Minimal polynomials of some beta-numbers
and Chebyshev polynomials

by

DOYONG KWON (Seoul)

1. Introduction. For $\beta > 1$, the $\beta$-transformation $T_\beta : x \mapsto \beta x \mod 1$ is multiplication by $\beta$ modulo 1. Then the $\beta$-expansion $d_\beta(x) = (x_i)_{i \geq 1}$ of $x \in [0,1]$ is defined by $x_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor$. While Rényi [15] introduced the $\beta$-transformation and proved that it is ergodic, Parry [13] found its invariant measure and characterized possible sequences that can be a $\beta$-expansion. It was shown that the $\beta$-expansion of 1 bounds, in a sense, the $\beta$-expansion of any other $x \in [0,1)$ in terms of lexicographic order. To be more precise, let us fix $\beta > 1$ and assume that $s = x_1x_2 \cdots$ is an infinite word over the alphabet $\{0,1,\ldots,\lfloor \beta \rfloor\}$. In addition, put

$$d_\beta^*(1) := \lim_{\varepsilon \downarrow 0} d_\beta(1 - \varepsilon);$$

we have $d_\beta^*(1) = (e_1 \cdots e_{m-1}(e_m - 1))^{\omega}$ if $d_\beta(1) = e_1 \cdots e_m 0^{\omega}$ and $d_\beta^*(1) = d_\beta(1)$ otherwise. Here $a^{\omega} := aa \cdots$. Then Parry showed that $s = d_\beta(x)$ for some $x \in [0,1)$ if and only if $\sigma^n(s) < d_\beta^*(1)$ for all $n \geq 0$, where $\sigma$ is the shift.

If $d_\beta(1)$ is eventually periodic, say $d_\beta(1) = e_1 \cdots e_n(e_{n+1} \cdots e_{n+p})^{\omega}$, then we call $\beta$ a beta-number. In this case, $\beta$ is the dominant root of

$$\left(x^{n+p} - \sum_{i=1}^{n+p} e_i x^{n+p-i}\right) - \left(x^n - \sum_{i=1}^{n} e_i x^{n-i}\right) = 0.$$ 

In particular, if $d_\beta(1)$ is finite, i.e., $d_\beta(1) = e_1 \cdots e_n 0^{\omega}$, then we call $\beta$ a simple beta-number and $\beta$ is the dominant root of

$$x^n - \sum_{i=1}^{n} e_i x^{n-i} = 0.$$ 

2000 Mathematics Subject Classification: Primary 11R06, 11R09; Secondary 37B10, 68R15.

Key words and phrases: beta-numbers, Chebyshev polynomials, Pisot numbers, Mahler measure.
These two polynomials above are called the beta-polynomial of $\beta$, or simply the $\beta$-polynomial if $\beta$ is clear from the context. The term characteristic polynomial is also used in the literature.

We focus on a special class of $\beta$-expansions. We study the real $\beta > 1$ for which $d^*_\beta(1)$ encodes a rational rotation on $\mathbb{R}/\mathbb{Z}$. In this case, $\beta$ is an algebraic integer. For the case of irrational rotations, see [4, 6]. An arithmetic study was also pursued there and the current work is in fact motivated by those papers.

Let $\alpha > 0$ and $\varrho \in [0,1]$. We consider the following two infinite sequences called lower and upper mechanical words with slope $\alpha$ and intercept $\varrho$: for $n \geq 0$,

\[
\begin{align*}
 s_{\alpha,\varrho}(n) &= \lfloor \alpha(n+1) + \varrho \rfloor - \lfloor \alpha n + \varrho \rfloor, \\
 s'_{\alpha,\varrho}(n) &= \lceil \alpha(n+1) + \varrho \rceil - \lceil \alpha n + \varrho \rceil.
\end{align*}
\]

Note that these are infinite words over the alphabet $A = \{\lceil \alpha \rceil - 1, \lceil \alpha \rceil\}$.

For an irrational $\alpha$, the words are aperiodic and called Sturmian words. On the other hand, a rational $\alpha$ produces purely periodic words, whose shortest period words are called Christoffel words. Our main concern is the rational case. We now describe it in more detail.

