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Evaluations of the Rogers–Ramanujan continued fraction $R(q)$ by modular equations

by

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1. Introduction. Let, for $|q| < 1$, the Rogers–Ramanujan continued fraction $R(q)$ be defined by

$$R(q) := \frac{q^{1/5}}{1} \, \underset{+}{} \, \frac{q}{1} \, \underset{+}{} \, \frac{q^2}{1} \, \underset{+}{} \, \frac{q^3}{1} \, \underset{+}{} \cdots \tag{1.1}$$

Also define

$$S(q) := -R(-q). \tag{1.2}$$

This famous continued fraction was introduced by L. J. Rogers in 1894 and rediscovered by S. Ramanujan in approximately 1912. In his first letter to G. H. Hardy [15, p. xxvii], [7, p. 29], Ramanujan gave the first non-elementary evaluations of $R(q)$ and $S(q)$, namely,

$$R(e^{-2\pi}) = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2} \tag{1.3}$$

and

$$S(e^{-\pi}) = \sqrt{\frac{5-\sqrt{5}}{2}} - \frac{\sqrt{5}-1}{2}. \tag{1.4}$$

In his second letter to Hardy [15, p. xxviii], [7, p. 57], Ramanujan further asserted that

$$R(e^{-2\pi\sqrt{5}}) = \frac{\sqrt{5}}{1+\left(5^{3/4}\left(\frac{\sqrt{5}-1}{2}\right)^{5/2}-1\right)^{1/5}} - \frac{\sqrt{5}+1}{2}. \tag{1.5}$$

These evaluations were first proved by G. N. Watson [17, 18], and K. G. Ramanathan [9] also established (1.5).

Ramanujan recorded other values for $R(q)$ and $S(q)$ in his first notebook [14] and in his "lost notebook" [16]. Several of these results were proved

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by Ramanathan [10, 11, 12]. He first made the evaluations of $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ for several rational numbers n in a uniform way by using Kronecker's limit formula, and he also established further evaluations not claimed by Ramanujan. However, Ramanathan's method cannot be applied to all the values of $R(q)$ stated by Ramanujan because $K = \mathbb{Q}(\sqrt{-n})$ satisfies the conditions usually imposed with regard to genera.

B. C. Berndt, H. H. Chan, and L.-C. Zhang [6] derived the first formulas for the explicit evaluations of $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ for positive rational numbers n in terms of Ramanujan–Weber class invariants. Also they have proved [6] that, for any rational number n, $R(e^{-\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ are units.

In this paper, we establish two theorems for evaluating $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ for any positive rational numbers n by using modular equations relating to degree 5 or 25. By using these theorems we will find simple proofs for some known values for $R(q)$, e.g., $R(e^{-6\pi})$. Also we will find new values for $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ for certain positive rational numbers n. For example,

$$\begin{aligned}R(e^{-\pi}) &= \tfrac{1}{2}\sqrt{40-25\sqrt[4]{5}+18\sqrt{5}-11\sqrt[4]{5^3}} - \tfrac{1}{4}(7-5\sqrt[4]{5}+3\sqrt{5}-\sqrt[4]{5^3})\\ &= \tfrac{1}{8}(3+\sqrt{5})(\sqrt[4]{5}-1)(\sqrt{10+2\sqrt{5}}-(3+\sqrt[4]{5})(\sqrt[4]{5}-1)).\end{aligned}$$

In Section 2, we present some modular equations discovered by Ramanujan and new modular equations which we found. We give proofs of the new modular equations.

In Section 3, we establish a theorem for evaluating $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ for any positive rational numbers n by using modular equations of degree 1, p, 25, and $25p$ where p is a positive integer, and establish old and new values of $R(q)$ and $S(q)$ by using that theorem. First, we define parameters J_n and D_n and then we find relations between J_n and J_{kn} and between D_n and D_{kn} for $k=p^2$, where p is an integer, and for rational n, by using modular equations. By using these values and Theorem 3.1, we determine values of $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ for certain positive rational numbers n.

In the final section, we establish a theorem for evaluating $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ for any positive rational numbers n by using modular equations of degree 1, p, 5, and $5p$ where p is a positive integer, and compute old and new values of $R(q)$ and $S(q)$ by using that Theorem 4.1. We use a method similar to that of Section 3, but with different parameters s_n and t_n. We use modular equations for finding relations between s_n and s_{kn} and between t_n and t_{kn} for $k = p^2$, where p is an integer and n is rational.

In summary, we give new proofs of values found by Ramanujan in Theorem 3.9(ii), Corollaries 3.3, 3.6(i), 3.8(i), 3.16(i), 3.18(i), 3.20(i), 4.3(ii), 4.6(i), 4.12(i), (iii), and 4.21(iii). The values in Theorems 3.21, 4.7(ii), (iv), Corollaries 3.6(ii), 3.8(ii), 3.10, 3.14, 3.16(ii), 3.18(ii), 3.20(ii), 3.22, 4.3(i), 4.6(ii), (iii), (iv), 4.10, 4.12(ii), (iv), 4.15, 4.17, 4.21(i), (ii), and (iv) are new. Further values of $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ can be calculated by using the theorems in this paper. We do not record these values here, because evidently they are not particularly elegant. For example, we can find $D_{3/25}$ and $D_{1/75}$ by using Theorem 3.24 with $n = 3/25$ and $n = 1/75$, and using D_3 and $D_{1/3}$ in Theorem 3.17, respectively, and so we can determine $S(e^{-\pi\sqrt{3/25}})$ and $S(e^{-\pi/\sqrt{75}})$ by using Theorem 3.1(ii). Since $D_{1/n} = D_n$ (see Remark 1), we can also determine $D_{25/3}$ and D_{75} and thereby explicitly calculate $S(e^{-\pi\sqrt{25/3}})$ and $S(e^{-5\pi\sqrt{3}})$ by using Theorem 3.1(ii). Furthermore, we can find $J_{1/25}$ and J_{25} with $n = 1$ in Theorem 3.23, and so by using Theorem 3.1(i), we can determine $R(e^{-2\pi/5})$ and $R(e^{-10\pi})$.

S.-Y. Kang [8] has recorded a table of all known values of the Rogers–Ramanujan continued fraction up until the time her paper was written in 1999.

Recall the reciprocity theorems for the Rogers–Ramanujan continued fraction stated by Ramanujan [16] in his second letter to Hardy and his second notebook [14], [1, p. 83], that is, if $\alpha, \beta > 0$ and $\alpha\beta = 1$, then

$$\left\{\frac{\sqrt{5}+1}{2} + R(e^{-2\pi\alpha})\right\}\left\{\frac{\sqrt{5}+1}{2} + R(e^{-2\pi\beta})\right\} = \frac{5+\sqrt{5}}{2}. \tag{1.6}$$

Second, if $\alpha, \beta > 0$ and $\alpha\beta = 1/5$, then [16, p. 364]

$$\left\{\left(\frac{\sqrt{5}+1}{2}\right)^5 + R^5(e^{-2\pi\alpha})\right\}\left\{\left(\frac{\sqrt{5}+1}{2}\right)^5 + R^5(e^{-2\pi\beta})\right\} = 5\sqrt{5}\left(\frac{\sqrt{5}+1}{2}\right)^5. \tag{1.7}$$

There are similar formulas for $S(q)$. First, if $\alpha, \beta > 0$ and $\alpha\beta = 1$, then

$$\left\{\frac{\sqrt{5}-1}{2} + S(e^{-\pi\alpha})\right\}\left\{\frac{\sqrt{5}-1}{2} + S(e^{-\pi\beta})\right\} = \frac{5-\sqrt{5}}{2}, \tag{1.8}$$

which can be found in Ramanujan's second notebook [14], [1, p. 83] and which was first proved by Ramanathan [9]. Second, if $\alpha, \beta > 0$ and $\alpha\beta = 1/5$, then

$$\left\{\left(\frac{\sqrt{5}-1}{2}\right)^5 + S^5(e^{-\pi\alpha})\right\}\left\{\left(\frac{\sqrt{5}-1}{2}\right)^5 + S^5(e^{-\pi\beta})\right\} = 5\sqrt{5}\left(\frac{\sqrt{5}-1}{2}\right)^5, \tag{1.9}$$

which was first established by Ramanathan [9].

We complete this section by defining, after Ramanujan,

$$f(-q) := (q;q)_\infty =: q^{-1/24}\eta(z), \qquad q = e^{2\pi iz},\ \operatorname{Im} z > 0, \tag{1.10}$$

where

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \qquad |q|<1,$$

and where $\eta(z)$ denotes the Dedekind eta-function.

2. Modular equations. This section is devoted to stating and proving certain modular equations which we will use in what follows. Most of them were first recorded by Ramanujan in his notebooks [14].

