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An improvement of an estimate for finite additive bases

by

G. Horváth (Dunaújváros)

1. Introduction. Let $\mathcal{A} = \{a_1, \ldots, a_k\}$ be a set of integers such that $0 \le a_1 < \cdots < a_k$, and let $\mathcal{A} + \mathcal{A} = \{a_l + a_m \mid a_l \in \mathcal{A},\, a_m \in \mathcal{A}\}$. If n is a natural number and $\{0, 1, \ldots, n\} \subseteq \mathcal{A} + \mathcal{A}$ then $\mathcal{A}$ is called a 2-*basis*. Let $k = k(n)$ be the smallest integer for which a 2-basis for n with k elements exists, and let $\mathcal{A}$ be such a minimal 2-basis.

Since $n + 1 \le |\mathcal{A} + \mathcal{A}| \le \binom{k}{2} + k = (k^2 + k)/2$, we have

$$\limsup_{n\to\infty} \frac{n}{k^2} \le \frac{1}{2}.$$

On the other hand, it is not hard to see that the set

$$\big\{0, 1, 2, \ldots, \big[\sqrt{n} - 1\big], \big[\sqrt{n}\,\big], 2\big[\sqrt{n}\,\big], 3\big[\sqrt{n}\,\big], \ldots, \big[\sqrt{n} + 1\big] \cdot \big[\sqrt{n}\,\big]\big\}$$

is a 2-basis for n with $2 \cdot [\sqrt{n}\,] + 1$ elements, thus

$$\liminf_{n\to\infty} \frac{n}{k^2} \ge \frac{1}{4}$$

(see Rohrbach [6]).

Mrose [5] proved that $\liminf_{n\to\infty} n/k^2 \ge 2/7 = 0.2857\ldots$.

Rohrbach [6] gave a nontrivial upper bound with combinatorial argument: $\limsup_{n\to\infty} n/k^2 \le 0.4992$. Moser [3] improved this estimate with analytic argument (0.4903), and later, Moser, Pounder and Riddell [4] showed that $\limsup_{n\to\infty} n/k^2 \le 0.4847$. W. Klotz [2] proved that $\limsup_{n\to\infty} n/k^2 \le 0.4802$.

Güntürk and Nathanson [1], using Fourier series for functions of two variables, showed that 0.4802 can be replaced by 0.4789. We will prove the following theorem (using Fourier series for functions of one variable):

Theorem. $\limsup_{n\to\infty} n/k^2 \le 0.4778$.

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2. Proof of the Theorem. Let n be a fixed large positive integer and let $F(z) = \sum_{l=1}^{k} z^{a_l}$ be the generating function of the sequence $\mathcal{A}$ (where $\mathcal{A}$ is a minimal basis for n). Then

$$\frac{1}{2}\left(F^2(z) + F(z^2)\right) = 1 + z + z^2 + \cdots + z^n + \sum_{j=0}^{2n} \delta(j) z^j, \tag{1}$$

where $\delta(j) \geq 0$ for all j, because $\mathcal{A}$ is a 2-basis for n.

By (1), for $z = 1$ we have

$$\frac{1}{2}\left(k^2 + k\right) = n + 1 + \sum_{j=0}^{2n} \delta(j). \tag{2}$$

Similarly to the proof of Moser, we will show that $\sum_{j=0}^{2n} \delta(j)$ is "large".

Let $z = e\left(\frac{t}{n+1}\right) = e^{2\pi i t/(n+1)}$, where t is a positive integer. For $(n+1) \nmid t$, we obtain $1 + z + z^2 + \cdots + z^n = 0$, thus by (1),

$$\begin{aligned} \sum_{j=0}^{2n} \delta(j) &\geq \left| \sum_{j=0}^{2n} \delta(j) e\left(\frac{jt}{n+1}\right) \right| \\ &= \frac{1}{2} \left| \left(\sum_{l=1}^{k} e\left(\frac{ta_l}{n+1}\right) \right)^2 + \sum_{l=1}^{k} e\left(\frac{2ta_l}{n+1}\right) \right| \geq \frac{1}{2} \left(\left| \sum_{l=1}^{k} e\left(\frac{ta_l}{n+1}\right) \right|^2 - k \right) \\ &= \frac{1}{2} \left(\left(\sum_{l=1}^{k} \cos \frac{2\pi t a_l}{n+1} \right)^2 + \left(\sum_{l=1}^{k} \sin \frac{2\pi t a_l}{n+1} \right)^2 \right) - \frac{k}{2}. \end{aligned} \tag{3}$$

We shall need the following lemma.

