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Multiplicative functions dictated by Artin symbols

by

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1. Introduction and statement of results. In a recent series of papers ([1], [3], [4], [5], [6] as a few examples), Granville and Soundararajan have introduced the notion of *pretentiousness* in analytic number theory. The idea of pretentiousness is to study generic complex-valued multiplicative functions of modulus bounded by 1 as an alternative to focusing on the zeros of L-functions. In this sense, this philosophy can be viewed as establishing a framework for the elementary proof of the prime number theorem due to Erdős and Selberg. Indeed, the theory has advanced well beyond that point—there are now pretentious proofs of many deep theorems in analytic number theory, including versions of the large sieve inequality [2], Linnik's theorem [2], and, due to Koukoulopoulos [10], a quantitative version of the prime number theorem for primes in arithmetic progressions of the same quality as can be obtained by traditional methods—and yet pretentiousness is still in its nascence. That is, even though they are essentially very basic objects, we still have much to learn about multiplicative functions.

Arguably the most striking multiplicative functions are Dirichlet characters. These are completely multiplicative functions defined on the primes via congruence relations, yet they also exhibit a strong additive structure—periodicity. One way in which this additional structure manifests itself is by the startling amount of cancellation exhibited by the summatory function. If $f(n)$ is any multiplicative function, let

$$S_f(X) := \sum_{n\le X} f(n).$$

Generically, if $|f(p)| \approx 1$, the best cancellation we can expect is what we would find in a random model, which would yield $S_f(X) = O(X^{1/2+\epsilon})$. However, if $\chi(n)$ is a non-principal Dirichlet character modulo q, we have

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the much stronger statement that $S_\chi(X) = O_q(1)$ as $X \to \infty$, where the subscript q indicates that the implied constant may depend on q.

Given this startling, albeit elementary, cancellation exhibited by Dirichlet characters, we may ask the following question.

QUESTION 1. *Given a completely multiplicative function $f(n)$ such that $|f(p)| \approx 1$ for almost all primes p, if $S_f(X) \ll X^{1/2-\delta}$ for some fixed $\delta > 0$, must $f(n)$ necessarily "come from" a Dirichlet character?*

Of course, this question is hopelessly vague, but a more precise version can be found in the author's work with J. Jung [7]. Even with a more precise formulation—requiring both that $|f(p)| \le 1$, so that $f(n)$ fits into the classical domain of pretentiousness, and, say, that $\sum_{n \le X} |f(n)|^2 \gg X$, so that $f(n)$ is not too small—Question 1 appears to be intractable at present. Not all is lost, however, as we are able to provide an answer for a certain, natural class of functions, which moreover seems like a not unreasonable place to look for conspiracies akin to the periodicity of Dirichlet characters.

This class of functions will be defined via the arithmetic of number fields, with Dirichlet characters arising from cyclotomic extensions. Thus, let $K/\mathbb{Q}$ be a finite Galois extension, not necessarily abelian, and let $\left(\frac{K/\mathbb{Q}}{\cdot}\right)$ denote the *Artin symbol*, defined so that for each rational prime p unramified in K, $\left(\frac{K/\mathbb{Q}}{p}\right)$ is the conjugacy class in $\mathrm{Gal}(K/\mathbb{Q})$ of elements acting like Frobenius modulo $\mathfrak{p}$ for some prime $\mathfrak{p}$ of K dividing p; recall that by the Chebotarev density theorem, each class occurs for a positive proportion of primes p. We let $\mathcal{S}_K$ denote the class of complex-valued completely multiplicative functions $f(n)$ with the following two properties.

First, we require that $|f(p)| \le 1$ for all primes p, with equality holding if p splits completely, so that f both fits into the context of pretentiousness and is of the same size as a Dirichlet character in absolute value. Secondly, generalizing the dependence of $\chi(p)$ only on the residue class of $p \pmod q$, we require $f(p)$ to depend only on the Artin symbol $\left(\frac{K/\mathbb{Q}}{p}\right)$. That is, if p_1 and p_2 are any two unramified primes such that

$$\left(\frac{K/\mathbb{Q}}{p_1}\right) = \left(\frac{K/\mathbb{Q}}{p_2}\right),$$

we must have $f(p_1) = f(p_2)$. We note that if $K = \mathbb{Q}(\zeta_m)$, the m-th cyclotomic extension, then $\mathcal{S}_K$ includes all Dirichlet characters modulo m, and by taking K to be a non-abelian extension, we can obtain other functions of arithmetic interest which are intrinsically different from Dirichlet characters; see the examples following Theorem 1.1. We are now interested in the following reformulation of Question 1 to the class of functions in $\mathcal{S}_K$.