THEOREM 2.1 (see [2, p. 206, Entry 53]). *Let*

$$P = \frac{f(-q)}{q^{1/6}f(-q^5)} \quad \textit{and} \quad Q = \frac{f(-q^2)}{q^{1/3}f(-q^{10})}.$$

Then

$$PQ + \frac{5}{PQ} = \left(\frac{P}{Q}\right)^3 + \left(\frac{Q}{P}\right)^3.$$

THEOREM 2.2 (see [2, p. 223, Entry 63]). *Let*

$$P = \frac{f(-q)}{q^{1/6}f(-q^5)} \quad \textit{and} \quad Q = \frac{f(-q^3)}{q^{1/2}f(-q^{15})}.$$

Then

$$(PQ)^3 + \left(\frac{5}{PQ}\right)^3 = \left(\frac{Q}{P}\right)^6 - 9\left(\frac{Q}{P}\right)^3 - 9\left(\frac{P}{Q}\right)^3 - \left(\frac{P}{Q}\right)^6. \tag{2.1}$$

THEOREM 2.3. *Let*

$$P = \frac{f(-q)}{q^{1/6}f(-q^5)} \quad \textit{and} \quad Q = \frac{f(-q^5)}{q^{5/6}f(-q^{25})}.$$

Then

$$(PQ)^2 + \left(\frac{5}{PQ}\right)^2 + 5\left(PQ + \frac{5}{PQ}\right) = \left(\frac{Q}{P}\right)^3 - 15.$$

Proof. From (11.7) and (11.8) [1, p. 268], we can deduce that

$$\frac{f^6(-q^5)}{q^5f^6(-q^{25})} = \frac{f^5(-q)}{q^5f^5(-q^{25})} + 5\frac{f^4(-q)}{q^4f^4(-q^{25})} + 15\frac{f^3(-q)}{q^3f^3(-q^{25})} + 25\frac{f^2(-q)}{q^2f^2(-q^{25})} + 25\frac{f(-q)}{qf(-q^{25})}.$$

Multiplying both sides by $q^3f^3(-q^{25})/f^3(-q)$, we complete the proof. ∎

THEOREM 2.4. *Let*

$$P = \frac{f(-q)}{q^{1/6}f(-q^5)} \quad \text{and} \quad Q = \frac{f(-q^7)}{q^{7/6}f(-q^{35})}.$$

Then

$$(PQ)^3 + \left(\frac{5}{PQ}\right)^3 = \left(\frac{Q}{P}\right)^4 - \left(\frac{P}{Q}\right)^4 - 7\left\{\left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3\right\} + 7\left\{\left(\frac{Q}{P}\right)^2 - \left(\frac{P}{Q}\right)^2\right\} + 14\left(\frac{Q}{P} + \frac{P}{Q}\right).$$

A proof of Theorem 2.4 was given by Berndt [4].

THEOREM 2.5 (see [2, pp. 212–213, Entry 58]). *Let*

$$P = \frac{f(-q^{1/5})}{q^{1/5}f(-q^5)} \quad \text{and} \quad Q = \frac{f(-q^{2/5})}{q^{2/5}f(-q^{10})}.$$

Then

$$PQ + \frac{25}{PQ} = \left(\frac{Q}{P}\right)^3 - 4\left(\frac{Q}{P}\right)^2 - 4\left(\frac{P}{Q}\right)^2 + \left(\frac{P}{Q}\right)^3$$

and

$$(PQ)^2 + 5PQ = P^3 - 2P^2Q - 2PQ^2 + Q^3.$$

THEOREM 2.6. *Let*

$$P = \frac{f(-q^{1/5})}{q^{1/5}f(-q^5)} \quad \text{and} \quad Q = \frac{f(-q^{3/5})}{q^{3/5}f(-q^{15})}.$$

Then

$$PQ + \frac{25}{PQ} = \left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 - 6\left(\frac{P}{Q} + \frac{Q}{P}\right) - 3\left(Q + \frac{5}{Q}\right) - 3\left(P + \frac{5}{P}\right) - 9.$$

Proof. Multiplying both sides of (2.1) by P^6Q^6 (in the notation of Theorem 2.2), solving for $Q^{12} - P^{12}$, and then squaring both sides, we find that

$$X^{24} + Y^{24} = 15625X^6Y^6 + 2250(X^{12}Y^6 + X^6Y^{12}) + 81(X^{18}Y^6 + X^6Y^{18}) + 414X^{12}Y^{12} + 18(X^{18}Y^{12} + X^{12}Y^{18}) + X^{18}Y^{18}, \tag{2.2}$$

where

$$X = \frac{f(-q)}{q^{1/6}f(-q^5)} \quad \text{and} \quad Y = \frac{f(-q^3)}{q^{1/2}f(-q^{15})}.$$

From (11.7) and (11.8) in Chapter 19 of [2, p. 268], we can deduce that

$$X^6 = P^5 + 5P^4 + 15P^3 + 25P^2 + 25P \tag{2.3}$$

and

$$Y^6 = Q^5 + 5Q^4 + 15Q^3 + 25Q^2 + 25Q. \tag{2.4}$$

By using (2.3) and (2.4) in (2.2), we find that $f(P,Q)g(P,Q) = 0$, where

$$g(P,Q) = P^4 - 25PQ - 15P^2Q - 6P^3Q - 15PQ^2 - 9P^2Q^2 - 3P^3Q^2 - 6PQ^3 - 3P^2Q^3 - P^3Q^3 + Q^4$$

and $f(P,Q)$ is a polynomial in P and Q of total degree 24, and $f(P,Q) > 0$ for $0 < q < 1$. So

$$g(P,Q) = 0. \tag{2.5}$$

By dividing (2.5) by P^2Q^2, we complete the proof. ∎

Another proof of Theorem 2.6 was given by Berndt [4] by using modular forms.

THEOREM 2.7. *Let*

$$P = \frac{f(-q^{1/5})}{q^{1/5}f(-q^5)} \quad \text{and} \quad Q = \frac{f(-q)}{qf(-q^{25})}.$$

Then

$$\left(\frac{Q}{P}\right)^3 - 25\left(\frac{Q}{P}\right)^2 - 125\left(\frac{Q}{P}\right) - 225 - 125\left(\frac{P}{Q}\right) - 25\left(\frac{P}{Q}\right)^2$$
$$= (PQ)^2 + \left(\frac{25}{PQ}\right)^2 + 25\left(PQ + \frac{25}{PQ}\right) + 5\left(P^2 + \frac{25}{P^2}\right)\left(Q + \frac{5}{Q}\right)$$
$$+ 5\left(Q^2 + \frac{25}{Q^2}\right)\left(P + \frac{5}{P}\right) + 15\left(P^2 + \frac{25}{P^2}\right) + 15\left(Q^2 + \frac{25}{Q^2}\right)$$
$$+ 75\left(P + \frac{5}{P}\right) + 75\left(Q + \frac{5}{Q}\right).$$

Proof. From Theorem 2.3, we find that

$$P^2 + \frac{25}{P^2} + 5\left(P + \frac{5}{P}\right) + 15 = \frac{f^6(-q)}{q^{2/5}f^3(-q^{1/5})f^3(-q^5)}, \tag{2.6}$$

$$Q^2 + \frac{25}{Q^2} + 5\left(Q + \frac{5}{Q}\right) + 15 = \frac{f^6(-q^5)}{q^2f^3(-q)f^3(-q^{25})}. \tag{2.7}$$

By multiplying (2.6) and (2.7), we find that

$$\left(P^2 + \frac{25}{P^2} + 5\left(P + \frac{5}{P}\right) + 15\right)\left(Q^2 + \frac{25}{Q^2} + 5\left(Q + \frac{5}{Q}\right) + 15\right)$$
$$= \frac{f^3(-q)f^3(-q^5)}{q^{12/5}f^3(-q^{1/5})f^3(-q^{25})} = \left(\frac{Q}{P}\right)^3.$$

So the proof is complete. ∎

3. Formulas and values for $R(q)$ and $S(q)$ from modular equations of degree $25p$. We shall use the following relation discovered by Ramanujan [1, p. 267, (11.5)], and proved by Watson [17]:

$$\frac{1}{R(q)} - 1 - R(q) = \frac{f(-q^{1/5})}{q^{1/5}f(-q^5)}. \tag{3.1}$$

Replacing q by $-q$, we have

$$\frac{1}{S(q)} + 1 - S(q) = \frac{f(q^{1/5})}{q^{1/5}f(q^5)}. \tag{3.2}$$

THEOREM 3.1. *For $q = e^{-2\pi\sqrt{n}}$, let*

$$J_n = \frac{f(-q^{1/5})}{\sqrt{5}\,q^{1/5}f(-q^5)}.$$

Then

(i) $$R(e^{-2\pi\sqrt{n}}) = \sqrt{c^2+1} - c, \quad \textit{where} \quad 2c = \sqrt{5}\,J_n + 1.$$

Similarly, for $q = e^{-\pi\sqrt{n}}$, let

$$D_n = \frac{f(q^{1/5})}{\sqrt{5}\,q^{1/5}f(q^5)}.$$

Then

(ii) $$S(e^{-\pi\sqrt{n}}) = \sqrt{d^2+1} - d, \quad \textit{where} \quad 2d = \sqrt{5}\,D_n - 1.$$

Proof. (i) From (3.1), we have

$$R^2(q) + (\sqrt{5}\,J_n + 1)R(q) - 1 = 0.$$

Solving for $R(q)$ and noting that $R(q) > 0$, we complete the proof.

(ii) Similarly, from (3.2), we have

$$S^2(q) + (\sqrt{5}\,D_n - 1)S(q) - 1 = 0.$$

Solving for $S(q)$ and noting that $S(q) > 0$, we complete the proof. ■

THEOREM 3.2. *We have*

(i) $$J_1 = 1,$$

(ii) $$D_1 = 1.$$

Proof. (i) From [1, p. 43, Entry 27(iii)], for $\alpha, \beta > 0$ and $\alpha\beta = \pi^2$,

$$e^{-\alpha/12}\alpha^{1/4}f(-e^{-2\alpha}) = e^{-\beta/12}\beta^{1/4}f(-e^{-2\beta}). \tag{3.3}$$

Setting $\alpha = \pi/5$ and $\beta = 5\pi$, we deduce that

$$J_1 = \frac{e^{2\pi/5}f(-e^{2\pi/5})}{\sqrt{5}\,f(-e^{-10\pi})} = 1.$$

(ii) From [1, p. 43, Entry 27(iv)], for $\alpha, \beta > 0$ and $\alpha\beta = \pi^2$,

$$e^{-\alpha/24}\alpha^{1/4}f(e^{-\alpha}) = e^{-\beta/24}\beta^{1/4}f(e^{-\beta}). \tag{3.4}$$

Setting $\alpha = \pi/5$ and $\beta = 5\pi$, we deduce that

$$D_1 = \frac{e^{\pi/5}f(e^{-\pi/5})}{\sqrt{5}\,f(q^{-5\pi})} = 1. \ \blacksquare$$

COROLLARY 3.3. *We have*

(i) $$R(e^{-2\pi}) = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2},$$

(ii) $$S(e^{-\pi}) = \sqrt{\frac{5-\sqrt{5}}{2}} - \frac{\sqrt{5}-1}{2}.$$

Proof. (i) We apply Theorem 3.1. From Theorem 3.2(i),

$$c = \tfrac{1}{2}(\sqrt{5}\,J_1 + 1) = \tfrac{1}{2}(\sqrt{5}+1).$$

Thus

$$\sqrt{c^2+1} = \sqrt{\tfrac{1}{4}(6+2\sqrt{5})+1} = \sqrt{\tfrac{1}{2}(5+\sqrt{5})}.$$

Applying Theorem 3.1(i), we complete the proof.

