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Fractional moments of Dirichlet L-functions

by

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1. Introduction. Mean-values of the type

$$I_k(T) := \int_0^T |\zeta(1/2+it)|^{2k}\,dt,$$

with positive non-integral values of k, have been investigated by a number of authors, including Ramachandra [5], [6], Conrey and Ghosh [1] and Heath-Brown [3]. In particular the above papers by Ramachandra show, under the Riemann Hypothesis, that

$$I_k(T) \gg_k T(\log T)^{k^2} \qquad (T \ge 2)$$

for all real $k \ge 0$, and that

$$I_k(T) \ll_k T(\log T)^{k^2} \qquad (T \ge 2)$$

for all real $k \in [0,2]$.

It is natural to ask about the corresponding problem for Dirichlet L-functions in q-aspect, that is, to investigate

$$M_k(q) := \sum_{\substack{\chi \,(\mathrm{mod}\, q)\\ \chi\ne\chi_0}} |L(1/2,\chi)|^{2k}$$

for positive real k. However rather little is known about this in general. The method of Rudnick and Soundararajan [7] enables one to show unconditionally that

$$M_k(q) \gg_k \phi(q)(\log q)^{k^2}$$

for rational $k \ge 1$, at least when q is prime. The method does not immediately apply to the case $0 < k < 1$ and it would be interesting to establish lower bounds in this range.

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In the reverse direction, Soundararajan [9, Section 4] shows under the Generalized Riemann Hypothesis that

$$M_k(q) \ll_{k,\varepsilon} \phi(q)(\log q)^{k^2+\varepsilon}$$

for any real $k \geq 0$ and any fixed $\varepsilon > 0$. One would conjecture that the true order of magnitude for $M_k(q)$ should be $\phi(q)(\log q)^{k^2}$. The present paper will prove upper bound results of exactly this order, motivated by the author's work [3]. We establish the following theorems.

Theorem 1. *Assuming the Generalized Riemann Hypothesis we have*

$$M_k(q) \ll_k \phi(q)(\log q)^{k^2}$$

for all $k \in (0,2)$.

Theorem 2. *Unconditionally we have*

$$M_k(q) \ll_k \phi(q)(\log q)^{k^2}$$

for any k *of the form* $k = 1/v$ *with* $v \in \mathbb{N}$.

Thus taking $v = 2$ we have

$$\sum_{\chi \,(\mathrm{mod}\, q)} |L(1/2,\chi)| \ll \phi(q)(\log q)^{1/4}$$

in particular.

The approach in [3] is based on a convexity theorem for mean-value integrals, which appears to have no analogue for character sums. We therefore work with integrals, and extract the sum $M_k(q)$ at the end. While we can give lower bounds for the integrals that occur, as well as upper bounds, it is not clear how to give a lower bound for $M_k(q)$ in terms of an integral.

It seems plausible that our approach might apply to other families of L-functions. One interesting case would be the fractional moments of L-functions with quadratic characters, in the form

$$\sum_{q \leq Q} \mu(2q)^2 \left| L\left(\frac{1}{2}, \left(\frac{*}{q}\right)\right)\right|^k \ll_k Q(\log Q)^{k(k+1)/2},$$

for example. However the estimation of the sum corresponding to $K(\sigma)$ will be more difficult than in the present paper, although the techniques used by Soundararajan [8, Section 5] seem likely to suffice. In addition, with the argument in its current form, a crude bound for the analogue of $J^*(\sigma)$ will need to be found.

This work arose from a number of conversations with Dr H. M. Bui, and would not have been undertaken without his prompting. It is a pleasure to acknowledge his contribution.