LEMMA. *Let* $0 < \beta < 1$, $0 < \varepsilon \leq (1-\beta)/2$ *and*

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 2\pi\beta, \\ 1 - \dfrac{x - 2\pi\beta}{2\pi\varepsilon(1-\beta-\varepsilon)} & \text{if } 2\pi\beta \leq x \leq 2\pi(\beta+\varepsilon), \\ 1 - \dfrac{1}{1-\beta-\varepsilon} & \text{if } 2\pi(\beta+\varepsilon) \leq x \leq (1-\varepsilon)2\pi, \\ 1 - \dfrac{2\pi - x}{2\pi\varepsilon(1-\beta-\varepsilon)} & \text{if } (1-\varepsilon)2\pi \leq x \leq 2\pi. \end{cases}$$

Then the Fourier series of f *is*

$$\sum_{t=1}^{\infty} \left(\frac{2}{\pi^2 \varepsilon(1-\beta-\varepsilon)} \cdot \frac{\sin(\pi t\varepsilon)\sin(\pi t(\beta+\varepsilon))\cos(\pi t\beta)}{t^2} \cos(tx) + \frac{2}{\pi^2 \varepsilon(1-\beta-\varepsilon)} \cdot \frac{\sin(\pi t\varepsilon)\sin(\pi t(\beta+\varepsilon))\sin(\pi t\beta)}{t^2} \sin(tx) \right).$$

Proof. Let $0 < d \le 1/2$ and

$$\varrho_d(x) = \begin{cases} 1 - |x|/d2\pi & \text{if } |x| \le d2\pi, \\ 0 & \text{if } d2\pi \le |x| \le \pi. \end{cases}$$

Then

$$f(x) = 1 - \frac{1-\beta}{2\varepsilon(1-\beta-\varepsilon)} \varrho_{(1-\beta)/2}(x - (1+\beta)\pi) + \frac{1}{1-\beta-\varepsilon} \cdot \frac{1-\beta-2\varepsilon}{2\varepsilon} \varrho_{(1-\beta-2\varepsilon)/2}(x-(1+\beta)\pi). \tag{4}$$

If we denote the Fourier series of the function $\varrho_d(x)$ by

$$u_0 + \sum_{t=1}^{\infty} u_t \cos(tx) + \sum_{t=1}^{\infty} v_t \sin(tx), \tag{5}$$

then

$$u_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varrho_d(x)\, dx = d, \tag{6}$$

and for $t > 0$,

$$u_t = \frac{1}{\pi} \int_{-\pi}^{\pi} \varrho_d(x) \cos(-tx)\, dx, \tag{7}$$

$$v_t = -\frac{1}{\pi} \int_{-\pi}^{\pi} \varrho_d(x) \sin(-tx)\, dx = 0. \tag{8}$$

By (7),

$$\begin{aligned} u_t &= \frac{2}{\pi} \int_0^{d2\pi} \left(1 - \frac{1}{d2\pi} x\right) \cos(-tx)\, dx \\ &= \frac{2}{\pi} \left[\frac{\sin(-tx)}{-t} \left(1 - \frac{1}{d2\pi} x\right) - \frac{\cos(-tx)}{t^2 d2\pi} \right]_0^{d2\pi} \\ &= \frac{2}{\pi} \left(\frac{-\cos(-td2\pi)}{t^2 d2\pi} + \frac{1}{t^2 d2\pi} \right) = \frac{1 - \cos(td2\pi)}{t^2 d\pi^2}. \end{aligned}$$

Therefore by (5), (6) and (8), we have

$$\varrho_d(x) = d + \sum_{t=1}^{\infty} \frac{1-\cos(td2\pi)}{t^2 d\pi^2} \cos(tx),$$

hence in view of (4),

$$f(x) = 1 - \frac{1-\beta}{2\varepsilon(1-\beta-\varepsilon)}\left(\frac{1-\beta}{2} + \sum_{t=1}^{\infty} \frac{1-\cos(t(1-\beta)\pi)}{t^2\frac{1-\beta}{2}\pi^2}\right.$$
$$\left.\times \cos(t(x-(1+\beta)\pi))\right) + \frac{1}{1-\beta-\varepsilon}\cdot\frac{1-\beta-2\varepsilon}{2\varepsilon}\left(\frac{1-\beta-2\varepsilon}{2}\right.$$
$$\left.+ \sum_{t=1}^{\infty} \frac{1-\cos(t(1-\beta-2\varepsilon)\pi)}{t^2\frac{1-\beta-2\varepsilon}{2}\pi^2}\cos(t(x-(1+\beta)\pi))\right)$$
$$= 1 - \frac{(1-\beta)^2}{4\varepsilon(1-\beta-\varepsilon)} + \frac{(1-\beta-2\varepsilon)^2}{4\varepsilon(1-\beta-\varepsilon)}$$
$$+ \sum_{t=1}^{\infty} \frac{1}{\varepsilon(1-\beta-\varepsilon)\pi^2 t^2}\left(\cos(t(1-\beta)\pi) - \cos(t(1-\beta-2\varepsilon)\pi)\right)$$
$$\times \cos(t(x-(1+\beta)\pi))$$
$$= \sum_{t=1}^{\infty} \frac{2\sin(t(1-\beta-\varepsilon)\pi)\sin(-t\varepsilon\pi)}{\varepsilon(1-\beta-\varepsilon)\pi^2 t^2}$$
$$\times \left(\cos(tx)\cos(t(1+\beta)\pi) + \sin(tx)\sin(t(1+\beta)\pi)\right)$$
$$= \sum_{t=1}^{\infty} \frac{2\sin(\pi t(\beta+\varepsilon))\sin(\pi t\varepsilon)}{\varepsilon(1-\beta-\varepsilon)\pi^2 t^2}\left(\cos(tx)\cos(\pi t\beta) + \sin(tx)\sin(\pi t\beta)\right),$$

