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Generalised Mertens and Brauer–Siegel theorems

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1. Introduction. In this article, we prove a generalisation of the Mertens theorem for prime numbers to number fields and algebraic varieties over finite fields, paying attention to the genus of the field (or the Betti numbers of the variety), in order to make it tend to infinity and thus to point out the link between it and the famous Brauer–Siegel theorem. Using this we deduce an explicit version of the generalised Brauer–Siegel theorem under GRH, and a unified proof of this theorem for asymptotically exact families of almost normal number fields.

The classical Brauer–Siegel theorem describes the asymptotic behaviour of the quantity hR (the product of the class number and the regulator) in a family of number fields with growing genus under the conditions that the genus grows much faster than the degree and assuming some additional properties like normality or the Generalised Riemann Hypothesis (GRH) to deal with the Siegel zeroes. These two hypotheses are of different nature: omitting the first changes the final result, while the second is a technical hypothesis. Tsfasman and Vlăduţ [9] were able to remove the first hypothesis, which led to the so called generalised Brauer–Siegel theorem, and Zykin [10] was able to replace “normality” by “almost normality” in the second hypothesis by using results of Stark and Louboutin. He also managed to generalise the Brauer–Siegel theorem to the case of smooth absolutely irreducible projective varieties over finite fields.

As for the Mertens theorem, proven by Mertens in the case of $\mathbb{Q}$, and much later generalised by Rosen [5] to both number and function fields, it can be regarded as the Brauer–Siegel theorem in the global field or variety constituting the family. An explicit Mertens theorem leads therefore to an explicit formulation of the generalised Brauer–Siegel theorem. We first recall the formulations of the (generalised) Brauer–Siegel theorem and Mertens theorem, then we prove their explicit versions for number fields and smooth

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projective absolutely irreducible varieties over finite fields, and finally we deduce the explicit generalised Brauer–Siegel theorem.

2. Around the Brauer–Siegel theorem. Let us now recall the notations and definitions involved in the generalized Brauer–Siegel theorem, and state it for global fields and smooth absolutely irreducible projective algebraic varieties (s.a.i.p.a.v.) over a finite field $\mathbb{F}_r$. Throughout this paper we will write (NF) and (V) to say that something is true in the case of number fields and s.a.i.p.a.v. respectively.

2.1. *Number field case.* Given a number field K, let ζ_K be the zeta function of K and $\varkappa_K$ be its residue at $s = 1$. The *genus* of K is $\log \sqrt{|\mathrm{Discr}(K)|}$. Denote by $\Phi_q(K)$ the number of places of K whose norm is equal to q. We will say that a number field L is *almost normal* if there exists a tower $L_0 \subset \cdots \subset L_n = L$ of fields such that L_{i+1} is normal over L_i for all i.

For any sequence $(K_i)_{i\in\mathbb{N}}$ of finite extensions of $\mathbb{Q}$ denote by g_i the genus of K_i and by n_i its degree. Recall that a sequence $(K_i)_{i\in\mathbb{N}}$ of number fields is said to be a *family* if K_i is not isomorphic to K_j for $i \neq j$ (see [9]). For any real number g, there are only a finite number of number fields whose genus does not exceed g. Therefore, in any family, the sequence (g_i) tends to $+\infty$. A family $(K_i)_{i\in\mathbb{N}}$ is said to be *asymptotically exact* if $\phi_q := \lim \Phi_q(K_i)/g_i$ exists for all prime powers q and if $\phi_\mathbb{R} := \lim r_1(K_i)/g_i$ and $\phi_\mathbb{C} := \lim r_2(K_i)/g_i$ exist, where $r_1(K_i)$ and $r_2(K_i)$ stand for the number of real and complex places of K_i respectively. We put $\phi_\infty = \phi_\mathbb{R} + 2\phi_\mathbb{C}$. Being asymptotically exact is not a restrictive property. In fact, every tower of global fields is asymptotically exact, and each family of number fields contains an asymptotically exact subfamily. In the classical Brauer–Siegel theorem, all the ϕ_q are zero because of the assumption $n_i/g_i \to 0$:

THEOREM 1 (Classical Brauer–Siegel). *Let $(K_i)_{i\in\mathbb{N}}$ be a family of number fields. Assume that the fields K_i are normal over $\mathbb{Q}$ or that* GRH *holds, and assume that $\lim_i n_i/g_i = 0$. Then $\log h_iR_i \sim g_i$.*

Using the class number formula

$$\varkappa_K = \frac{2^{r_1}(2\pi)^{r_2}}{w|d_K|^{1/2}}\, hR,$$

this result can be reformulated in this way:

$$\lim_i \frac{\log \varkappa_{K_i}}{g_i} = 0.$$

Suppressing the second hypothesis leads to the Tsfasman–Vlăduţ Brauer–Siegel theorem (T-V Brauer–Siegel). This time, the ϕ_q are not always zero:

THEOREM 2 (T-V Brauer–Siegel (2002)). *Let $(K_i)_{i\in\mathbb{N}}$ be an asymptotically exact family of number fields. Assume that either* GRH *holds, or (K_i) is* *a family of almost normal number fields. Then the limit* $\kappa = \lim_i \log(\varkappa_{K_i})/g_i$ *exists and satisfies*

$$\kappa = \sum_q \phi_q \log\left(\frac{q}{q-1}\right) < \infty,$$

where the sum is taken over all powers of prime numbers.

In [9], Tsfasman and Vlăduţ proved this theorem without the assumption of GRH for asymptotically good families of almost normal number fields (this means $\lim n_i/g_i > 0$), and Zykin [10] proved it also for asymptotically bad families. In order to get this result, we have to deal with two inequalities, but one of them is always satisfied:

THEOREM 3 (Brauer–Siegel Inequality). *Let* $\mathcal{K} = (K_i)_{i\in\mathbb{N}}$ *be an asymptotically exact family of number fields. Then*

$$\limsup_i \frac{\log \varkappa_{K_i}}{g_i} \le \sum_q \phi_q \log\left(\frac{q}{q-1}\right) < \infty.$$

The difficulties come from the second inequality

$$\sum_q \phi_q \log\left(\frac{q}{q-1}\right) \le \liminf_i \frac{\log \varkappa_{K_i}}{g_i},$$

which requires technical assumptions.