Let $\alpha = p/q > 0$ with $\gcd(p, q) = 1$ and $b = \lceil \alpha \rceil$. Then $A = \{b-1, b\}$ and the lengths of the Christoffel words, say $t_{p,q}$, $t'_{p,q}$, are $q$. So $s_{\alpha,0} = t_{p,q}$ and $s'_{\alpha,0} = (t'_{p,q})^\omega$. We also have

\[
t_{p,q} = (b-1)z_{p,q}b, \quad t'_{p,q} = bz_{p,q}(b-1),
\]

for some word $z_{p,q}$, called a central word. We see that $z_{p,q}$ is a palindrome, i.e., $z_{p,q}$ is equal to its reversal (see [12]). The motivation of this paper comes from the next proposition. For any word $w = a_0a_1 \cdots a_{n-1}$ with $a_i \in \mathbb{Z}$, we denote by $\overrightarrow{w}$ the vector $(a_0, \ldots, a_{n-1}) \in \mathbb{Z}^n$.

**Proposition 1.1 ([4, 6]).** For $\alpha > 0$, there exists a unique $\beta > 1$ for which $d^*_\beta(1) = s'_{\alpha,0}$. Define $\Delta : (0, \infty) \to (1, \infty)$ by $\Delta(\alpha) = \beta$. Then

(a) At an irrational $\alpha > 0$, $\Delta$ is continuous, and $\Delta(\alpha)$ is a transcendental number.

(b) $\Delta$ is left-continuous but not right-continuous at every rational.

(c) If $\alpha = p/q$ with $\gcd(p, q) = 1$ and $b = \lceil \alpha \rceil$, then the $\Delta(\alpha)$-polynomial is

\[
x^q - b\overrightarrow{z_{p,q}b} \cdot (x^{q-1}, x^{q-2}, \ldots, 1).
\]

(d) For the same rational $\alpha$ as above, let $\Delta(\alpha+) := \lim_{x \to \alpha+} \Delta(x)$. Then the $\Delta(\alpha+)$-polynomial is

\[
x^{q+1} - b\overrightarrow{z_{p,q}b} \cdot (x^q, x^{q-1}, \ldots, x) - x + 1.
\]
For a rational $\alpha > 0$, $\Delta(\alpha)$ is called a lower self-Christoffel number, and the right limit $\Delta(\alpha+)$ an upper self-Christoffel number.

We are now in a position to state our main theorem.

**Theorem.** Let $p_0/q_0$ be a fixed rational with $0 < p_0 \leq q_0$ and $\gcd(p_0, q_0) = 1$. For a positive integer $b$, suppose $p/q = b - 1 + p_0/q_0$. Then there exists an effectively computable $B$ such that for all $b \geq B$, we have:

(a) $x^q - b_{p,q} b \cdot (x^{q-1}, x^{q-2}, \ldots, 1)$ is irreducible over $\mathbb{Q}$,

(b) $x^{q+1} - b_{p,q} b \cdot (x^q, x^{q-1}, \ldots, x) - x + 1$ is irreducible over $\mathbb{Q}$.

Thus the polynomials mentioned are eventually the minimal polynomials of $\Delta(p/q)$ and $\Delta(p/q+)$. The above theorem was in fact proved in [7] but we did not know from which value $b$ on the polynomials were irreducible. Here we give another proof involving Chebyshev polynomials. The new proof enables us to effectively find $B$ for which $b \geq B$ implies that the corresponding polynomials are irreducible over $\mathbb{Q}$. The effective procedure for finding $B$ is given in Section 4. These quantitative results have some connections with other fields of number theory, which will be discussed in Section 4.

**2. Preliminaries.** In this section we briefly review some concepts and known results to be used in the main proof. First we recall some definitions in number theory.