which completes the proof of the lemma.

Let $A(y) = |\{a_l \in \mathcal{A} \mid a_l \le y\}|$. Then by the lemma,

$$A(\beta n) - \frac{\beta+\varepsilon}{1-\beta-\varepsilon}\left(k - A(\beta n)\right) \le \sum_{l=1}^{k} f\left(\frac{2\pi a_l}{n+1}\right) \tag{9}$$
$$= \sum_{l=1}^{k} \frac{2}{\pi^2\varepsilon(1-\beta-\varepsilon)} \sum_{t=1}^{\infty}\left(\frac{\sin(\pi t\varepsilon)\sin(\pi t(\beta+\varepsilon))}{t^2}\right.$$
$$\left.\times \left(\cos(\pi t\beta)\cos\frac{2\pi t a_l}{n+1} + \sin(\pi t\beta)\sin\frac{2\pi t a_l}{n+1}\right)\right)$$
$$= \frac{2}{\pi^2\varepsilon(1-\beta-\varepsilon)} \sum_{t=1}^{\infty}\left(\frac{\sin(\pi t\varepsilon)\sin(\pi t(\beta+\varepsilon))}{t^2}\right.$$
$$\left.\times \left(\cos(\pi t\beta)\sum_{l=1}^{k}\cos\frac{2\pi t a_l}{n+1} + \sin(\pi t\beta)\sum_{l=1}^{k}\sin\frac{2\pi t a_l}{n+1}\right)\right)$$

$$\leq \frac{2}{\pi^2\varepsilon(1-\beta-\varepsilon)} \sum_{\substack{t=1\\(n+1)\nmid t}}^{\infty} \frac{|\sin(\pi t\varepsilon)\sin(\pi t(\beta+\varepsilon))|}{t^2} \sqrt{\cos^2(\pi t\beta)+\sin^2(\pi t\beta)}$$

$$\times \sqrt{\left(\sum_{l=1}^{k}\cos\frac{2\pi t a_l}{n+1}\right)^2 + \left(\sum_{l=1}^{k}\sin\frac{2\pi t a_l}{n+1}\right)^2} + \frac{2}{\pi^2\varepsilon(1-\beta-\varepsilon)}$$

$$\times \sum_{s=1}^{\infty} \frac{|\sin(\pi s(n+1)\varepsilon)\sin(\pi s(n+1)(\beta+\varepsilon))|}{s^2(n+1)^2}\,|\cos(\pi s(n+1)\beta)|k.$$

It follows from (3) and (9) that

$$\frac{1}{1-\beta-\varepsilon}\,(A(\beta n)-(\beta+\varepsilon)k)$$

$$\leq \frac{2}{\pi^2\varepsilon(1-\beta-\varepsilon)} \sum_{\substack{t=1\\(n+1)\nmid t}}^{\infty} \frac{|\sin(\pi t\varepsilon)\sin(\pi t(\beta+\varepsilon))|}{t^2} \sqrt{k+2\sum_{j=0}^{2n}\delta(j)}$$

$$+ \frac{2}{\pi^2\varepsilon(1-\beta-\varepsilon)}\cdot\frac{k}{(n+1)^2}\sum_{s=1}^{\infty}\frac{1}{s^2},$$

which implies that

$$A(\beta n) \leq (\beta+\varepsilon)k \tag{10}$$

$$+ \frac{1}{\pi^2\varepsilon}\left(\sum_{t=1}^{\infty}\frac{|2\sin(\pi t\varepsilon)\sin(\pi t(\beta+\varepsilon))|}{t^2}\right)\sqrt{k+2\sum_{j=0}^{2n}\delta(j)} + \frac{1}{3\varepsilon}\cdot\frac{k}{(n+1)^2}.$$

