term in the inner sum is just $+1$, and this, by virtue of the outer sum, contributes $N - M$ to the total. Now change the order of summation to sum first over the Gram points $g_{2\nu}$. The conditions of summation are $g_{2N} \geq g_{2\nu} \geq g_{2(M+1)}$ and $g_{2\nu} \geq 2\pi n^2$. Define $\tau = \max\{2\pi n^2, g_{2(M+1)}\}$; then equation (18) becomes

$$\sum_{\nu=M+1}^{N} Z_1(g_{2\nu}) = N - M + \sum_{2\leq n\leq\sqrt{g_{2N}/2\pi}} \frac{1}{\sqrt{n}} \sum_{\tau\leq g_{2\nu}\leq g_{2N}} \cos(g_{2\nu}\log n).$$

The inner sum is of the form

$$\sum \cos\{2\pi\phi(\nu)\},$$

where $\phi(\nu) = \frac{g_{2\nu}\log n}{2\pi}$. In order that $\phi(\nu)$ be continuous, define t_ν to satisfy

$$\theta(t_\nu) = \nu\pi,$$

where ν need not be integral. Naturally this definition coincides with that of g_ν when ν is indeed an integer. Thus, to apply Lemma 3.1, a bound on $\phi'(\nu)$ is sought. Since

$$\phi'(\nu) = \frac{\log n}{2\pi}\frac{dt_{2\nu}}{d\nu}, \qquad \theta'(t_{2\nu})\frac{dt_{2\nu}}{d\nu} = 2\pi, \tag{19}$$

the derivative of $\phi(\nu)$ is related to that of $\theta(t_{2\nu})$, viz.

$$\phi'(\nu) = \frac{\log n}{\theta'(t_{2\nu})}. \tag{20}$$

As deduced in (9), when t is large, $\theta'(t) \sim \frac{1}{2}\log t$. Thus $\theta'(t_{2\nu})$ can be bounded below by

$$\theta'(t_{2\nu}) > \tfrac{1}{3}\log t_{2\nu}, \tag{21}$$

for sufficiently large ν. Since $\theta'(t_{2\nu})$ is bounded below, $\phi'(\nu)$ can be bounded above by

$$\phi'(\nu) < \frac{\log n}{2}\frac{3}{\log t_{2\nu}} < \frac{3}{4},$$

since the bounds of summation show that $t_{2\nu} \geq 2\pi n^2$ and so $\log t_{2\nu} > 2\log n$ for all ν. Also, via equations (20) and (21),

$$\phi''(\nu) = -2\pi\log n\,\frac{\theta''(t_{2\nu})}{\{\theta'(t_{2\nu})\}^3} < 0, \tag{22}$$

since by (9), $\theta''(t_{2\nu}) > 0$ for sufficiently large ν. Now the bound in Lemma 3.1 is used: when M is large enough, i.e. for large τ,

$$\sum_{\tau\leq g_{2\nu}\leq t_{2N}} \cos(g_{2\nu}\log n) = \int_{\tau\leq t_{2\nu}\leq t_{2N}} \cos\{2\pi\phi(\nu)\}\,d\nu + O(1).$$

Now

$$\int_{\tau\le t_{2\nu}\le t_{2N}} \cos\{2\pi\phi(\nu)\}\,d\nu = \int_{\tau\le t_{2\nu}\le t_{2N}} \frac{d\{\sin 2\pi\phi(\nu)\}}{d\nu}\,\frac{d\nu}{2\pi\phi'(\nu)}, \tag{23}$$

and integrating the right-hand side of the above equation by parts gives

$$\frac{\{\sin 2\pi\phi(\nu)\}}{2\pi\phi'(\nu)}\bigg|_{\tau}^{t_{2N}} + \frac{1}{2\pi}\int_{\tau\le t_{2\nu}\le t_{2N}} \frac{\{\sin 2\pi\phi(\nu)\}\phi''(\nu)}{\{\phi'(\nu)\}^2}\,d\nu.$$

The integrand has modulus at most

$$\left|\frac{\phi''(\nu)}{\phi'(\nu)^2}\right| = -\frac{\phi''(\nu)}{\phi'(\nu)^2},$$

where the equality comes from (22). This when integrated is $\{\phi'(\nu)\}^{-1}$. By equations (9) and (19) there exists a constant A such that

$$\phi'(\nu) = \frac{\log n}{2\theta'(t_{2\nu})} \ge \frac{\log n}{2\theta'(t_{2N})} \ge \frac{A\log n}{\log t_{2N}},$$

and so both the boundary term and the integral in (23) are

$$O\left(\frac{\log t_{2N}}{\log n}\right),$$

and therefore

$$\sum_{\tau\le g_{2\nu}\le t_{2N}} \cos\{2\pi\phi(\nu)\} = O\left(\frac{\log g_{2N}}{\log n}\right).$$

Finally returning to $\sum Z_1(g_{2\nu})$ for a fixed M,

$$\sum_{\nu=M+1}^{N} Z_1(g_{2\nu}) = N - M + O\left(\log g_{2N} \sum_{2\le n\le\sqrt{g_{2N}/2\pi}} \frac{1}{\sqrt{n}\log n}\right),$$

where the error term is, by partial summation, $O(\{g_{2N}\}^{1/4})$. With the use of Lemma 2.1 it follows that

$$\sum_{\nu=M+1}^{N} Z_1(g_{2\nu}) = N + O(N^{1/4}\log^{-1/4} N).$$

Now writing $Z(g_{2\nu}) = 2Z_1(g_{2\nu}) + O(\{g_{2\nu}\}^{-1/4})$, one obtains

$$\sum_{\nu=M+1}^{N} Z(g_{2\nu}) = 2N + O(N^{1/4}\log^{-1/4} N) + O\left(\sum_{\nu=M+1}^{N}\left(\frac{2\nu}{\log 2\nu}\right)^{-1/4}\right),$$

the last term of which is, by partial summation, $O(N^{3/4}\log^{1/4} N)$. Thus

$$\sum_{\nu=M+1}^{N} Z(g_{2\nu}) \sim 2N,$$

which shows that $Z(g_{2\nu})$ is positive for infinitely many ν. A similar argument leads to $\sum_{\nu=M+1}^{N} Z_1(g_{2\nu+1}) = N + O(N^{1/4}\log^{-1/4} N)$, which shows that

$$\sum_{\nu=M+1}^{N} Z(g_{2\nu+1}) \sim -2N,$$

whence $Z(g_{2\nu+1})$ is negative for infinitely many values of ν. Together, these statements prove that there are infinitely many intervals $(g_n, g_{n+1}]$ which contain an odd zero of $\zeta\left(\frac{1}{2}+it\right)$.

3.1. An estimate for the frequency with which the Weak Gram Law is true. There is another result of Titchmarsh [34] which is of interest here. The sum of $Z(g_\nu)Z(g_{\nu+1})$ is shown to be negative for infinitely many values of ν. Certainly this leads to the same result as above, but what is achieved is a lower bound on the number of Gram intervals which contain a zero of $\zeta\left(\frac{1}{2}+it\right)$. The result is

$$\sum_{\nu=M+1}^{N} Z(g_\nu)Z(g_{\nu+1}) \sim -2N(\gamma+1),$$

where γ is Euler's constant. Let N^- denote the number of negative terms in the above sum, and, as is standard, let $\mu = \mu\left(\frac{1}{2}\right)$ be the infimum of all ξ for which $\zeta\left(\frac{1}{2}+it\right) = O(t^\xi)$. Then

$$AN < N^- \max_{M+1\le\nu\le N} |Z(g_\nu)Z(g_{\nu+1})| < AN^-(g_{N+1}^{2\mu}), \tag{24}$$

and by Lemma 2.1, the expression on the right of (24) is less than

$$AN^-\left(\frac{N}{\log N}\right)^{2\mu},$$

which finally shows that

$$N^- > AN^{1-2\mu}(\log N)^{2\mu}.$$

Let $G(T)$ denote the number of intervals in $(0,T)$ for which $(g_n, g_{n+1}]$ contains an odd zero of $\zeta\left(\frac{1}{2}+it\right)$. It follows that

$$G(T) > AT^{1-2\mu}\log T,$$

and thus the proportion of Gram intervals up to height T which contain an odd zero of $\zeta\left(\frac{1}{2}+it\right)$ is bounded below by

$$\frac{A}{T^{2\mu}\log T}.$$

So even on the Lindelöf hypothesis where one can take $\mu = 0$, this method of Titchmarsh will not show that the Weak Gram Law is true a positive proportion of the time.