Among algebraic integers, a Pisot number is $\alpha > 1$ all of whose conjugates lie inside the unit circle. Suppose $g(x) = a_n x^n + \cdots + a_1 x + a_0 = a_n \prod_{i=1}^n (x - \alpha_i) \in \mathbb{Z}[x]$ with $a_n \neq 0$. Then the Mahler measure of $g$ is the positive number defined by

$$M(g) = |a_n| \prod_{i=1}^n \max\{1, |\alpha_i|\}.$$ 

In particular, cyclotomic polynomials have Mahler measure 1. The most famous problem on Mahler measures, posed by Lehmer [10], is whether or not 1 is an accumulation point of the set $\{M(g) \mid g \in \mathbb{Z}[x]\}$. Though it is still open, an answer can be given for a special class of polynomials. Let $f \in \mathbb{R}[x]$ and $\deg(f) = n$. If $f$ satisfies $f(x) = x^n f(x^{-1})$ then we say that $f$ is reciprocal. Now the Mahler measures of nonreciprocal polynomials cannot be less than the smallest Pisot number:

**Theorem 2.1 ([17]).** Let $p(x) \in \mathbb{Z}[x]$ and let $\theta_0 = 1.32472 \ldots$ be the smallest Pisot number, i.e., the real zero of $x^3 - x - 1$. If $M(p) < \theta_0$, then $p(x)$ is a reciprocal polynomial.

Let $T_n$ and $U_n$ be the $n$th Chebyshev polynomials of the first and the second kind respectively. Among many equivalent definitions we adopt the
following simple equations, which are suitable for our proof:

\[ T_n(\cos \theta) := \cos n\theta, \quad U_n(\cos \theta) := \frac{\sin(n + 1)\theta}{\sin \theta}. \]

For instance, \( T_2(x) = 2x^2 - 1 \) and \( U_3(x) = 8x^3 - 4x. \)

In the analysis of our polynomials, we shall encounter some reciprocal polynomials, and they will be converted into more convenient ones via a transformation introduced in \([3]\).

Suppose that \( p(z) = \sum_{i=0}^{2n} a_i z^i \in \mathbb{R}[z] \) is a nonzero reciprocal polynomial with \( a_i = a_{2n-i}, \ i = 0, \ldots, n. \) If \( a_{2n} = a_{2n-1} = \cdots = a_{n+k+1} = 0 \) but \( a_{n+k} \neq 0, \) then

\[
p(z) = \sum_{i=0}^{2n} a_i z^i = z^n \left[ a_{n+k} \left( z^k + \frac{1}{z^k} \right) + \cdots + a_{n+1} \left( z + \frac{1}{z} \right) + a_n \right]
\]

\[
= a_{n+k} z^n \prod_{i=1}^{k} \left( z + \frac{1}{z} - \alpha_i \right) = a_{n+k} z^{n-k} \prod_{i=1}^{k} (z^2 - \alpha_i z + 1)
\]

for some \( \alpha_i \in \mathbb{C}, \ i = 1, \ldots, k. \) Given a reciprocal polynomial \( p \) as above, the Chebyshev transform \( T \) of \( p \) is defined by

\[
T p(x) := a_{n+k} \prod_{i=1}^{k} (x - \alpha_i).
\]

**Theorem 2.2** ([3]). The Chebyshev transform \( T \) is a linear isomorphism of the space of real reciprocal polynomials, \( \{ p(z) = \sum_{i=0}^{2n} a_i z^i \in \mathbb{R}[z] : a_i = a_{2n-i}, \ i = 0, \ldots, n \} \), into the space of real polynomials of degree at most \( n. \)

Note that

\[
|p(e^{i\theta})| = |T p(2 \cos \theta)|.
\]

Chebyshev polynomials now naturally appear via Chebyshev transforms.

**Lemma 2.3** ([9]). If \( p(z) = z^{2n} + 1 \) and \( q(z) = z^{2n} + z^{2n-2} + \cdots + z^2 + 1, \) then \( T p(x) = 2T_n(x/2) \) and \( T q(x) = U_n(x/2). \)

The following geometric study on beta-polynomials will be useful below. The reader should note that all the bounds given in the proposition tend to 1 as \( \beta \) increases.

**Proposition 2.4** ([7]).

(a) If \( \beta \) is an upper self-Christoffel number and \( \gamma \neq \beta \) is a zero of the \( \beta \)-polynomial, then

\[
|\gamma| \leq \frac{\beta + \sqrt{\beta^2 + 4\beta}}{2\beta}.
\]
(b) If \( \beta \) is a lower self-Christoffel number and \( \gamma \neq \beta \) is a zero of the \( \beta \)-polynomial, then

\[
\frac{2\beta + 1 - \sqrt{8\beta + 1}}{2\beta} \leq |\gamma| \leq \frac{2\beta + 1 + \sqrt{8\beta + 1}}{2\beta}.
\]

3. Proof. Through this section, the notations appearing in the Theorem will be used without explicit mention. First we consider lower self-Christoffel numbers.