2.2. *Case of algebraic varieties over a finite field.* Consider an algebraic variety X of dimension d, defined over a finite field $\mathbb{F}_r$. Suppose that X is smooth, projective and absolutely irreducible and let $|X|$ denote the set of its closed points. For $\mathfrak{p} \in |X|$ and $k(\mathfrak{p})$ its residue field, let $\deg(\mathfrak{p}) = [k(\mathfrak{p}) : \mathbb{F}_r]$. For $m \ge 1$ define the Φ-numbers as before:

$$\Phi_{r^m} := \#\{\mathfrak{p} \in |X| \mid \deg(\mathfrak{p}) = m\}.$$

Put $\overline{X} = X \otimes \mathbb{F}$ where $\mathbb{F} = \overline{\mathbb{F}}_r$ is the algebraic closure of $\mathbb{F}_r$. Let ℓ be a prime different from p. Let $b_i = \dim H^i(\overline{X}, \mathbb{Q}_\ell)$ for $0 \le i \le 2d$ be the Betti numbers for the ℓ-adic etale cohomology of X. As X is smooth, they do not depend on ℓ and satisfy $b_i = b_{2d-i}$ by Poincaré duality. Let $b_X = \max_{i=0,\dots,2d} b_i$. In the case of dimension 1, $b_0 = b_2 = 1$ and $b_1 = g$, so $b_X = \max(g, 1)$. In this theory the quantity b_X will play the role of the genus of number fields (and function fields). Since the asymptotic theory of varieties of dimension higher than 1 is not yet well understood, we do not know exactly which quantity is the exact analogue of the genus. We choose b_X because it was easier to compute the sums, but it might happen that the sum of b_i's or a sum with coefficients depending on r could make a better choice. However, unless we want to increase r or d unboundedly, all these choices are equivalent.

By the famous Deligne–Grothendieck theorem, the zeta function of X satisfies

$$Z(X,t)=\prod_{i=0}^{2d}P_i(t)^{(-1)^{i+1}},\quad \text{where}\quad P_i(t)=\prod_{j=1}^{b_i}(1-\omega_{i,j}r^{i/2}t),$$

$\omega_{i,j}$ being algebraic numbers of modulus 1 and $P_0(t)=1-t$, $P_{2d}(t)=1-r^dt$. We will consider $\zeta_X(s)=Z(X,r^{-s})$ and $\varkappa_X=\operatorname{Res}_{s=d}\zeta_X$.

Let us fix the dimension d, and let $(X_i)_{i\in\mathbb{N}}$ be a family of s.a.i.p.a.v. of dimension d. We say that the family $(X_i)_{i\in\mathbb{N}}$ is *asymptotically exact* if $b_{X_i}\to\infty$ and, for all $m\geq 1$, the limit $\phi_{r^m}=\lim_i \Phi_{r^m}(X_i)/b_{X_i}$ exists.

We can now formulate a generalisation of the Brauer–Siegel theorem for varieties of dimension d. It was proved by Tsfasman and Vlăduţ in the function field case [9], and by Zykin (unpublished) in the case of $d>1$, using a different definition of b_X.

THEOREM 4. *Let $(X_i)_{i\in\mathbb{N}}$ be an asymptotically exact family of s.a.i.p.a.v. of dimension d defined over $\mathbb{F}_r$. Then $\kappa=\lim_i\log(\varkappa_{X_i})/b_{X_i}$ exists and satisfies*

$$\kappa=\sum_{m=1}^{\infty}\phi_{r^m}\log\left(\frac{r^{dm}}{r^{dm}-1}\right).$$

Unfortunately, we do not know any reasonable interpretation of the residue of the zeta function at $s=d$, such as we have for $s=1$ through the class number formula in the number field and function field cases.

3. Mertens theorem and its relation to the generalised Brauer–Siegel theorem. If one wants to get an explicit version of the generalized Brauer–Siegel equality, one needs to know what happens explicitly between the residue $\varkappa_{K_i}$ and $\sum_{q\leq x}\phi_q\log\frac{q}{q-1}$ at the finite steps of the family. This is given by the Mertens theorem.

THEOREM 5 (Mertens). *For any number field K and any s.a.i.p.a.v. X, one has, as $N,x\to\infty$:*

$$\text{(V)}\qquad \prod_{\substack{P\in|X|\\ \deg(P)\leq N}}\left(1-\frac{1}{\mathcal{N}P^d}\right)=\frac{e^{-\gamma_X}}{N}+\mathcal{O}_X\left(\frac{1}{N^2}\right),$$

$$\text{(NF)}\qquad \prod_{\substack{P\in P_f(K)\\ \mathcal{N}P\leq x}}\left(1-\frac{1}{\mathcal{N}P}\right)=\frac{e^{-\gamma_K}}{\log x}+\mathcal{O}_K\left(\frac{1}{\log^2 x}\right),$$

$$\text{(NF\&GRH)}\qquad \prod_{\substack{P\in P_f(K)\\ \mathcal{N}P\leq x}}\left(1-\frac{1}{\mathcal{N}P}\right)=\frac{e^{-\gamma_K}}{\log x}+\mathcal{O}_K\left(\frac{1}{\sqrt{x}}\right),$$

where

$$\text{(V)} \qquad \gamma_X = \gamma + \log(\varkappa_X \log r),$$
$$\text{(NF)} \qquad \gamma_K = \gamma + \log \varkappa_K,$$

$P_f(K)$ *being the set of the non-archimedean places of* K, *and* $\mathcal{N}P$ *denoting the absolute norm of the place* P.