3.2. Moser's work. In the concluding paragraph of his paper, Titchmarsh [op. cit.] notes that the following argument may be used to improve the estimate on the number of Gram intervals in which the Weak Gram Law is true.

Denote by $\sum'$ a sum taken over values of ν for which $Z(g_\nu)Z(g_{\nu+1})$ is negative, and denote by N' the number of negative terms in the sum. Then

$$(25)\qquad AN < \sum_{\nu=M+1}^{N} \{-Z(g_\nu)Z(g_{\nu+1})\} \le \sum{}' \{-Z(g_\nu)Z(g_{\nu+1})\}$$

$$\le \Big[\sum{}' 1 \sum{}' \{Z(g_\nu)Z(g_{\nu+1})\}^2\Big]^{1/2} \le (N')^{1/2}\Big[\sum_{\nu=M+1}^{N} \{Z(g_\nu)Z(g_{\nu+1})\}^2\Big]^{1/2}.$$

This last sum is similar to that which arises in the computation of the integral of $\left|\zeta\left(\frac{1}{2}+it\right)\right|^4$ as given in e.g. [37, Ch. VII]. Titchmarsh then makes the conjecture that

$$(26)\qquad \sum_{\nu=M+1}^{N} \{Z(g_\nu)Z(g_{\nu+1})\}^2 = O(N\log^A N),$$

and adds:

> . . . but there are additional complications, and the conjecture has not been verified.

Moser studied sums of this sort in a series of papers [26–29]. In [29] it is shown that

$$(27)\qquad \sum_{\nu=M+1}^{N} \{Z(g_\nu)\}^4 = O(N\log^4 N),$$

whence, by the Cauchy–Schwarz inequality, the conjecture of Titchmarsh in (26) follows with $A = 4$. This then shows, in the notation of (25), that

$$N' \ge \frac{AN}{\log^4 N},$$

or that the proportion of Gram intervals up to height T which contain an odd zero of $\zeta\left(\frac{1}{2}+it\right)$ is bounded below by

$$\frac{A}{\log^3 T}.$$

This bound on the proportion of Gram intervals cannot be improved further by these methods, since Lavrik [21] showed that in fact

$$\sum_{\nu=M+1}^{N} \{Z(g_\nu)\}^4 \sim \frac{N}{2\pi}\log^4 N.$$

3.3. Atkinson's result. In the course of his extensive study of the mean-value properties of the zeta-function, Atkinson [1] pursued the above ideas of Titchmarsh in the following continuous analogue. For a fixed $\alpha > 0$, define t_α, as a function of t, by

$$\theta(t_\alpha) - \theta(t) = \alpha. \tag{28}$$

Atkinson then proved the following

THEOREM 3.2 (Atkinson). *If T_0 is a positive constant then*

$$\begin{aligned}\int_{T_0}^{T} Z(t)Z(t_\alpha)\,dt &= e^{i\alpha}\int_{T_0}^{T} \zeta\big(\tfrac{1}{2}+it_\alpha\big)\zeta\big(\tfrac{1}{2}-it\big)\,dt \\ &= \frac{\sin\alpha}{\alpha}T\log T + O(T\log^{3/4}T).\end{aligned} \tag{29}$$

With the particular choice of $\alpha = \frac{3}{2}\pi$, it follows that

$$\int_{T/2}^{T} Z(t)Z(t_\alpha)\,dt \sim -\frac{1}{3\pi}T\log T. \tag{30}$$

Let $Q(T)$ denote the set of points t in $\big(\frac{1}{2}T, T\big)$ such that $Z(t)Z(t_\alpha) < 0$. Thus $\zeta\big(\frac{1}{2}+it\big)$ must vanish in the interval (t, t_α) and so, by (30),

$$\int_{Q(T)} |Z(t)Z(t_\alpha)|\,dt > AT\log T$$

for T sufficiently large. Two applications of Cauchy's inequality give

$$\begin{aligned}AT^2\log^2 T &\le \operatorname{meas} Q(T)\int_{Q(T)} |Z(t)Z(t_\alpha)|^2\,dt \\ &\le \operatorname{meas} Q(T)\Big\{\int_{T/2}^{T} |Z(t)|^4\,dt\int_{T/2}^{T} |Z(t_\alpha)|^4\,dt\Big\}^{1/2}.\end{aligned} \tag{31}$$

The first integral on the right can be estimated using the fourth-power moment, due to Ingham [18], viz.

$$\int_{T/2}^{T} |Z(t)|^4\,dt = \int_{T/2}^{T} \big|\zeta\big(\tfrac{1}{2}+it\big)\big|^4\,dt = O(T\log^4 T). \tag{32}$$

To handle the second integral in (31) note that since $\theta(t_\alpha) - \theta(t) = \alpha$ it follows that

$$\frac{dt}{dt_\alpha} = \frac{\theta'(t_\alpha)}{\theta'(t)} = O(1),$$

so that

$$\int\limits_{T/2}^{T} |Z(t_\alpha)|^4\,dt = \int\limits_{T/2}^{T} \left|\zeta\left(\tfrac{1}{2}+it_\alpha\right)\right|^4 dt_\alpha\,\frac{dt}{dt_\alpha} = O(T\log^4 T). \tag{33}$$

Hence (31)–(33) give

$$\operatorname{meas} Q(T) \geq \frac{AT}{\log^2 T}.$$

If there are N' zeroes of $\zeta\left(\frac{1}{2}+it\right)$ for $\frac{1}{2}T \leq t \leq T$, then, since each zero contributes an interval of length $O(\log^{-1} T)$ to $Q(T)$, it follows that

$$\operatorname{meas} Q(T) < \frac{AN'}{\log T},$$

and thence that

$$N' > \frac{AT}{\log T}.$$

A variant of this method of Atkinson's was used in 2005 by Hall [15] as part of his extensive studies on the existence of large gaps between the zeroes of $\zeta\left(\frac{1}{2}+it\right)$. Specifically, Hall [op. cit., Thm. 4] shows that, uniformly for any interval $\alpha \ll \log T$,

$$\begin{aligned}&\int\limits_{0}^{T} Z(t)Z\left(t+\frac{\alpha}{\log T}\right)dt \\ &= \frac{\sin\alpha/2}{\alpha/2}T\log T + (2\gamma-1-\log 2\pi)T\cos\alpha/2 + O\left(\frac{\alpha T}{\log T}+T^{1/2}\log T\right).\end{aligned} \tag{34}$$

The similarities to (29) can be easily seen. For, by (28) and the mean-value theorem,

$$\alpha = \theta(t_\alpha) - \theta(t) = \theta'(c)(t_\alpha - t)$$

for some $c \in (t, t_\alpha)$. Since $\theta'(c) \sim \frac{1}{2}\log c$, by (9) it follows that

$$t_\alpha = t + \frac{2\alpha\{1+o(1)\}}{\log T}.$$

This shows that (34) is a refinement of Atkinson's result (29).