Let

$$S(\beta,\varepsilon)=\sum_{t=1}^{\infty}\frac{|2\sin(\pi t\varepsilon)\sin(\pi t(\beta+\varepsilon))|}{t^2} \tag{11}$$

and $0<\tau<1/2$ (the value of τ will be chosen later). If $\sum_{j=0}^{2n}\delta(j)\geq \tau k^2$, then by (2),

$$n+1 \leq \left(\frac{1}{2}-\tau\right)k^2+\frac{1}{2}\,k. \tag{12}$$

If $\sum_{j=0}^{2n}\delta(j)\leq \tau k^2$, then by (10),

$$A(\beta n) \leq (\beta+\varepsilon)k + \frac{1}{\pi^2\varepsilon}\,S(\beta,\varepsilon)\sqrt{\frac{1}{k}+2\tau}\cdot k + \frac{1}{3\varepsilon}\cdot\frac{1}{(n+1)^2}\,k. \tag{13}$$

Let $0<\mu<1/2$. Since $a_l>n/2$, $a_m>n/2$ implies that $a_l+a_m>n$, and also $(1/2-\mu)n<a_l\leq n/2$, $(1/2+\mu)n<a_m\leq n$ implies that $a_l+a_m>n$, we have

$$(14)\quad n+1 \le \frac{k^2+k}{2} - \frac{(k-A(n/2))(k-A(n/2)+1)}{2} - \left(A\left(\frac{n}{2}\right) - A\left(\left(\frac{1}{2}-\mu\right)n\right)\right)\left(k - A\left(\left(\frac{1}{2}+\mu\right)n\right)\right)$$
$$\le \frac{1}{2}k^2 + \frac{1}{2}k - \frac{1}{2}\left(k - A\left(\frac{n}{2}\right)\right)^2 - \left(A\left(\frac{n}{2}\right) - A\left(\left(\frac{1}{2}-\mu\right)n\right)\right)\left(k - A\left(\left(\frac{1}{2}+\mu\right)n\right)\right),$$

which can be written as

$$(15)\quad -\frac{1}{2}\left(A\left(\frac{n}{2}\right)\right)^2 + A\left(\left(\frac{1}{2}+\mu\right)n\right)A\left(\frac{n}{2}\right) + \frac{1}{2}k + A\left(\left(\frac{1}{2}-\mu\right)n\right)\left(k - A\left(\left(\frac{1}{2}+\mu\right)n\right)\right)$$
$$= -\frac{1}{2}\left(A\left(\frac{n}{2}\right) - A\left(\left(\frac{1}{2}+\mu\right)n\right)\right)^2 + \frac{1}{2}\left(A\left(\left(\frac{1}{2}+\mu\right)n\right)\right)^2 + \frac{1}{2}k + A\left(\left(\frac{1}{2}-\mu\right)n\right)\left(k - A\left(\left(\frac{1}{2}+\mu\right)n\right)\right).$$

By (13), for $\beta = 1/2-\mu$ and $\varepsilon = \varepsilon_1$ $(0<\varepsilon_1 \le 1/4+\mu/2)$ we have

$$(16)\quad A\left(\left(\frac{1}{2}-\mu\right)n\right) \le \left(\frac{1}{2}-\mu+\varepsilon_1\right)k + \frac{1}{\pi^2\varepsilon_1} S\left(\frac{1}{2}-\mu, \varepsilon_1\right)\sqrt{\frac{1}{k}+2\tau}\cdot k + \frac{1}{3\varepsilon_1}\cdot\frac{1}{(n+1)^2}\,k.$$

For $\beta = 1/2+\mu$ and $\varepsilon = \varepsilon_2$ $(0<\varepsilon_2<1/4-\mu/2)$, by (13),

$$(17)\quad A\left(\left(\frac{1}{2}+\mu\right)n\right) \le \left(\frac{1}{2}+\mu+\varepsilon_2\right)k + \frac{1}{\pi^2\varepsilon_2} S\left(\frac{1}{2}+\mu, \varepsilon_2\right)\sqrt{\frac{1}{k}+2\tau}\cdot k + \frac{1}{3\varepsilon_2}\cdot\frac{1}{(n+1)^2}\,k.$$

From (13), for $\beta = 1/2$ and $\varepsilon = \varepsilon_0$ $(0<\varepsilon_0<1/4)$ we get

$$(18)\quad A\left(\frac{n}{2}\right) \le \left(\frac{1}{2}+\varepsilon_0\right)k + \frac{1}{\pi^2\varepsilon_0} S\left(\frac{1}{2}, \varepsilon_0\right)\sqrt{\frac{1}{k}+2\tau}\cdot k + \frac{1}{3\varepsilon_0}\cdot\frac{1}{(n+1)^2}\,k.$$