QUESTION 2. *Suppose $f \in \mathcal{S}_K$ is such that $S_f(X) = O_f(X^{1/2-\delta})$ as $X \to \infty$ for some fixed $\delta > 0$. Must $f(n)$ coincide with a Dirichlet character? That is, must $f(p) = \chi(p)$ for all but finitely many primes?*

Modifying techniques of Soundararajan [12] that were developed to show that degree 1 elements of the Selberg class arise from Dirichlet L-functions, we are able to answer this question in the affirmative.

THEOREM 1.1. *If $f \in \mathcal{S}_K$ is such that $S_f(X) = O_f(X^{1/2-\delta})$, then $f(p) = \chi(p)$ for all unramified primes p, where χ is a Dirichlet character of conductor dividing the discriminant of K.*

TWO REMARKS. First, as mentioned above, the author and Jung [7] recently asked a more general version of Question 1 in their work on pretentiously detecting power cancellation in the partial sums of multiplicative functions. It is unfortunate that while Question 1 fits nicely into the pretentious philosophy, the proof of Theorem 1.1 is highly non-pretentious, as it relies critically on L-function arguments. However, we still consider this proof to be of merit, as it highlights the interface between pretentious questions and techniques relying on L-functions.

Second, as the proof will show, the conditions on the class $\mathcal{S}_K$ are not optimal. We have chosen the definition of $\mathcal{S}_K$ that we did for aesthetic purposes, but in fact, the class can be expanded in a few ways. First, we do not actually require that $f(n)$ is completely multiplicative, only that $f(p^2)$ is also determined by the Artin symbol $\left(\frac{K/\mathbb{Q}}{p}\right)$ and that $f(p^k)$, $k \geq 3$, can be suitably bounded independent of p. We also need to assume nothing about the size of $|f(p)|$ for primes p that do not split completely in K. In particular, the condition that $f(p)$ is determined by the Artin symbol implies that square root cancellation is still the expected random order, so that the question remains interesting. It would also be nice to completely remove the restriction on those primes which split completely, but this would likely require greater understanding of a particular extension of the Selberg class. At present, following techniques used in the proof of Theorem 1.1, it would likely be possible to allow $|f(p)| \leq 1$ for primes that split completely. However, we consider within this setting the case of $|f(p)| = 1$ to be the most interesting, as this is the only case in which there are $f(n)$ with $S_f(X) \ll X^{1/2-\delta}$; that is, if $f(p)$ is dictated by Artin symbols and $0 < |f(p)| < 1$ for primes p that split completely in K, then $S_f(X) \not\ll X^{1/2-\delta}$. We also note that, if we do not bound $f(p)$ at all, then coefficients of Artin L-functions associated with K are also permitted, and, assuming they are automorphic, they would also likely exhibit more than square root cancellation.

We conclude this section with three examples of functions in some $\mathcal{S}_K$ which we believe to be of arithmetic interest.

Example 1. Let $F(x) = x^3+x^2-x+1$, and let K be the splitting field of $F(x)$, which has Galois group $G \cong S_3$ and discriminant $-21296 = -2^4 \cdot 11^3$. Let $\rho(p)$ denote the number of inequivalent solutions to the congruence $F(x) \equiv 0 \pmod p$, and define the function $f \in \mathcal{S}_K$ by

$$f(p) = \begin{cases} -1 & \text{if } p \nmid 22 \text{ and } \rho(p) = 0, \\ 0 & \text{if } p \mid 22 \text{ or } \rho(p) = 1, \\ 1 & \text{if } p \nmid 22 \text{ and } \rho(p) = 3. \end{cases}$$

There is a unique Dirichlet character χ in $\mathcal{S}_K$, which corresponds to the alternating character of S_3, and is given by $\chi(p) = \left(\frac{-11}{p}\right)$. Alternatively, we can write $\chi(p)$ in terms of $\rho(p)$ by

$$\chi(p) = \begin{cases} -1 & \text{if } p \neq 11 \text{ and } \rho(p) = 1, \text{ or } p = 2, \\ 1 & \text{if } p \nmid 22 \text{ and } \rho(p) = 0 \text{ or } 3, \\ 0 & \text{if } p = 11. \end{cases}$$

