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An arithmetic criterion for the values of the exponential function

by

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1. Introduction. The motivation for the present work comes from the following conjecture due to S. Schanuel (see [1], Historical notes of Chapter III):

CONJECTURE 1. *Let l be a positive integer and let $y_1, \ldots, y_l \in \mathbb{C}$ be linearly independent over $\mathbb{Q}$. Then*

$$\mathrm{tr.deg}_{\mathbb{Q}}\mathbb{Q}(y_1, \ldots, y_l, e^{y_1}, \ldots, e^{y_l}) \geq l.$$

This conjecture is known to be true when $l = 1$ (Hermite–Lindemann theorem) and when $y_1, \ldots, y_l \in \overline{\mathbb{Q}}$ (Lindemann–Weierstrass theorem), where $\overline{\mathbb{Q}}$ denotes the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. There are other evidences for this conjecture, but the general case is open, including the algebraic independence of e and π (take $y_1 = 1$ and $y_2 = \pi i$).

Here, we will show that this conjecture is equivalent to the following algebraic statement where the symbol $\mathcal{D}$ stands for the derivation:

$$\mathcal{D} = \frac{\partial}{\partial X_0} + X_1 \frac{\partial}{\partial X_1}$$

in the field $\mathbb{C}(X_0, X_1)$, and where the *height* of a polynomial $P \in \mathbb{C}[X_0, X_1]$ is defined as the maximum of the absolute values of its coefficients.

CONJECTURE 2. *Let l be a positive integer, let $y_1, \ldots, y_l \in \mathbb{C}$ be linearly independent over $\mathbb{Q}$ and let $\alpha_1, \ldots, \alpha_l \in \mathbb{C}^\times$. Moreover, let s_0, s_1, t_0, t_1, u be positive numbers satisfying*

$$(1)\quad \max\{1, t_0, 2t_1\} < \min\{s_0, 2s_1\}, \quad \max\{s_0, s_1+t_1\} < u < \tfrac{1}{2}(1+t_0+t_1).$$

Assume that, for any sufficiently large positive integer N, there exists a nonzero polynomial $P_N \in \mathbb{Z}[X_0, X_1]$ with partial degree $\leq N^{t_0}$ in X_0, partial

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degree $\le N^{t_1}$ *in* X_1 *and height* $\le e^N$ *which satisfies*

$$\Big|(\mathcal{D}^k P_N)\Big(\sum_{j=1}^{l} m_j y_j, \prod_{j=1}^{l} \alpha_j^{m_j}\Big)\Big| \le \exp(-N^u),$$

for any integers $k, m_1, \ldots, m_l \in \mathbb{N}$ *with* $k \le N^{s_0}$ *and* $\max\{m_1, \ldots, m_l\} \le N^{s_1}$. *Then* $\mathrm{tr.deg}_{\mathbb{Q}}\mathbb{Q}(y_1, \ldots, y_l, \alpha_1, \ldots, \alpha_l) \ge l$.

Note that this arithmetic statement is similar to the present criteria of algebraic independence (see for example [2] and [3]). It suggests, we hope, a reasonable approach toward Schanuel's conjecture. We will show that if it is true for some positive integer l and some choice of parameters s_0, s_1, t_0, t_1, u satisfying (1), then Schanuel's conjecture is true for this value of l. This follows from a general construction of an auxiliary function due to Michel Waldschmidt (Theorem 3.1 of [4]). Conversely, we will show that, if Conjecture 1 is true for some positive integer l, then Conjecture 2 is also true for the same value of l and for any choice of parameters satisfying (1). In particular, Conjecture 2 is true in the case $l = 1$. Moreover, if, for fixed l, Conjecture 2 is true for at least one choice of parameters satisfying (1), then it is true for all of them. We prove the reverse implication as a consequence of the following criterion concerning the values of the exponential function.

THEOREM 1. *Let* $(y, \alpha) \in \mathbb{C}\times\mathbb{C}^\times$, *and let* s_0, s_1, t_0, t_1, u *be positive numbers satisfying the inequalities* (1). *Then the following conditions are equivalent*:

(a) *there exists an integer* $d \ge 1$ *such that* $\alpha^d = e^{dy}$;

(b) *for any sufficiently large positive integer* N, *there exists a nonzero polynomial* $Q_N \in \mathbb{Z}[X_0, X_1]$ *with partial degree* $\le N^{t_0}$ *in* X_0, *partial degree* $\le N^{t_1}$ *in* X_1 *and height* $\le e^N$ *such that*

$$|(\mathcal{D}^k Q_N)(my, \alpha^m)| \le \exp(-N^u)$$

for any $k, m \in \mathbb{N}$ *with* $k \le N^{s_0}$ *and* $m \le N^{s_1}$.