Put \( f(x) = x^q - b_{p,q}b \cdot (x^{q-1}, x^{q-2}, \ldots, 1) \) and suppose that \( f(x) = g(x)h(x) \) over \( \mathbb{Q} \) and \( g(\beta) \neq 0 = h(\beta) \). Then Theorem 2.1 together with Proposition 2.4 implies that \( g \) is eventually reciprocal as \( b \) increases. If \( \gamma \) is a zero of \( g \), then so is \( \gamma - 1 \). Thus \( \gamma q \rightarrow_{p,q} b(b \cdot (\gamma q - 1, \gamma q - 2, \ldots, 1) = \gamma q - 1 \gamma - 1, \)

where we use the fact that \( b_{p,q}b \) is a palindrome. So \( \gamma q + 1 = 1 \).

Assume that \( q = 2n + 1 \) and \( \gamma_j = e^{i\theta_j} = e^{2j\pi i/(q+1)} \) is a zero of \( g \) for some \( j = 1, \ldots, q \). We then consider the polynomial

\[
r(z) := z^q - f(z) = b_{p,q}b \cdot (z^{2n}, z^{2n-1}, \ldots, 1)
\]

\[
= b(z^{2n} + z^{2n-1} + \ldots + 1) - \sum_{k=1}^{n} a_k z^k (z^{2n-2k} + 1), \quad a_k \in \{0, 1/2, 1\}.
\]

Together with the equation

\[
z^{2n} + z^{2n-1} + \ldots + 1 = (z^{2n} + z^{2n-2} + \ldots + z^2 + 1) + z(z^{2n-2} + z^{2n-4} + \ldots + z^2 + 1),
\]

Lemma 2.3 guarantees that the Chebyshev transform of \( r(z) \) is

\[
Tr(x) = b \left( U_n \left( \frac{x}{2} \right) + U_{n-1} \left( \frac{x}{2} \right) \right) - \sum_{k=1}^{n} 2a_k T_{n-k} \left( \frac{x}{2} \right).
\]

To compute \( Tr(2\cos \theta_j) \), we use the fact that \( T_{n-k}(\cos \theta_j) = \cos(n-k)\theta_j \) and that

\[
U_n(\cos \theta_j) + U_{n-1}(\cos \theta_j) = \frac{\sin(n+1)\theta_j + \sin n\theta_j}{\sin \theta_j}
\]

\[
= 2 \frac{\sin \left( \frac{(2n+1)\theta_j}{2} \right) \cos \frac{\theta_j}{2}}{\sin \frac{\theta_j}{2}} = \frac{\sin \left( \frac{(2n+1)\theta_j}{2} \right)}{\sin \frac{\theta_j}{2}}.
\]

Thus one finds that

\[
1 = |Tr(2\cos \theta_j)| \geq b \left| \frac{\sin \left( \frac{(2n+1)\theta_j}{2} \right)}{\sin \frac{\theta_j}{2}} \right| - \sum_{k=1}^{n} 2a_k.
\]
Since $0 < \theta_j/2 < \pi$ and $(2n+1)\theta_j/2 = (2n+1)j\pi/(2n+2)$ cannot be an integer multiple of $\pi$ for any $j = 1, \ldots, 2n+1$, this leads to a contradiction as $b$ increases.

If $q = 2n$, then

$$r(z) = z^q - f(z) = b z_p q b \cdot (z^{2n-1}, z^{2n-2}, \ldots, 1)$$

$$= b (z^{2n-1} + z^{2n-2} + \cdots + 1) - \sum_{k=1}^{n-1} a_k z^k (z^{2n-2k-1} + 1), \quad a_k \in \{0, 1\}.$$