Since $f(p) \neq \chi(p)$ for those primes p such that $\rho(p) = 0$ or 1 and since such primes occur a positive proportion of the time by the Chebotarev density theorem, we should not expect to see more than square root cancellation in the partial sums of $f(n)$ by Theorem 1.1, and indeed, we find the following:

X	$S_f(X)$	$\lvert S_f(X)\rvert/\sqrt{X}$	$\lvert S_f(X)\rvert/(X/\log^{7/6} X)$
10	0	0	0
10^2	-4	0.4	0.238
10^3	-12	0.379	0.114
10^4	-102	1.02	0.136
10^5	-736	2.327	0.127
10^6	-5757	5.757	0.123

In the next example, we discuss the apparent convergence in the fourth column.

Example 2. The astute reader may object to the above example by noting that $f(n)$ as constructed has mean $-1/6$ on the primes as a consequence of the Chebotarev density theorem. The Selberg–Delange method [13, Section II.5] predicts that, for a multiplicative function $g(n)$ of mean z on the primes, the summatory function $S_g(X)$ should be of the order

$$\frac{X(\log X)^{z-1}}{\Gamma(z)}.$$

In particular, for $f(n)$ as in the previous example, we would expect that $S_f(X)$ should have order $X/(\log X)^{7/6}$, which indeed matches the data more closely (and explains the fourth column). Notice, however, that the predicted main term is 0 when the mean on the primes is 0 or -1 (or any non-positive integer, but recall that we are inside the unit disc). The latter possibility essentially corresponds to the Möbius function $\mu(n)$, so the most interesting case occurs when the mean on the primes is 0. It is a simple exercise to see that any such function $g \in \mathcal{S}_K$ (where K is as in Example 1) arises as the "twist" of $\chi(n)$—we must have $g(p) = \omega\chi(p)$ for all primes $p \nmid 22$ and some ω satisfying $|\omega| = 1$. Taking $g(p) = i\chi(p)$ for all primes p, we compute the following:

X	$S_g(X)$	$\lvert S_g(X)\rvert/\sqrt{X}$
10	$1+i$	0.447
10^2	$2+i$	0.224
10^3	$6+2i$	0.2
10^4	$13+6i$	0.143
10^5	$36+50i$	0.195
10^6	$-260+215i$	0.337

Here, the fact that $S_g(X)$ is not $O(X^{1/2-\delta})$ for some $\delta > 0$ is less apparent than was the case in Example 1 (there is even more fluctuation than is visible in the limited information above—for example, $S_g(810000)/\sqrt{810000} \approx 0.059$), but nevertheless, since $g(n)$ does not coincide with a Dirichlet character, Theorem 1.1 guarantees that the partial sums are not $O(X^{1/2-\delta})$.

EXAMPLE 3. Let $F(x) = x^4 + 3x + 3$, and let K be the splitting field of $F(x)$, which has Galois group $G \cong D_4$ and discriminant $1750329 = 3^6 \cdot 7^4$. There are five conjugacy classes of G, three of which, each of order two, can be determined by exploiting the quadratic subfields $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{-7})$. To distinguish the remaining two conjugacy classes, each of which consists of a single element, we exploit the factorization of $F(x)$ modulo p. As in the previous examples, let $\rho(p)$ denote the number of inequivalent solutions to the congruence $F(x) \equiv 0 \pmod{p}$, and additionally define $l(p)$ to be the pair $\left(\left(\frac{-3}{p}\right), \left(\frac{-7}{p}\right)\right)$. We now consider $f \in \mathcal{S}_K$ defined by

$$f(p) = \begin{cases} 1 & \text{if } l(p) = (1,1) \text{ and } \rho(p) = 4, \\ -1 & \text{if } l(p) = (1,1) \text{ and } \rho(p) = 0, \\ 1 & \text{if } l(p) = (-1,-1), \\ \zeta_3 & \text{if } l(p) = (-1,1), \\ \zeta_3^2 & \text{if } l(p) = (1,-1), \\ 0 & \text{if } p \mid 21. \end{cases}$$

We note that $f(n)$ is neither a Dirichlet character nor its twist—each of the three Dirichlet characters $\left(\frac{-3}{\cdot}\right)$, $\left(\frac{-7}{\cdot}\right)$, and $\left(\frac{21}{\cdot}\right)$ has the same value on the singleton conjugacy classes, and these are the unique characters in $\mathcal{S}_K$—yet it has mean 0 on the primes. We find the following:

