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Note on the Erdős–Graham theorem

by

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The Erdős–Graham theorem [1] provides a necessary and sufficient condition for an asymptotic basis of the positive integers to possess an exact order. My concern in this note is to link the property of having an exact (asymptotic) order with the classical notion of basis for (all) the positive integers in the sense of Šnirel'man [3, 4].

1. Notation, definitions, and the Erdős–Graham condition. Let $\mathcal{B}$ be a sequence of nonnegative integers.

Definitions.

(i) We say that $\mathcal{B}$ is an *asymptotic basis of order* h, where h is a positive integer, if every sufficiently large integer can be expressed as a sum of at most h integers in $\mathcal{B}$.

(ii) If $\mathcal{B}$ is an asymptotic basis, we say that it possesses an *exact order* h (where h is a positive integer) if every sufficiently large integer can be expressed as the sum of *exactly* h integers in $\mathcal{B}$.

(iii) We say that $\mathcal{B}$ is a *Šnirel'man basis of order* h (see [3, 4]) if every positive integer can be expressed as the sum of *at most* h integers in $\mathcal{B}$.

Remarks. When $0 \in \mathcal{B}$, which is usually assumed in the modern versions of Šnirel'man's theory (see [2]), we may replace in (iii) above "at most h" by "exactly h". Also note that a Šnirel'man basis necessarily contains 1.

Notation. Let now $\mathcal{A}$ be a sequence of positive integers $a_1 < a_2 < \cdots$, $\mathcal{A}' := \mathcal{A} \cup \{0, 1\}$, and $\mathcal{A}'' := \mathcal{A} \cup \{1\}$. Put

$$d := \gcd\{a_k \mid k \geq 1\} \quad \text{and} \quad D := \gcd\{a_{k+1} - a_k \mid k \geq 1\}.$$

With these definitions, remarks, and notation, the Erdős–Graham theorem [1] can be stated as follows.

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Theorem EG. *$\mathcal{A}$ is an asymptotic basis with an exact order $\Leftrightarrow$ $\mathcal{A}$ is an asymptotic basis and $D = 1$.*

The object of this note is to establish the following.

Theorem. *$\mathcal{A}$ is an asymptotic basis and $D = 1$ $\Leftrightarrow$ $\mathcal{A}'$ is a Šnirel'man basis and $D = 1$.*

2. Proof of the Theorem

Further notation. The assertion "$\mathcal{A}$ is an asymptotic basis of order h" can be expressed as: "there is an integer n_0 such that $\{n \geq n_0\} \subset \bigcup_{i=1}^{h} i\mathcal{A} =: h \star \mathcal{A}$". Here and below, the symbols $:=$ and $=:$ as usual indicate that the meaning of the (new) expression on the side of the colon is defined by the (already known) expression on the other side.

Proof of $\Rightarrow$. There are positive integers h and n_0 such that $\{n \geq n_0\} \subset h \star \mathcal{A}$, whence $\{n \geq 1\} \subset (M \star \mathcal{A}'') \cup \{0\} = M\mathcal{A}'$, where $M := \max\{h, n_0 - 1\}$.

Proof of $\Leftarrow$. We clearly have, in general, $d \mid D$; hence as $D = 1$, we have $d = 1$ as well. There is thus an integer t such that $\gcd\{a_1, \ldots, a_t\} = 1$, whence there are integral constants c_i with $\sum c_i a_i = 1$.

Put now $A := a_1 \cdots a_t$ and $b_i := c_i + k_i A$, where k_i is the smallest nonnegative integer with $b_i > 0$. Then $\sum b_i a_i = kA + 1$ for some nonnegative integer k. Hence if $N := \sum b_i a_i + kA = 2kA + 1$, then $N + 1 = 2 \sum b_i a_i = 2kA + 2$, and there is a positive integer h_1 with $\{N, N+1\} \subset h_1 \star \mathcal{A}$. This implies that $\{2N, 2N+1, 2N+2\} \subset 2h_1 \star \mathcal{A}$, and in general, for every $N_r := rN$ and $h_r := rh_1$ $(r \geq 1)$, $\{N_r, N_r + 1, \ldots, N_r + r\} \subset h_r \star \mathcal{A}$.

By hypothesis $\mathcal{A}'$ is a Šnirel'man basis: there is an integer s such that $\{n \geq 1\} \subset (s\mathcal{A}') \setminus \{0\} = s \star \mathcal{A}''$. Thus if $n \geq N_s$, then $n = N_s + (n - N_s) = N_s + (\text{at most } s \text{ 1s}) + (\text{at most } s \text{ elements of } \mathcal{A}) \in (h_s + s) \star \mathcal{A}$.

References

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