2. Mean-value integrals. Throughout our argument we will write $v = 1$ for the proof of Theorem 1, and $v = k^{-1}$ in handling Theorem 2. In both cases the primary mean-value integral we will work with is

$$J(\sigma, \chi) := \int_{-\infty}^{\infty} |L(\sigma + it, \chi)|^{2k} |W(\sigma + it)|^6 \, dt,$$

where the weight function $W(s)$ is defined by

$$W(s) := \frac{q^{\delta(s-1/2)} - 1}{(s - 1/2)\log q},$$

with $\delta > 0$ to be specified later, see (6) and (7). We emphasize that, for the rest of this paper, all constants implied by the Vinogradov $\ll$ symbol will be uniform in σ for the ranges specified. However they will be allowed to depend on the values of k and δ, so that the symbol $\ll$ should be read as $\ll_{k,\delta}$ throughout.

In addition to the integral $J(\sigma, \chi)$ we will use

$$K(\sigma, \chi) := \int_{-\infty}^{\infty} |S(\sigma + it, \chi)|^{2} |W(\sigma + it)|^6 \, dt,$$

where

$$S(s) := \sum_{n \le q} d_k(n) \chi(n) n^{-s}.$$

Notice here that a little care is needed in defining $d_k(n)$ when k is not an integer (see [3, §2]).

When χ is a non-principal character the function $L(s, \chi)$ is entire. Moreover, if we assume the Generalized Riemann Hypothesis then there are no zeros for $\sigma > 1/2$, so that one can define a holomorphic extension of

$$L(s, \chi)^k = \sum_{m=1}^{\infty} d_k(m) \chi(m) m^{-s} \qquad (\sigma > 1)$$

in the half-plane $\sigma > 1/2$. Having defined $L(s, \chi)^k$ in this way we now set

$$G(\sigma, \chi) := \int_{-\infty}^{\infty} |L(\sigma + it, \chi)^k - S(\sigma + it, \chi)|^2 |W(\sigma + it)|^6 \, dt \qquad (\sigma > 1/2).$$

This integral will be used in the proof of Theorem 1, while for the unconditional Theorem 2 we will employ

$$H(\sigma, \chi) := \int_{-\infty}^{\infty} |L(\sigma + it, \chi) - S(\sigma + it, \chi)^v|^{2/v} |W(\sigma + it)|^6 \, dt.$$

In addition to $J(\sigma,\chi)$, $K(\sigma,\chi)$, $G(\sigma,\chi)$ and $H(\sigma,\chi)$ we will consider their averages over non-principal characters,

$$J(\sigma) := \sum_{\substack{\chi \,(\mathrm{mod}\, q)\\ \chi\neq\chi_0}} J(\sigma,\chi), \qquad K(\sigma) := \sum_{\substack{\chi \,(\mathrm{mod}\, q)\\ \chi\neq\chi_0}} K(\sigma,\chi),$$

$$G(\sigma) := \sum_{\substack{\chi \,(\mathrm{mod}\, q)\\ \chi\neq\chi_0}} G(\sigma,\chi), \qquad H(\sigma) := \sum_{\substack{\chi \,(\mathrm{mod}\, q)\\ \chi\neq\chi_0}} H(\sigma,\chi).$$

To derive estimates relating values of these integrals we begin with the following convexity estimate of Gabriel [2, Theorem 2].

Lemma 1. *Let F be a complex-valued function which is regular in the strip $\alpha < \Re(z) < \beta$, and continuous for $\alpha \le \Re(z) \le \beta$. Suppose that $|F(z)|$ tends to zero as $|\Im(z)| \to \infty$, uniformly for $\alpha \le \Re(z) \le \beta$. Then for any $\gamma \in [\alpha,\beta]$ and any $a > 0$ we have*

$$I(\gamma) \le I(\alpha)^{(\beta-\gamma)/(\beta-\alpha)} I(\beta)^{(\gamma-\alpha)/(\beta-\alpha)}$$

where

$$I(\eta) := \int_{-\infty}^{\infty} |F(\eta+it)|^a \, dt.$$

The inequality should be interpreted appropriately if any of the integrals diverge. From Lemma 1 we will deduce the following variant.