(ii) From Theorem 3.2(ii),

$$d = \tfrac{1}{2}(\sqrt{5}D_1 - 1) = \tfrac{1}{2}(\sqrt{5}-1).$$

Thus

$$\sqrt{d^2+1} = \sqrt{\tfrac{1}{4}(6-2\sqrt{5})+1} = \sqrt{\tfrac{1}{2}(5-\sqrt{5})}.$$

Applying Theorem 3.1(ii), we complete the proof. $\blacksquare$

REMARK 1. We note that it is easily seen from the definition of J_n and (3.3) that $J_{1/n} = 1/J_n$. Also we note that it is easily seen from the definition of D_n and (3.4) that $D_{1/n} = 1/D_n$.

THEOREM 3.4. *If* J_n *is as defined in Theorem* 3.1, *then*

(i) $$5\left(J_nJ_{4n} + \frac{1}{J_nJ_{4n}}\right) = \left(\frac{J_{4n}}{J_n}\right)^3 + \left(\frac{J_n}{J_{4n}}\right)^3 - 4\left\{\left(\frac{J_{4n}}{J_n}\right)^2 + \left(\frac{J_n}{J_{4n}}\right)^2\right\},$$

(ii) $$\sqrt{5}(J_nJ_{4n})^2 + \sqrt{5}\,J_nJ_{4n} = J_n^3 + J_{4n}^3 - 2J_nJ_{4n}(J_n + J_{4n}).$$

Proof. The theorem follows directly from Theorem 2.5 and the definition of J_n. $\blacksquare$

REMARK 2. Theorem 3.4 implies that if we know J_n, then we can compute J_{4n} or $J_{n/4}$, that is, if we know $R(e^{-2\pi\sqrt{n}})$, then we can also determine $R(e^{-4\pi\sqrt{n}})$ or $R(e^{-\pi\sqrt{n}})$.

THEOREM 3.5. *We have*

$$J_2 = \tfrac{1}{2}(a + b + \sqrt{a^2 + b^2 - 2/3})$$

where $a = (\sqrt{5} + \sqrt{30}/9)^{1/3}$ *and* $b = (\sqrt{5} - \sqrt{30}/9)^{1/3}$.

Proof. Putting $n = 1/2$ in (ii) of Theorem 3.4, setting $A = J_2 + J_2^{-1}$, and recalling that $J_{1/n} = 1/J_n$, we find that

$$2\sqrt{5} = A^3 - 3A - 2A = A^3 - 5A.$$

Since A is real-valued,

$$A = \left(\sqrt{5} + \frac{\sqrt{30}}{9}\right)^{1/3} + \left(\sqrt{5} - \frac{\sqrt{30}}{9}\right)^{1/3}.$$

Hence

$$J_2^2 - \left\{\left(\sqrt{5} + \frac{\sqrt{30}}{9}\right)^{1/3} + \left(\sqrt{5} - \frac{\sqrt{30}}{9}\right)^{1/3}\right\} J_2 + 1 = 0,$$

which gives the result. ∎

REMARK 3. If $J_n + 1/J_n = A$, then $J_{1/n} + 1/J_{1/n} = A$ since $J_{1/n} = 1/J_n$. So J_n and $J_{1/n}$ are the solutions of the equation $x^2 - Ax + 1 = 0$. Since J_n is increasing in n, $J_n > J_{1/n}$ when $n \geq 1$. Thus we conclude that

$$J_n = \tfrac{1}{2}(A + \sqrt{A^2 - 4}) \quad \text{and} \quad J_{1/n} = \tfrac{1}{2}(A - \sqrt{A^2 - 4}).$$

EXAMPLE 1. Using Theorem 3.5 and Remark 3, we find that

$$J_{1/2} = \tfrac{1}{2}(a + b - \sqrt{a^2 + b^2 - 2/3}),$$

where $a = (\sqrt{5} + \sqrt{30}/9)^{1/3}$ and $b = (\sqrt{5} - \sqrt{30}/9)^{1/3}$.

COROLLARY 3.6. *We have*

(i) $$R(e^{-2\pi\sqrt{2}}) = \sqrt{c^2 + 1} - c, \quad \textit{where} \quad 2c = \sqrt{5}\, J_2 + 1,$$

and J_2 *is given in Theorem* 3.5. *Furthermore*,

(ii) $$R(e^{-\pi\sqrt{2}}) = \sqrt{c^2 + 1} - c, \quad \textit{where} \quad 2c = \sqrt{5}\, J_{1/2} + 1.$$

Proof. For the proof of (i), use Theorems 3.1 and 3.5; for the proof of (ii), use Theorem 3.1 and Example 1 above. ∎

THEOREM 3.7. *We have*

(i) $$J_4 = \tfrac{1}{2}(3 + \sqrt[4]{5} + \sqrt{5} + \sqrt[4]{5^3}) = \frac{\sqrt[4]{5} + 1}{\sqrt[4]{5} - 1},$$

(ii) $$J_{1/4} = \tfrac{1}{2}(3 - \sqrt[4]{5} + \sqrt{5} - \sqrt[4]{5^3}) = \frac{\sqrt[4]{5} - 1}{\sqrt[4]{5} + 1}.$$

Proof. For the proof of (i), setting $n = 1$ in (i) of Theorem 3.4, using Theorem 3.2(i), and putting $A = J_4 + J_4^{-1}$, we find that

$$5A = A^3 - 3A - 4(A^2 - 2).$$

Thus

$$(A + 2)(A^2 - 6A + 4) = 0.$$

Since J_n is positive and increasing in n, we have $J_4 > J_2 > 2$. Hence $A = 3 + \sqrt{5}$ and

$$\begin{aligned} J_4 &= \tfrac{1}{2}(3 + \sqrt{5} + \sqrt{10 + 6\sqrt{5}}) = \tfrac{1}{2}(3 + \sqrt{5} + \sqrt[4]{5} + \sqrt[4]{5^3}) \\ &= \frac{(3 + \sqrt{5} + \sqrt[4]{5} + \sqrt[4]{5^3})(\sqrt[4]{5} - 1)}{2(\sqrt[4]{5} - 1)} = \frac{\sqrt[4]{5} + 1}{\sqrt[4]{5} - 1}. \end{aligned}$$

For the proof of (ii), use Remarks 1 and 3 in the result of J_4. ∎

Corollary 3.8. *We have*

(i) $$R(e^{-4\pi}) = \sqrt{c^2 + 1} - c, \quad \textit{where} \quad 2c = \sqrt{5}\, J_4 + 1,$$

(ii) $$R(e^{-\pi}) = \sqrt{c^2 + 1} - c, \quad \textit{where} \quad 2c = \sqrt{5}\, J_{1/4} + 1.$$

Proof. Parts (i) and (ii) follow from Theorems 3.1 and 3.7. ∎

Theorem 3.9. *We have*

(i) $$J_{16} = \tfrac{1}{4}(2 + \sqrt[4]{20})(17 + 11\sqrt[4]{5} + 7\sqrt{5} + 5\sqrt[4]{5^3}),$$

(ii) $$R(e^{-8\pi}) = \sqrt{c^2 + 1} - c, \quad \textit{where} \quad 2c = \sqrt{5}\, J_{16} + 1.$$

Proof. We know the value of J_4 from Theorem 3.7, and so by using (ii) of Theorem 3.4 with $n = 4$, we can find the value of J_{16}. It follows that the value of $R(e^{-8\pi})$ can be found by Theorem 3.1(i). Now we shall show how to find the value of J_{16} by applying Theorem 3.4(ii). Let $n = 4$ in Theorem 3.4(ii) to deduce that

$$J_{16}^3 - (\sqrt{5}\, J_4^2 + 2J_4)J_{16}^2 - (\sqrt{5}\, J_4 + 2J_4^2)J_{16} + J_4^3 = 0.$$

Now putting $J_4 = \frac{1}{2}(3 + \sqrt[4]{5} + \sqrt{5} + \sqrt[4]{5^3})$ in the preceding equation, we find that

$$\begin{aligned} &\tfrac{1}{2}(J_{16} - 1)\{2J_{16}^2 - 2(17 + 11\sqrt[4]{5} + 7\sqrt{5} + 5\sqrt[4]{5^3})J_{16} \\ &\qquad\qquad - (63 + 43\sqrt[4]{5} + 29\sqrt{5} + 19\sqrt[4]{5^3})\} = 0. \end{aligned}$$

Since $J_{16} > 1$,

$$\begin{aligned} J_{16} &= \frac{1}{2}(17 + 11\sqrt[4]{5} + 7\sqrt{5} + 5\sqrt[4]{5^3} + \sqrt{1210 + 810\sqrt[4]{5} + 542\sqrt{5} + 362\sqrt[4]{5^3}}\,) \\ &= \frac{1}{2}\left\{17 + 11\sqrt[4]{5} + 7\sqrt{5} + 5\sqrt[4]{5^3} + \frac{1}{\sqrt{2}}(25 + 17\sqrt[4]{5} + 11\sqrt{5} + 7\sqrt[4]{5^3})\right\} \end{aligned}$$

$$= \frac{1}{2}(17 + 11\sqrt[4]{5} + 7\sqrt{5} + 5\sqrt[4]{5^3})\left(1 + \frac{\sqrt[4]{5}}{\sqrt{2}}\right)$$
$$= \frac{1}{4}(2 + \sqrt[4]{20})(17 + 11\sqrt[4]{5} + 7\sqrt{5} + 5\sqrt[4]{5^3}).$$

So we complete the proof of (i). Part (ii) follows from Theorem 3.1(i) and part (i). ■

REMARK 4. In his first notebook [13], Ramanujan recorded the value $R(e^{-8\pi}) = \sqrt{c^2+1} - c$, where

$$2c = 1 + \sqrt{5}\,\frac{3 + \sqrt{2} - \sqrt{5} + \sqrt[4]{20}}{3 + \sqrt{2} - \sqrt{5} - \sqrt[4]{20}}.$$

The first proof was given by Berndt and Chan [3, 5].