We will distinguish two cases. If the right-hand side of (18) is not greater than $A((1/2+\mu)n)$ (the first case), then replacing in (15) $A(n/2)$ by the right-hand side of (18), by (14), we obtain

$$\begin{aligned}
n+1 \le{}& \frac{1}{2}\,k^2+\frac{1}{2}\,k\\
&-\frac{1}{2}\biggl(\frac{1}{2}-\varepsilon_0-\frac{1}{\pi^2\varepsilon_0}\,S\Bigl(\frac{1}{2},\varepsilon_0\Bigr)\sqrt{\frac{1}{k}+2\tau}-\frac{1}{3\varepsilon_0}\cdot\frac{1}{(n+1)^2}\biggr)^2 k^2\\
&-\biggl(\Bigl(\frac{1}{2}+\varepsilon_0+\frac{1}{\pi^2\varepsilon_0}\,S\Bigl(\frac{1}{2},\varepsilon_0\Bigr)\sqrt{\frac{1}{k}+2\tau}+\frac{1}{3\varepsilon_0}\cdot\frac{1}{(n+1)^2}\Bigr)k\\
&-A\Bigl(\Bigl(\frac{1}{2}-\mu\Bigr)n\Bigr)\biggr)\Bigl(k-A\Bigl(\Bigl(\frac{1}{2}+\mu\Bigr)n\Bigr)\Bigr).
\end{aligned}$$

Hence in view of (16) and (17),

$$\begin{aligned}
n+1 \le{}& \frac{1}{2}\,k^2+\frac{1}{2}\,k\\
&-\frac{1}{2}\biggl(\frac{1}{2}-\varepsilon_0-\frac{1}{\pi^2\varepsilon_0}\,S\Bigl(\frac{1}{2},\varepsilon_0\Bigr)\sqrt{\frac{1}{k}+2\tau}-\frac{1}{3\varepsilon_0}\cdot\frac{1}{(n+1)^2}\biggr)^2 k^2\\
&-\biggl(\mu+\varepsilon_0-\varepsilon_1+\frac{1}{\pi^2}\sqrt{\frac{1}{k}+2\tau}\,\Bigl(\frac{S(1/2,\varepsilon_0)}{\varepsilon_0}-\frac{S(1/2-\mu,\varepsilon_1)}{\varepsilon_1}\Bigr)\\
&+\Bigl(\frac{1}{3\varepsilon_0}-\frac{1}{3\varepsilon_1}\Bigr)\cdot\frac{1}{(n+1)^2}\biggr)\\
&\times\biggl(\frac{1}{2}-\mu-\varepsilon_2-\frac{1}{\pi^2\varepsilon_2}\,S\Bigl(\frac{1}{2}+\mu,\varepsilon_2\Bigr)\sqrt{\frac{1}{k}+2\tau}-\frac{1}{3\varepsilon_2}\cdot\frac{1}{(n+1)^2}\biggr)k^2
\end{aligned}\tag{19}$$

(if the right-hand side of (17) is less than or equal to k).

If the right-hand side of (18) is greater than $A((1/2+\mu)n)$ (the second case), then we may suppose that $k-A(n/2)\le\sqrt{2\tau}\cdot k$, i.e.,

$$A\Bigl(\frac{n}{2}\Bigr)\ge(1-\sqrt{2\tau})\,k,\tag{20}$$

otherwise (14) would imply $n+1\le\frac{1}{2}k^2+\frac{1}{2}k-\frac{1}{2}(k-A(n/2))^2\le(1/2-\tau)k^2+\frac{1}{2}k$, which is identical with (12). So by (14), (18) and (16),

$$\begin{aligned}
n+1 \le{}& \frac{1}{2}\,k^2+\frac{1}{2}\,k\\
&-\frac{1}{2}\biggl(\frac{1}{2}-\varepsilon_0-\frac{1}{\pi^2\varepsilon_0}\,S\Bigl(\frac{1}{2},\varepsilon_0\Bigr)\sqrt{\frac{1}{k}+2\tau}-\frac{1}{3\varepsilon_0}\cdot\frac{1}{(n+1)^2}\biggr)^2 k^2\\
&-\biggl(\frac{1}{2}-\sqrt{2}\cdot\sqrt{\tau}+\mu-\varepsilon_1-\frac{1}{\pi^2\varepsilon_1}\,S\Bigl(\frac{1}{2}-\mu,\varepsilon_1\Bigr)\sqrt{\frac{1}{k}+2\tau}-\frac{1}{3\varepsilon_1}\cdot\frac{1}{(n+1)^2}\biggr)\\
&\times\biggl(\frac{1}{2}-\varepsilon_0-\frac{1}{\pi^2\varepsilon_0}\,S\Bigl(\frac{1}{2},\varepsilon_0\Bigr)\sqrt{\frac{1}{k}+2\tau}-\frac{1}{3\varepsilon_0}\cdot\frac{1}{(n+1)^2}\biggr)k^2
\end{aligned}\tag{21}$$

(provided that the right-hand side of (18) is less than or equal to k).