Lemma 2. *Let f and g be complex-valued functions which are regular in the strip $\alpha < \Re(z) < \beta$, and continuous for $\alpha \le \Re(z) \le \beta$. Let b and c be positive real numbers. Suppose that $|f(z)|^b |g(z)|^c$ and $|g(z)|$ tend to zero as $|\Im(z)| \to \infty$, uniformly for $\alpha \le \Re(z) \le \beta$. Set*

$$I(\eta) := \int_{-\infty}^{\infty} |f(\eta+it)|^b |g(\eta+it)|^c \, dt.$$

Then for any $\gamma \in [\alpha,\beta]$ we have

$$I(\gamma) \le I(\alpha)^{(\beta-\gamma)/(\beta-\alpha)} I(\beta)^{(\gamma-\alpha)/(\beta-\alpha)}. \tag{1}$$

To deduce Lemma 2 from Lemma 1 we choose a rational number $p/q > c/b$, and apply Lemma 1 with $F = f^q g^p$ and $a = b/q$. Since

$$|F| = (|f|^b |g|^c)^{q/b} |g|^{p-cq/b}$$

with $p - cq/b > 0$, we deduce that $|F|$ tends to zero as $|\Im(z)| \to \infty$, uniformly for $\alpha \le \Re(z) \le \beta$. We then obtain an inequality of the same shape as (1), but with the exponent c replaced by bp/q. Lemma 2 then follows on choosing a sequence of rationals p_n/q_n tending downwards to c/b.

We now apply Lemma 2 to $J(\sigma,\chi)$. When $\sigma = 3/2$ we have

$$W(s) \ll q^{\delta}/(1+|t|)$$

whence we trivially obtain

$$J(3/2,\chi) \ll q^{6\delta}.$$

An immediate application of Lemma 2 therefore yields

$$J(\sigma,\chi) \ll J(1/2,\chi)^{3/2-\sigma} q^{6\delta(\sigma-1/2)}$$

for $1/2 \le \sigma \le 3/2$, whence we trivially deduce that

$$J(\sigma) \ll J(1/2)^{3/2-\sigma} q^{6\delta(\sigma-1/2)},$$

by Hölder's inequality. Since

$$J^f \le \left(\frac{\log q}{q}\right)^{1-f}\left(\frac{q}{\log q} + J\right) \ll q^{-(1-\delta)(1-f)}\left(\frac{q}{\log q} + J\right) \tag{2}$$

for any $J \ge 0$ and any $f \in [0,1]$, we conclude as follows.

LEMMA 3. *We have*

$$J(\sigma) \ll q^{-(1-7\delta)(\sigma-1/2)}\left(\frac{q}{\log q} + J\left(\frac{1}{2}\right)\right)$$

for $1/2 \le \sigma \le 3/2$.

To obtain a second estimate involving $J(\sigma,\chi)$ we use Lemma 2 to show that if $1/2 \le \sigma \le 3/4$ and $1-\sigma \le \gamma \le \sigma$ then

$$J(\gamma,\chi) \le J(\sigma,\chi)^{(\gamma-1+\sigma)/(2\sigma-1)} J(1-\sigma,\chi)^{(\sigma-\gamma)/(2\sigma-1)}.$$

An application of Hölder's inequality then shows that

$$J(\gamma) \le J(\sigma)^{(\gamma-1+\sigma)/(2\sigma-1)} J(1-\sigma)^{(\sigma-\gamma)/(2\sigma-1)}.$$