COROLLARY 3.10. *We have*

(i) $$J_{1/16} = \tfrac{1}{4}(2 - \sqrt[4]{20})(17 - 11\sqrt[4]{5} + 7\sqrt{5} - 5\sqrt[4]{5^3}),$$

(ii) $$R(e^{-\pi/2}) = \sqrt{c^2+1} - c, \quad \textit{where} \quad 2c = \sqrt{5}\,J_{1/16} + 1.$$

Proof. For the proof of (i), use Theorem 3.9 and Remark 3. Then part (ii) follows from Theorem 3.1 and part (i).

THEOREM 3.11. *We have*

$$5\left(J_nJ_{9n} + \frac{1}{J_nJ_{9n}}\right) = \left(\frac{J_{9n}}{J_n}\right)^2 + \left(\frac{J_n}{J_{9n}}\right)^2 - 6\left(\frac{J_{9n}}{J_n} + \frac{J_n}{J_{9n}}\right) - 3\sqrt{5}\left(J_{9n} + \frac{1}{J_{9n}}\right) - 3\sqrt{5}\left(J_n + \frac{1}{J_n}\right) - 9.$$

Proof. The result follows directly from Theorem 2.6 and the definition of J_n. ■

REMARK 5. By Theorem 3.11, we can compute J_{9n} or $J_{n/9}$ if we know J_n, i.e., if we know $R(e^{-2\pi\sqrt{n}})$, then the value of $R(e^{-6\pi\sqrt{n}})$ or $R(e^{-2\pi\sqrt{n}/3})$ can be computed.

THEOREM 3.12. *We have*

$$5\left(D_nD_{9n} + \frac{1}{D_nD_{9n}}\right) = \left(\frac{D_{9n}}{D_n}\right)^2 + \left(\frac{D_n}{D_{9n}}\right)^2 - 6\left(\frac{D_{9n}}{D_n} + \frac{D_n}{D_{9n}}\right) + 3\sqrt{5}\left(D_{9n} + \frac{1}{D_{9n}}\right) + 3\sqrt{5}\left(D_n + \frac{1}{D_n}\right) - 9.$$

Proof. Replacing q by $-q$ in Theorem 2.6 and using the definition of D_n yields the assertion. ■

REMARK 6. By Theorem 3.12, if we know D_n, then we can find D_{9n} or $D_{n/9}$, which implies that if we know $S(e^{-\pi\sqrt{n}})$, then we can compute $S(e^{-3\pi\sqrt{n}})$ or $S(e^{-\pi\sqrt{n}/3})$.

THEOREM 3.13. *We have*

$$\text{(i)} \qquad J_3 = \tfrac{1}{2}(1 + \sqrt[3]{10} + \sqrt{5 + 2\sqrt[3]{10} + \sqrt[3]{10^2}}),$$

$$\text{(ii)} \qquad J_{1/3} = \tfrac{1}{2}(-1 - \sqrt[3]{10} + \sqrt{5 + 2\sqrt[3]{10} + \sqrt[3]{10^2}}).$$

Proof. Letting $n = 1/3$ in Theorem 3.11 and putting $A = J_3^2 + J_3^{-2}$, we find that

$$10 = (A^2 - 2) - 6A - 6\sqrt{5}\sqrt{A+2} - 9,$$

since $J_{1/n} = 1/J_n$. Hence

$$(A-3)(A^3 - 9A^2 - 33A - 27) = 0.$$

Since $A > J_3^2 > J_2^2 > 3$ and A is real,

$$A = 3 + 2\sqrt[3]{10} + \sqrt[3]{10^2} = (1 + \sqrt[3]{10})^2 + 2.$$

Now, since $(J_3 - J_3^{-1})^2 = J_3^2 + J_3^{-2} - 2 = (1 + \sqrt[3]{10})^2$ and $J_3 - J_3^{-1} > 0$, it follows that $J_3 - J_3^{-1} = 1 + \sqrt[3]{10}$, from which we complete the proof of (i). Since $J_{1/3} = 1/J_3$, we can easily deduce (ii) from the foregoing equality. ∎

COROLLARY 3.14. *We have*

$$\text{(i)} \qquad R(e^{-2\sqrt{3}\pi}) = \sqrt{c^2+1} - c, \quad \textit{where} \quad 2c = \sqrt{5}\,J_3 + 1,$$

$$\text{(ii)} \qquad R(e^{-2\pi/\sqrt{3}}) = \sqrt{c^2+1} - c, \quad \textit{where} \quad 2c = \sqrt{5}\,J_{1/3} + 1.$$

Proof. Parts (i) and (ii) follow from Theorems 3.1 and 3.13. ∎

THEOREM 3.15. *We have*

$$\text{(i)} \qquad J_9 = \tfrac{1}{4}\{11 + 3\sqrt{5} + 5\sqrt{3} + 3\sqrt{15} + \sqrt[4]{60}(4 + 2\sqrt{3} + \sqrt{5} + \sqrt{15})\}$$
$$= \frac{\sqrt[4]{60} + 2 - \sqrt{3} + \sqrt{5}}{\sqrt[4]{60} - 2 + \sqrt{3} - \sqrt{5}},$$

$$\text{(ii)} \qquad J_{1/9} = \tfrac{1}{4}\{11 + 3\sqrt{5} + 5\sqrt{3} + 3\sqrt{15} - \sqrt[4]{60}(4 + 2\sqrt{3} + \sqrt{5} + \sqrt{15})\}$$
$$= \frac{\sqrt[4]{60} - 2 + \sqrt{3} - \sqrt{5}}{\sqrt[4]{60} + 2 - \sqrt{3} + \sqrt{5}}.$$

Proof. Setting $n = 1$ and $J_9 + J_9^{-1} = A$ in Theorem 3.11 and using Theorem 3.2(i), we have

$$5A = A^2 - 2 - 6A - 3\sqrt{5}A - 6\sqrt{5} - 9.$$

Hence

$$A = \tfrac{1}{2}\{(11 + 3\sqrt{5}) \pm \sqrt{30(7 + 3\sqrt{5})}\} = \tfrac{1}{2}\{(11 + 3\sqrt{5}) \pm \sqrt{15}(3 + \sqrt{5})\}.$$

Since $A > 0$, $J_9 + J_9^{-1} = \frac{1}{2}(11 + 3\sqrt{5} + 5\sqrt{3} + 3\sqrt{15})$. Thus

$$\begin{aligned} J_9 &= \tfrac{1}{4}\{11 + 3\sqrt{5} + 5\sqrt{3} + 3\sqrt{15} + 2\sqrt{(90 + 50\sqrt{3} + 39\sqrt{5} + 24\sqrt{15})}\} \\ &= \tfrac{1}{4}\{11 + 3\sqrt{5} + 5\sqrt{3} + 3\sqrt{15} + \sqrt[4]{60}(4 + 2\sqrt{3} + \sqrt{5} + \sqrt{15})\}, \end{aligned}$$

and using Remark 3, we find that

$$\begin{aligned} J_{1/9} &= \tfrac{1}{4}\{11 + 3\sqrt{5} + 5\sqrt{3} + 3\sqrt{15} - 2\sqrt{(90 + 50\sqrt{3} + 39\sqrt{5} + 24\sqrt{15})}\} \\ &= \tfrac{1}{4}\{11 + 3\sqrt{5} + 5\sqrt{3} + 3\sqrt{15} - \sqrt[4]{60}(4 + 2\sqrt{3} + \sqrt{5} + \sqrt{15})\}. \end{aligned}$$

Also

$$\{11+3\sqrt{5}+5\sqrt{3}+3\sqrt{15}+\sqrt[4]{60}(4+2\sqrt{3}+\sqrt{5}+\sqrt{15})\}\{\sqrt[4]{60}-2+\sqrt{3}-\sqrt{5}\} = 8 - 4\sqrt{3} + 4\sqrt{5} + 4\sqrt[4]{60}.$$

Thus

$$J_9 = \frac{8 - 4\sqrt{3} + 4\sqrt{5} + 4\sqrt[4]{60}}{4(\sqrt[4]{60} - 2 + \sqrt{3} - \sqrt{5})} = \frac{\sqrt[4]{60} + 2 - \sqrt{3} + \sqrt{5}}{\sqrt[4]{60} - 2 + \sqrt{3} - \sqrt{5}}.$$

Using Remark 1, we complete the proof. ■

COROLLARY 3.16. *We have*

(i) $$R(e^{-6\pi}) = \sqrt{c^2 + 1} - c, \quad \textit{where} \quad 2c = \sqrt{5}\, J_9 + 1,$$

(ii) $$R(e^{-2\pi/3}) = \sqrt{c^2 + 1} - c, \quad \textit{where} \quad 2c = \sqrt{5}\, J_{1/9} + 1.$$

Proof. Parts (i) and (ii) follow from Theorems 3.1 and 3.15. ■

Berndt and Chan [3, 5] gave another proof of (i).