Thus we have one of the estimates (12), (19) and (21). Let $\mu = \varepsilon_1 = \varepsilon_2 = 1/12$, $\varepsilon_0 = 1/14$ and $\sqrt{\tau} = 0.149$, i.e., $\tau = 0.022201$. Then $1/2 - \tau < 0.4778$, so (12) implies the theorem.

By (11),

$$S\Big(\frac{5}{12}, \frac{1}{12}\Big) = \sum_{t=1}^{\infty} \frac{2\big|\sin\frac{\pi t}{12}\big| \cdot \big|\sin\frac{\pi t}{2}\big|}{t^2} \tag{22}$$
$$= 2\Big(\sin\frac{\pi}{12}\sum_{s=0}^{\infty}\Big(\frac{1}{(12s+1)^2} + \frac{1}{(12s+11)^2}\Big) + \frac{\sqrt{2}}{2}\sum_{s=0}^{\infty}\Big(\frac{1}{(12s+3)^2} + \frac{1}{(12s+9)^2}\Big) + \sin\frac{5\pi}{12}\sum_{s=0}^{\infty}\Big(\frac{1}{(12s+5)^2} + \frac{1}{(12s+7)^2}\Big)\Big),$$

where

$$\frac{\sqrt{2}}{2}\sum_{s=0}^{\infty}\Big(\frac{1}{(12s+3)^2} + \frac{1}{(12s+9)^2}\Big) \tag{23}$$
$$= \frac{\sqrt{2}}{2}\Big(\sum_{s=1}^{\infty}\frac{1}{(3s)^2} - \sum_{s=1}^{\infty}\frac{1}{(6s)^2}\Big)$$
$$= \frac{\sqrt{2}}{2}\cdot\frac{\pi^2}{6}\Big(\frac{1}{9} - \frac{1}{36}\Big) = \frac{\sqrt{2}\,\pi^2}{144} < 0.0969287.$$

Since $\sum_{s=1}^{m}\frac{1}{s^2} > \frac{m(2m-1)}{3(2m+1)^2}\,\pi^2$, it follows that

$$\sum_{s=0}^{\infty}\Big(\frac{1}{(12s+1)^2} + \frac{1}{(12s+11)^2}\Big)$$
$$\le \sum_{s=0}^{M}\Big(\frac{1}{(12s+1)^2} + \frac{1}{(12s+11)^2}\Big) + \sum_{s=M+1}^{\infty}\Big(\frac{1}{(12s)^2} + \frac{1}{(12s+6)^2}\Big)$$
$$= \sum_{s=0}^{M}\Big(\frac{1}{(12s+1)^2} + \frac{1}{(12s+11)^2}\Big) + \frac{1}{36}\Big(\frac{\pi^2}{6} - \sum_{s=1}^{2M+1}\frac{1}{s^2}\Big)$$
$$\le \sum_{s=0}^{M}\Big(\frac{1}{(12s+1)^2} + \frac{1}{(12s+11)^2}\Big) + \frac{1}{36}\Big(\frac{\pi^2}{6} - \frac{(2M+1)(4M+1)}{3(4M+3)^2}\,\pi^2\Big)$$
$$= \sum_{s=0}^{M}\Big(\frac{1}{(12s+1)^2} + \frac{1}{(12s+11)^2}\Big) + \frac{\pi^2}{216}\cdot\frac{12M+7}{(4M+3)^2}$$

and (by aid of computer) we find that for $M = 79$ this is less than 1.02342. Similarly,

$$\sum_{s=0}^{\infty}\left(\frac{1}{(12s+5)^2}+\frac{1}{(12s+7)^2}\right) \le \sum_{s=0}^{M}\left(\frac{1}{(12s+5)^2}+\frac{1}{(12s+7)^2}\right)+\frac{\pi^2}{216}\cdot\frac{12M+7}{(4M+3)^2},$$

and for $M = 84$, the right-hand side of this inequality is less than 0.0737. Furthermore, $\sin\frac{\pi}{12} < 0.25881905$ and $\sin\frac{5\pi}{12} < 0.96592583$, by (22) and (23) we have

$$S\left(\frac{5}{12},\frac{1}{12}\right) \le 2(0.25881905\cdot 1.02342 + 0.0969287 + 0.96592583\cdot 0.0737) < 0.866. \tag{24}$$