The function field and number field cases are due to Rosen, who proved them following the classical proof of the Mertens theorem [2]. But he paid no attention to the behaviour of the constants in field extensions. Unfortunately we did not know about his work before having finished ours. Mireille Car also proposed in [1] a different proof in the case of function fields. In the number field case, we also follow the classical Mertens proof with small variations in order to get an explicit version of this theorem, which takes into account the genus and the degree of K. In the case of varieties over finite fields, we present a natural proof using explicit formulae. We prove in fact the following sharper results:

Without assuming GRH, we have to deal with exceptional zeroes. A real zero ϱ of ζ_K is said to be *exceptional* if $1-(8g)^{-1} \le \varrho < 1$. A number field has at most one exceptional zero. A real zero ϱ is a *Siegel zero* if $1-(32g)^{-1} \le \varrho < 1$.

THEOREM 6. *Let* K *be a number field. Then*

$$\text{(NF)} \qquad \sum_{q \le x} \Phi_q \log\left(\frac{q}{q-1}\right) = \log\log x + \gamma + \log \varkappa_K + \tau_1(x) + \frac{1}{1-\varrho}\,\tau_2(x),$$

and there exist effective constants C, C_1, C_2 *such that*, *for all* $x \ge Cng^2$,

$$|\tau_1(x)| \le C_1 \frac{1}{\log x},$$

$$|\tau_2(x)| \begin{cases} \le C_2 \dfrac{1}{\log x} & \text{if } K \text{ has an exceptional zero } \varrho, \\ = 0 & \text{otherwise.} \end{cases}$$

The condition on x does not allow us to have explicit results as in the case where GRH holds, but these results, combined with Theorem 3, lead us to a unified proof of the Brauer–Siegel theorem, and to other nice results around the Brauer–Siegel theorem and the family of ϕ_q's.

COROLLARY 1. *Let* (K_i) *be an asymptotically exact family of almost normal number fields. Then the limit* $\lim_{i\to\infty} \log(\varkappa_{K_i})/g_{K_i} = \kappa$ *exists and satisfies*

$$\sum_q \phi_q \log\left(\frac{q}{q-1}\right) = \kappa,$$

the sum being taken over all prime powers q.

We cannot suppress the hypothesis of normality, because of exceptional zeroes that can appear in the family. But we can say something more in the general case:

PROPOSITION 1. *Let (K_i) be an asymptotically exact family of number fields.*

(i) *Assume that* $\lim_i n_i \log(n_i)/g_i = 0$. *Then κ exists and equals* 0.
(ii) *Assume that the family (K_i) is asymptotically good* (*i.e.* $\phi_\infty > 0$), *and there are infinitely many Siegel zeroes in the family. Then*

$$\sum_q \phi_q \log\left(\frac{q}{q-1}\right) \leq \phi_\infty \log\left(\frac{e}{\phi_\infty}\right).$$

If we assume GRH, then there is no condition on x, and we have the following result, which leads to an explicit version of the generalised Brauer–Siegel theorem.

THEOREM 7 (GRH Mertens theorem). *Assume that* GRH *holds. For any number field K and any s.a.i.p.a.v. X, one has, as $N, x \to \infty$:*

$$\text{(V)} \qquad \sum_{m=1}^{N} \Phi_{r^m} \log\left(\frac{r^{dm}}{r^{dm}-1}\right) = \log N + \gamma + \log(\varkappa_X \log r) + \mathcal{O}\left(\frac{1}{N}\right) + b_X \mathcal{O}\left(\frac{r^{-N/2}}{N}\right),$$

$$\text{(NF\&GRH)} \qquad \sum_{q \leq x} \Phi_q \log\left(\frac{q}{q-1}\right) = \log\log x + \gamma + \log \varkappa_K + n_K \mathcal{O}\left(\frac{\log x}{\sqrt{x}}\right) + g_K \mathcal{O}\left(\frac{1}{\sqrt{x}}\right),$$

where the $\mathcal{O}$ constants are effective and depend neither on X, nor on K.

COROLLARY 2. *Let (K_i) be an asymptotically exact family of number fields, and (X_i) a family of s.a.p.a.i.v. of dimension d. Assume* GRH *in the number field case. Then* $\lim_{i\to\infty} \log(\varkappa_{K_i})/g_{K_i} = \kappa$ (*resp.* $\lim_{i\to\infty} \log(\varkappa_{X_i})/g_{X_i} = \kappa$) *exists, and*

$$\text{(V)} \qquad \sum_{q \leq r^N} \phi_q \log\left(\frac{q}{q-1}\right) = \kappa + \mathcal{O}\left(\frac{r^{-N/2}}{N}\right),$$

$$\text{(NF\&GRH)} \qquad \sum_{q \leq x} \phi_q \log\left(\frac{q}{q-1}\right) = \kappa + \mathcal{O}\left(\frac{\log x}{\sqrt{x}}\right).$$

4. Proof of the Mertens theorem

4.1. *Proof in the number field case.* To prove the Mertens theorem for number fields, we follow the nice proof of the classical Mertens theorem of [2] as Rosen does in his article, but we use another counting function for prime ideals. In addition, we need a precise version of the Mertens theorem, so we will have to do the work once again, sketching Rosen's proofs.