If $n = 1$, then the sum is understood to be zero. Define a polynomial $r_1(z)$ by

$$r_1(z) := \frac{r(z)}{z + 1} = b (z^{2n-2} + z^{2n-4} + \cdots + z^2 + 1)$$

$$- \sum_{k=1}^{n-1} a_k z^k (z^{2n-2k-2} - z^{2n-2k-3} + \cdots - z + 1).$$

Its Chebyshev transform is

$$\mathcal{T} r_1(x) = b U_{n-1} \left( \frac{x}{2} \right) - \sum_{k=1}^{n-1} a_k \left( U_{n-k-1} \left( \frac{x}{2} \right) - U_{n-k-2} \left( \frac{x}{2} \right) \right).$$

If $\gamma_j = e^{i\theta_j} = e^{2j\pi i/(q+1)}$ is a zero of $g$ for some $j = 1, \ldots, q$, then

$$(2) \quad U_{n-k-1}(\cos \theta_j) - U_{n-k-2}(\cos \theta_j) = \frac{\sin(n-k)a_j - \sin(n-k-1)a_j}{\sin \theta_j}$$

$$= \frac{2 \cos (2n-2k-1)a_j \sin \theta_j}{\sin \theta_j} = \frac{\cos (2n-2k-1)a_j}{\cos \frac{\theta_j}{2}}.$$

Since $\gamma_j \neq -1$, there is a constant $M$ so that

$$M \geq |\mathcal{T} r_1(2 \cos \theta_j)| \geq b \frac{\sin n\theta_j}{|\sin \theta_j|} - \sum_{k=1}^{n} \frac{a_k}{|\cos \frac{\theta_j}{2}|}.$$

For any $j = 1, \ldots, 2n$, the number $n\theta_j = 2n j \pi/(2n+1)$ is never an integer multiple of $\pi$ and the number $\theta_j/2 = j\pi/(2n+1)$ cannot be $\pi/2$. Now we get a contradiction for every sufficiently large $b$. This proves part (a) of the Theorem.

The crucial ingredient in the previous proof is to split a given beta-polynomial into a reciprocal polynomial and a monomial. This technique also works for upper self-Christoffel numbers.

Put $f(x) = x^{q+1} - b z_p q b \cdot (x^q, x^{q-1}, \ldots, x) - x + 1$ and suppose that $f(x) = g(x)h(x)$ over $\mathbb{Q}$ and $g(\beta) \neq 0 = h(\beta)$. Then the same reasoning as before shows that $g$ is eventually reciprocal as $b$ increases. If $\gamma$ is a zero of $g$, the
then so is $\gamma^{-1}$. Hence  
\[
\gamma = \gamma^{q+1} - b\overline{z_{p,q}b} \cdot (\gamma^q, \gamma^{q-1}, \ldots, \gamma) + 1 \\
= \gamma^{q+1}[\gamma^{-q-1} - b\overline{z_{p,q}b} \cdot (\gamma^{-q}, \gamma^{-q+1}, \ldots, \gamma^{-1}) + 1] = \gamma^{q+1}\gamma^{-1} = \gamma^q,
\]
which gives $\gamma^{q-1} = 1$.

First, we suppose that $q = 2n - 1$ and $\gamma_j = e^{i\theta_j} = e^{2j\pi/(q-1)}$ is a zero of $g$ for some $j = 1, \ldots, q - 2$. We then put
\[
r(z) := f(z) + z = z^{2n} - b\overline{z_{p,q}b} \cdot (z^{2n-1}, z^{2n-2}, \ldots, z) + 1 \\
= z^{2n} + z^{2n-1} + \cdots + 1 - (b + 1)z(z^{2n-2} + \cdots + 1) \\
+ \sum_{k=1}^{n} a_k z^{2n-2k} + 1, \quad a_k \in \{0, 1/2, 1\}.
\]
Similar arguments to those above yield the Chebyshev transform
\[
T r(x) = U_n\left(\frac{x}{2}\right) + U_{n-1}\left(\frac{x}{2}\right) - (b + 1)\left(U_{n-1}\left(\frac{x}{2}\right) + U_{n-2}\left(\frac{x}{2}\right)\right) \\
+ \sum_{k=1}^{n} 2a_k T_{n-k}\left(\frac{x}{2}\right).
\]
Using $T_{n-k}(\cos \theta_j) = \cos(n - k)\theta_j$ and (1) one thus finds that
\[
1 = |T r(2\cos \theta_j)| \geq (b + 1) \left|\frac{\sin \left(\frac{(2n-1)\theta_j}{2}\right)}{\sin \frac{\theta_j}{2}}\right| - \left|\frac{\sin \left(\frac{(2n+1)\theta_j}{2}\right)}{\sin \frac{\theta_j}{2}}\right| - \sum_{k=1}^{n} 2a_k.
\]
Since $0 < \theta_j / 2 < \pi$ and $(2n - 1)\theta_j / 2 = (2n - 1)j\pi/(2n - 2)$ cannot be an integer multiple of $\pi$ for any $j = 1, \ldots, 2n - 3$, we eventually get a contradiction as $b$ increases.