To handle $J(1-\sigma,\chi)$ we will use the functional equation for $L(s,\chi)$. If ψ is primitive, with conductor q_1, this yields

$$L(1-\sigma+it,\psi) \ll (1+|t|)^{\sigma-1/2} q_1^{\sigma-1/2} |L(\sigma+it,\psi)|$$

for $1/2 \le \sigma \le 3/4$ say. Thus if ψ induces a character χ modulo q we will have

$$L(1-\sigma+it,\chi) \ll (1+|t|)^{\sigma-1/2} q_1^{\sigma-1/2} \rho |L(\sigma+it,\chi)|$$

with

$$\rho = \prod_{p|q_2} \frac{|1-\psi(p)p^{-\sigma-it}|}{|1-\psi(p)p^{\sigma-1-it}|},$$

where $q_2 = q/q_1$. Thus

$$\log \rho \le (2\sigma-1)\sum_{p|q_2} \frac{\log p}{p^{1-\sigma}-1}.$$

However

$$\sum_{p|m} \frac{\log p}{p^{1/4}-1} \le \tfrac{1}{2}\log m$$

for all sufficiently large m, whence $\rho \ll q_2^{\sigma-1/2}$. We therefore conclude that

$$L(1-\sigma+it,\chi) \ll (1+|t|)^{\sigma-1/2} q^{\sigma-1/2} |L(\sigma+it,\chi)|$$

when $1/2 \le \sigma \le 3/4$, for any character χ modulo q, whether primitive or not.

We now deduce that

$$J(1-\sigma,\chi) \ll q^{2k(\sigma-1/2)} \int_{-\infty}^{\infty} |L(\sigma+it,\chi)|^{2k} (1+|t|)^{2k(\sigma-1/2)} |W(1-\sigma+it)|^6 \, dt.$$

The presence of the factor $(1+|t|)^{2k(\sigma-1/2)}$ is inconvenient. However, since $0<k<2$ we have

$$(1+|t|)^{2k(\sigma-1/2)} |W(1-\sigma+it)|^6 \ll (\log q)^{-6} |t|^{-2}$$

for $|t| \ge 1$ and $1/2 \le \sigma \le 1$. It follows that

$$J(1-\sigma,\chi) \ll q^{2k(\sigma-1/2)} (J(\sigma,\chi) + (\log q)^{-6} J^*(\sigma,\chi)),$$

where

$$J^*(\sigma,\chi) := \int_{-\infty}^{\infty} |L(\sigma+it,\chi)|^{2k} \frac{dt}{1+t^2}.$$

Thus

$$J(1-\sigma) \ll q^{2k(\sigma-1/2)} (J(\sigma) + (\log q)^{-6} J^*(\sigma))$$

with

$$J^*(\sigma) := \sum_{\substack{\chi \,(\mathrm{mod}\, q) \\ \chi \ne \chi_0}} \int_{-\infty}^{\infty} |L(\sigma+it,\chi)|^{2k} \frac{dt}{1+t^2}.$$

Finally we observe that

$$J(\sigma)^{(\gamma-1+\sigma)/(2\sigma-1)} \{J(\sigma) + (\log q)^{-6} J^*(\sigma)\}^{(\sigma-\gamma)/(2\sigma-1)} \le J(\sigma) + (\log q)^{-6} J^*(\sigma).$$

On comparing our results we therefore conclude that

$$J(\gamma) \ll q^{k(\sigma-\gamma)} (J(\sigma) + (\log q)^{-6} J^*(\sigma)). \tag{3}$$

We now have to consider $J^*(\sigma)$. It was shown by Montgomery [4, Theorem 10.1] that

$$\sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \int_{-T}^{T} |L(1/2+it,\chi)|^4 \, dt \ll \phi(q) T (\log qT)^4$$

for $T \ge 2$, where $\sum^*$ indicates that only primitive characters are to be considered. (It should be noted that there is a misprint in the statement of [4, Theorem 10.1], in that $L(1/2+it,\chi)$ should be replaced by $L(\sigma+it,\chi)$. However we are only interested in the case $\sigma = 1/2$. Moreover, in the proof of [4, Theorem 10.1], at the top of page 83, the reference to Theorem 6.3 should be to Theorem 6.5.)