4. The function $S(t)$

4.1. Introduction. Whenever t is not an ordinate of a zero of $\zeta(s)$, define

$$S(t) = \frac{1}{\pi}\arg\zeta\left(\tfrac{1}{2}+it\right), \tag{35}$$

and write $S(t) = \lim_{\epsilon\to 0^+} S(t+\epsilon)$ if t coincides with an ordinate of a zero. The argument is determined by continuous variation along the straight lines connecting $2, 2+it, \frac{1}{2}+it$; and $S(0)$ is defined to be zero. The peculiarities in the definition of $S(t)$ stem from the use of Littlewood's result concerning the number of zeroes of an analytic function inside a rectangle (see e.g. [24], and [37, Ch. IX, §3]). The study of the function $S(t)$ is useful in understanding the distribution of the zeroes of the zeta-function, and the connexion between the two is shown in

Theorem 4.1 (Backlund [2]). *With $S(T)$ defined by* (35) *and $N(T)$ defined as the number of zeroes of $\zeta(s)$ for $0 < t < T$, we have*

$$N(T) = \pi^{-1}\theta(T) + S(T) + 1 = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).$$

Proof. See e.g. [37, pp. 212–213]. ∎

4.2. Basic properties of $S(t)$. Two of the simplest properties of the function $S(t)$ are

$$S(T) = O(\log T) \tag{36}$$

and

$$\int_0^T S(t)\,dt = O(\log T). \tag{37}$$

The former is due to von Mangoldt (see e.g. [37, Thm. 9.4]) and the latter is due to Littlewood (see e.g. [37, pp. 221–222]). Note that neither result necessarily implies the other: (36) guarantees the growth of $S(T)$ must be suitably slow; (37) ensures that the average value of $S(T)$ is zero.

It is also known that $S(T)$ takes large values infinitely often. The first result of this type was proved by Bohr and Landau in 1913, viz.

Theorem 4.2 (Bohr–Landau). *On the assumption of the Riemann hypothesis each of the inequalities*

$$S(T) > (\log T)^{1/2-\epsilon}, \qquad S(T) < -(\log T)^{1/2-\epsilon}$$

has solutions for arbitrary large values of T.

Proof. See [3]. It can be shown (see e.g. [6, p. 202]) that $|S(T)|$ is unbounded if one merely assumes that the number of zeroes off the critical line is finite. ∎

4.3. The failure of Gram's Law. This section shows that Gram's Law fails infinitely often. It is shown that Gram's Law induces a degree of constancy in $S(T)$, which shows that $S(T)$ is bounded. With an ancillary argument, this contradicts Theorem 4.2. The approach here is modelled on that given by Titchmarsh [35]. To begin, one needs

Lemma 4.3 (Titchmarsh). *The equation $S(t) = 0$ is satisfied for infinitely many values of t.*

Proof. Arrange the zeroes $\beta_n + i\gamma_n$ of $\zeta(s)$ in order of non-decreasing ordinates, where, as is customary, a zero with multiplicity m is included m times. Then for $t \in [\gamma_n, \gamma_{n+1})$ the function $N(t)$ is constant and indeed $N(t) = n$. By (9) the function $\theta(t)$ is increasing for sufficiently large n, and therefore, by Theorem 4.1 the function $S(t)$ is decreasing over the interval. One can approximate $S(t)$ by a linear function $l(t)$ which takes the values $S(\gamma_n)$ and $\lim_{r\uparrow\gamma_{n+1}} S(r)$ at γ_n and γ_{n+1} respectively. Then over the interval $(\gamma_n, \gamma_{n+1}]$ it follows that

$$\begin{aligned} l(t) - S(t) &= \{\lim_{r\uparrow\gamma_{n+1}} S(r) - S(\gamma_n)\}\frac{t-\gamma_n}{\gamma_{n+1}-\gamma_n} - \{S(t) - S(\gamma_n)\} \\ &= -\pi^{-1}\{\theta(\gamma_{n+1}) - \theta(\gamma_n)\}\frac{t-\gamma_n}{\gamma_{n+1}-\gamma_n} + \pi^{-1}\{\theta(t) - \theta(\gamma_n)\}, \end{aligned} \tag{38}$$

where the second equality is deduced from Theorem 4.1, the continuity of $\theta(t)$, and the fact that $N(t)$ is constant over the interval. Now two applications of the mean-value theorem give

$$l(t) - S(t) = -\pi^{-1}\{\theta'(\xi_1)(t-\gamma_n) - \theta'(\xi_2)(t-\gamma_n)\},$$

with $\gamma_n < \xi_1 < \gamma_{n+1}$ and $\gamma_n < \xi_2 < t$. Another application of the mean-value theorem gives

$$l(t) - S(t) = \pi^{-1}\theta''(\xi)(\xi_2 - \xi_1)(t-\gamma_n),$$

with $\xi_1 < \xi < \xi_2$. By (9),

$$\theta''(t) \sim \frac{1}{2t},$$

and given the definitions of ξ, ξ_1 and ξ_2 it follows that

$$l(t) - S(t) = O\left(\frac{(\gamma_{n+1}-\gamma_n)^2}{\gamma_n}\right) = O\left(\frac{1}{\gamma_n}\right), \tag{39}$$

since the distance $\gamma_{n+1} - \gamma_n$ is bounded (see e.g. [37, Ch. IX, §1]). Suppose now that $S(t) \geq 0$ for $t > t_0$, i.e. that there are no sign changes in $S(t)$ past $t = t_0$. Since for any $\epsilon > 0$, there is least one zero at γ_n, we see that

$$N(\gamma_n) - N(\gamma_n - \epsilon) \geq 1.$$

This, by virtue of Theorem 4.1 and the fact that $\theta(t)$ is a continuous function, implies

$$S(\gamma_n) - S(\gamma_n - \epsilon) \geq 1.$$

From the assumption that $S(t) \geq 0$ for sufficiently large t, it follows that $S(\gamma_n) \geq 1$ for sufficiently large n. Thus integrating (39) with respect to t gives

$$\int_{\gamma_n}^{\gamma_{n+1}} S(t)\,dt = \int_{\gamma_n}^{\gamma_{n+1}} l(t)\,dt + O\left(\frac{\gamma_{n+1}-\gamma_n}{\gamma_n}\right).$$

After integrating (38) with respect to t, the above becomes

$$\int_{\gamma_n}^{\gamma_{n+1}} S(t)\,dt = \frac{1}{2}(\gamma_{n+1}-\gamma_n)\{S(\gamma_n)+\lim_{r\uparrow\gamma_{n+1}} S(r)\} + O\left(\frac{\gamma_{n+1}-\gamma_n}{\gamma_n}\right)$$
$$\geq \frac{1}{2}(\gamma_{n+1}-\gamma_n) + O\left(\frac{\gamma_{n+1}-\gamma_n}{\gamma_n}\right) \geq \frac{1}{4}(\gamma_{n+1}-\gamma_n)$$

for sufficiently large n. If $\gamma_{n+1}=\gamma_n$ then both sides of the inequality are zero. So for n_0 sufficiently large, summing both sides gives

$$\int_{\gamma_{n_0}}^{\gamma_N} S(t)\,dt \geq \frac{1}{4}(\gamma_N-\gamma_{n_0}),$$

which contradicts Littlewood's result in (37). A similar contradiction is obtained after the assumption that $S(t)\leq 0$ for sufficiently large t; whence $S(t)$ must change sign infinitely often. Since $S(t)$ decreases continuously and only increases by jumps at the zeroes of the zeta-function, it follows that $S(t)=0$ for infinitely many values of t. ∎