Again the proof that (a) implies (b) follows from Waldschmidt's construction. To prove the reverse implication, we establish a new interpolation lemma for holomorphic functions $F(z_1, z_2)$ of two complex variables. This interpolation lemma takes into account not only the values of F on a subgroup of $\mathbb{C}^2$ of rank 2, but also the values of its derivatives in the direction of a nonzero point $\mathbf{w} = (w_1, w_2)$ of $\mathbb{C}^2$. The corresponding derivation is denoted by

$$D_{\mathbf{w}} = w_1 \frac{\partial}{\partial z_1} + w_2 \frac{\partial}{\partial z_2}.$$

To state this result, we need to fix additional notation. We define

$$B(0, R) = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \le R, |z_2| \le R\}$$

for any $R > 0$. For a continuous function $F : B(0,R) \to \mathbb{C}$, we put

$$|F|_R = \sup\{|F(z_1, z_2)| : |z_1| = |z_2| = R\}.$$

By the maximum modulus principle, when F is holomorphic in the interior of $B(0,R)$, this coincides with the supremum of $|F|$ on $B(0,R)$.

THEOREM 2. *Let $\{\mathbf{u}, \mathbf{w}\}$ be a basis of $\mathbb{C}^2$, let $\mathbf{v} \in \mathbb{C}^2$ and let a be the complex number for which $\mathbf{v} - a\mathbf{u} \in \mathbb{C}\mathbf{w}$. Then there exists a constant $c \geq 1$ which depends only on $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, and which satisfies $\mathbf{u}, \mathbf{v} \in B(0,c)$ and the following property: For any integer $N \geq 1$ with*

$$\min\{|m + na| : m, n \in \mathbb{Z},\ 0 < \max\{|m|, |n|\} < N\} \geq 2^{-N}, \tag{2}$$

for any pair of real numbers r, R with $R \geq 2r$ and $r \geq cN$, and for any continuous function $F : B(0,R) \to \mathbb{C}$ which is holomorphic inside $B(0,R)$, we have

$$|F|_r \leq \left(\frac{cr}{N}\right)^{N^2} \max_{\substack{0 \leq k < N^2 \\ 0 \leq m,n < N}} \left\{\frac{1}{k!}|D_{\mathbf{w}}^k F(m\mathbf{u} + n\mathbf{v})| N^k\right\} + \left(\frac{cr}{R}\right)^{N^2} |F|_R.$$

The condition (2) is satisfied for infinitely many values of N if $a \notin \mathbb{Q}$ (see Lemma 4 below). It can be shown that such a condition is necessary in the above result. However, for a function F satisfying $D_{\mathbf{w}}^k F(m\mathbf{u} + n\mathbf{v}) = 0$ for $0 \leq k < N^2$ and $0 \leq m, n < N$, Theorem 2 gives

$$|F|_r \leq \left(\frac{cr}{R}\right)^{N^2} |F|_R,$$

and it is not clear that a Diophantine condition like (2) is needed any more. We refer the reader to Chapter 7 of [5] for related conjectures and results concerning the growth of holomorphic functions vanishing at points of finitely generated subgroups of $\mathbb{C}^n$.

The organization of this paper is as follows. In Section 2 below, we establish a first interpolation formula. The proof of Theorem 2 follows in Section 3, using this formula. Finally, the proof of Theorem 1 and the equivalence between the two conjectures are established in Sections 4 and 5 respectively.

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2. A first interpolation formula. We fix a point $(a, b) \in \mathbb{C}^2$ and a positive integer N. We put $L = N^2$ and we assume that a satisfies the condition (2) in the statement of Theorem 2, that is, $|m + na| \geq 2^{-N}$ for any pair $(m, n) \in \mathbb{Z}^2$ with $0 < \max\{|m|, |n|\} < N$.