If χ is an imprimitive character modulo q, induced by a primitive character ψ with conductor q_1, then

$$|L(1/2+it,\chi)|^4 \le |L(1/2+it,\psi)|^4 \prod_{p\mid q,\, p\nmid q_1} (1+p^{-1/2})^4.$$

Thus if $\sum^{(1)}$ indicates summation over all characters χ modulo q for which the conductor has a given value q_1, we will have

$$\sum_{\chi}^{(1)} \int_{-T}^{T} |L(1/2+it,\chi)|^4\,dt \ll \phi(q_1)T(\log q_1T)^4 \prod_{p\mid q,\, p\nmid q_1} (1+p^{-1/2})^4.$$

If we now sum for $q_1 \mid q$ we obtain

$$\sum_{\chi \,(\mathrm{mod}\, q)} \int_{-T}^{T} |L(1/2+it,\chi)|^4\,dt \ll T(\log qT)^4 f(q),$$

where

$$f(q) = \sum_{q_1\mid q} \phi(q_1) \prod_{p\mid q,\, p\nmid q_1} (1+p^{-1/2})^4.$$

The function f is multiplicative, with

$$f(p^e) = (1+p^{-1/2})^4 + \phi(p) + \phi(p^2) + \cdots + \phi(p^e) = p^e(1+O(p^{-3/2})).$$

Thus $f(q) \ll q$ and we conclude that

$$\sum_{\chi \,(\mathrm{mod}\, q)} \int_{-T}^{T} |L(1/2+it,\chi)|^4\,dt \ll qT(\log qT)^4.$$

We may now deduce that if $f(s) = L(s,\chi)^2 s^{-1}$ then

$$\sum_{\chi \,(\mathrm{mod}\, q)} \int_{-\infty}^{\infty} |f(1/2+it)|^2\,dt \ll q(\log q)^4.$$

Moreover the trivial bound $L(s,\chi) \ll 1$ for $\sigma = 3/2$ shows that

$$\sum_{\chi \,(\mathrm{mod}\, q)} \int_{-\infty}^{\infty} |f(3/2+it)|^2\,dt \ll q.$$

We can therefore apply Lemma 1, together with Hölder's inequality, to deduce that

$$\sum_{\chi \,(\mathrm{mod}\, q)} \int_{-\infty}^{\infty} |f(\sigma+it)|^2\,dt \ll q(\log q)^4$$

uniformly for $1/2 \le \sigma \le 3/2$. A final application of Hölder's inequality then implies that

$$J^*(\sigma) \ll q(\log q)^4.$$

We can now insert this into (3) and deduce

LEMMA 4. *We have*

$$J(\gamma) \ll q^{k(\sigma-\gamma)}\left(\frac{q}{\log q} + J(\sigma)\right)$$

for $1/2 \le \sigma \le 1$ *and* $1-\sigma \le \gamma \le \sigma$.

We now turn our attention to $G(\sigma,\chi)$ and $H(\sigma,\chi)$. By Lemma 2 we have

$$G(\sigma,\chi) \le G(1/2,\chi)^{3/2-\sigma}G(3/2,\chi)^{\sigma-1/2} \qquad (1/2 \le \sigma \le 3/2)$$

for non-principal characters χ modulo q. We then find via Hölder's inequality that

$$G(\sigma) \le G(1/2)^{3/2-\sigma}G(3/2)^{\sigma-1/2}. \tag{4}$$

Since

$$W(3/2+it) \ll q^{\delta}(1+|t|)^{-1}$$

we see that

$$G(3/2,\chi) \ll q^{6\delta}\int_{-\infty}^{\infty} |L(3/2+it,\chi)^k - S(3/2+it,\chi)|^2 \,\frac{dt}{1+|t|^2}.$$

However

$$L(3/2+it,\chi)^k - S(3/2+it,\chi) = \sum_{n>q} d_k(n)\chi(n)n^{-3/2-it}$$

whence

$$\int_{-\infty}^{\infty} |L(3/2+it,\chi)^k - S(3/2+it,\chi)|^2 \,\frac{dt}{1+|t|^2}$$
$$= \pi \sum_{m,n>q} d_k(m)d_k(n)\chi(m)\overline{\chi(n)}\min(m^{-1/2}n^{-5/2}, n^{-1/2}m^{-5/2}).$$