By (11),

$$\begin{aligned} S\left(\frac{7}{12},\frac{1}{12}\right) &= \sum_{t=1}^{\infty}\frac{2\left|\sin\frac{\pi t}{12}\right|\cdot\left|\sin\frac{2\pi t}{3}\right|}{t^2} \\ &= \sqrt{3}\left(\sin\frac{\pi}{12}\sum_{s=0}^{\infty}\left(\frac{1}{(12s+1)^2}+\frac{1}{(12s+11)^2}\right)\right. \\ &\quad+\frac{1}{2}\sum_{s=0}^{\infty}\left(\frac{1}{(12s+2)^2}+\frac{1}{(12s+10)^2}\right) \\ &\quad+\frac{\sqrt{3}}{2}\sum_{s=0}^{\infty}\left(\frac{1}{(12s+4)^2}+\frac{1}{(12s+8)^2}\right) \\ &\quad\left.+\sin\frac{5\pi}{12}\sum_{s=0}^{\infty}\left(\frac{1}{(12s+5)^2}+\frac{1}{(12s+7)^2}\right)\right), \end{aligned}$$

where

$$\sum_{s=0}^{\infty}\left(\frac{1}{(12s+4)^2}+\frac{1}{(12s+8)^2}\right)=\frac{1}{16}\cdot\frac{\pi^2}{6}\left(1-\frac{1}{9}\right)=\frac{\pi^2}{108}<0.0913853$$

and

$$\sum_{s=0}^{\infty}\left(\frac{1}{(12s+2)^2}+\frac{1}{(12s+10)^2}\right) \le \sum_{s=0}^{83}\left(\frac{1}{(12s+2)^2}+\frac{1}{(12s+10)^2}\right)+\frac{\pi^2}{216}\cdot\frac{12\cdot 83+7}{(4\cdot 83+3)^2}<0.2744.$$

Hence

$$S\left(\frac{7}{12},\frac{1}{12}\right)\le\sqrt{3}\Big(0.25881905\cdot 1.02342+\frac{1}{2}\cdot 0.2744 + \frac{\sqrt{3}}{2}\cdot 0.0913853+0.96592583\cdot 0.0737\Big)<0.95681. \tag{25}$$

Again by (11),

$$\begin{aligned} S\left(\frac{1}{2},\frac{1}{14}\right) &= \sum_{t=1}^{\infty}\frac{2\left|\sin\frac{\pi t}{14}\right|\cdot\left|\sin\frac{4\pi t}{7}\right|}{t^2} \\ &= 2\sin\frac{\pi}{14}\sin\frac{4\pi}{7}\sum_{s=0}^{\infty}\left(\frac{1}{(14s+1)^2}+\frac{1}{(14s+13)^2}\right) \\ &\quad + 2\left(\sin\frac{\pi}{7}\right)^2\sum_{s=0}^{\infty}\left(\frac{1}{(14s+2)^2}+\frac{1}{(14s+12)^2}\right) \\ &\quad + 2\sin\frac{3\pi}{14}\sin\frac{2\pi}{7}\sum_{s=0}^{\infty}\left(\frac{1}{(14s+3)^2}+\frac{1}{(14s+11)^2}\right) \\ &\quad + 2\left(\sin\frac{2\pi}{7}\right)^2\sum_{s=0}^{\infty}\left(\frac{1}{(14s+4)^2}+\frac{1}{(14s+10)^2}\right) \\ &\quad + 2\sin\frac{5\pi}{14}\sin\frac{\pi}{7}\sum_{s=0}^{\infty}\left(\frac{1}{(14s+5)^2}+\frac{1}{(14s+9)^2}\right) \\ &\quad + 2\left(\sin\frac{3\pi}{7}\right)^2\sum_{s=0}^{\infty}\left(\frac{1}{(14s+6)^2}+\frac{1}{(14s+8)^2}\right), \end{aligned} \tag{26}$$

where

$$\begin{aligned} &\sum_{s=0}^{\infty}\left(\frac{1}{(14s+1)^2}+\frac{1}{(14s+13)^2}\right) \\ &\quad\le \sum_{s=0}^{M}\left(\frac{1}{(14s+1)^2}+\frac{1}{(14s+13)^2}\right)+\sum_{s=M+1}^{\infty}\left(\frac{1}{(14s)^2}+\frac{1}{(14s+7)^2}\right) \\ &\quad\le \sum_{s=0}^{M}\left(\frac{1}{(14s+1)^2}+\frac{1}{(14s+13)^2}\right)+\frac{1}{49}\left(\frac{\pi^2}{6}-\frac{(2M+1)(4M+1)}{3(4M+3)^2}\,\pi^2\right) \\ &\quad= \sum_{s=0}^{M}\left(\frac{1}{(14s+1)^2}+\frac{1}{(14s+13)^2}\right)+\frac{\pi^2}{294}\cdot\frac{12M+7}{(4M+3)^2}, \end{aligned}$$

which is less than 1.0171 (let $M=103$). Similarly, setting $M=90$, $M=90$, $M=87$, $M=89$ and $M=92$ in the estimates of the other series of the right-hand side of (26), respectively, we get

$$(27)\quad S\left(\frac{1}{2},\frac{1}{14}\right) \le 0.4338837392\cdot 1.0171 + 0.3765101982\cdot 0.26765 + 0.9749279122\cdot 0.1297 + 1.222520934\cdot 0.08255 + 0.7818314825\cdot 0.0622 + 1.900968868\cdot 0.05314 < 0.9191.$$