For each triple of integers (m, n, k) with $0 \le m, n < N$ and $0 \le k < L$, we define

$$g_{m,n,k}(z, w) = (w - nb)^k \prod_{(m',n') \neq (m,n)} \frac{z - m' - n'a}{m + na - m' - n'a}$$

where the product on the right hand side is over all pairs of integers (m', n') with $0 \le m', n' < N$ and $(m', n') \neq (m, n)$. By construction, these polynomials have the following interpolation property:

$$\left(\frac{\partial}{\partial w}\right)^{k'} g_{m,n,k}(m' + n'a, n'b) = \begin{cases} k! & \text{if } (m', n', k') = (m, n, k), \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

They also satisfy:

Lemma 1. *For each triple* (m, n, k) *as above, we have*

$$|g_{m,n,k}|_N \le c_1^L ((1 + |b|)N)^k$$

where $c_1 = 8e(2 + |a|)$.

Proof. Fix an integer s with $0 \le s < N$ and consider the set

$$I_s = \{m + na - m' - sa : m' = 0, 1, \ldots, N - 1\}.$$

The elements of I_s are N complex numbers which differ from one another by an integer. Let x_0 be an element of I_s whose real part has minimal absolute value. It is possible to order the remaining elements $x_1, \ldots, x_{N-1}$ of I_s so that the absolute values of their real parts are respectively bounded from below by $1/2, 2/2, \ldots, (N - 1)/2$. Since $|x_0| \ge 2^{-N}$ if $x_0 \neq 0$, this implies

$$\prod_{x \in I_s \setminus \{0\}} |x| \ge 2^{-N} \frac{(N - 1)!}{2^{N-1}} \ge 2^{-N} e^{-N} \left(\frac{N}{2}\right)^{N-1} \ge \left(\frac{N}{8e}\right)^N.$$

So, for the denominator of $g_{m,n,k}$, we get

$$\prod_{(m',n') \neq (m,n)} |m + na - m' - n'a| = \prod_{s=0}^{N-1} \prod_{x \in I_s \setminus \{0\}} |x| \ge \left(\frac{N}{8e}\right)^L.$$

Using this lower bound, we deduce

$$\begin{aligned} |g_{m,n,k}|_N &\le (N + n|b|)^k (2N + N|a|)^{L-1} \left(\frac{8e}{N}\right)^L \\ &\le (8e(2 + |a|))^L ((1 + |b|)N)^k. \end{aligned}$$

Lemma 2. *For each* $(r, s) \in \mathbb{N}^2$, *define*

$$f_{r,s}(z, w) = \sum_{\substack{0 \le m,n < N \\ 0 \le k < L}} g_{m,n,k}(z, w)(m + na)^r \binom{s}{k} (nb)^{s-k}.$$

Then $f_{r,s}(z,w) = z^r w^s$ *whenever* $\max\{r,s\} < L$. *Moreover*, *for any* $(r,s) \in \mathbb{N}^2$,

$$|f_{r,s}|_N \le (2c_1)^L((1+|a|+2|b|)N)^{r+s}.$$

Proof. For the first assertion, consider the vector subspace V of $\mathbb{C}[z,w]$ consisting of all polynomials of partial degree $< L$ in z and partial degree $< L$ in w. By virtue of (3), the L^2 functions $g_{m,n,k}$ form a basis of V with the dual basis given by the linear functionals

$$\varphi_{m,n,k}(g) = (1/k!)(\partial/\partial w)^k g(m+na, nb).$$

When $\max\{r,s\} < L$, the polynomial $z^r w^s$ belongs to V and its image under $\varphi_{m,n,k}$ is the same as that of $f_{r,s}$ for $0 \le m,n < N$ and $0 \le k < L$. So, the two polynomials must be equal. For the second assertion, we use Lemma 1. It gives

$$\begin{aligned} |f_{r,s}|_N &\le \sum_{\substack{0\le m,n<N\\ 0\le k<L}} |g_{m,n,k}|_N (m+n|a|)^r \binom{s}{k} (n|b|)^{s-k} \\ &\le c_1^L \sum_{0\le m,n<N} (m+n|a|)^r((1+|b|)N+n|b|)^s \\ &\le c_1^L N^2((1+|a|)N)^r((1+2|b|)N)^s. \end{aligned}$$

The conclusion follows if we use $N^2 = L \le 2^L$.