Now suppose that Gram's Law fails only finitely many times. Then there exists some n_1 such that for every $n\geq n_1$ there is exactly one zero in the Gram interval $(g_n, g_{n+1}]$. So if $g_n < t \leq g_{n+1}$, then the number of zeroes on the critical line up to height t satisfies

$$N_0(t) = n - n_1 = \pi^{-1}\theta(g_{n+1}) - 1 - n_1 \geq \pi^{-1}\theta(t) - 1 - n_1. \tag{40}$$

The previous result of infinitely many zeroes of $S(t)$ is now used. Let t^* denote a sequence tending to infinity with $S(t^*)=0$, and hence $N(t^*)=\pi^{-1}\theta(t^*)+1$. Then

$$N_0(t^*) > N(t^*) + O(1).$$

Therefore the number of complex zeroes of $\zeta(s)$ not on the line $\sigma=\frac{1}{2}$ is finite. So for all values of t,

$$N(t) = N_0(t) + O(1) > \pi^{-1}\theta(t) + O(1),$$

which, along with Theorem 4.1, implies that $S(t)$ is bounded below. But by the weaker form of Theorem 4.2, a finite number of exceptions to the Riemann hypothesis implies that $S(t)$ assumes arbitrarily large negative values. This contradiction shows that Gram's Law must fail infinitely often. The argument also shows that the Weak Gram Law fails infinitely often: one replaces the first "=" in (40) with "$\geq$".

4.4. Improvements. The link between Gram's Law and the function $S(t)$ has been explored in the above proof. Suppose that for $t\in(g_n, g_{n+1}]$

there are k zeroes of $\zeta(s)$ (not necessarily on the critical line). Then by Theorem 4.1,

$$S(g_{n+1}) - S(g_n) = N(g_{n+1}) - N(g_n) - \pi^{-1}\{\theta(g_{n+1}) - \theta(g_n)\} = k - 1.$$

Moreover, $S(t)$ is integral if, and only if, $t = g_n$ for some n. Note that if Gram's Law is true over an interval then it need not follow that $k = 1$, owing to the possible presence of zeroes off the critical line. Nevertheless in intervals with $k = 1$ it follows that $S(g_{n+1}) = S(g_n)$ and $|S(t) - S(g_n)| \leq 1$ for $t \in (g_n, g_{n+1}]$. This in turn induces some constancy in the function $S(t)$. Theorem 4.2 has been used to show that $S(t)$ takes arbitrarily large values under the assumption that the number of zeroes off the critical line is finite. This additional assumption will not be needed henceforth, since Selberg [31] showed that, independently of any unproven hypotheses,

$$S(t) = \Omega_{\pm}\left(\frac{(\log t)^{1/3}}{(\log\log t)^{7/3}}\right). \tag{41}$$

4.5. Selberg's approximation. Selberg proved the following theorem [31, Thm. 4].

THEOREM 4.4 (Selberg). *If $T^{\alpha} < H \leq T$, where α is fixed and $\frac{1}{2} < \alpha \leq 1$, and, for m a positive integer, $T^{(\alpha-1/2)/m} \leq x \leq H^{1/m}$, then*

$$\int_T^{T+H} \left| S(t) + \frac{1}{\pi}\sum_{p<x} \frac{\sin(t\log p)}{\sqrt{p}} \right|^{2m} dt \leq c(m)H,$$

where $c(m)$ depends on m but not on T.

The factor $c(m)$ has been improved over the years: in Selberg's paper [op. cit.] it was not calculated explicitly. Fujii [7] calculated that $c(m) \leq (Am)^{4m}$, and this result, which was also proved by Ghosh [11], follows more or less directly from Selberg's original arguments. Tsang [39] showed that $c(m) \leq (Am)^{2m}$, where the improvement comes from a repeated application of Selberg's density theorem, specifically Lemma 5.2 of [38]. Karatsuba in [19] proved that in Theorem 4.4 one could take (3) $H = T^{27/82+\epsilon}$, where, say, $0 < \epsilon < 0.001$. With this, Karatsuba and Korolëv [20] placed a bound on the explicit constant in the result of Tsang, to show that $c(m) < (\epsilon^{-3}e^{37}\pi^{-2}m^2)^m$. A different approach is due to Goldston [12]. On the assumption of the Riemann hypothesis and introducing a different weight to that used by Selberg, Goldston showed that $c(m) \geq (A\log m)^m$.

Improving either the upper or lower bound on $c(m)$ seems a difficult task, but one of great interest.

(3) See also the remarks in §6.4.

5. A positive proportion of failures

5.1. Introduction. This section is concerned with the number of failures of Gram's Law between T and $2T$, denoted by $N_F(T)$. It has been shown in §4.3 that $N_F(T) \to \infty$. Corollary 5.4 will show that the Weak Gram Law fails a positive proportion of the time, and hence so too does Gram's Law.

The results in this section are dependent on the following theorem concerning the "shifted moments" of the function $S(t)$.

THEOREM 5.1 (Tsang). *Let $a > \frac{1}{2}, T^a < H \le T$ and $0 < h < 1$. Then, for any positive integer m,*

$$\int_T^{T+H} \{S(t+h) - S(t)\}^{2m}\, dt = \frac{(2m)!}{(2\pi^2)^m m!} H\{\log(2 + h\log T)\}^m + O\{H(Am)^m\{m^m + (\log(2 + h\log T))^{m-1/2}\}\}.$$

Proof. See [39, Thm. 4]. ■

The case $m = 1$ was first shown by Fujii in [8], and indeed this was shown in greater generality than Theorem 5.1. Fujii considered Dirichlet L-functions and the function $S(t, \chi)$ defined in an analogous way to (35).

5.2. An auxiliary approach to Gram's Law. It is convenient to introduce the following notation. For $j = 0, 1, \ldots,$ let F_j denote a Gram interval $(g_n, g_{n+1}]$ in which j zeroes are located, *whether or not these zeroes lie on the critical line.* Furthermore, let $N_{F_j}(T)$ denote the number of F_j intervals between heights T and $2T$. Thus an F_1 interval is one in which Gram's Law is true, but the converse need not be so. Results concerning Gram's Law will be deduced from

THEOREM 5.2. *Let $N_G(T) = N_{F_0}(T) + N_{F_2}(T) + N_{F_3}(T) + \cdots$, that is, $N_G(T)$ is the number of non-F_1 intervals between T and $2T$. Then $N_G(T) \gg T\log T$ for sufficiently large T.*

Proof. Consider the case $m = 1$ of Theorem 5.1; for simplicity (4), take $H = T$, and write

$$I(T) = \int_T^{2T} |S(t+h) - S(t)|^2\, dt = \pi^{-2} T\log(3 + h\log T) + O[T\{\log(3 + h\log T)\}^{1/2}]. \tag{42}$$

This becomes an asymptotic relationship, i.e.

$$I(T) \sim \pi^{-2} T\log(3 + h\log T),$$

(4) See §6.4 for taking $H < T$.

if $h \log T \to \infty$. Henceforth $h = C_0(\log T)^{-1}$, where C_0 is a constant that is chosen to be sufficiently large. Initially, C_0 is chosen to be large enough to ensure the dominance of the main term in (42) over the error term. Thus, for some $\delta = \delta(C_0) > 0$,

$$(\pi^{-2} - \delta)T \log(3 + h \log T) \le I(T) \le (\pi^{-2} + \delta)T \log(3 + h \log T). \tag{43}$$

To prove Theorem 5.2, consider that if $[T, 2T + h]$ were covered by F_1 intervals, then for all $t \in [T, 2T + h]$, it would follow that $|S(t + h) - S(t)| \le 2$. Thence $I(T) \le 4T$, which is "too small", i.e. there is a contradiction to (43).