THEOREM 3.17. *We have*

(i) $$D_3 = \frac{\sqrt{5} + 1}{2},$$

(ii) $$D_{1/3} = \frac{\sqrt{5} - 1}{2}.$$

Proof. Letting $n = 1/3$ and $B = D_3^2 + D_3^{-2}$ in Theorem 3.12, and using the fact that $D_{1/n} = 1/D_n$ we have

$$10 = (B^2 - 2) - 6B + 6\sqrt{5}\sqrt{B + 2} - 9.$$

Hence

$$(B - 3)(B^3 - 9B^2 - 33B - 27) = 0.$$

Since $D_3 < J_3$ and B is real-valued, $B = 3$ by the proof of Theorem 3.13. Now the assertion follows from

$$D_3 + D_3^{-1} = \sqrt{D_3^2 + D_3^{-2} + 2} = \sqrt{5}. \ ■$$

COROLLARY 3.18. *We have*

(i) $$S(e^{-\sqrt{3}\pi}) = \tfrac{1}{4}\{\sqrt{6(5+\sqrt{5})} - (3+\sqrt{5})\},$$

(ii) $$S(e^{-\pi/\sqrt{3}}) = \tfrac{1}{4}\{\sqrt{6(5-\sqrt{5})} - (3-\sqrt{5})\}.$$

Proof. (i) From Theorem 3.17(i),

$$d = \frac{1}{2}(\sqrt{5}\,D_3 - 1) = \frac{1}{2}\left(\frac{5+\sqrt{5}}{2} - 1\right) = \frac{1}{4}(3+\sqrt{5}),$$

which implies that $\sqrt{d^2+1} = \frac{1}{4}\sqrt{30+6\sqrt{5}}$. Now apply Theorem 3.1(ii).

(ii) From Theorem 3.17(ii),

$$d = \tfrac{1}{2}(\sqrt{5}\,D_{1/3} - 1) = \tfrac{1}{4}(3-\sqrt{5})$$

which implies that $\sqrt{d^2+1} = \frac{1}{4}\sqrt{30-6\sqrt{5}}$. Now apply Theorem 3.1(ii) again. ∎

THEOREM 3.19. *We have*

(i) $$\begin{aligned} D_9 &= \tfrac{1}{4}\{11 - 5\sqrt{3} - 3\sqrt{5} + 3\sqrt{15} + \sqrt[4]{60}(4 - 2\sqrt{3} - \sqrt{5} + \sqrt{15})\} \\ &= \frac{\sqrt[4]{60} + 2 + \sqrt{3} - \sqrt{5}}{\sqrt[4]{60} - 2 - \sqrt{3} + \sqrt{5}}, \end{aligned}$$

(ii) $$\begin{aligned} D_{1/9} &= \tfrac{1}{4}\{11 - 5\sqrt{3} - 3\sqrt{5} + 3\sqrt{15} - \sqrt[4]{60}(4 - 2\sqrt{3} - \sqrt{5} + \sqrt{15})\} \\ &= \frac{\sqrt[4]{60} - 2 - \sqrt{3} + \sqrt{5}}{\sqrt[4]{60} + 2 + \sqrt{3} - \sqrt{5}}. \end{aligned}$$

Proof. Set $n = 1$ and $B = D_9 + D_9^{-1}$ in Theorem 3.12. Then, using Theorem 3.2(ii), we find that

$$B^2 - (11 - 3\sqrt{5})B - (11 - 6\sqrt{5}) = 0.$$

Hence

$$B = \tfrac{1}{2}\{11 - 3\sqrt{5} + \sqrt{30(7 - 3\sqrt{5})}\} = \tfrac{1}{2}(11 - 3\sqrt{5} - 5\sqrt{3} + 3\sqrt{15}).$$

From this we deduce that

$$\begin{aligned} D_9 &= \tfrac{1}{4}(11 - 3\sqrt{5} - 5\sqrt{3} + 3\sqrt{15} + 2\sqrt{90 - 50\sqrt{3} - 39\sqrt{5} + 24\sqrt{15}}) \\ &= \tfrac{1}{4}\{11 - 3\sqrt{5} - 5\sqrt{3} + 3\sqrt{15} + \sqrt[4]{60}(4 - 2\sqrt{3} - \sqrt{5} + \sqrt{15})\}. \end{aligned}$$

Now apply the same argument as in Theorem 3.15 for computing J_9 and $J_{1/9}$ to conclude that

$$D_9 = \frac{\sqrt[4]{60} + 2 + \sqrt{3} - \sqrt{5}}{\sqrt[4]{60} - 2 - \sqrt{3} + \sqrt{5}}$$

and we can easily deduce (ii). ∎

COROLLARY 3.20. *We have*

$$\text{(i)}\qquad S(e^{-3\pi}) = \sqrt{d^2+1}-d, \quad \textit{where} \quad 2d=\sqrt{5}\,D_9-1,$$

$$\text{(ii)}\qquad S(e^{-\pi/3}) = \sqrt{d^2+1}-d, \quad \textit{where} \quad 2d=\sqrt{5}\,D_{1/9}-1.$$

Proof. Parts (i) and (ii) follow from Theorems 3.1(ii) and 3.19. ■

THEOREM 3.21. *We have*

$$\text{(i)}\qquad D_{27} = 2+\sqrt{5}+(1+\sqrt[6]{5})(20+9\sqrt{5})^{1/3},$$

$$\text{(ii)}\qquad S(e^{-3\sqrt{3}\pi}) = \sqrt{d^2+1}-d, \quad \textit{where} \quad 2d=\sqrt{5}\,D_{27}-1.$$

Proof. By multiplying $D_n^2D_{9n}^2$ on both sides of Theorem 3.12, we can deduce the equation

$$D_{9n}^4-(5D_n^3-3\sqrt{5}\,D_n^2+6D_n)D_{9n}^3+3(\sqrt{5}\,D_n^3-3D_n^2+\sqrt{5}\,D_n)D_{9n}^2 \\ -(6D_n^3-3\sqrt{5}\,D_n^2+5D_n)D_{9n}+D_n^4=0.$$

With $n=3$ and $D_3=(\sqrt{5}+1)/2$ in the above equation, we deduce that

$$D_{27}^4-\tfrac{1}{2}(11+7\sqrt{5})D_{27}^3+3(3+\sqrt{5})D_{27}^2-(7+4\sqrt{5})D_{27}+\tfrac{1}{2}(7+3\sqrt{5}) \\ =\left(D_{27}-\frac{\sqrt{5}-1}{2}\right)\left\{D_{27}^3-3(2+\sqrt{5})D_{27}^2+\tfrac{3}{2}(3+\sqrt{5})D_{27}-\tfrac{1}{2}(11+5\sqrt{5})\right\} \\ =0.$$

Since $D_{27}>1$ and is real-valued, we find, upon solving the cubic equation above, that

$$\begin{aligned} D_{27} &= 2+\sqrt{5}+\tfrac{1}{2}(260+116\sqrt{5}-4\sqrt{1230+550\sqrt{5}})^{1/3} \\ &\quad +\tfrac{1}{2}(260+116\sqrt{5}+4\sqrt{1230+550\sqrt{5}})^{1/3} \\ &= 2+\sqrt{5}+\tfrac{1}{2}(160+72\sqrt{5})^{1/3}+\tfrac{1}{2}(360+160\sqrt{5})^{1/3} \\ &= 2+\sqrt{5}+(20+9\sqrt{5})^{1/3}+\{\sqrt{5}(9\sqrt{5}+20)\}^{1/3} \\ &= 2+\sqrt{5}+(20+9\sqrt{5})^{1/3}(1+\sqrt[6]{5}). \end{aligned}$$

So we complete the proof of (i). Part (ii) follows from Theorem 3.1(ii) and part (i). ■

COROLLARY 3.22. *We have*

$$S(e^{-\sqrt{3}\pi/9}) = \sqrt{d^2+1}-d, \quad \textit{where} \quad 2d=\sqrt{5}\,D_{1/27}-1,$$

and

$$D_{1/27} = -2+\sqrt{5}+(-20+9\sqrt{5})^{1/3}-\tfrac{1}{2}(3+\sqrt{5})(-20+9\sqrt{5})^{2/3}.$$

Proof. Apply Theorem 3.21(i) and $D_{1/27}=1/D_{27}$, and then use Theorem 3.1(ii). ■

THEOREM 3.23. *We have*

$$\left(\frac{J_{25n}}{J_n}\right)^3 - 25\left\{\left(\frac{J_{25n}}{J_n}\right)^2 + \left(\frac{J_n}{J_{25n}}\right)^2\right\} - 125\left(\frac{J_{25n}}{J_n} + \frac{J_n}{J_{25n}}\right) - 225$$
$$= 25\{(J_nJ_{25n})^2 + (J_nJ_{25n})^{-2}\} + 125\{J_nJ_{25n} + (J_nJ_{25n})^{-1}\}$$
$$+ 25\sqrt{5}(J_n^2 + J_n^{-2})(J_{25n} + J_{25n}^{-1}) + 25\sqrt{5}(J_n + J_n^{-1})(J_{25n}^2 + J_{25n}^{-2})$$
$$+ 75(J_n^2 + J_n^{-2}) + 75(J_{25n}^2 + J_{25n}^{-2}) + 75\sqrt{5}(J_n + J_n^{-1})$$
$$+ 75\sqrt{5}(J_{25n} + J_{25n}^{-1}).$$

Proof. The result follows directly from Theorem 2.7 and the definition of J_n. ■

REMARK 7. Theorem 3.23 implies that if we know J_n, then we can compute J_{25n} or $J_{n/25}$, that is, if $R(e^{-2\pi\sqrt{n}})$ is known, then so is $R(e^{-10\pi\sqrt{n}})$ or $R(e^{-2\pi\sqrt{n}/5})$.