Let K be a number field, $n = [K : \mathbb{Q}]$ and $g = \frac{1}{2}\log|\mathrm{Discr}(K)|$. Put $\pi(x) := \#\{P \in P(K) \,|\, \mathcal{N}P \le x\}$. One can estimate $\pi(x)$ by the following bound due to Lagarias and Odlyzko, and improved by Serre [6]:

Consider the Li-function defined by

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t}.$$

Theorem 8 (Prime ideals theorem).

$$\text{(NF)} \qquad \pi(x) = \mathrm{Li}(x) + \varDelta(x),$$

where, for all x such that

$$\text{(C1)} \qquad \log x \ge c_3 n g^2,$$

we have

$$|\varDelta(x)| \le \mathrm{Li}(x^{\varrho}) + c_1 x \exp\left(-c_2 n^{-1/2} \log^{1/2} x\right),$$

the term $\mathrm{Li}(x^{\varrho})$ *being only present if* ζ_K *has an exceptional zero* ϱ*. Under* GRH, *one has a stronger result for all* $x \ge 2$:

$$\text{(NF\&GRH)} \qquad |\varDelta(x)| \le c x^{1/2}(2g + n\log x).$$

First, we will give an asymptotic expression for $\sum \frac{1}{\mathcal{N}P}$:

Proposition 2.

$$\sum_{\mathcal{N}P \le x} \frac{1}{\mathcal{N}P} = \log\log x + B + o(1).$$

Proof. We have

$$C(x) = \sum_{\mathcal{N}P \le x} \frac{1}{\mathcal{N}P} = C(2) + \int_2^x \frac{d\pi(t)}{t} = C(2) + \int_2^x \frac{dt}{t\log t} + \int_2^x \frac{d\varDelta(t)}{t},$$

thus

$$\sum_{\mathcal{N}P \le x} \frac{1}{\mathcal{N}P} = \int_2^x \frac{dt}{t\log t} + \frac{\varDelta(x)}{x} + \int_2^x \frac{\varDelta(t)\,dt}{t^2}.$$

The prime ideals theorem implies that

$$\int_2^x \frac{dt}{t\log t} + \frac{\Delta(x)}{x} = \log\log x - \log\log 2 + o(1) \qquad \text{as } x \to \infty.$$

It remains to prove that $\int_2^x (|\Delta(t)|/t^2)\,dt$ is convergent, which is the case if $\int_2^x \mathrm{Li}(t^{\varrho})t^{-2}\,dt$ and $\int_2^x c_1 t \exp(-c_2 n^{-1/2}\log^{1/2} t)t^{-2}\,dt$ are convergent.

Since $\varrho < 1$ and

$$\mathrm{Li}(t^{\varrho}) \sim \frac{t^{\varrho}}{\log t^{\varrho}}, \qquad \text{as } t \to \infty,$$

the integral $\int_2^x \mathrm{Li}(t^{\varrho})t^{-2}\,dt$ is convergent.

In order to prove the convergence of the second integral, we need the following lemma:

LEMMA 1. *For all x such that*

$$\log x \geq 32^2 c_2^{-2} n \log^2\left(\frac{n^{1/2}}{c_2}\right), \tag{C2}$$

we have

$$\exp(-c_2 n^{-1/2} \log^{1/2} x) \leq \log^{-2} x.$$

Proof of Lemma 1. Put $y = \log^{1/2} x$ and $a = n^{1/2}/c_2$. Consider $f(y) = y^4 \exp(-y/a)$. We have to prove that $f(y) < 1$ if y is large enough. One can easily see that f is decreasing for $y \geq 4a$. Assume first that $a \leq e$. Then $y = 16a$ satisfies $f(y) \leq 1$. Indeed,

$$f(16a) = 2^{16} a^4 e^{-16} = \frac{2^{16}}{e^{12}} \frac{a^4}{e^4} \leq 1.$$

If $a > e$, then $y = 32a \log a$ fits. Indeed,

$$f(y) = 32^4 a^4 \log^4(a) e^{-32 \log a} = \frac{2^{20}}{a^{24}} \frac{\log^4 a}{a^4} \leq 1,$$

finishing the proof. ∎

Therefore we have $\exp(-c_2 n^{-1/2} \log^{1/2} t) = \mathcal{O}(\log^{-2} t)$, thus

$$\int_2^x \frac{t \exp(-c_2 n^{-1/2} \log^{1/2} t)\, dt}{t^2} \quad \text{is convergent}$$

and we get the expansion of Proposition 2. ∎

We now make the residue appear by calculating the constant term B.

PROPOSITION 3.

$$B = \sum_P \left\{ \log\left(1 - \frac{1}{\mathcal{N}P}\right) + \frac{1}{\mathcal{N}P} \right\} + \gamma + \log \varkappa_K.$$

Proof. For a complete proof, we refer to the article of Rosen [5]; let us still give a sketch. Write $C(x) = \log\log x + B + \varepsilon(x)$. For $\delta > 0$ define

$$g(\delta) = \sum_P \frac{1}{\mathcal{N}P^{1+\delta}}, \qquad f(\delta) = g(\delta) - \log \zeta(1+\delta).$$

After some computation using the Abel transform, we find $g(\delta) = B - \gamma - \log \delta + \mathcal{O}(\delta)$. Comparing with $\log \zeta_K(1+\delta) = -\log \delta + \log \varkappa_K + \mathcal{O}(\delta)$, we get $f(\delta) = B - \gamma - \log \varkappa_K + \mathcal{O}(\delta)$. Taking the limit as $\delta \to 0$, we obtain the assertion. ∎

We finally conclude that

$$\sum_{\mathcal{N}P \leq x} \log\left(\frac{\mathcal{N}P}{\mathcal{N}P - 1}\right) = \log\log x + \gamma + \log \varkappa_K + o(1).$$

Let us now estimate the error term

$$\varepsilon(x) = \Delta(x)x^{-1} - \int_x^\infty \Delta(t)t^{-2}\,dt,$$

as a function of n and g.

