

On the constant factor in Vinogradov's Mean Value Theorem

by

GÉZA KÓS (Budapest)

1. Introduction. Vinogradov's Mean Value Theorem gives an upper bound for the integral

$$(1) \quad J_k(b, Q) = \int_0^1 \dots \int_0^1 \left| \sum_{x=1}^Q e(\alpha_1 x + \alpha_2 x^2 + \dots + \alpha_k x^k) \right|^{2b} d\alpha_1 \dots d\alpha_k$$

for positive integers k , b and Q . The theorem has many applications in estimating Weyl sums and in several problems of number theory, such as Waring's problem and estimating the value of the Riemann zeta function.

The integral $J_k(b, Q)$ has a particular interpretation: it is the number of solutions of the system

$$(2) \quad \begin{aligned} x_1 + x_2 + \dots + x_b &= x_{b+1} + x_{b+2} + \dots + x_{2b}, \\ x_1^2 + x_2^2 + \dots + x_b^2 &= x_{b+1}^2 + x_{b+2}^2 + \dots + x_{2b}^2, \\ &\vdots \\ x_1^k + x_2^k + \dots + x_b^k &= x_{b+1}^k + x_{b+2}^k + \dots + x_{2b}^k, \end{aligned}$$

where the variables x_1, \dots, x_{2b} are from the set $\{1, 2, \dots, Q\}$.

Vinogradov's idea was to estimate Weyl sums using the "average" of all Weyl sums $J_k(b, Q)$, and estimate $J_k(b, Q)$ as the number of solutions of (2).

Both Vinogradov's theorem and its improvements have the form

$$(3) \quad J_k(b, Q) < C_{k,b} Q^{2b-k(k+1)/2+\delta_{k,b}}$$

where the magnitude order of $\delta_{k,b}$ is $\frac{1}{2}k^2 \exp(-cb/k^2)$ and the difference is only in the constants. (See for example [2] and [3].) The magnitude order of $C_{k,b}$ is roughly $\exp(ck^{3+\varepsilon})$.

Some lower bounds for $J_k(b, Q)$ have been done by estimating the integrand near the origin. The goal of these considerations was to determine the

correct exponent of Q , and the results were written as

$$(4) \quad J_k(b, Q) > c_{k,b} Q^{2b-k(k+1)/2}$$

where $c_{k,b} \ll 1/b^k$.

Hua [1] also proved an asymptotics

$$J_k(b, Q) = (1 + o(1)) d_{k,b} Q^{2b-k(k+1)/2}$$

for fixed $b > (3 + o(1))k^2 \log k$ and $Q \rightarrow \infty$. Recently, Wooley [4] proved the same for $b > (1 + o(1))k^2 \log k$ as well.

In this paper we show that the constant $c_{k,b}$ in (4) can be increased to roughly k^{ck^2}/b^k .

2. The new lower bound. The main result is the following theorem.

THEOREM 2.1. *We have*

$$J_k(b, Q) > \frac{1! \cdot 2! \cdot \dots \cdot k!}{(b+1)^k} Q^{2b-k(k+1)/2}.$$

Proof. The idea is to use binomial coefficients instead of powers. The system

$$(5) \quad \begin{aligned} \binom{x_1}{1} + \dots + \binom{x_b}{1} &= \binom{x_{b+1}}{1} + \dots + \binom{x_{2b}}{1}, \\ \binom{x_1}{2} + \dots + \binom{x_b}{2} &= \binom{x_{b+1}}{2} + \dots + \binom{x_{2b}}{2}, \\ &\vdots \\ \binom{x_1}{k} + \dots + \binom{x_b}{k} &= \binom{x_{b+1}}{k} + \dots + \binom{x_{2b}}{k} \end{aligned}$$

is equivalent to (2), but the terms are smaller.

For arbitrary integers a_1, \dots, a_k , denote by $\psi(a_1, \dots, a_k)$ the number of solutions of the system

$$\binom{x_1}{1} + \dots + \binom{x_b}{1} = a_1, \quad \dots, \quad \binom{x_1}{k} + \dots + \binom{x_b}{k} = a_k.$$

Then

$$\begin{aligned} Q^{2b} &= \left(\sum_{x_1=1}^Q \dots \sum_{x_b=1}^Q 1 \right)^2 = \left(\sum_{a_1=0}^{\binom{Q}{1}} \dots \sum_{a_k=0}^{\binom{Q}{k}} \psi(a_1, \dots, a_k) \right)^2 \\ &\leq \left(\sum_{a_1=0}^{\binom{Q}{1}} \dots \sum_{a_k=0}^{\binom{Q}{k}} 1 \right) \left(\sum_{a_1=0}^{\binom{Q}{1}} \dots \sum_{a_k=0}^{\binom{Q}{k}} \psi^2(a_1, \dots, a_k) \right) \end{aligned}$$

$$< \prod_{\nu=1}^k \left(b \binom{Q}{\nu} + 1 \right) \cdot J_k(b, Q) \leq \frac{(b+1)^k}{1! \cdot 2! \cdot \dots \cdot k!} Q^{k(k+1)/2} J_k(b, Q),$$

and the proof is complete.

Recently the author has found an expression $H_k(b, Q)$ such that if k and Q are fixed and $b \rightarrow \infty$, then $J_k(b, Q) = (1 + o(1))H_k(b, Q)$. Due to the length and complicated nature of the argument, it will appear in a subsequent paper.

References

- [1] L. K. Hua, *Additive Theory of Prime Numbers*, Amer. Math. Soc., Providence, RI, 1965, Theorem 15.
- [2] I. M. Vinogradov, *The Method of Trigonometric Sums in Number Theory*, Nauka, Moscow, 1986 (in Russian).
- [3] T. D. Wooley, *On Vinogradov's mean value theorem*, *Mathematika* 40 (1993), 379–399.
- [4] —, *Some remarks on Vinogradov's mean value theorem and Tarry's problem*, *Monatsh. Math.* 122 (1996), 265–273, Theorem 3.

Computer and Automation Research Institute
 Kende u. 13-17
 H-1111 Budapest, Hungary
 E-mail: kosgeza@cs.elte.hu

*Received on 15.5.1998
 and in revised form on 18.9.2000*

(3384)