THEOREM 3.24. *We have*

$$\left(\frac{D_{25n}}{D_n}\right)^3 - 25\left\{\left(\frac{D_{25n}}{D_n}\right)^2 + \left(\frac{D_n}{D_{25n}}\right)^2\right\} - 125\left(\frac{D_{25n}}{D_n} + \frac{D_n}{D_{25n}}\right) - 225$$
$$= 25\{(D_nD_{25n})^2 + (D_nD_{25n})^{-2}\} + 125\{D_nD_{25n} + (D_nD_{25n})^{-1}\}$$
$$- 25\sqrt{5}(D_n^2 + D_n^{-2})(D_{25n} + D_{25n}^{-1}) - 25\sqrt{5}(D_n + D_n^{-1})(D_{25n}^2 + D_{25n}^{-2})$$
$$+ 75(D_n^2 + D_n^{-2}) + 75(D_{25n}^2 + D_{25n}^{-2}) - 75\sqrt{5}(D_n + D_n^{-1})$$
$$- 75\sqrt{5}(D_{25n} + D_{25n}^{-1}).$$

Proof. Replace q by $-q$ in Theorem 2.7, set $q = e^{-\pi\sqrt{n}}$, and use the definition of D_n to achieve the result. ■

REMARK 8. By Theorem 3.24, if we know D_n, then we can find D_{25n} or $D_{n/25}$, which implies that if we know $S(e^{-\pi\sqrt{n}})$, then we can compute $S(e^{-5\pi\sqrt{n}})$ or $S(e^{-\pi\sqrt{n}/5})$.

4. Formulas and values for $R(q)$ and $S(q)$ from modular equations of degree $5p$. In this section, we shall need the following relations stated by Ramanujan [1, p. 267, (11.6)], and proved by Watson [17]:

$$\frac{1}{R^5(q)} - 11 - R^5(q) = \frac{f^6(-q)}{qf^6(-q^5)}. \tag{4.1}$$

Replacing q by $-q$, we have

$$\frac{1}{S^5(q)} + 11 - S^5(q) = \frac{f^6(q)}{qf^6(q^5)}. \tag{4.2}$$

THEOREM 4.1. (i) *For* $q = e^{-2\pi\sqrt{n/5}}$, *let*

$$s_n = \frac{f^6(-q)}{5\sqrt{5}\, q f^6(-q^5)}.$$

Then

$$R^5(e^{-2\pi\sqrt{n/5}}) = \sqrt{a^2+1} - a, \quad \textit{where} \quad 2a = 5\sqrt{5}\, s_n + 11.$$

(ii) *Also for* $q = e^{-\pi\sqrt{n/5}}$, *let*

$$t_n = \frac{f^6(q)}{5\sqrt{5}\, q f^6(q^5)}.$$

Then

$$S^5(e^{-\pi\sqrt{n/5}}) = \sqrt{b^2+1} - b, \quad \textit{where} \quad 2b = 5\sqrt{5}\, t_n - 11.$$

Proof. (i) From (4.1), we have

$$R^{10}(q) + (5\sqrt{5}\, s_n + 11) R^5(q) - 1 = 0.$$

Solving for $R^5(q)$ and using the fact that $R^5(q) > 0$, we complete the proof.

(ii) From (4.2), we find that

$$S^{10}(q) + (5\sqrt{5}\, t_n - 11) S^5(q) - 1 = 0.$$

The result follows upon solving for $S^5(q)$ and noting that $S^5(q) > 0$. ∎

THEOREM 4.2. *We have*

(i) $$s_1 = 1, \quad s_{1/n} = 1/s_n,$$

(ii) $$t_1 = 1, \quad t_{1/n} = 1/t_n.$$

Proof. The results (i) follow from (3.3), and the results (ii) follow from (3.4). ∎

COROLLARY 4.3. *We have*

(i) $$R^5(q^{-2\pi/\sqrt{5}}) = \tfrac{1}{2}\{\sqrt{10(25+11\sqrt{5})} - (5\sqrt{5}+11)\},$$

(ii) $$S^5(q^{-\pi/\sqrt{5}}) = \tfrac{1}{2}\{\sqrt{10(25-11\sqrt{5})} - (5\sqrt{5}-11)\}.$$

Proof. Set $n = 1$ in Theorem 4.1 and use the values $s_1 = 1$ and $t_1 = 1$, respectively, from Theorem 4.2. ∎

THEOREM 4.4. *We have*

$$\sqrt{5}\{(s_n s_{4n})^{1/6} + (s_n s_{4n})^{-1/6}\} = \left(\frac{s_n}{s_{4n}}\right)^{1/2} + \left(\frac{s_{4n}}{s_n}\right)^{1/2}.$$

Proof. The result follows directly from Theorem 2.1 upon setting $q = e^{-2\pi\sqrt{n/5}}$ and using the definition of s_n. ∎

Remark 9. Using Theorem 4.4, we can compute s_{4n} or $s_{n/4}$ if s_n is given. That is, we can compute $R^5(e^{-4\pi\sqrt{n/5}})$ or $R^5(e^{-\pi\sqrt{n/5}})$ if we know $R^5(e^{-2\pi\sqrt{n/5}})$.

Theorem 4.5. *We have*

(i) $$s_2 = \sqrt{5}+2, \qquad s_{1/2} = \sqrt{5}-2,$$

(ii) $$s_4 = \left(\frac{1+\sqrt{5}+\sqrt{2}\sqrt{1+\sqrt{5}}}{2}\right)^3, \qquad s_{1/4} = \left(\frac{1+\sqrt{5}-\sqrt{2}\sqrt{1+\sqrt{5}}}{2}\right)^3.$$

Proof. (i) Letting $n = 1/2$ in Theorem 4.4 and using Theorem 4.2, we find that $2\sqrt{5} = s_2 + s_2^{-1}$. Since s_2 and $s_{1/2}$ are the solutions of $x^2 - 2\sqrt{5}x + 1 = 0$ and s_n is increasing in n, we conclude that $s_2 = \sqrt{5}+2$ and $s_{1/2} = \sqrt{5}-2$.

(ii) Letting $n = 1$ in Theorem 4.4, using Theorem 4.2, and setting $A = s_4^{1/3} + s_4^{-1/3}$, we find that $\sqrt{5}\sqrt{A+2} = (A-1)\sqrt{A+2}$. Hence $A = 1+\sqrt{5}$. Since $s_4^{1/3}$ and $s_{1/4}^{1/3}$ are the solutions of $x^2 - (1+\sqrt{5})x + 1 = 0$ and s_n is increasing in n, we conclude that

$$s_4^{1/3} = \tfrac{1}{2}(1+\sqrt{5}+\sqrt{2(1+\sqrt{5})}),$$
$$s_{1/4}^{1/3} = \tfrac{1}{2}(1+\sqrt{5}-\sqrt{2(1+\sqrt{5})}). \ ■$$

Corollary 4.6. *We have*

(i) $$R^5(e^{-2\pi\sqrt{2/5}}) = 3\sqrt{10(5+2\sqrt{5})} - (18+5\sqrt{5}),$$

(ii) $$R^5(e^{-2\pi/\sqrt{10}}) = 3\sqrt{10(5-2\sqrt{5})} - (18-5\sqrt{5}),$$

(iii) $$R^5(e^{-4\pi/\sqrt{5}}) = \sqrt{a^2+1} - a, \quad \textit{where} \quad 2a = 5\sqrt{5}\,s_4 + 11,$$

(iv) $$R^5(e^{-\pi/\sqrt{5}}) = \sqrt{a^2+1} - a, \quad \textit{where} \quad 2a = 5\sqrt{5}\,s_{1/4} + 11.$$

Proof. The results follow from Theorems 4.1 and 4.5. ■

Theorem 4.7. *We have*

(i) $$s_8 = \left\{\frac{(3+\sqrt{5})(1+\sqrt{2})}{2}\right\}^3 = 63 + 45\sqrt{2} + 28\sqrt{5} + 20\sqrt{10},$$

(ii) $$R^5(e^{-4\pi\sqrt{2/5}}) = \tfrac{1}{2}\{3\sqrt{10(22310 + 15775\sqrt{2} + 9977\sqrt{5} + 7055\sqrt{10})} - (711 + 500\sqrt{2} + 315\sqrt{5} + 225\sqrt{10})\},$$

(iii) $$s_{1/8} = \left\{\frac{(3-\sqrt{5})(\sqrt{2}-1)}{2}\right\}^3 = -63 + 45\sqrt{2} + 28\sqrt{5} - 20\sqrt{10},$$

(iv) $$R^5(e^{-\pi/\sqrt{10}}) = \tfrac{1}{2}\{3\sqrt{10(22310 - 15775\sqrt{2} - 9977\sqrt{5} + 7755\sqrt{10})} - (711 - 500\sqrt{2} - 315\sqrt{5} + 225\sqrt{10})\}.$$

Proof. Multiplying both sides of Theorem 4.4 by $(s_n s_{4n})^{1/2}$, we find that

$$\sqrt{5}(s_n s_{4n})^{2/3} + \sqrt{5}(s_n s_{4n})^{1/3} = s_{4n} + s_n.$$

Now letting $n = 2$ in the above equation and using the value $s_2 = \sqrt{5}+2 = ((1+\sqrt{5})/2)^3$ from Theorem 4.5, we find that

$$s_8 - \frac{3\sqrt{5}+5}{2} s_8^{2/3} - \frac{\sqrt{5}+5}{2} s_8^{1/3} + \sqrt{5}+2 = 0.$$

That is,

$$\tfrac{1}{4}(2s_8^{1/3} + 1 - \sqrt{5})\{2s_8^{2/3} - (6+2\sqrt{5})s_8^{1/3} - (7+3\sqrt{5})\} = 0.$$

Since $s_8^{1/3} > s_2^{1/3} = (1+\sqrt{5})/2$,

$$s_8^{1/3} = \tfrac{1}{2}(3+\sqrt{5}+\sqrt{28+12\sqrt{5}}) = \tfrac{1}{2}(3+\sqrt{5})(1+\sqrt{2}).$$

Hence

$$s_8 = \tfrac{1}{8}(3+\sqrt{5})^3(1+\sqrt{2})^3 = 63 + 45\sqrt{2} + 28\sqrt{5} + 20\sqrt{10},$$

which proves (i). And since $s_{1/8} = 1/s_8$, we find that

$$s_{1/8} = \left\{\frac{(3-\sqrt{5})(\sqrt{2}-1)}{2}\right\}^3 = -63 + 45\sqrt{2} + 28\sqrt{5} - 20\sqrt{10},$$

which proves (iii). By using (i) and (iii) of Theorem 4.1, we deduce (ii) and (iv), respectively. ∎

THEOREM 4.8. *We have*

(i) $$5\sqrt{5}\{(s_n s_{9n})^{1/2} + (s_n s_{9n})^{-1/2}\} = \frac{s_{9n}}{s_n} - \frac{s_n}{s_{9n}} - 9\left\{\left(\frac{s_{9n}}{s_n}\right)^{1/2} + \left(\frac{s_n}{s_{9n}}\right)^{1/2}\right\},$$

(ii) $$5\sqrt{5}\{(t_n t_{9n})^{1/2} + (t_n t_{9n})^{-1/2}\} = \frac{t_{9n}}{t_n} - \frac{t_n}{t_{9n}} + 9\left\{\left(\frac{t_{9n}}{t_n}\right)^{1/2} + \left(\frac{t_n}{t_{9n}}\right)^{1/2}\right\}.$$

Proof. (i) Setting $q = e^{-2\pi\sqrt{n/5}}$ in Theorem 2.2 and using the definition of s_n in Theorem 4.1, we derive the desired result.