PROPOSITION 4. *There are computable constants such that*:

$$\text{(GRH)} \quad |\varepsilon(x)| \le cx^{-1/2}(6g + 3n\log x + 2n) \quad \textit{for any } x \ge 2,$$

$$|\varepsilon(x)| \le c_4 \frac{1}{\varrho \log x}(1+(1-\varrho)^{-1}) + 2c_1 \log^{-1} x \quad \textit{for any } x \gg 1,$$

$x \gg 1$ *meaning that x must satisfy conditions* (C1) *and* (C2), *the term involving ϱ being present only if ζ_K has an exceptional zero.*

Proof. Assuming GRH, we obtain directly

$$|\varepsilon(x)| \le cx^{-1/2}(2g + n\log x) + \int_x^\infty ct^{-3/2}(2g + n\log t)\,dt,$$

$$|\varepsilon(x)| \le cx^{-1/2}(2g + n\log x) + 2cx^{-1/2}(n\log x + 2n + 2g),$$

and finally

$$\text{(GRH)} \quad |\varepsilon(x)| \le cx^{-1/2}(6g + 3n\log x + 4n).$$

If we do not believe in GRH, we have to use the prime ideal theorem again: for x satisfying (C1),

$$|\Delta(x)| \le \mathrm{Li}(x^\varrho) + c_1 x \exp(-c_2 n^{-1/2} \log^{1/2} x).$$

Consider first the term $\Delta_1 := \mathrm{Li}(x^\varrho)$. Put

$$\varepsilon_1(x) = \Delta_1(x)x^{-1} - \int_x^\infty \Delta_1(t)t^{-2}\,dt.$$

Let c_4 be a constant such that $\mathrm{Li}(x) \le c_4 x \log^{-1} x$ (for example $(1-\log 2)^{-1}$). Then

$$|\varepsilon_1(x)|/c_4 \le \frac{1}{\varrho \log x} + \int_x^\infty \frac{dt}{\varrho t^{2-\varrho} \log t}.$$

We can thus easily bound the first error term by

$$|\varepsilon_1(x)|/c_4 \le \frac{1}{\varrho \log x}(1 + (1-\varrho)^{-1}).$$

We now deal with the second error term

$$\Delta_2(x) = c_1 x \exp(-c_2\, n^{-1/2} \log^{1/2} x).$$

Using Lemma 1, for x satisfying condition (C2), we have

$$\exp(-c_2 n^{-1/2}\log^{1/2} x) \le \log^{-2} x.$$

Put

$$\varepsilon_2(x) = \Delta_2(x)x^{-1} - \int_x^\infty \Delta_2(t)t^{-2}\,dt.$$

Thus, for x satisfying (C1) and (C2) (note that (C2) is very weak as compared to (C1)) and $x \ge e$, we obtain

$$\varepsilon_2(x) \le c_1(\log^{-2} x + \log^{-1} x) \le 2c_1 \log^{-1} x. \ \blacksquare$$

End of proof of the Mertens theorem. Let us start with the equality

$$\sum_{\mathcal{N}P\le x} \frac{1}{\mathcal{N}P} = \log\log x + B_K + \varepsilon_K(x).$$

One has

$$\sum_{\mathcal{N}P\le x} \log\left(\frac{\mathcal{N}P}{\mathcal{N}P-1}\right) = \log\log x + \sum_{\mathcal{N}P>x}\left\{\log\left(1-\frac{1}{\mathcal{N}P}\right)+\frac{1}{\mathcal{N}P}\right\} + \gamma + \log \varkappa_K + \varepsilon_K(x).$$

We can bound the remainder term in the following way:

$$\left|\sum_{\mathcal{N}P>x}\left\{\log\left(1-\frac{1}{\mathcal{N}P}\right)+\frac{1}{\mathcal{N}P}\right\}\right| \le \sum_{\mathcal{N}P>x}\frac{1}{\mathcal{N}P^2}.$$

This sum can be calculated easily under GRH using the prime ideal theorem:

$$D(x) = \sum_{\mathcal{N}P>x}\frac{1}{\mathcal{N}P^2} = \int_x^\infty \frac{dt}{t^2\log t} + \int_x^\infty t^{-2}\,d\Delta(t) \le \frac{1}{x\log x} + \frac{|\Delta(x)|}{x^2} + 2\int_x^\infty |\Delta(t)|t^{-3}\,dt,$$

therefore

$$\text{(GRH)} \qquad D(x) \le \frac{1}{x\log x} + \frac{10g+3n\log x}{3x\sqrt{x}} + \frac{2n}{x} \qquad \text{for any } x \ge 2.$$

Without GRH, we can use the bound for $\pi(x)$ (see [6]) valid for

$$\text{(C3)} \qquad \log x \ge c_5 g \log 2g \log\log 12g,$$

namely

$$\pi(x) \le c_6 x \log^{-1} x.$$

We have

$$D(x) = \sum_{\mathcal{N}P>x}\frac{1}{\mathcal{N}P^2} = \int_x^\infty \frac{d\pi(t)}{t^2}$$

and, for x sufficiently large,

$$D(x) = -\frac{\pi(x)}{x^2} + 2\int_x^\infty \pi(t)t^{-3}\,dt \le 2c_6\int_x^\infty t^{-2}\log^{-1} t\,dt \le \frac{2c_6}{x\log x}.$$

Putting all this together, we obtain the following. For x satisfying conditions (C1)–(C3),

$$\sum_{\mathcal{N}P\le x}\log\left(\frac{\mathcal{N}P}{\mathcal{N}P-1}\right) = \log\log x + \gamma + \log\varkappa_K + \mathcal{O}\left(\frac{1}{\log x}\right) + \frac{1}{1-\varrho}\,\mathcal{O}\left(\frac{1}{\log x}\right),$$

where the ϱ-term is only present if K has an exceptional zero. The classical Mertens theorem follows by an easy application of the Taylor expansion.

Under GRH, we obtain a stronger result for $x \ge 2$:

$$\sum_{\mathcal{N}P\le x}\log\left(\frac{\mathcal{N}P}{\mathcal{N}P-1}\right) = \log\log x + \gamma + \log\varkappa_K + n\mathcal{O}\left(\frac{\log x}{\sqrt{x}}\right) + g\mathcal{O}\left(\frac{1}{\sqrt{x}}\right),$$

which leads to the Mertens theorem under GRH.

4.2. *Proof in the case of algebraic varieties.* We first establish the Mertens theorem in the case of smooth absolutely irreducible projective algebraic varieties. The generalised Brauer–Siegel theorem follows immediately from it.