Let the sequences $\{i_n\}$ and $\{j_n\}$ index the Gram points such that F_1 intervals cover $(g_{i_n}, g_{j_n}]$ and there are no F_1 intervals in $(g_{j_n}, g_{i_{n+1}}]$. Also let $k_n = i_{n+1} - j_n$, that is, the number of consecutive non-F_1 intervals, whence $\sum_n k_n = N_G(T)$.

It is clear that the relative locations of t and $t + h$ will determine the bound on $|S(t + h) - S(t)|$, viz. if $g_{i_n} \le t < t + h \le g_{j_n}$ then it follows that $|S(t + h) - S(t)| \le 2$. This leads to the definition

$$J := \{t \in [T, 2T] : \exists n \text{ such that } g_{i_n} \le t < t + h \le g_{j_n}\},$$

whence $\int_J |S(t + h) - S(t)|^2 \, dt \le 4T$.

Now let $\bar{J}$ be the complement of J in $[T, 2T]$. If t belongs to $\bar{J}$, then either $t \in [g_{i_n}, g_{j_n}]$ and $t + h > g_{j_n}$, or $t \in (g_{j_n}, g_{i_{n+1}}]$. The former condition implies $g_{j_n} \ge t > g_{j_n} - h$ and so in any case $g_{j_n} - h < t \le g_{i_{n+1}}$. These intervals may overlap in $[T, 2T]$ and indeed

$$\bar{J} \subset \bigcup_n (g_{j_n} - h, g_{i_{n+1}}].$$

Whether or not these intervals are disjoint is of no consequence, for Lemma 2.1 gives

$$\operatorname{meas} \bar{J} \ll \sum_n \left(h + \frac{k_n}{\log T}\right) \ll \left(h + \frac{1}{\log T}\right) N_G(T). \tag{44}$$

Ultimately an estimate on this number $N_G(T)$ is sought and hence the imposition of a lower bound of (44) would be useful. Returning to (43), it is seen that

$$(\pi^{-2} - \delta)T \log(3 + h \log T) \le I(T) \le 4T + \int_{\bar{J}} |S(t + h) - S(t)|^2 \, dt. \tag{45}$$

Currently $h = C_0(\log T)^{-1}$ and C_0 is chosen to be sufficiently large such that the main term in (42) dominates the error term. If, in addition to this, C_0 is taken large enough to make the quantity $(\pi^{-2} - \delta)T \log(3 + h \log T)$ larger than, say, $5T$, then (45) gives

$$T \le \int_{\bar{J}} |S(t + h) - S(t)|^2 \, dt. \tag{46}$$

An application of Cauchy's inequality gives

$$\int_{\bar{J}} |S(t+h) - S(t)|^2\,dt \le \Big(\int_{\bar{J}} |S(t+h) - S(t)|^4\,dt\Big)^{1/2} \cdot \Big(\int_{\bar{J}} dt\Big)^{1/2}$$
$$= (\operatorname{meas} \bar{J})^{1/2} \cdot \Big(\int_{\bar{J}} |S(t+h) - S(t)|^4\,dt\Big)^{1/2},$$

whence, via (46), it follows that

$$T \le (\operatorname{meas} \bar{J})^{1/2} \cdot \Big(\int_{\bar{J}} |S(t+h) - S(t)|^4\,dt\Big)^{1/2}.$$

The right side of the above inequality can be estimated by taking $m = 2$ in Theorem 5.1, whence

$$T \ll (\operatorname{meas} \bar{J})^{1/2} T^{1/2} \log(3 + h \log T) \ll (\operatorname{meas} \bar{J})^{1/2} T^{1/2}, \tag{47}$$

since $h \ll C_0 (\log T)^{-1}$. Together (44) and (47) show that

$$T \ll \Big(\frac{1 + C_0}{\log T}\Big) N_G(T),$$

which proves the theorem. ∎

5.3. The Weak Gram Law. Theorem 5.2 can now be used to address the failure of Gram's Law. From Lemma 2.1 one may write

$$N_{F_0}(T) + N_{F_1}(T) + N_{F_2}(T) + \cdots = N_g(2T) - N_g(T) = \frac{T}{2\pi} \log T + O(T). \tag{48}$$

Since all the zeroes between heights T and $2T$ fall within Gram intervals,

$$N_{F_1}(T) + \cdots + k N_{F_k}(T) + \cdots = N(2T) - N(T) = \frac{T}{2\pi} \log T + O(T), \tag{49}$$

on using Theorem 4.1. The subtraction of equation (48) from (49) gives

$$O(T) = -N_{F_0}(T) + N_{F_2}(T) + 2N_{F_3}(T) + \cdots + (k-1) N_{F_k}(T) + \cdots$$
$$\ge -N_{F_0}(T) + N_{F_2}(T) + N_{F_3}(T) + \cdots + N_{F_k}(T) + \cdots,$$

whence, upon an addition of $2N_{F_0}(T)$ to both sides, and an invocation of Theorem 5.2, it is seen that

$$2N_{F_0}(T) + O(T) \ge N_{F_0}(T) + N_{F_2}(T) + N_{F_3}(T) + \cdots \ge A\{N_g(2T) - N_g(T)\},$$

so that

$$\frac{N_{F_0}(T)}{N_g(2T) - N_g(T)} \ge \frac{A}{2} + O\Big(\frac{1}{\log T}\Big),$$

which proves

THEOREM 5.3. *For sufficiently large T there is a positive proportion of Gram intervals between T and $2T$ which do not contain a zero of $\zeta(s)$.*

Following a fortiori from the above theorem is

COROLLARY 5.4. *For sufficiently large T there is a positive proportion of failures of the Weak Gram Law, and therefore of Gram's Law, between T and $2T$.*

Since the number of F_0 intervals is certainly less than the total number of violations of Gram's Law, the order of $N_{F_0}(T)$ is exactly determined, viz. $N_{F_0}(T) \asymp T \log T$. There is little else ([5]) to be said about the nature of F_0 intervals, so it is natural to now turn to the remaining cases: those Gram intervals which contain at least one zero of $\zeta\big(\frac{1}{2}+it\big)$.

6. A positive proportion of successes

6.1. Introduction. Recall the result of Titchmarsh given in §3 that the Weak Gram Law is true infinitely often. What is actually shown in this proof is that there is an infinite number of Gram intervals which contain an *odd number* of zeroes. The work of Moser from §3.2 shows that the proportion of Gram intervals between T and $2T$ which contain an odd number of zeroes of $\zeta\big(\frac{1}{2}+it\big)$ is greater than $A(\log T)^{-3}$. This is here improved via

THEOREM 6.1. *There exists a K such that, for sufficiently large T, there is a positive proportion of Gram intervals between T and $2T$ which contain at least one zero and not more than K zeroes of $\zeta\big(\frac{1}{2}+it\big)$. In particular, the Weak Gram Law is true a positive proportion of the time.*

It is worthwhile to note that Gram's Law has yet to be shown to be true infinitely often. It is difficult to investigate the quantities $N_{F_k}(T)$ for "small" k, since the induced behaviour in $S(t)$ is virtually undetectable. Indeed, using the shifted moments of $S(t)$ one is unable to distinguish a collection of F_1 intervals from a sequence of alternating F_0 and F_2 intervals. The proof of Theorem 6.1 is therefore based on showing that F_k intervals are rare when k is large.