It follows that

$$\sum_{\chi \,(\mathrm{mod}\, q)} \int_{-\infty}^{\infty} |L(3/2+it,\chi)^k - S(3/2+it,\chi)|^2 \,\frac{dt}{1+|t|^2}$$
$$= \pi\phi(q) \sum_{\substack{m,n>q \\ q\mid m-n,\ (mn,q)=1}} d_k(m)d_k(n)\min(m^{-1/2}n^{-5/2}, n^{-1/2}m^{-5/2}).$$

To estimate this double sum we use the fact that $d_k(n) \ll_\varepsilon n^\varepsilon$ for any fixed $\varepsilon > 0$. This leads to the bound

$$\sum_{\substack{m,n>q\\ q\mid m-n}} d_k(m)d_k(n)\min(m^{-1/2}n^{-5/2}, n^{-1/2}m^{-5/2}) \ll_\varepsilon q^{2\varepsilon-2}.$$

It therefore follows that

$$\sum_{\chi\,(\mathrm{mod}\,q)} \int_{-\infty}^{\infty} |L(3/2+it,\chi)^k - S(3/2+it,\chi)|^2 \frac{dt}{1+|t|^2} \ll_\varepsilon q^{2\varepsilon-1}.$$

Inserting this bound into (4) we obtain

$$G(\sigma) \ll_\varepsilon G(1/2)^{3/2-\sigma} q^{(\sigma-1/2)(6\delta+2\varepsilon-1)}.$$

Using (2) again, we see that

$$G(\sigma) \ll_\varepsilon q^{1-2\sigma+(7\delta+2\varepsilon)(\sigma-1/2)}\left(\frac{q}{\log q} + G(1/2)\right)$$

for $\sigma \in [1/2, 3/2]$. The positive number ε is at our disposal, and we choose it to be $\varepsilon = \delta/2$, whence

$$G(\sigma) \ll q^{-(1-4\delta)(2\sigma-1)}\left(\frac{q}{\log q} + G(1/2)\right).$$

The treatment of $H(\sigma,\chi)$ is similar. This time, since $k = 1/v$, we have

$$H(3/2,\chi) \le \Big\{\int_{-\infty}^{\infty} |W(3/2+it)|^6\,dt\Big\}^{1-k} \times \Big\{\int_{-\infty}^{\infty} |L(3/2+it,\chi) - S(3/2+it,\chi)^v|^2 |W(3/2+it)|^6\,dt\Big\}^k$$

by Hölder's inequality. The first integral on the right is trivially $O(q^{6\delta})$. Moreover

$$L(3/2+it,\chi) - S(3/2+it,\chi)^v = \sum_{n>q} a_k(n)\chi(n)n^{-3/2-it}$$

with certain coefficients $a_k(n) \ll_\varepsilon n^\varepsilon$. The argument then proceeds as before, on noting that

$$\sum_{\substack{m,n>q\\ q\mid m-n}} a_k(m)a_k(n)\min(m^{-1/2}n^{-5/2}, n^{-1/2}m^{-5/2}) \ll_\varepsilon q^{2\varepsilon-2}.$$

It follows that

$$\sum_{\chi\,(\mathrm{mod}\,q)} \int_{-\infty}^{\infty} |L(3/2+it,\chi) - S(3/2+it,\chi)^v|^2 |W(3/2+it)|^6\,dt \ll q^{6\delta+2\varepsilon-1}.$$

We then deduce, by the same line of argument as before, that

$$H(\sigma) \ll q^{-(k-4\delta)(2\sigma-1)}\left(\frac{q}{\log q} + H(1/2)\right)$$

for $\sigma \in [1/2, 3/2]$.

We record these results formally in the following lemma.