If $q = 2n$, then
\[
r(z) = f(z) + z = z^{2n+1} - b\overline{z_{p,q}b} \cdot (z^{2n}, z^{2n-1}, \ldots, z) + 1 \\
= z^{2n+1} + z^{2n} + \cdots + 1 - (b + 1)z(z^{2n-1} + \cdots + 1) \\
+ \sum_{k=1}^{n} a_k z^{2n-2k} + 1, \quad a_k \in \{0, 1\}.
\]
This can be rewritten in the form
\[
r_1(z) := \frac{r(z)}{z + 1} \\
= z^{2n} + z^{2n-2} + \cdots + z^2 + 1 - (b + 1)z(z^{2n-2} + z^{2n-4} + \cdots + z^2 + 1) \\
+ \sum_{k=1}^{n} a_k z^{2n-2k} - z^{2n-2k-1} + \cdots - z + 1.
\]
Hence we find
\[ T r_1(x) = U_n \left( \frac{x}{2} \right) - (b + 1)U_{n-1} \left( \frac{x}{2} \right) + \sum_{k=1}^{n} a_k \left( U_{n-k} \left( \frac{x}{2} \right) - U_{n-k-1} \left( \frac{x}{2} \right) \right). \]

If \( \gamma_j = e^{i \theta_j} = e^{2j\pi i/(q-1)} \) is a zero of \( g \) for some \( j = 1, \ldots, q - 2 \), then one can verify as in (2) that
\[ U_{n-k}(\cos \theta_j) - U_{n-k-1}(\cos \theta_j) = \cos \left( \frac{2n-2k+1}{2} \theta_j \right). \]

Since \( \gamma_j \neq -1 \), there is a constant \( M \) so that
\[ M \geq |T r_1(2 \cos \theta_j)| \geq (b + 1) \left| \frac{\sin n \theta_j}{\sin \theta_j} - \frac{\sin(n + 1) \theta_j}{\sin \theta_j} - \sum_{k=1}^{n} a_k \left| \cos \frac{\theta_j}{2} \right| \right|. \]

For any \( j = 1, \ldots, 2n - 2 \), the number \( n \theta_j = 2nj\pi/(2n - 1) \) is never an integer multiple of \( \pi \) and the number \( \theta_j/2 = j\pi/(2n - 1) \) cannot be \( \pi/2 \). Now we get a contradiction for every sufficiently large \( b \).

4. Discussion and further studies. Suppose that an integer \( b \geq 1 \) is divisible by a prime \( p \) but not by \( p^2 \). Then the Eisenstein criterion shows that the beta-polynomial of \( \Delta((bq - 1)/q) \) is irreducible for any \( q \geq 1 \). Now we introduce another interesting connection between prime numbers and irreducibility for self-Christoffel numbers. This is related to Mersenne primes, i.e., primes of the form \( 2^n - 1 \). A famous open question in number theory is:

Are there infinitely many Mersenne primes?

We need a classical irreducibility criterion for polynomials in \( \mathbb{Z}[x] \), which appears for example in [14].

**Theorem 4.1.** Let \( \beta_1, \ldots, \beta_n \) be the zeros of some \( f(x) \in \mathbb{Z}[x] \) of degree \( n \). If there exists an integer \( b \) such that \( f(b) \) is prime, \( f(b - 1) \neq 0 \) and

\[ \Re(\beta_i) < b - 1/2 \quad \text{for} \quad 1 \leq i \leq n, \]

then \( f(x) \) is irreducible over \( \mathbb{Q} \).