X	$S_f(X)$	$\lvert S_f(X)\rvert/\sqrt{X}$
10	0	0
10^2	$4.5 - 2.598i$	0.520
10^3	$-11 + 6.928i$	0.411
10^4	$0.5 - 2.598i$	0.026
10^5	$-34 - 71.014i$	0.249
10^6	$-21 + 124.708i$	0.126

As with Example 2, we find the fact that $S_f(X)$ is not $O(X^{1/2-\delta})$ to be not entirely clear, yet it is guaranteed to be so. In this case, there is even more fluctuation in the values of $|S_f(X)|/\sqrt{X}$. For example, when $X = 7.61 \cdot 10^5$, we have $|S_f(X)|/\sqrt{X} \approx 0.012$, yet when $X = 7.69 \cdot 10^5$, we have $|S_f(X)|/\sqrt{X} \approx 0.186$. Thus, without knowledge of Theorem 1.1, it would be difficult to guess the correct order of $S_f(X)$, although if one were forced to speculate, $O(X^{1/2})$ would probably be the most reasonable guess. In fact, under the generalized Riemann hypothesis, we have $S_f(X) = O_\epsilon(X^{1/2+\epsilon})$ for all $\epsilon > 0$, so that by Theorem 1.1, square root cancellation is the truth in this case.

2. Proof of Theorem 1.1. We begin by recalling the setup in which we are working. $K/\mathbb{Q}$ is a finite Galois extension with Galois group $G := \mathrm{Gal}(K/\mathbb{Q})$, and $f \in \mathcal{S}_K$ if and only if $|f(p)| \le 1$ for all primes p, with equality holding if p splits completely in K, and $f(p_1) = f(p_2)$ for all unramified primes p_1 and p_2 such that

$$\left(\frac{K/\mathbb{Q}}{p_1}\right) = \left(\frac{K/\mathbb{Q}}{p_2}\right).$$

Looking only at unramified primes, we can therefore regard f as a class function of G, and as such, it can be decomposed [11, p. 520] as

$$f = \sum_{\chi \in \mathrm{Irr}(G)} a_\chi \chi, \tag{2.1}$$

where $\mathrm{Irr}(G)$ denotes the set of characters associated to the irreducible representations of G, and each $a_\chi \in \mathbb{C}$. We remark that, because we have disregarded any information coming from the ramified primes, we lose all control over the values f assumes on such primes. In the proof to come, we will show that $f(p) = \chi(p)$ for all unramified primes p. To do so, that is, to establish Theorem 1.1, we will show that $a_\chi = 1$ for some one-dimensional χ and that $a_{\chi'} = 0$ for all other χ'. We will do so incrementally: first we establish that each a_χ is, in fact, rational, and then, using techniques due to Soundararajan [12] developed to study elements of the Selberg class of degree 1, we prove the result.

Let $L(s,f)$ denote the Dirichlet series associated to $f(n)$, so that we have

$$L(s,f) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left(1 - \frac{f(p)}{p^s}\right)^{-1},$$

recalling that $f(n)$ is completely multiplicative. By matching the coefficients of p^{-s} in each Euler factor, the decomposition (2.1) then guarantees the Euler product factorization

$$L(s,f) = \prod_\chi L(s,\chi)^{a_\chi} \prod_p (1 + O(p^{-2s})),$$

valid in the region of absolute convergence $\Re(s) > 1$, and where $L(s,\chi)$ is the Artin L-function associated to the representation attached to χ. In fact, we will need to go further with this factorization. The coefficient of p^{-2s} in the Euler product is again essentially a class function of G, so it can be decomposed as a linear combination of the characters χ, and we obtain

$$L(s,f) = \prod_\chi L(s,\chi)^{a_\chi} \prod_\chi L(2s,\chi)^{b_\chi} A(s), \tag{2.2}$$

where $A(s)$ is analytic and non-zero in the region $\Re(s) > 1/3$.