LEMMA 5. *For $\sigma \in [1/2, 3/2]$ we have*

$$G(\sigma) \ll q^{-(1-4\delta)(2\sigma-1)}\left(\frac{q}{\log q} + G(1/2)\right),$$

$$H(\sigma) \ll q^{-(k-4\delta)(2\sigma-1)}\left(\frac{q}{\log q} + H(1/2)\right).$$

We end this section by considering $K(\sigma)$. We have

$$K(\sigma) \le \sum_{\chi\,(\mathrm{mod}\,q)} K(\sigma,\chi) = \sum_{m,n\le q} \frac{d_k(m)d_k(n)}{(mn)^\sigma} S(m,n)I(m,n),$$

where

$$S(m,n) = \sum_{\chi\,(\mathrm{mod}\,q)} \chi(m)\overline{\chi(n)}, \qquad I(m,n) = \int_{-\infty}^{\infty} \left(\frac{n}{m}\right)^{it} |W(\sigma+it)|^6\,dt.$$

Evaluating the sum $S(m,n)$ we find that

$$\sum_{m,n\le q} \frac{d_k(m)d_k(n)}{(mn)^\sigma} S(m,n)I(m,n) = \phi(q) \sum_{\substack{m,n\le q\\ q\mid m-n,\ (mn,q)=1}} \frac{d_k(m)d_k(n)}{(mn)^\sigma} I(m,n)$$

$$= \phi(q) \sum_{\substack{n\le q\\ (n,q)=1}} \frac{d_k(n)^2}{n^{2\sigma}} \int_{-\infty}^{\infty} |W(\sigma+it)|^6\,dt.$$

We then observe that

$$\sum_{\substack{n\le q\\ (n,q)=1}} \frac{d_k(n)^2}{n^{2\sigma}} \le \sum_{n\le q} \frac{d_k(n)^2}{n} \ll (\log q)^{k^2},$$

and that

$$\int_{-\infty}^{\infty} |W(\sigma+it)|^6\,dt \ll q^{3\delta(2\sigma-1)}(\log q)^{-1}.$$

These bounds allow us to conclude as follows.

LEMMA 6. *For $1/2 \le \sigma \le 3/2$ we have*

$$K(\sigma) \ll \phi(q)q^{3\delta(2\sigma-1)}(\log q)^{k^2-1}.$$

3. Proof of the theorems. By definition of $G(\sigma,\chi)$ and $H(\sigma,\chi)$ we have

$$J(\sigma) \ll K(\sigma) + G(\sigma)$$

under the Generalized Riemann Hypothesis, and

$$J(\sigma) \ll K(\sigma) + H(\sigma)$$

unconditionally. In view of Lemma 5 these produce

$$J(\sigma) \ll K(\sigma) + q^{-(1-4\delta)(2\sigma-1)}\left(\frac{q}{\log q} + G(1/2)\right)$$

and

$$J(\sigma) \ll K(\sigma) + q^{-(k-4\delta)(2\sigma-1)}\left(\frac{q}{\log q} + H(1/2)\right)$$

respectively. However we also have

$$G(1/2) \ll K(1/2) + J(1/2) \quad \text{and} \quad H(1/2) \ll K(1/2) + J(1/2)$$

from the definitions again, so that

$$J(\sigma) \ll K(\sigma) + q^{-(1-4\delta)(2\sigma-1)}\left(\frac{q}{\log q} + K(1/2) + J(1/2)\right)$$

and

$$J(\sigma) \ll K(\sigma) + q^{-(k-4\delta)(2\sigma-1)}\left(\frac{q}{\log q} + K(1/2) + J(1/2)\right)$$

in the two cases respectively.