For any sequence (v_n) such that the radius ϱ of convergence of the series $\sum v_n t^n$ is strictly positive, put

$$\psi_{m,v}(t) = \sum_{n=1}^{\infty} v_{mn}t^{mn},$$

and $\psi_v(t) = \psi_{1,v}(t)$. For $t < r^{-d}\varrho$, we have the explicit formulae:

THEOREM 9 (Explicit formula, see [3]).

$$\sum_{f=1}^{\infty} f\Phi_{r^f}\psi_{f,v} = \psi_v(t) + \psi_v(r^d t) + \sum_{i=1}^{2d-1}(-1)^i\sum_{j=1}^{b_i}\psi_v(r^{i/2}\omega_{i,j}t).$$

Choose $N \in \mathbb{N}$ and take $v_n(N) = 1/n$ if $n \le N$, and 0 otherwise. Applying the explicit formula with $t = r^{-d}$, we get

$$S_0(N) = S_1(N) + S_2(N) + S_3(N),$$

where

$$S_0(N) = \sum_{n=1}^{N} n^{-1} r^{-dn} \sum_{m|n} m\Phi_{r^{dm}},$$
$$S_1(N) = \sum_{n=1}^{N} \frac{1}{n},$$
$$S_2(N) = \sum_{n=1}^{N} \frac{1}{nr^{dn}},$$
$$S_3(N) = \sum_{i=1}^{2d-1} (-1)^i \sum_{j=1}^{b_i} \sum_{n=1}^{N} \frac{1}{n} (r^{i/2-d}\omega_{i,j})^n.$$

LEMMA 2.

$$0 \le \sum_{f=1}^{N} \Phi_{r^f} \log\left(\frac{r^{df}}{r^{df}-1}\right) - S_0(N) \le \frac{8}{Nr^{dN/2}} + \frac{6b}{Nr^{(d+1/2)N/2}}.$$

Proof of Lemma 2. Let us first transform the expression of S_0:

$$S_0(N) = \sum_{f=1}^{N} \sum_{m=1}^{E(N/f)} f\Phi_{r^f} r^{-dfm} (fm)^{-1} = \sum_{f=1}^{N} \Phi_{r^f} \sum_{m=1}^{E(N/f)} \frac{1}{r^{dfm} m}.$$

Then we evaluate S_0:

$$\begin{aligned} 0 \le \sum_{f=1}^{N} \Phi_{r^f} \log\left(\frac{r^{df}}{r^{df}-1}\right) - S_0(N) &= \sum_{f=1}^{N} \Phi_{r^f} \left(\log\left(\frac{r^{df}}{r^{df}-1}\right) - \sum_{m=1}^{E(N/f)} \frac{1}{r^{dfm} m}\right) \\ &= \sum_{f=1}^{N} \Phi_{r^f} \sum_{m=E(N/f)+1}^{\infty} \frac{1}{r^{dfm} m}. \end{aligned}$$

As $1/m \le 1/(E(N/f)+1)$, we get

$$0 \le \sum_{f=1}^{N} \Phi_{r^f} \log\left(\frac{r^{df}}{r^{df}-1}\right) - S_0(N) \le \sum_{f=1}^{N} \frac{\Phi_{r^f}}{(E(N/f)+1)(r^{df})^{E(N/f)}(r^{df}-1)}.$$

In order to deal with Φ_{r^f} we use

$$\Phi_{r^f} \le \frac{r^{df} + 1 + \sum_{i=1}^{2d-1} r^{if/2} b_i}{f}.$$

Let $b = b_X = \max_i(b_i)$. We obtain

$$\begin{aligned} \Phi_{r^f} &\le \frac{1}{f} \left(r^{df} + 1 + b \sum_{i=1}^{2d-1} r^{if/2}\right) \le \frac{1}{f} \left(r^{df} + 1 + br^{f/2} \frac{r^{(2d-1)f/2} - 1}{r^{f/2} - 1}\right) \\ &\le \frac{1}{f} (r^{df} + 1 + 2br^{df-f/2}). \end{aligned}$$

Thus

$$0 \le \sum_{f=1}^{N} \Phi_{rf} \log\left(\frac{r^{df}}{r^{df}-1}\right) - S_0(N) \le \frac{1}{N}\sum_{f=1}^{N} \frac{(r^{df}+1+2br^{df-f/2})(r^{df}-1)^{-1}}{r^{dfE(N/f)}}.$$

We split our sum in two in the following way: for $f > E(N/2)$ where $E(N/f) = 1$, and for $f \le E(N/2)$ where we use $fE(N/f) \le N - f$. Hence

$$\begin{aligned} 0 \le \sum_{f=1}^{N} \Phi_{rf} \log\left(\frac{r^{df}}{r^{df}-1}\right) - S_0(N) &\le \frac{1}{N}\sum_{f=1}^{E(N/2)} \frac{2+4br^{-f/2}}{r^{d(N-f)}} \\ &\quad + \frac{1}{N}\sum_{f>E(N/2)}^{N} \frac{2+4br^{-f/2}}{r^{df}} \\ &\le \frac{8+12br^{-N/4}}{Nr^{dN/2}}. \end{aligned}$$

We finally obtain the following inequality:

$$0 \le \sum_{f=1}^{N} \Phi_{rf} \log\left(\frac{r^{df}}{r^{df}-1}\right) - S_0(N) \le \frac{8}{Nr^{dN/2}} + \frac{6b}{Nr^{(d+1/2)N/2}}. \ ■$$

In order to estimate S_1 we use the following well-known inequality (see [2]):

LEMMA 3.

$$\frac{1}{N(N+1)} \le S_1(N) - \log N - \gamma \le \frac{1}{N}.$$

LEMMA 4.

$$0 \le \log\left(\frac{r^d}{r^d-1}\right) - S_2(N) = \sum_{n=N+1}^{\infty} \frac{1}{nr^{dn}} \le \frac{1}{r^{dN}(N+1)(r^d-1)}.$$