(ii) Replacing q by $-q$ in Theorem 2.2, letting $q = e^{-\pi\sqrt{n/5}}$, and using the definition of t_n in Theorem 4.1, we complete the proof. ∎

REMARK 10. By Theorem 4.8, if we know s_n and t_n, then we can find s_{9n} or $s_{n/9}$ and t_{9n} or $t_{n/9}$, respectively, which implies that if we know $R^5(e^{-2\pi\sqrt{n/5}})$, then we can find $R^5(e^{-6\pi\sqrt{n/5}})$ or $R^5(e^{-2\pi\sqrt{n}/(3\sqrt{5})})$, and if we know $S^5(e^{-\pi\sqrt{n/5}})$, then we can find $S^5(e^{-3\pi\sqrt{n/5}})$ or $S^5(e^{-\pi\sqrt{n}/(3\sqrt{5})})$.

THEOREM 4.9. *We have*

(i) $$s_3 = \tfrac{1}{2}(11+5\sqrt{5}), \qquad s_{1/3} = \tfrac{1}{2}(5\sqrt{5}-11),$$

(ii) $$s_9 = 104 + 60\sqrt{3} + 45\sqrt{5} + 26\sqrt{15},$$

(iii) $$s_{1/9} = 104 - 60\sqrt{3} + 45\sqrt{5} - 26\sqrt{15}.$$

Proof. (i) Setting $n = 1/3$ in (i) of Theorem 4.8 and using the equality $s_{1/n} = 1/s_n$ from Theorem 4.2, we have

$$10\sqrt{5} = s_3^2 - s_3^{-2} - 9(s_3 + s_3^{-1}).$$

Now letting $A = s_3 + s_3^{-1}$, we deduce that

$$10\sqrt{5} = \sqrt{A^4 - 4A^2} - 9A.$$

Thus

$$A^4 - 85A^2 - 180\sqrt{5}\,A - 500 = (A+\sqrt{5})(A-5\sqrt{5})(A+2\sqrt{5})^2 = 0.$$

Since $A > 0, A = 5\sqrt{5}$. Thus

$$s_3 = \tfrac{1}{2}(5\sqrt{5} + \sqrt{125-4}) = \tfrac{1}{2}(5\sqrt{5}+11) \quad \text{and} \quad s_{1/3} = \tfrac{1}{2}(5\sqrt{5}-11)$$

since $s_{1/3} = 1/s_3$.

(ii) and (iii). Let $n = 1$ in (i) of Theorem 4.8 and use the value $s_1 = 1$ to find that

$$5\sqrt{5}(s_9^{1/2} + s_9^{-1/2}) = s_9 - s_9^{-1} - 9(s_9^{1/2} + s_9^{-1/2}).$$

Since $s_9^{1/2} + s_9^{-1/2} > 0$, by dividing both sides by $s_9^{1/2} + s_9^{-1/2}$, we have

$$5\sqrt{5} + 9 = s_9^{1/2} - s_9^{-1/2}.$$

So

$$\begin{aligned} s_9 &= \tfrac{1}{4}(9 + 5\sqrt{5} + \sqrt{210 + 90\sqrt{5}})^2 = \tfrac{1}{4}\{9 + 5\sqrt{5} + \sqrt{15}(3+\sqrt{5})\}^2 \\ &= \tfrac{1}{4}(9 + 5\sqrt{3} + 5\sqrt{5} + 3\sqrt{15})^2 = 104 + 60\sqrt{3} + 45\sqrt{5} + 26\sqrt{15} \end{aligned}$$

and, since $s_{1/9} = 1/s_9$,

$$s_{1/9} = 104 - 60\sqrt{3} + 45\sqrt{5} - 26\sqrt{15}. \ \blacksquare$$

COROLLARY 4.10. *We have*

(i) $$R^5(e^{-2\pi\sqrt{3/5}}) = \tfrac{1}{4}\{-147 - 55\sqrt{5} + \sqrt{1470(25+11\sqrt{5})}\},$$

(ii) $$R^5(e^{-2\pi/\sqrt{15}}) = \tfrac{1}{4}\{-147 + 55\sqrt{5} + \sqrt{1470(25-11\sqrt{5})}\},$$

(iii) $$R^5(e^{-6\pi/\sqrt{5}}) = \sqrt{a^2+1} - a, \quad \textit{where} \quad 2a = 5\sqrt{5}\,s_9 + 11,$$

(iv) $$R^5(e^{-2\pi/(3\sqrt{5})}) = \sqrt{a^2+1} - a, \quad \textit{where} \quad 2a = 5\sqrt{5}\,s_{1/9} + 11,$$

where $s_{1/9}$ and s_9 are given in Theorem 4.9.

Proof. These results follow from Theorems 4.1 and 4.9. $\blacksquare$

THEOREM 4.11. *We have*

(i) $$t_3 = \frac{\sqrt{5}+1}{2}, \qquad t_{1/3} = \frac{\sqrt{5}-1}{2},$$

(ii) $$t_9 = 104 + 60\sqrt{3} - 45\sqrt{5} - 26\sqrt{15},$$

(iii) $$t_{1/9} = 104 - 60\sqrt{3} - 45\sqrt{5} + 26\sqrt{15}.$$

Proof. (i) Letting $n = 1/3$ in Theorem 4.8(ii) and using the relation $t_{1/3} = 1/t_3$, we find that

$$10\sqrt{5} = t_3^2 - t_3^{-2} + 9(t_3 + t_3^{-1}).$$

If $B = t_3 + t_3^{-1}$, then $10\sqrt{5} = B\sqrt{B^2-4} + 9B$. Hence

$$B^4 - 85B^2 + 180\sqrt{5}\,B - 500 = (B - \sqrt{5})(B + 5\sqrt{5})(B - 2\sqrt{5})^2 = 0.$$

Since t_n is increasing and positive, and $t_5 < 3$ (we will see this later in Theorem 4.16), $B = \sqrt{5}$. Therefore $t_3 = (\sqrt{5}+1)/2$ and $t_{1/3} = (\sqrt{5}-1)/2$ since $t_{1/3} = 1/t_3$.

(ii) and (iii). Letting $n = 1$ in Theorem 4.8(ii) and recalling that $t_1 = 1$, we deduce that

$$5\sqrt{5}(t_9^{1/2} + t_9^{-1/2}) = t_9 - t_9^{-1} + 9(t_9^{1/2} + t_9^{-1/2}).$$

By dividing both sides by $t_9^{1/2} + t_9^{-1/2}$, we find that

$$t_9^{1/2} - t_9^{-1/2} = 5\sqrt{5} - 9.$$

Hence

$$\begin{aligned} t_9 &= \tfrac{1}{4}(5\sqrt{5} - 9 + \sqrt{210 - 90\sqrt{5}})^2 = \tfrac{1}{4}(5\sqrt{5} - 9 + 3\sqrt{15} - 5\sqrt{3})^2 \\ &= 104 + 60\sqrt{3} - 45\sqrt{5} - 26\sqrt{15} \end{aligned}$$

and, since $t_{1/9} = 1/t_9$,

$$t_{1/9} = 104 - 60\sqrt{3} - 45\sqrt{5} + 26\sqrt{15}. \ \blacksquare$$

COROLLARY 4.12. *We have*

(i) $$S^5(e^{-\pi\sqrt{3/5}}) = \tfrac{1}{4}\{-5\sqrt{5} - 3 + \sqrt{30(5+\sqrt{5})}\},$$

(ii) $$S^5(e^{-\pi/\sqrt{15}}) = \tfrac{1}{4}\{5\sqrt{5} - 3 + \sqrt{30(5-\sqrt{5})}\},$$

(iii) $$S^5(e^{-3\pi/\sqrt{5}}) = \sqrt{b^2+1} - b, \quad \textit{where} \quad 2b = 5\sqrt{5}\,t_9 - 11,$$

(iv) $$S^5(e^{-\pi/(3\sqrt{5})}) = \sqrt{b^2+1} - b, \quad \textit{where} \quad 2b = 5\sqrt{5}\,t_{1/9} - 11,$$

where $t_{1/9}$ and t_9 are given in Theorem 4.11.

Proof. These results follow from Theorems 4.1 and 4.11. $\blacksquare$

Theorem 4.13. *We have*

(i) $$5\{(s_n s_{25n})^{1/3}+(s_n s_{25n})^{-1/3}\}+5\sqrt{5}\{(s_n s_{25n})^{1/6}+(s_n s_{25n})^{-1/6}\} = \left(\frac{s_{25n}}{s_n}\right)^{1/2}-15,$$

(ii) $$5\{(t_n t_{25n})^{1/3}+(t_n t_{25n})^{-1/3}\}-5\sqrt{5}\{(t_n t_{25n})^{1/6}+(t_n t_{25n})^{-1/6}\} = \left(\frac{t_{25n}}{t_n}\right)^{1/2}-15.$$

Proof. (i) This follows from Theorem 2.3 and the definition of s_n in Theorem 4.1.