We consider the beta-polynomials of \( \Delta(1/q) \) and \( \Delta(1/q+) \). Denote them by \( f^l_q(x) \) and \( f^u_q(x) \) respectively, and take \( b = 2 \). Then \( f^l_q(1) = f^u_q(1) = -1 \). If \( q \geq 3 \) then \( \Delta(1/q) < 3/2 \), and if \( q \geq 4 \) then \( \Delta(1/q+) < 3/2 \). Since beta-numbers are Perron numbers [11], condition (4) holds for lower (resp. upper) self-Christoffel numbers whenever \( q \geq 3 \) (resp. \( q \geq 4 \)). We also note that
\[ f^l_q(2) = 2^{q-1} - 1, \quad f^u_q(2) = 2^q - 3. \]

Now one can say the following:
1. If there exist infinitely many Mersenne primes, then $f_q^1(x)$ is irreducible for infinitely many $q \geq 3$.

2. If there exist infinitely many primes of the form $2^n - 3$, then $f_q^2(x)$ is irreducible for infinitely many $q \geq 4$.

Using a computer the author checked the irreducibility of beta-polynomials of self-Christoffel numbers. To be more precise, suppose $\alpha = \frac{p}{q} + b - 1$, where $1 \leq p < q \leq 200$, $\gcd(p, q) = 1$ and $1 \leq b \leq 150$. For given $1/2 \leq p/q \leq 199/200$, the irreducibility of beta-polynomials of $\Delta(\alpha)$ and $\Delta(\alpha+)$ was checked for $1 \leq b \leq 150$. When some beta-polynomial is reducible, the case was recorded as “$(p, q, b)$” in a row. As a result, a table was obtained, which comprises 5642 (resp. 3422) reducible cases for lower (resp. upper) self-Christoffel numbers. This is too huge to be included here. We refer to [8] instead. For example, if $\alpha = \frac{6}{11} + b - 1$ then the beta-polynomial of $\Delta(\alpha)$ is irreducible for all $1 \leq b \leq 4 \leq b \leq 150$, and the beta-polynomial of $\Delta(\alpha+)$ is irreducible for all $2 \leq b \leq 150$. On the other hand, if $\alpha = \frac{4}{9} + b - 1$ then the beta-polynomials of both $\Delta(\alpha)$ and $\Delta(\alpha+)$ are irreducible for all $1 \leq b \leq 150$. We will use this table below implicitly.

Suppose $\alpha = \frac{p}{q} + b - 1$. If we follow the proof of the Theorem described in Section 3, then we can find effectively the smallest possible constant $B$ for which $b \geq B$ implies that the beta-polynomial of $\Delta(\alpha)$ or $\Delta(\alpha+)$ is irreducible. This procedure is demonstrated in the next example.

Example 4.2. Let $\alpha = \frac{4}{7} + b - 1$ and let us find $B$ for the upper self-Christoffel number $\Delta(\alpha+)$. First we must determine $B_1$ so that the Mahler measure of $g$ is less than $\theta_0$ given in Theorem 2.1. One can see that it is enough that

$$\left(\frac{B_1 + \sqrt{B_1^2 + 4B_1}}{2B_1}\right)^5 < \theta_0.$$  

So $B_1 \geq 17$. Second we also find $B_2$ which contradicts inequality (3). Computation shows that $B_2 \geq 11$. Let $B_3 = \max\{B_1, B_2\} = 17$. Then for $b \geq B_3$ the beta-polynomial of $\Delta((4/7 + b - 1)+)$ is irreducible. One readily notes that the constant $B_3$ works not only for $4/7$ but for all $1/7, 2/7, \ldots, 6/7$. To find the smallest $B$ instead of $B_3$, we consult the table of [8]. It suffices to check whether or not $(1, 7, b), (2, 7, b), \ldots, (6, 7, b)$ for $1 \leq b \leq 16$ are on the list. But none of them is, even for $1 \leq b \leq 150$. Now we can state the following (i.e., in this case the beta-polynomial of $\Delta(\alpha+)$ is irreducible for every $b \geq B = 1$):

For any integer $p \geq 1$ which is not a multiple of 7, the beta-polynomial of $\Delta(p/7+)$ is irreducible.