Recall that Artin L-functions factor as products of integral powers of Hecke L-functions, so that for each non-trivial χ, the function $L(2s,\chi)$ is analytic and non-zero in some neighborhood of the region $\Re(s) \geq 1/2$, and for the trivial character χ_0, we see that $L(2s,\chi_0) = \zeta(2s)$ is again analytic and non-zero in a neighborhood of $\Re(s) \geq 1/2$, except at $s = 1/2$. Thus, the product

$$\prod_\chi L(2s,\chi)^{b_\chi} A(s)$$

is analytic and non-zero in some neighborhood of $\Re(s) \geq 1/2$, except possibly at $s = 1/2$. Now, the condition

$$\sum_{n \leq X} f(n) = O(X^{1/2-\delta})$$

guarantees that $L(s,f)$ is analytic in the region $\Re(s) > 1/2 - \delta$, so by the above, it must be the case that

$$\prod_\chi L(s,\chi)^{a_\chi}$$

is analytic in a neighborhood of $\Re(s) \geq 1/2$, except possibly at $s = 1/2$. In particular, we must have

$$\operatorname{ord}_{s=s_0} L(s,f) = \sum_\chi a_\chi \operatorname{ord}_{s=s_0} L(s,\chi) \tag{2.3}$$

for any $s \neq 1/2$ with $\Re(s) \geq 1/2$.

Recall now that we wish to show that each a_χ is rational. This would follow from (2.3) above if there are $n := \#\mathrm{Irr}(G)$ choices $s_1, \ldots, s_n$ such that the matrix

$$(\operatorname{ord}_{s=s_i} L(s, \chi_j)), \qquad 1 \le i, j \le n,$$

is invertible over $\mathbb{Q}$. This follows by suitably adapting the proof of Lemma 2 of [8] by Kaczorowski and Perelli, who prove the statement provided that each $L(s, \chi_j)$ is in the Selberg class; potential poles in the critical strip pose little trouble for their method which depends almost entirely on the shape of the functional equation. One can also argue directly and show that, if there are no such $s_1, \ldots, s_n$, then there are integers n_χ, not all zero, such that the product

$$\prod_\chi L(s, \chi)^{n_\chi}$$

is holomorphic and non-vanishing away from $s = 1/2$. This product therefore behaves like a degree 0 L-function with conductor 1, and so must in fact be constant and equal to 1. The linear independence of characters implies that this contradicts the assumption that not all n_χ are 0.

At this stage, we are able to prove the theorem. The advantage gained by knowing that each a_χ is rational is that the function

$$F(s) := \prod_\chi L(s, \chi)^{a_\chi}$$

enjoys nice analytic properties. In particular, apart from a possible branch along the ray $(-\infty, 1/2]$, it will be holomorphic. To see this, let k be the denominator of the a_χ, and note that, from (2.3), we must have

$$\prod_\chi \Lambda(s, \chi)^{k a_\chi} = (s - 1/2)^m h(s)^k$$

for some entire function $h(s)$. Ignoring the branch, $F(s)$ essentially behaves as an L-function of degree $\sum a_\chi \dim \chi$. However, we note that this is also the evaluation of $f(p)$ at a prime that splits completely in K (or, equivalently, at the identity of $\mathrm{Gal}(K/\mathbb{Q})$) by (2.1). Thus, it is a rational number of absolute value 1, and so equals either 1 or -1. However, there are no holomorphic L-functions of negative degree, as can be seen, for example, by a zero counting argument (which can be modified simply to account for the possible branch), and so the degree must be 1. Moreover, it is known that a degree 1 element of the Selberg class must come from a Dirichlet L-function, a fact which is originally due to Kaczorowski and Perelli [9] and was reproved by Soundararajan [12]. However, as before, $F(s)$ does not satisfy the axioms of the Selberg class, as its gamma factor may have negative exponents, so we must modify Soundararajan's proof to our situation.

There are only two key components in Soundararajan's proof—an approximate functional equation for $F(s)$ and control of the gamma factors on the line $\Re(s)=1/2$. The proof of the approximate functional equation naturally requires the analytic properties of $F(s)$, and as it may have a branch in our situation, we must modify the proof slightly; we do so in Lemma 2.1 below. On the other hand, the control over the gamma factors is provided from our assumption on the degree, so in particular, the same estimates hold. We state these estimates in (2.4) and we give the idea of Soundararajan's proof below, after the proof of Lemma 2.1.

LEMMA 2.1. *For any $t\in\mathbb{R}$ such that $|t|\ge 2$ and any $X>1$, we have*

$$F(1/2+it)=\sum_{n=1}^{\infty}\frac{a(n)}{n^{1/2+it}}e^{-n/X}+O(X^{-1+\epsilon}(1+|t|)^{1+\epsilon}+X^{1/2+\epsilon}e^{-|t|}).$$