We are now ready to prove:

PROPOSITION 1. *Let* (a,b), N *and* L *be as above. Let* $R \ge 2(1+|a|+2|b|)N$, *and let* $F(z,w)$ *be a complex-valued function which is continuous on* $B(0,R)$ *and holomorphic inside. Put*

$$A = \max\left\{\frac{1}{k!}\left|\frac{\partial^k F}{\partial w^k}(m+na, nb)\right| N^k : 0 \le m,n < N \text{ and } 0 \le k < L\right\}.$$

Then

$$|F|_N \le c_2^L A + (c_2 N/R)^L |F|_R \quad \text{with } c_2 = 8(1+2c_1)(1+|a|+2|b|).$$

Proof. Define $\mathbf{T} = \{(\xi,\zeta) \in \mathbb{C}^2 : |\xi| = |\zeta| = R\}$. For any continuous function $G : \mathbf{T} \to \mathbb{C}$, we put

$$\langle F, G\rangle = \frac{1}{(2\pi i)^2} \int_{\mathbf{T}} F(\xi,\zeta)G(\xi,\zeta)\, d\xi\, d\zeta.$$

This integral satisfies $|\langle F,G\rangle| \le R^2 |F|_R |G|_R$ where $|G|_R$ denotes the supremum of $|G|$ on the torus $\mathbf{T}$. On the other hand, Cauchy's integral formulas give

$$F(z,w) = \left\langle F, \frac{1}{(\xi - z)(\zeta - w)}\right\rangle$$

for any point (z,w) in the interior of $B(0,R)$. For a triple (m,n,k) of integers with $0 \le m,n < N$ and $0 \le k < L$, they also give

$$\frac{1}{k!}\cdot\frac{\partial^k F}{\partial w^k}(m+na,nb) = \langle F, D_{m,n,k}\rangle$$

where

$$D_{m,n,k}(\xi,\zeta) := \frac{1}{(\xi-m-na)(\zeta-nb)^{k+1}}$$

since $(m+na,nb)$ belongs to the interior of $B(0,R)$. We claim that

$$\frac{1}{(\xi-z)(\zeta-w)} = \sum_{\substack{0\le m,n<N\\0\le k<L}} g_{m,n,k}(z,w)D_{m,n,k}(\xi,\zeta) + U(z,w,\xi,\zeta) \tag{4}$$

where the remainder U satisfies $|U(z,w,\xi,\zeta)| \le (c_2N)^L R^{-L-2}$ for any $(z,w,\xi,\zeta) \in B(0,N)\times\mathbf{T}$. If we take this for granted, then, multiplying both sides of (4) by $F(\xi,\zeta)$ and integrating over $\mathbf{T}$, we get, by linearity of the integral,

$$|F|_N \le \sum_{\substack{0\le m,n<N\\0\le k<L}} |g_{m,n,k}|_N \left|\frac{1}{k!}\cdot\frac{\partial^k F}{\partial w^k}(m+na,nb)\right| + \left(\frac{c_2N}{R}\right)^L |F|_R.$$

Using Lemma 1, we deduce

$$|F|_N \le c_1^L \sum_{\substack{0\le m,n<N\\0\le k<L}} (1+|b|)^k A + \left(\frac{c_2N}{R}\right)^L |F|_R \le c_2^L A + \left(\frac{c_2N}{R}\right)^L |F|_R$$

and the proposition is proved.

To prove the claim, we use the developments

$$\frac{1}{(\xi-z)(\zeta-w)} = \sum_{r,s\ge0}\frac{z^r w^s}{\xi^{r+1}\zeta^{s+1}}$$

and

$$D_{m,n,k}(\xi,\zeta) = \sum_{r,s\ge0}\frac{(m+na)^r\binom{s}{k}(nb)^{s-k}}{\xi^{r+1}\zeta^{s+1}}$$

which converge absolutely and represent these functions whenever $(z,w)\in B(0,N)$ and $(\xi,\zeta)\in\mathbf{T}$. Using Lemma 2, we deduce that the function U defined by (4) is given by

$$U(z,w,\xi,\zeta) = \sum_{\max\{r,s\}\ge L}\frac{z^r w^s - f_{r,s}(z,w)}{\xi^{r+1}\zeta^{s+1}}$$