6.2. Proof of Theorem 6.1. If, in Theorem 5.1, h is suitably small such that $h \log T \ll 1$, and $H = T$, then

$$\int_T^{2T} |S(t+h) - S(t)|^{2m}\, dt \ll (Am)^{2m} T.$$

Suppose now that the interval $(g_n, g_{n+1}]$ contains k zeroes, for some $k \geq 0$, and that $S(g_n) = \lambda$. Since $S(t)$ cannot decrease by more than 1 over a Gram interval it therefore follows that

$$S(g_{n-1}) \leq \lambda + 1, \qquad S(g_{n+1}) = \lambda + k - 1, \qquad S(g_{n+2}) \geq \lambda + k - 2.$$

([5]) One possibility is to calculate these constants, although any practically useful results are not achievable via these methods (cf. §6.4).

Take $h = 4\pi/\log T$ so that, by Lemma 2.1, h is asymptotically twice the length of a Gram interval. Then $|S(t+h) - S(t)| > k - 2$ over an interval of length $\gg (\log T)^{-1}$. So if there are $N_{F_k}(T)$ intervals between T and $2T$ it follows that

$$T(Am)^{2m} \gg \int_T^{2T} |S(t+h) - S(t)|^{2m}\, dt \gg \frac{(k-2)^{2m} N_{F_k}(T)}{\log T},$$

whence

$$\frac{N_{F_k}(T)}{T\log T} \ll \left(\frac{Am}{k-2}\right)^{2m}.$$

One now chooses an $m > 0$ depending on k, to minimise the right side of the above equation. Write

$$F(m) = F(k, m) = \left(\frac{Am}{k-2}\right)^{2m},$$

whence

$$F'(m) = 0 \ \Leftrightarrow\ m = m^* := \frac{k-4}{Ae},$$

and it is easily seen (by e.g. the second derivative test) that this value of m is indeed minimal. It follows that

$$F(m^*) \ll e^{-Ak}.$$

Now Theorem 5.1 is valid only for integral m, so consider

$$m' \in \left\{\left[\frac{k-4}{Ae}\right], \left[\frac{k-4}{Ae}\right] + 1\right\},$$

where, as usual, $[x]$ denotes the greatest integer not exceeding x. Both of these above terms differ by not more than $\frac{1}{2}$ from m^*, and it is easily seen that

$$F(m') \ll e^{-Ak},$$

whence

$$\frac{N_{F_k}(T)}{T\log T} \ll e^{-Ak}. \tag{50}$$

Now denote by $\widehat{F_j}$ a Gram interval with j zeroes of $\zeta(s)$, at least one of which is on the critical line. Then, if $N_{\widehat{F_j}}(T)$ is the number of $\widehat{F_j}$ intervals between T and $2T$ it is clear that

$$N_{\widehat{F_j}}(T) \leq N_{F_j}(T). \tag{51}$$

Since a positive proportion of zeroes lie on the critical line there is a constant A' such that

(52)
$$0 < A' < \frac{N_0(2T) - N_0(T)}{T\log T} \le \frac{N_{\widehat{F_1}}(T) + 2N_{\widehat{F_2}}(T) + \cdots + kN_{\widehat{F_k}}(T) + \cdots}{T\log T},$$

and by (50) and (51) the series on the right-hand side is convergent. So, if δ is any small positive number, choose K so large that the sum

$$(T\log T)^{-1} \sum_{k=K+1}^{\infty} kN_{\widehat{F_k}}(T)$$

is less than $A' - \delta$. Then

$$0 < \delta < \frac{\sum_{k=1}^{K} kN_{\widehat{F_k}}(T)}{T\log T} < K\frac{\sum_{k=1}^{K} N_{\widehat{F_k}}(T)}{T\log T}, \tag{53}$$

whence follows Theorem 6.1. ∎

6.3. The work of Selberg and Fujii. Selberg [32, p. 198] writes:

> By a more detailed investigation of the variation of the amplitude of $\zeta(\frac{1}{2}+it)$, I have succeeded in proving that there exist absolute positive constants K and N_0, such that for $N > N_0$, $1 \le \nu \le N$, the numbers $\zeta(\frac{1}{2} + it_{\nu-1})$ and $\zeta(\frac{1}{2} + it_\nu)$ are of different sign in more than KN cases, and of the same sign in more than KN cases.

The first statement (concerning the same parity of $\zeta\big(\frac{1}{2} + it_{\nu-1}\big)$ and $\zeta\big(\frac{1}{2} + it_\nu\big)$) is equivalent to there being a positive proportion of $F_1 + F_3 + \cdots$ intervals. If one applies formulas (50)–(53) then this statement is seen to be stronger than Theorem 6.1. The second statement follows directly from Theorem 5.3. It would be interesting to discover the method by which Selberg arrived at these results, and unfortunately no proof is given in [32].

Fujii [9] states that

$$\frac{N_{F_0}(T)}{N_g(T)} \gg A, \qquad \sum_{k=2}^{\infty} \frac{N_{F_k}(T)}{N_g(T)} \gg A. \tag{54}$$

The first is equivalent to Theorem 5.3, although it is unclear how this is derived in [8]. There, the sum

$$\sum_{m\le M} \left\{ S\left(\frac{2\pi\alpha(m+1)}{\log(m+1)}\right) - S\left(\frac{2\pi\alpha m}{\log m}\right) \right\}^2$$

is considered, for $\alpha \ll \log M$. Since this is more or less a discrete version of the integral

$$\int_0^T \{S(t+h) - S(t)\}^2\, dt,$$

it is likely that methods of §5 would suffice. The second result in (54) is slightly different from Theorem 6.1 in that the sum is infinite.

Based on predictions from Random Matrix Theory, Fujii [op. cit.] makes the following

CONJECTURE 6.2 (Fujii).

$$\frac{N_{F_0}(T)}{N_g(T)} \to 0.17, \quad \frac{N_{F_1}(T)}{N_g(T)} \to 0.74, \quad \frac{N_{F_2}(T)}{N_g(T)} \to 0.13. \tag{55}$$

This then predicts that Gram's Law should be true approximately 74% of the time. Note that although the sum of the three numbers in (55) exceeds unity, the figures can be compared with the calculations of van de Lune et al. [42], viz. up to the $M = 1\,500\,000\,000$th Gram point,

$$\frac{N_{F_0}(g_M)}{M} = 0.1378\ldots, \quad \frac{N_{F_1}(g_M)}{M} = 0.7261\ldots, \quad \frac{N_{F_2}(g_M)}{M} = 0.1342\ldots.$$

The results in Conjecture 6.2 rest on Montgomery's pair-correlation conjecture (see e.g. [25]), and the following

CONJECTURE 6.3 (Gallagher–Mueller [10]). *Assume the pair-correlation conjecture. Then, as $T \to \infty$ and $\alpha \to 0$,*

$$\int_0^T \left| S\left(t + \frac{2\pi\alpha}{\log T}\right) - S(t)\right|^2 dt \sim (\alpha - \alpha^2 + o(\alpha^2))T.$$

A discussion on the relationship between the pair-correlation conjecture and the moments of $S(t+h) - S(t)$ is beyond the scope of this article, however pursuing this connection will be of interest in future work.

6.4. Concluding remarks. Intuitively one might expect $N_{F_k}(T)$ to be steadily decreasing with k (which would be an improvement to the estimate in (50)). If such a relation could be shown it would therefore follow that there is a positive proportion of intervals in which Gram's Law is valid. However the details of such an approach are at present unknown.