Some calculations tempt us to conjecture that all lower self-Christoffel numbers are Pisot numbers. But this is false as explained below.
Flatto et al. [5] considered the dynamical zeta-function

\[
\zeta_\beta(z) = \exp\left(\sum_{n=1}^{\infty} \frac{P_n}{n} z^n\right)
\]

for the \(\beta\)-transformation \(T_\beta\), where \(P_n\) is the number of fixed points of \(T_\beta^n\). They wrote it in terms of the \(\beta\)-expansions of 1:

1. If \(\beta\) is not a simple beta-number with \(d_\beta(1) = e_1e_2\cdots\), then

\[
\zeta_\beta(z) = 1 - \sum_{i=1}^{\infty} e_iz^i.
\]

2. If \(\beta\) is a simple beta-number with \(d_\beta(1) = e_1\cdots e_n\), then

\[
\zeta_\beta(z) = 1 - z^n - \sum_{i=1}^{n} e_iz^i.
\]

In both cases, \(\zeta_\beta(z)\) is meromorphic in the open unit disk. It has a simple pole at \(z = 1/\beta\) and no other pole in \(\{z : |z| \leq 1/\beta\}\). Then \(M(\beta)\) was defined by the second smallest modulus of the poles of \(\zeta_\beta(z)\). If no pole other than \(1/\beta\) exists in the open unit disk, then \(M(\beta) = 1\). The main interest was the behavior of the function \(M(\beta)\). The next theorem hints at the possible abundance of non-Pisot lower self-Christoffel numbers.

**Theorem 4.3** ([5]). There exists \(\varepsilon > 0\) such that \(M(\beta) < 1\) for every \(\beta \in (1, 1 + \varepsilon)\).

In particular, if the Mersenne prime conjecture is true then there are infinitely many non-Pisot lower self-Christoffel numbers. There still remains a question in this direction.

**Question 1.** Let \(p/q\) be a fixed rational with \(0 < p \leq q\) and \(\gcd(p, q) = 1\). If \(\alpha = b - 1 + p/q\) for \(b \in \mathbb{N}\), is \(\Delta(\alpha)\) eventually a Pisot number as \(b\) increases?

Together with the Theorem, an affirmative answer to Question 1 would produce many concrete examples of irreducible Pisot type beta-substitutions. See [1] for details.

While our main interest in this paper is irreducible beta-polynomials, Boyd [2] focused on reducible beta-polynomials. Specifically, he dealt with the situation where \(\beta\) is a Pisot number. It is well known that every Pisot number is a beta-number [16]. Let \(\beta\) be a Pisot number and \(f(x)\) be the \(\beta\)-polynomial. Now we suppose that \(f(x) = g(x)h(x)\) is a nontrivial factorization over \(\mathbb{Q}\), where \(h(x)\) is the minimal polynomial of \(\beta\). Boyd called \(g(x)\) the *complementary factor*. Although extensive calculations made some mathematicians suspect that \(g(x)\) should be a product of cyclotomic polynomials, Boyd showed by systematic computation that there exist many Pisot
numbers $\beta$ for which their complementary factors are noncyclotomic or even nonreciprocal.

Motivated by Boyd’s work, the author checked the complementary factors of all reducible cases listed in [8] for both lower (5642) and upper (3422) self-Christoffel numbers. Surprisingly enough, all the complementary factors are (either cyclotomic polynomials or) products of cyclotomic polynomials.

For lower self-Christoffel numbers in [8], all irreducibles of the complementary factors are among the $\Phi_n$ with $n = 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 26, 28$, or $30$, where $\Phi_n$ is the $n$th cyclotomic polynomial. For upper self-Christoffel numbers in [8], all irreducibles of the complementary factors are among the $\Phi_n$ with $n = 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 14$.

A natural question arises.

**Question 2.** For $\beta$ a lower or an upper self-Christoffel number, if the $\beta$-polynomial is reducible, is its complementary factor a product of cyclotomic polynomials?

**References**


Department of Mathematics
Yonsei University
134 Shinchon-dong, Seodaemun-gu
Seoul 120-749, Republic of Korea
E-mail: doyong@yonsei.ac.kr

Received on 21.3.2007
and in revised form on 12.9.2007