Proof of Lemma 4. S_2 is the partial summation of the entire function $\log\left(\frac{r^d}{r^d-1}\right)$. The inequality comes from the estimation of the remainder term. ■

Let us recall first that

$$\log(\varkappa_X \log r) - \log\left(\frac{r^d}{r^d-1}\right) = \sum_{i=1}^{2d-1} (-1)^{i+1} \sum_{j=1}^{b_i} \log(1 - r^{i/2-d}\omega_{i,j}).$$

Compute now S_3:

LEMMA 5.

$$\left| S_3(N) - \sum_{i=1}^{2d-1} (-1)^{i+1} \sum_{j=1}^{b_i} \log(1 - r^{i/2-d}\omega_{i,j}) \right| \le \frac{b}{(r^{1/2}-1)(N+1)(r^{N/2}-1)}.$$

Proof of Lemma 5. Consider

$$R(N) = \Big| S_3(N) - \sum_{i=1}^{2d-1} (-1)^{i+1} \sum_{j=1}^{b_i} \log(1 - r^{i/2-d}\omega_{i,j}) \Big|.$$

One has

$$R(N) = \Big| \sum_{i=1}^{2d-1} (-1)^i \sum_{j=1}^{b_i} \sum_{n=N+1}^{\infty} \frac{1}{n} (r^{i/2-d}\omega_{i,j})^n \Big|,$$

therefore

$$R(N) \le \sum_{i=1}^{2d-1} \sum_{j=1}^{b_i} \frac{1}{N+1} \sum_{n=N+1}^{\infty} (r^{i/2-d})^n \le \frac{1}{N+1} \sum_{i=1}^{2d-1} \frac{b_i (r^{i/2-d})^{N+1}}{1 - r^{i/2-d}},$$

and

$$R(N) \le \frac{b}{(r^{1/2}-1)(N+1)r^{dN}} \sum_{i=1}^{2d-1} r^{iN/2},$$

which leads to the result. ∎

Putting everything together. We deduce that for N large enough,

$$\log \prod_{f=1}^{N} \left(1 - \frac{1}{r^{df}}\right)^{\Phi_{rf}} = -\log N - \gamma + \log\left(1 - \frac{1}{r^d}\right) - \log\left(1 - \frac{1}{r^d}\right) - \log(\varkappa_X \log r) + \mathcal{O}\left(\frac{1}{N}\right) + b\mathcal{O}\left(\frac{r^{-N/2}}{N}\right),$$

where the $\mathcal{O}$ constants do not depend on X. ∎

5. Proof of the generalised Brauer–Siegel theorem

5.1. *Without GRH.* If we do not believe in the generalised Riemann hypothesis, we have to take into account the conditions on x which forbid us to take the limit. Consider an asymptotically exact family (K_i) of number fields, and divide it into three subfamilies. The first one consists of the fields that have no exceptional zeroes, the second of the fields that do have exceptional zeroes but no Siegel zero, and the last of the fields that have a Siegel zero. If one of the subfamilies is finite, we omit it.

Let us focus on the second and third families, the case of the first one being much easier because of the absence of the ϱ-term (or take $\varrho = 0$ in the following). Let us specialise the Mertens theorem to $x = e^{Cng^2(1-\varrho)^{-1}}$, where C is large enough to allow x satisfy all the three conditions. Thus, for g large enough and an explicit constant M,

$$(*) \qquad \Big| \sum_{q \le e^{Cng^2(1-\varrho)^{-1}}} \frac{\Phi_q}{g} \log\left(\frac{q}{q-1}\right) - \frac{\log \varkappa}{g} \Big| \le M \frac{\log g}{g} - \frac{\log(1-\varrho)}{g}.$$

LEMMA 6. *Consider the family* (K_i) *and its exceptional zeroes* ϱ_i. *Suppose that*

$$\lim_i \log(1-\varrho_i)/g_i = 0.$$

Then κ *exists and*

$$\kappa = \sum_q \phi_q \log\left(\frac{q}{q-1}\right).$$

Let us assume the lemma. Look first at the second subfamily still denoted by (K_i). Each ζ_{K_i} has an exceptional zero satisfying $1-(8g)^{-1} \le \varrho < 1-(32g)^{-1}$, thus $(1-\varrho)^{-1} \le 32g$. Taking the logarithm, we see that this family satisfies the condition of the lemma.

The case of the third subfamily, which is still denoted by (K_i) for convenience, is not so easy, because ϱ can get very close to 1. In order to control the ϱ-term, we need to assume that the fields are almost normal (or some additional condition as below). Indeed, thanks to Stark we know that a Siegel zero ϱ of an almost normal number field K is also a Siegel zero of a subextension of K of degree 2 over $\mathbb{Q}$ (see [7]). In addition, we can estimate $(1-\varrho)^{-1}$ as follows [4]:

LEMMA 7. *Let* K *be a number field of degree* $n_K > 1$. *Then*

$$\frac{1}{1-\varrho_K} \le \varkappa_K^{-1}\left(\frac{g_K}{n_K}\right)^{n_K}.$$

Let (k_i) be a family of quadratic extensions of $\mathbb{Q}$ having the same Siegel zeroes as (K_i). Lemma 7 yields

$$-\log(1-\varrho_{K_i}) = -\log(1-\varrho_{k_i}) \le -\log\varkappa_{k_i} + 2\log\left(\frac{g_{k_i}}{2}\right).$$

Thus we obtain

$$0 < -\frac{\log(1-\varrho_{K_i})}{g_{K_i}} \le -\frac{\log\varkappa_{k_i}}{g_{K_i}} + 2g_{K_i}^{-1}\log\left(\frac{g_{k_i}}{2}\right).$$

As $k_i \subset K_i$, we have $g_{K_i} \ge g_{k_i}$; both terms of the right side of the inequality tend to zero as $i \to \infty$. The second term, if positive, can be bounded by $g_{k_i}^{-1}\log\kappa_i$ and we use the classical Brauer–Siegel theorem for quadratic fields which says that it tends to 0. We then apply Lemma 7 to deduce the generalised Brauer–Siegel theorem.