In Theorem 5.1 it is possible to take H as small as T^a where a is any fixed number greater than $\frac{1}{2}$. This restriction comes from the allowance made for potential zeroes off the critical line, in particular Selberg's zero-density theorem [31, Thm. 1]. This was proved for intervals of the type $(T, T + T^{1/2+\epsilon})$. Selberg remarked that such a density theorem should be valid for shorter intervals, and Karatsuba and Korolëv [20, Ch. I] extended Selberg's theorem to show that it was valid over the range $(T, T + T^{27/82+\epsilon})$. The constant $\frac{27}{82}$ comes from the estimate of the order of growth of $\left|\zeta\left(\frac{1}{2}+it\right)\right|$. Indeed, if one writes, as is customary, $\mu(\sigma)$ as the lower bound on the number ξ such that

$$\zeta(\sigma + it) = O(t^{\xi}),$$

then Titchmarsh [37, Ch. V, §18] proved that $\mu\left(\frac{1}{2}\right) \leq \frac{27}{164}$. One can estimate the exponential sums which appear in Selberg's work using the same techniques as estimating the sums endemic in the calculation of $\mu\left(\frac{1}{2}\right)$. In particular, the work of Huxley [17] shows that one may take $\mu\left(\frac{1}{2}\right) = \frac{32}{205}$, and thus the conclusions of Theorems 4.4, 5.1, 5.2, 5.3 and 6.1 should be expected to hold over the shorter range $(T, T + T^{2\mu(1/2)+\epsilon})$.

The explicit constant given by Karatsuba and Korolëv (see §4.5) provides an upper bound on the constant K in Theorem 6.1. Using the result that at least two-fifths of the zeroes of the zeta-function lie on the critical line (see e.g. [5]) one can show that $K \leq 10^9$. The proportion of Gram intervals in which Gram's Law is false could be similarly computed. Since there is little chance to prove that K can be small—e.g. 2 or 3—there is not much more to be said on this point.

An interesting problem would be to calculate how short an interval must be before one is guaranteed to find, not a positive proportion of failures, but *just one failure* of Gram's Law (or the Weak Gram Law).

7. Extensions to Gram's Law

7.1. Introduction. Interest in the relationship between the function $S(t)$ and Gram intervals diminished after the proof of the infinite failure of Gram's Law. However, large scale computations into the values of the zeta-function revealed a phenomenon which required further investigation. Patterns emerged in the distribution of zeroes of $\zeta\left(\frac{1}{2} + it\right)$ over unions of Gram intervals. It was observed that some intervals contain too few roots of the zeta-function but these are "balanced out" by nearby intervals which contain more than one zero. Recall the counterexamples of Gram's Law found by Hutchinson: two roots are found over a union of two Gram intervals. It is fitting that the computation of values of $\zeta\left(\frac{1}{2} + it\right)$ should refuel interest in Gram's Law given that it was empirical observation which prompted the initial study in the early 20th century.

7.2. Gram blocks and Rosser's Rule. Rosser, Yohe and Schoenfeld [30] gave the following

DEFINITION 7.1 (Gram blocks). The interval $(g_n, g_{n+l}]$ is a *Gram block* of length l if $(-1)^j Z(g_j) > 0$ for $j = n$ and for $j = n + l$, but $(-1)^j Z(g_j) \leq 0$ for $n < j < n + l$. Furthermore, define the intervals $(g_n, g_{n+1}]$ and $(g_{n+l-1}, g_{n+l}]$ as *exterior intervals*; the remaining intervals are defined as *interior*.

It follows from the above definition that a Gram block of length 1 is an F_{2m+1} interval for some non-negative integer m. Also from Definition 7.1 it is easily seen that when $k \geq 2$ a Gram block $(g_n, g_{n+k}]$ has an odd zero of

$\zeta\left(\frac{1}{2}+it\right)$ in each of its interior intervals. If additional zeroes occur in the Gram block, it follows that there must be an even number in each interval.

For convenience the endpoints of a Gram block are referred to as *good*, since at these points g_n one has $(-1)^n Z(g_n) > 0$, which is what one expects from Gram's Law—cf. (13). Consequently, the interior Gram points of a Gram block are termed "bad", since at these points $Z(g_n)$ has the "wrong" sign.

Upon the framework of Gram blocks, Rosser, Yohe and Schoenfeld [op. cit.] proposed

DEFINITION 7.2 (The Rosser Rule). The *Rosser Rule* is said to hold in a Gram block of length l if this block contains exactly l zeroes of $\zeta\left(\frac{1}{2}+it\right)$.

Note that, just as in the definition of Gram's Law (Definition 2.2) the above definition is not concerned with the potentiality of zeroes off the critical line. In keeping with the definition of the Weak Gram Law, define the *Weak Rosser Rule* to be true over a Gram block of length l if this block contains *at least* l zeroes of $\zeta\left(\frac{1}{2}+it\right)$.

The calculations of Rosser et al. [op. cit.] show that the first 3 500 000 complex zeroes of $\zeta(s)$ lie on the critical line and that the Rosser Rule is true up to this height. However, in subsequent calculations, failures have been observed, and it will be shown in the following section that the Rosser Rule fails infinitely often.

7.3. The failure of Rosser's Rule. The first exception to the Rosser Rule is at the 13 999 825th Gram point. This corresponds to a height $t \approx 5\,346\,000$ which falls outside the calculations of Rosser et al. Further failures are slight, and it is seen in the calculations of Gourdon [13] that up to the first 10^{13} zeroes of $\zeta\left(\frac{1}{2}+it\right)$ there are approximately 32 violations of the Rosser Rule per million zeroes. Indeed this section, which is based on the argument of Lehman [22], shows that the Rosser Rule fails infinitely often. It is shown that the Rosser Rule implies that $S(t)$ is bounded below on a Gram block and therefore that $S(t)$ is bounded below for all t. However this cannot be reconciled with equation (41).

THEOREM 7.3. *The Weak Rosser Rule fails infinitely often, and therefore so too does the Rosser Rule.*

Proof. Suppose there are only finitely many failures to the Weak Rosser Rule. Let T_0 be the point beyond which the Weak Rosser Rule holds. Now consider a Gram block $(g_n, g_{n+k}]$ where $g_n > T_0$. From Theorem 4.1 it follows that

$$S(g_{n+k}) - S(g_n) = N(g_{n+k}) - N(g_n) - k.$$

Given that there are at least k zeroes in a Gram block of length k, it follows that

$$S(g_{n+k}) \geq S(g_n).$$

This process is now repeated on the Gram block with g_n as its right endpoint, that is, a Gram block of length l, say, and $S(g_n) \geq S(g_{n-l})$. This can be continued to show inductively that

$$S(g_n) \geq S(T_0) \tag{56}$$

over all good Gram points g_n. The function $S(t)$ has now been bounded over the good Gram points, and similarly a bound over the bad Gram points can be achieved. Let g_m and g_{m+1} be consecutive bad Gram points, i.e. $(-1)^m Z(g_m), (-1)^{m+1} Z(g_{m+1}) \leq 0$. Thus there must be at least one zero in this interval. There may be other roots, but in either case, since

$$S(g_{m+1}) - S(g_m) = N(g_{m+1}) - N(g_m) - 1,$$

we get

$$S(g_{m+1}) \geq S(g_m). \tag{57}$$

It follows from equations (56) and (57) that $S(g_\nu)$ is increasing at each Gram point g_ν. Thus the lowest value attainable by $S(g_\nu)$ is at the first bad Gram point where $S(t)$ drops by 1 over the interval. Hence for all Gram points,

$$S(g_\nu) \geq S(T_0) - 1.$$

Since $S(t)$ only attains integral values at the Gram points, and since $S(t)$ is continuous from the right, this proves that

$$S(t) \geq S(T_0) - 2$$

for all $t > T_0$. So ([6]) $S(t)$ is ultimately bounded below, which contradicts the theorem of Selberg given in (41). ∎

7.4. The number of Gram blocks in an interval. It will be of use to have at hand the following

LEMMA 7.4. *If $N_{GB}(T)$ denotes the number of Gram blocks between T and $2T$, then*

$$N_{GB}(T) \asymp T \log T.$$

Proof. It is clear that $N_{GB}(T) \leq N_g(2T) - N_g(T) \ll T \log T$, by Lemma 2.1. On the other hand, each exterior interval of a Gram block corresponds to an F_{2m} interval for some non-negative integer m. In particular,