If we now call on Lemma 6 then we find that

$$\begin{aligned} J(\sigma) &\ll \phi(q) q^{3\delta(2\sigma-1)}(\log q)^{k^2-1} + q^{-(1-4\delta)(2\sigma-1)}\left(\frac{q}{\log q} + J(1/2)\right) \\ &\ll q^{4\delta(2\sigma-1)}(\phi(q)(\log q)^{k^2-1} + q^{1-2\sigma} J(1/2)) \end{aligned}$$

under the Generalized Riemann Hypothesis, since

$$\frac{q}{\log q} \ll \phi(q)(\log q)^{k^2-1} \tag{5}$$

for $0 < k < 2$. Similarly we have

$$J(\sigma) \ll q^{4\delta(2\sigma-1)}(\phi(q)(\log q)^{k^2-1} + q^{k(1-2\sigma)} J(1/2))$$

unconditionally.

Finally we apply Lemma 4 with $\gamma = \frac{1}{2}$ and use (5) again to deduce that

$$J(\sigma) \ll q^{4\delta(2\sigma-1)}(\phi(q)(\log q)^{k^2-1} + q^{-(2-k)(\sigma-1/2)} J(\sigma))$$

under the Generalized Riemann Hypothesis. Similarly we may derive the unconditional bound

$$J(\sigma) \ll q^{4\delta(2\sigma-1)}(\phi(q)(\log q)^{k^2-1} + q^{-k(\sigma-1/2)} J(\sigma)).$$

We are now ready to choose our value of δ. For Theorem 1 we take

$$\delta = (2-k)/10, \tag{6}$$

and for Theorem 2 we choose

$$\delta = k/10. \tag{7}$$

Then in either case we will have

$$J(\sigma) \ll q^{4\delta(2\sigma-1)}\phi(q)(\log q)^{k^2-1} + q^{-\delta(2\sigma-1)}J(\sigma).$$

We write c_k for the implied constant in this last estimate, and note that c_k depends only on k. We then take

$$\sigma = \sigma_0 := \frac{1}{2} + \frac{\kappa}{\log q}$$

with

$$\kappa = (2\delta)^{-1}\max(1, \log 2c_k).$$

These choices ensure that

$$c_k q^{-\delta(2\sigma_0-1)} \le 1/2,$$

and hence imply that

$$J(\sigma_0) \ll q^{4\delta(2\sigma_0-1)}\phi(q)(\log q)^{k^2-1} \ll \phi(q)(\log q)^{k^2-1}.$$

Finally, we may apply Lemma 4 to deduce the following

Lemma 7. *With σ_0 as above we have*

$$J(\gamma) \ll \phi(q)(\log q)^{k^2-1}$$

uniformly for $1-\sigma_0 \le \gamma \le \sigma_0$.

All that remains is to bound $M_k(q)$ from above, using averages of $J(\gamma)$. Since $|L(s,\chi)|^{2k}$ is subharmonic we have

$$|L(1/2,\chi)|^{2k} \le \frac{1}{2\pi}\int_0^{2\pi} |L(1/2+re^{i\theta},\chi)|^{2k}\,d\theta.$$

We now multiply by r and integrate for $0 \le r \le R$ to show that

$$|L(1/2,\chi)|^{2k} \le \frac{1}{\mathrm{Meas}(D)}\int_D |L(1/2+z,\chi)|^{2k}\,dA,$$

where $D = D(0,R)$ is the disc of radius R about the origin, and dA is the measure of area. We take

$$R = \frac{\min(\kappa, \delta^{-1})}{\log q},$$

so that if $z \in D$ then $1-\sigma_0 \le \Re(1/2+z) \le \sigma_0$ and $|W(1/2+z)| \gg 1$. It follows that

$$\int_D |L(1/2+z,\chi)|^{2k}\,dA \ll \int_{1-\sigma_0}^{\sigma_0} J(\gamma,\chi)\,d\gamma$$

whence

$$M_k(q) \ll \frac{1}{\mathrm{Meas}(D)} \int_{1-\sigma_0}^{\sigma_0} J(\gamma)\,d\gamma.$$

Since $\mathrm{Meas}(D) \gg (\log q)^{-2}$ we now deduce from Lemma 7 that

$$M_k(q) \ll \phi(q)(\log q)^{k^2},$$

as required.

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