We still have to prove the preceding lemma.

Proof of Lemma 6. Put

$$f_g(q) = \frac{\Phi_q}{g}\log\left(\frac{q}{q-1}\right)\delta_g(q),$$

where $\delta_g(q) = 1$ if $q \leq e^{Cng^2(1-\varrho)^{-1}}$, and 0 otherwise. Now let the genus tend to infinity (g_{K_i} being g_i again). As

$$\sup_{i \gg 1} \sum_q f_{g_i}(q) \leq \sup_i \frac{\log \varkappa_{K_i}}{g_i} + 1,$$

this last quantity being well defined because of the basic inequality of [9], we can apply the Fatou lemma to obtain

$$\begin{aligned}\sum_q \phi_q \log\left(\frac{q}{q-1}\right) &= \sum_q \liminf_{i\to\infty} f_{g_i}(q) \\ &\leq \liminf_{i\to\infty} \sum_q f_{g_i}(q) = \liminf_{i\to\infty} \frac{\log \varkappa_{K_i}}{g_i}.\end{aligned}$$

Combining this result with the Brauer–Siegel inequality (Theorem 3), we deduce the existence of the limit of $\log(\varkappa_{K_i})/g_i$, which equals $\sum_q \phi_q \log\left(\frac{q}{q-1}\right)$. ■

Note that in the case of the first subfamily, the proof becomes easier, because we do not have to deal with the ϱ-term. Specialising to $x = e^{Cng^2}$ in the Mertens theorem (the bound in $(*)$ being then $M(\log g)/g$) and suppressing the ϱ-term in $f_g(q)$ $(q > e^{Cng^2})$, we obtain the generalised Brauer–Siegel theorem. ■

Let us now prove Proposition 1. First, we recall the following key lemma of Stark.

LEMMA 8 ([7, Lemma 8]). *Let k be a number field of degree $n_k > 1$. Assume that there is a $\beta \in \mathbb{R}$ such that*

$$1 - \frac{1}{8n_k! g_k} \leq \beta < 1$$

and $\zeta_k(\beta) = 0$. Then there is a quadratic subfield F of k such that $\zeta_F(\beta) = 0$.

Assume as before that (K_i) has an infinite number of Siegel zeroes which do not satisfy the condition of Lemma 8.

Let us split the family (K_i) as before into three subfamilies: the first containing the fields that do not have an exceptional zero, the second consisting of the fields that have zeroes that do not satisfy the condition of Lemma 8, and the last one consisting of those whose Siegel zeroes satisfy the condition of the lemma. If one of these families is finite, we omit it. The first and third cases have already been treated before, so let us consider the second subfamily. Let us call it (K_i) again. We still have to bound $g^{-1}\log\left(\frac{1}{1-\varrho}\right)$. The exceptional zeroes satisfy

$$\log\left(\frac{1}{1-\varrho}\right) \leq 8 + \log n! + \log g.$$

As $\log n! \le n \log n$, we deduce that

$$\frac{1}{g}\log\left(\frac{1}{1-\varrho}\right) \le \frac{n\log n}{g} + m\,\frac{\log g}{g},$$

where m is an explicit positive constant. Therefore this quantity tends to 0 and this completes the proof.

Suppose now that the family (K_i) is asymptotically good, and that an infinite number of K_i's admit a Siegel zero. Then, by Louboutin's lemma,

$$\frac{1}{g_i}\log\left(\frac{1}{1-\varrho}\right) \le -\frac{\log \varkappa_i}{g_i} + \frac{n_i}{g_i}\log\left(e\,\frac{g_i}{n_i}\right).$$

This leads to the result, since $\phi_\infty = \lim(n_i/g_i)$. ■

5.2. *Assuming GRH.* In the number field case, let (K_i) be a family of fields with $g_i \to \infty$. Starting with the Mertens theorem, dividing by g_i, taking $g_i \to \infty$ (we can do this because this time there is no condition on x), we obtain the Brauer–Siegel theorem. Indeed, the preceding subsection shows that the limit of $\log(\varkappa_{K_i})/g_i$ exists, and the asymptotical result follows directly.

In the variety case, let (X_i) be a family of smooth absolutely irreducible projective algebraic varieties over $\mathbb{F}_r$. We either assume the result of Zykin, or use the bounds for Φ_{r^f} that we needed for the Mertens theorem, to prove that the series

$$\sum \phi_{r^m}\log\left(\frac{r^{dm}}{r^{dm}-1}\right)$$

is convergent, and that

$$\lim_{b\to\infty}\frac{1}{b}\sum_{m=1}^{f(b)}\Phi_{r^m}\log\left(\frac{r^{dm}}{r^{dm}-1}\right) = \sum_{m=1}^{\infty}\phi_{r^m}\log\left(\frac{r^{dm}}{r^{dm}-1}\right)$$

for any function f of b satisfying

$$\lim_{b\to\infty} f(b) = \infty, \qquad \lim_{b\to\infty}\frac{f(b)}{b} = 0.$$

This is a bit technical but not hard (for the case $d = 1$ see [8]).

Using this result in the Mertens theorem (we put $N = f(b)$, divide by b and make $b \to \infty$) shows that the limit of $\log(\varkappa_{X_i})/b_{X_i}$ exists. We now divide by b_{X_i} (for any N) in the Mertens theorem and make $b_{X_i} \to \infty$ to obtain our version of the Brauer–Siegel theorem for varieties.

One could likely obtain similar results in the non-smooth case, using virtual Betti numbers. We hope to do this in further work. Let us conclude by the following remark. The explicit Mertens theorem is much more interesting than its application to the generalised Brauer–Siegel theorem, because it contains more information, and can therefore be useful, for example, if we would like to look at the problem in the classical way, focusing attention on the residues $\varkappa_i$ instead of the convergent series, and consider the limit of $\varkappa_i/g_i$ in the tower.

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