Proof. Consider the integral

$$I:=\int_{(1)}F(1/2+it+w)X^{w}\Gamma(w)\,dw.$$

On the one hand, replacing F by its Dirichlet series and directly computing, we find that I is given by

$$I=\sum_{n=1}^{\infty}\frac{a(n)}{n^{1/2+it}}e^{-n/X}.$$

On the other hand, moving the line of integration to the left and letting $\tilde{F}(w)$ denote the integrand, we find that

$$I=F(1/2+it)+\int_{-1+\epsilon-i\infty}^{-1+\epsilon-(t+1/2)i}\tilde{F}(w)\,dw+\int_{-1+\epsilon-(t-1/2)i}^{-1+\epsilon+i\infty}\tilde{F}(w)\,dw+\int_{\mathcal{C}}\tilde{F}(w)\,dw,$$

where $\mathcal{C}$ denotes the contour formed by the union of the segments $[-1+\epsilon-(t-1/2)i,1/2+\epsilon-(t+1/2)i]$, $[1/2+\epsilon-(t+1/2)i,1/2+\epsilon-(t-1/2)i]$, and $[1/2+\epsilon-(t-1/2)i,-1+\epsilon-(t-1/2)i]$. Notice that $\mathcal{C}$ is bounded away from the branch cut of $F(s)$, whence its contribution can be bounded as $O(X^{1/2+\epsilon}e^{-|t|})$. The contribution from the two vertical contours is handled exactly as in Soundararajan's proof, and yields a contribution of $O(X^{-1+\epsilon}(1+|t|)^{1+\epsilon})$. ∎

As mentioned above, we must also have some control over the gamma factors on the line $\Re(s)=1/2$. This is straightforward, as Stirling's formula implies, if $G(s)=\prod_{\chi}\Gamma(s,\chi)^{a_\chi}$, that there are constants $B,C\in\mathbb{R}$ such that

$$\frac{\overline{G}(1/2-it)}{G(1/2+it)} = e^{-it\log\frac{t}{2e}+iB+\frac{\pi i}{4}}C^{-it}(1+O(t^{-1})). \tag{2.4}$$

(For the reader closely following Soundararajan's line of reasoning, we do not have a t^{iA} term because, for each χ, all $\mu_{\chi,j}$ are real.) This is not the most natural representation, but it turns out to be convenient for the proof. Now, the idea of Soundararajan's proof is to consider, for any real $\alpha > 0$, the quantities

$$\mathcal{F}(\alpha,T) := \frac{1}{\sqrt{\alpha}}\int_{\alpha T}^{2\alpha T} F(1/2+it)e^{it\log\frac{t}{2\pi e\alpha}-\frac{\pi i}{4}}\,dt,$$
$$\mathcal{F}(\alpha) := \lim_{T\to\infty}\frac{1}{T}\mathcal{F}(\alpha,T).$$

Armed with Lemma 2.1, one can evaluate $\mathcal{F}(\alpha)$ in two ways, either using the functional equation or not. The first method shows that $\mathcal{F}(\alpha)=0$ unless $\pi\alpha Cq^2\in\mathbb{Z}$, where $q=\prod_\chi q_\chi^{a_\chi}$ and C is as in (2.4), in which case it is, essentially, the coefficient $a(\pi\alpha Cq^2)$. The second method, on the other hand, shows that $\mathcal{F}(\alpha)$ is periodic with period 1, whence $\pi Cq^2\in\mathbb{Z}$ and the coefficients $a(n)$ are periodic modulo $\pi Cq^2 =: q_0\in\mathbb{Z}$. As remarked above, the proof of this follows Soundararajan's [12] exactly, with the only modification necessary being the replacement of his approximate functional equation with ours, Lemma 2.1. In fact, this argument extends to show that, if we modify the definition of the Selberg class to allow rational exponents on the gamma factors and for there to be finitely many lapses of holomorphicity, then still, the only degree 1 elements are those coming from the traditional Selberg class.

To conclude the proof of Theorem 1.1, we note that since the coefficients $F(s)$ are periodic modulo q_0 and are also multiplicative, it must be the case that, away from primes dividing q_0, they coincide with a Dirichlet character $\chi_{q_0} \pmod{q_0}$. Thus, we have

$$\prod_\chi L(s,\chi)^{a_\chi} \doteq L(s,\chi_{q_0}),$$

where $\doteq$ means that equality holds up to a finite product over primes. By linear independence of characters, the only way this can happen is if $\chi=\chi_{q_0}$ and $a_\chi=1$ for some χ and $a_{\chi'}=0$ for all others. Moreover, this shows that the Euler factors absorbed into the $\doteq$ come only from the ramified primes. This is exactly what we wished to show, so the result follows.

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