(ii) By replacing q by $-q$ in Theorem 2.3 and using the definition of t_n in Theorem 4.1, we complete the proof. ∎

Remark 11. By Theorem 4.13, if we know s_n and t_n, then we can find s_{25n} or $s_{n/25}$ and t_{25n} or $t_{n/25}$, respectively, which implies that if we know $R^5(e^{-2\pi\sqrt{n/5}})$ and $S^5(e^{-\pi\sqrt{n/5}})$, then we can find $R^5(e^{-2\sqrt{5n}\pi})$ or $R^5(e^{-2\pi\sqrt{n}/(5\sqrt{5})})$ and $S^5(e^{-\sqrt{5n}\pi})$ or $S^5(e^{-\pi\sqrt{n}/(5\sqrt{5})})$, respectively.

Theorem 4.14. *We have*

$$s_5 = 25+10\sqrt{5} \quad \textit{and} \quad s_{1/5}=\frac{1}{25}(5-2\sqrt{5}).$$

Proof. Letting $n=1/5$ in Theorem 4.13(i) and using the equality $s_{1/5}=1/s_5$, we find that $10+10\sqrt{5}=s_5-15$, and so $s_5=25+10\sqrt{5}$. Furthermore, $s_{1/5}=1/s_5=\frac{1}{25}(5-2\sqrt{5})$. ∎

Corollary 4.15. *We have*

$$R^5(e^{-2\pi/5})=\sqrt{\frac{45+9\sqrt{5}}{2}}-\frac{\sqrt{5}+9}{2}.$$

Proof. This follows from Theorem 4.1 with $s_{1/5}=\frac{1}{25}(5-2\sqrt{5})$. ∎

Theorem 4.16. *We have*

$$t_5 = 25-10\sqrt{5} \quad \textit{and} \quad t_{1/5}=\frac{1}{25}(5+2\sqrt{5}).$$

Proof. Letting $n=1/5$ in (ii) of Theorem 4.13 and using the equality $t_{1/5}=1/t_5$, we find that $10-10\sqrt{5}=t_5-15$, and so $t_5=25-10\sqrt{5}$ and $t_{1/5}=1/t_5=\frac{1}{25}(5+2\sqrt{5})$. ∎

Corollary 4.17. *We have*

$$S^5(e^{-\pi/5})=\tfrac{3}{2}\sqrt{2(5-\sqrt{5})}-\tfrac{1}{2}(-9+\sqrt{5}).$$

Proof. By Theorem 4.1 with $t_{1/5}=\frac{1}{25}(5+2\sqrt{5})$. ∎

THEOREM 4.18. *We have*

(i) $$5\sqrt{5}\{(s_ns_{49n})^{1/2}+(s_ns_{49n})^{-1/2}\}$$
$$=\left(\frac{s_{49n}}{s_n}\right)^{2/3}-\left(\frac{s_n}{s_{49n}}\right)^{2/3}-7\left\{\left(\frac{s_{49n}}{s_n}\right)^{1/2}+\left(\frac{s_n}{s_{49n}}\right)^{1/2}\right\}$$
$$+7\left\{\left(\frac{s_{49n}}{s_n}\right)^{1/3}-\left(\frac{s_n}{s_{49n}}\right)^{1/3}\right\}+14\left\{\left(\frac{s_{49n}}{s_n}\right)^{1/6}+\left(\frac{s_n}{s_{49n}}\right)^{1/6}\right\}$$

and

(ii) $$5\sqrt{5}\{(t_nt_{49n})^{1/2}+(t_nt_{49n})^{-1/2}\}$$
$$=\left(\frac{t_{49n}}{t_n}\right)^{2/3}-\left(\frac{t_n}{t_{49n}}\right)^{2/3}+7\left\{\left(\frac{t_{49n}}{t_n}\right)^{1/2}+\left(\frac{t_n}{t_{49n}}\right)^{1/2}\right\}$$
$$+7\left\{\left(\frac{t_{49n}}{t_n}\right)^{1/3}-\left(\frac{t_n}{t_{49n}}\right)^{1/3}\right\}-14\left\{\left(\frac{t_{49n}}{t_n}\right)^{1/6}+\left(\frac{t_n}{t_{49n}}\right)^{1/6}\right\}.$$

Proof. These equations follow from Theorem 2.4 and the definitions of s_n and t_n in Theorem 4.1, respectively. ∎

REMARK 12. By Theorem 4.18, if we know s_n and t_n, then we can find s_{49n} or $s_{n/49}$ and t_{49n} or $t_{n/49}$, respectively, which implies that if we know $R^5(e^{-2\pi\sqrt{n/5}})$ and $S^5(e^{-\pi\sqrt{n/5}})$, then we can find $R^5(e^{-14\pi\sqrt{n/5}})$ or $R^5(e^{-2\pi\sqrt{n}/(7\sqrt{5})})$ and $S^5(e^{-7\pi\sqrt{n/5}})$ or $S^5(e^{-\pi\sqrt{n}/(7\sqrt{5})})$, respectively.

THEOREM 4.19. *We have*
$$s_7=\frac{1}{216}(3\sqrt{5}+a+b+\sqrt{57+6\sqrt{5}(a+b)+a^2+b^2})^3,$$
$$s_{1/7}=\frac{1}{216}(-3\sqrt{5}-a-b+\sqrt{57+6\sqrt{5}(a+b)+a^2+b^2})^3,$$
where $a=(54\sqrt{5}-6\sqrt{21})^{1/3}$ *and* $b=(54\sqrt{5}+6\sqrt{21})^{1/3}$.

Proof. Letting $n=1/7$ in (i) of Theorem 4.18, we have
$$10\sqrt{5}=s_7^{4/3}-s_7^{-4/3}-7(s_7+s_7^{-1})$$
$$+7(s_7^{2/3}-s_7^{-2/3})+14(s_7^{1/3}+s_7^{-1/3}).$$
Let $A=s_7^{1/3}+s_7^{-1/3}$. Then
$$10\sqrt{5}=(A^3-2A)\sqrt{A^2-4}-7(A^3-3A)+7A\sqrt{A^2-4}+14A.$$
Hence
$$7A^3-35A+10\sqrt{5}=(A^3+5A)\sqrt{A^2-4}.$$

Now by squaring both sides we have

$$A^8 - 43A^6 + 475A^4 - 140\sqrt{5}\,A^3 - 1325A^2 + 700\sqrt{5} - 500$$
$$= (A-\sqrt{5})(A^2-5)(A+2\sqrt{5})^2(A^3 - 3\sqrt{5}\,A^2 + 7A - \sqrt{5}) = 0.$$

Since $A > 0$ and $A > s_5^{1/3} > \sqrt{5}$ by Theorem 4.14, A satisfies the equation $A^3 - 3\sqrt{5}\,A^2 + 7A - \sqrt{5} = 0$. Now since A is real-valued, we have

$$A = \sqrt{5} + \tfrac{1}{3}a + \tfrac{1}{3}b,$$

where $a = (54\sqrt{5} - 6\sqrt{21})^{1/3}$ and $b = (54\sqrt{5} + 6\sqrt{21})^{1/3}$. Since $s_{1/7} = 1/s_7$, it follows that $s_{1/7}$ and $1/s_7$ are the solutions of the equation

$$x^2 - \left(\sqrt{5} + \tfrac{1}{3}a + \tfrac{1}{3}b\right)x + 1 = 0.$$

Hence we deduce the results by using $s_7 > s_{1/7}$. ■

THEOREM 4.20. *We have*

$$t_7 = 2 + \sqrt{5} \quad \textit{and} \quad t_{1/7} = \sqrt{5} - 2.$$

Proof. Letting $n = 1/7$ in (ii) of Theorem 4.18, we have

$$10\sqrt{5} = t_7^{4/3} - t_7^{-4/3} + 7(t_7 + t_7^{-1})$$
$$+ 7(t_7^{2/3} - t_7^{-2/3}) - 14(t_7^{1/3} + t_7^{-1/3}).$$

Put $B = t_7^{1/3} + t_7^{-1/3}$. Then, by the same argument as in the proof of Theorem 4.19,

$$(B+\sqrt{5})(B^2-5)(B-2\sqrt{5})^2(B^3 + 3\sqrt{5}\,B^2 + 7B + \sqrt{5}) = 0.$$

Since B is positive and t_n is increasing in n, $B = t_7^{1/3} + t_7^{-1/3} < 2t_9^{1/3} < 2\sqrt{5}$ from Theorem 4.11. Thus we find that $B = \sqrt{5}$. Hence

$$t_7 = \left(\frac{\sqrt{5}+1}{2}\right)^3 = 2 + \sqrt{5} \quad \text{and} \quad t_{1/7} = \left(\frac{\sqrt{5}-1}{2}\right)^3 = \sqrt{5} - 2. \blacksquare$$

COROLLARY 4.21. *We have*

(i) $$R^5(e^{-2\pi\sqrt{7/5}}) = \sqrt{a^2+1} - a, \quad \textit{where} \quad 2a = 5\sqrt{5}\,s_7 + 11$$

and s_7 *is given in Theorem* 4.19,

(ii) $$R^5(e^{-2\pi/\sqrt{35}}) = \sqrt{a^2+1} - a, \quad \textit{where} \quad 2a = 5\sqrt{5}\,s_{1/7} + 11$$

and $s_{1/7}$ *is given in Theorem* 4.19,

(iii) $$S^5(e^{-\pi\sqrt{7/5}}) = \sqrt{35(5+2\sqrt{5})} - (7 + 5\sqrt{5}),$$

(iv) $$S^5(e^{-\pi/\sqrt{35}}) = \sqrt{35(5-2\sqrt{5})} - (7 - 5\sqrt{5}).$$

Proof. The results follow from Theorem 4.1 with the values in Theorems 4.19 and 4.20. ■

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