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([6]) Here, a proof that the Rosser Rule implies only finitely many exceptions to the Riemann hypothesis can be given (see [6, pp. 180–181]). Then the theorem of Bohr and Landau (Theorem 4.2) may be applied. But since the result of Selberg in (41) is independent of any assumptions of the Riemann hypothesis, this proof can be made considerably shorter than that in §4.3.

each F_0 interval is an exterior interval for some Gram block. So certainly,

$$N_{GB}(T) \gg N_{F_0}(T) \gg T \log T,$$

by virtue of Theorem 5.3. ∎

A method to visualise the lengths of Gram blocks is as follows. Calculate the sign of $Z(g_n)$ at each of the Gram points $g_1, \dots, g_N$, say, for some N. Write down this progression of signs, and suppose for an example that it is

$$- + - + - - + - + - - + - + + +. \tag{58}$$

According to Definition 7.1 one can read off that (58) comprises 4 Gram blocks of length 1, followed by 1 Gram block of length 6, followed by 3 Gram blocks of length 1, followed by 1 Gram block of length 2. Thus one can distinguish Gram blocks of length $k \geq 2$ as those commencing with two identical signs and concluding with two identical signs (which need not be the same in both instances). The distribution of these signs is related to the average number of Gram points per Gram block, which will be referred to as λ.

By a simple combinatorial argument it can be shown that if the signs of $Z(g_n)$ were positive or negative with equal probability, then $\lambda = 2$. Up to height g_n for $n = 7 \cdot 10^7$, Brent [4] calculated that $\lambda = 1.1873$, and that λ was increasing slowly with n. Brent then conjectured [op. cit., p. 1368] that, as $n \to \infty$, we have $\lambda \to l$ for some $l \leq 2$.

Since each Gram block has length at least 1, it is clear that $\lambda \geq 1$. Also, by Lemma 7.4,

$$\lambda = \frac{N_g(2T) - N_g(T)}{N_{GB}(T)} \ll \frac{N_g(T)}{T \log T} = 1,$$

whence

THEOREM 7.5. *If $\lambda = \lambda(T)$ is the average length of a Gram block lying between T and $2T$, then there exists a positive constant A such that*

$$1 \leq \lambda(T) \leq A.$$

Due to the size of the implicit constants used in the above methods (cf. §6.4), there is little plausibility that the above theorem can be refined to give a bound on λ which is less than 2. However this result still shows that the average length of a Gram block cannot be too large.

7.5. A positive proportion of failures. For $j = 0, 1, \dots,$ define a B_j *Gram block* to be one of length k that contains $k + j - 2$ zeroes. Also, write $N_{B_j}(T)$ as the number of B_j Gram blocks between T and $2T$. Note that, following the discussion after Definition 7.1, there are no Gram blocks of type B_{2l+1} for any integer l.

Consider $(g_n, g_{n+k}]$, a B_2 Gram block of length $k \geq 2$, and write $S(g_n) = \lambda$. Since $S(t)$ can decrease by at most one over the length of a Gram interval, it follows that $S(g_{n+2}) \geq \lambda - 2$, whence $S(t) \geq \lambda - 2$ for all $t \in (g_n, g_{n+k}]$. Also, the remaining two zeroes must lie in the same Gram interval, whence $S(t) \leq \lambda + 2$ for all $t \in (g_n, g_{n+k}]$. Thus if t and $t+h$ lie in a connected union of B_2 Gram blocks then $|S(t+h) - S(t)| \leq 4$. The argument in §5.2 can now be applied mutatis mutandis to prove

THEOREM 7.6. *Let $N_B(T) = N_{B_0}(T) + N_{B_4}(T) + N_{B_6}(T) + \cdots$, that is, $N_B(T)$ is the number of non-B_2 Gram blocks between T and $2T$. Then $N_B(T) \gg T \log T$ for sufficiently large T.*

7.6. Further failures. The frequency of the failure of the Rosser Rule can now be discussed using the methods of §5.3. Take the difference between the number of Gram points and the number of zeroes between T and $2T$; it follows from the definition of the $N_{B_j}(T)$ that

$$\{N_g(2T) - N_g(T)\} - \{N(2T) - N(T)\} = 2N_{B_0}(T) - 2N_{B_4}(T) - 4N_{B_6}(T) - \cdots .$$

Since, as before, the left side is $O(T)$ it follows that

$$\begin{aligned} N_{B_0}(T) &= O(T) + N_{B_4}(T) + 2N_{B_6}(T) + \cdots \\ &> O(T) + N_{B_4}(T) + N_{B_6}(T) + \cdots, \end{aligned}$$

after the addition of $N_{B_0}(T)$ to each side and the invocation of Theorem 7.6, one arrives at

$$N_{B_0}(T) \gg T \log T.$$

When combined with Lemma 7.4 this proves

THEOREM 7.7. *For sufficiently large T there is a positive proportion of Gram blocks between T and $2T$ which contain two fewer zeroes of $\zeta(s)$ than their length.*

Following a fortiori from the above theorem is

COROLLARY 7.8. *For sufficiently large T there is a positive proportion of failures of the Weak Rosser Rule, and therefore of Rosser's Rule, between T and $2T$.*

7.7. Difficulties with successes. One might hope to be able to adapt the arguments of §6 to show that there is a positive proportion of successes to the Weak Rosser Rule. Such an argument would need to show that the number of zeroes on the critical line contained in B_0 blocks is suitably small. With this is mind, let $M_{j,k}(T)$ denote the number of Gram blocks between T and $2T$ with length j and containing k zeroes of $\zeta(s)$; and let

$M_j(T) = \sum_k M_{j,k}(T)$. Consider the array

$$\begin{array}{lllll} N_{B_0}(T) = & & M_{2,0}(T) + M_{3,1}(T) & + M_{4,2}(T) & + \cdots, \\ N_{B_2}(T) = M_{1,1}(T) & + M_{2,2}(T) + M_{3,3}(T) & + M_{4,4}(T) & + \cdots, \\ N_{B_4}(T) = M_{1,3}(T) & + M_{2,4}(T) + M_{3,5}(T) & + M_{4,6}(T) & + \cdots, \\ \vdots \quad \vdots & \vdots \quad \vdots & \vdots \\ N_{B_k}(T) = M_{1,k-1}(T) + M_{2,k}(T) + M_{3,k+1}(T) + M_{4,k+3}(T) + \cdots, \\ \vdots \quad \vdots & \vdots \quad \vdots & \vdots \end{array}$$

To achieve a result analogous to Theorem 6.1 one needs to truncate this array both horizontally (by showing that Gram blocks with large length are rare) and vertically (by showing that the presence of many zeroes in a Gram block is rare). The latter can be achieved using the result given in (50), viz.

$$\frac{N_{F_k}(T)}{T \log T} \ll e^{-Ak}.$$

In the horizontal direction, note that for large K,

$$N_{GB}(T) = \sum_{j \geq K} M_j(T) + \sum_{j<K} M_j(T),$$

and

$$N_g(T) \geq K \sum_{j \geq K} M_j(T) + \sum_{j<K} M_j(T).$$

It therefore follows that

$$\frac{\sum_{j \geq K} M_j(T)}{T \log T} \ll \frac{1}{K}.$$

But then one sees that the proportion of zeroes contained in the $N_{B_0}(T)$ Gram blocks is $\ll 1$. This proportion needs to be $o(1)$ if the analysis contained in (52) and (53) is to be applied. Future research will comprise a detailed investigation into this problem.

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