On the other hand, by (26),

$$\begin{aligned}(28)\quad S\left(\frac{1}{2},\frac{1}{14}\right) &\ge 2\sin\frac{\pi}{14}\sin\frac{4\pi}{7}\sum_{s=0}^{1}\left(\frac{1}{(14s+1)^2}+\frac{1}{(14s+13)^2}\right)\\ &\quad + 2\left(\sin\frac{\pi}{7}\right)^2\sum_{s=0}^{1}\left(\frac{1}{(14s+2)^2}+\frac{1}{(14s+12)^2}\right)\\ &\quad + 2\sin\frac{3\pi}{14}\sin\frac{2\pi}{7}\sum_{s=0}^{2}\left(\frac{1}{(14s+3)^2}+\frac{1}{(14s+11)^2}\right)\\ &\quad + 2\left(\sin\frac{2\pi}{7}\right)^2\sum_{s=0}^{4}\left(\frac{1}{(14s+4)^2}+\frac{1}{(14s+10)^2}\right)\\ &\quad + 2\sin\frac{5\pi}{14}\sin\frac{\pi}{7}\sum_{s=0}^{5}\left(\frac{1}{(14s+5)^2}+\frac{1}{(14s+9)^2}\right)\\ &\quad + 2\left(\sin\frac{3\pi}{7}\right)^2\sum_{s=0}^{3}\left(\frac{1}{(14s+6)^2}+\frac{1}{(14s+8)^2}\right)\\ &\ge 0.43388\cdot 1.01 + 0.37651\cdot 0.26 + 0.97492\cdot 0.125\\ &\quad + 1.22252\cdot 0.08 + 0.78183\cdot 0.06 + 1.90096\cdot 0.05\\ &> 0.8976.\end{aligned}$$

Now, by (19), (27), (28), (24) and (25), for sufficiently large n ($k(n)\to\infty$ as $n\to\infty$) we obtain

$$\begin{aligned} n+1 &\le \frac{1}{2}k^2 + \frac{1}{2}k\\ &\quad - \frac{1}{2}\left(\frac{3}{7} - \frac{14}{\pi^2}\cdot 0.9191\cdot\sqrt{\frac{1}{k}+2\cdot 0.022201} - \frac{14}{3}\cdot\frac{1}{(n+1)^2}\right)^2 k^2\\ &\quad - \left(\frac{1}{14}+\frac{1}{\pi^2}\cdot\sqrt{\frac{1}{k}+2\cdot 0.022201}\,(14\cdot 0.8976 - 12\cdot 0.866) + \frac{2}{3}\cdot\frac{1}{(n+1)^2}\right)\\ &\quad \times\left(\frac{1}{3} - \frac{12}{\pi^2}\cdot 0.95681\cdot\sqrt{\frac{1}{k}+2\cdot 0.022201} - 4\cdot\frac{1}{(n+1)^2}\right)k^2\\ &\le \frac{1}{2}k^2 + \frac{1}{2}k - \frac{1}{2}(0.153845)^2k^2 - 0.11785\cdot 0.08819\cdot k^2 < 0.4778k^2,\end{aligned}$$

thus (19) implies the theorem.

By (21), (27) and (24),

$$
\begin{aligned}
n+1 \le{}& \frac{1}{2}\,k^2 + \frac{1}{2}\,k \\
&- \frac{1}{2}\left(\frac{3}{7} - \frac{14}{\pi^2}\cdot 0.9191\cdot\sqrt{\frac{1}{k} + 2\cdot 0.022201} - \frac{14}{3}\cdot\frac{1}{(n+1)^2}\right)^2 k^2 \\
&- \left(\frac{1}{2} - \sqrt{2}\cdot 0.149 - \frac{12}{\pi^2}\cdot 0.866\cdot\sqrt{\frac{1}{k} + 2\cdot 0.022201} - 4\cdot\frac{1}{(n+1)^2}\right) \\
&\times\left(\frac{3}{7} - \frac{14}{\pi^2}\cdot 0.9191\cdot\sqrt{\frac{1}{k} + 2\cdot 0.022201} - \frac{14}{3}\cdot\frac{1}{(n+1)^2}\right) k^2 \\
\le{}& \frac{1}{2}\,k^2 + \frac{1}{2}\,k - \frac{1}{2}(0.153845)^2 k^2 - 0.067407\cdot 0.153845\cdot k^2 \\
<{}& 0.4778k^2,
\end{aligned}
$$

therefore (21) also implies the theorem, which completes the proof.

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Department of Mathematical Analysis
Institute of Mathematics
College of Dunaújváros
Táncsics M. u. 1/A.
H-2401, Dunaújváros, Hungary
E-mail: ghorvath@mail.duf.hu

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