for $(z,w,\xi,\zeta)\in B(0,N)\times\mathbf{T}$. For those values of (z,w,ξ,ζ), we get

$$\begin{aligned}|U(z,w,\xi,\zeta)| &\le \sum_{\max\{r,s\}\ge L}\frac{N^{r+s}+|f_{r,s}|_N}{R^{r+s+2}}\\ &\le \biggl(\frac{1+(2c_1)^L}{R^2}\biggr)\sum_{\max\{r,s\}\ge L}\biggl(\frac{(1+|a|+2|b|)N}{R}\biggr)^{r+s}\\ &\le 8\biggl(\frac{1+(2c_1)^L}{R^2}\biggr)\biggl(\frac{(1+|a|+2|b|)N}{R}\biggr)^{L},\end{aligned}$$

since $(1+|a|+2|b|)N/R\le 1/2$. This proves the claim and thus completes the proof of the proposition.

3. Proof of Theorem 2.

We first prove:

LEMMA 3. *Let L be a positive integer, let r_0, r and R be positive numbers with $r\ge r_0$ and $R\ge 2r$, and let $F(z,w)$ be a complex-valued function which is continuous on $B(0,R)$ and holomorphic inside. Then*

$$|F|_r\le\binom{L+1}{2}\biggl(\frac{r}{r_0}\biggr)^L|F|_{r_0}+(2L+4)\biggl(\frac{r}{R}\biggr)^L|F|_R.$$

Proof. Since $r<R$, the Taylor expansion of F around $(0,0)$ converges normally in $B(0,r)$ and we get

$$|F|_r\le\sum_{j,k\ge 0}\frac{1}{j!k!}\biggl|\frac{\partial^{j+k}F}{\partial z^j\partial w^k}(0,0)\biggr|r^{j+k}.$$

Using Cauchy's inequalities, we deduce

$$|F|_r\le\sum_{j+k<L}\biggl(\frac{r}{r_0}\biggr)^{j+k}|F|_{r_0}+\sum_{j+k\ge L}\biggl(\frac{r}{R}\biggr)^{j+k}|F|_R.$$

The conclusion follows using $\sum_{j+k\ge L}2^{L-j-k}=2L+4$. Note that a sharper inequality with the factor $2L+4$ replaced by $\sqrt{L}+1$ follows from Lemma 3.4 of [4].

Proof of Theorem 2. Let $T:\mathbb{C}^2\to\mathbb{C}^2$ be the linear map for which $T(1,0)=\mathbf{u}$ and $T(0,1)=\mathbf{w}$, and let (a,b) be the point of $\mathbb{C}^2$ for which $T(a,b)=\mathbf{v}$. Let c_3, c_4 be positive constants such that $c_3\|\mathbf{z}\|\le\|T(\mathbf{z})\|\le c_4\|\mathbf{z}\|$ for any $\mathbf{z}\in\mathbb{C}^2$, where $\|\ \|$ denotes the maximum norm. Assume that N is a positive integer satisfying the condition (2) in the statement of Theorem 2. Choose real numbers r and R with

$$R\ge 2r\quad\text{and}\quad r\ge\max\{c_3,(1+|a|+2|b|)c_4\}N.$$

Choose also a continuous function $F : B(0,R) \to \mathbb{C}$ which is holomorphic in the interior of $B(0,R)$. Put $L = N^2$ and $G = F \circ T$. Then G is defined and continuous on $B(0, R/c_4)$. It is holomorphic in the interior of this ball and satisfies

$$(5) \qquad |F|_{c_3 N} \le |G|_N \quad \text{and} \quad |G|_{R/c_4} \le |F|_R.$$

On the other hand, $T(m+na, nb) = m\mathbf{u} + n\mathbf{v}$ for any $(m,n) \in \mathbb{Z}^2$. Since $T(0,1) = \mathbf{w}$, this implies

$$(6) \qquad \frac{\partial^k G}{\partial w^k}(m+na, nb) = D^k_{\mathbf{w}} F(m\mathbf{u} + n\mathbf{v}) \qquad \text{for any integer } k \ge 0.$$

Let B be the maximum of the numbers $|D^k_{\mathbf{w}} F(m\mathbf{u}+n\mathbf{v})| N^k / k!$ with $0 \le k < L$ and $0 \le m, n < N$. For any choice of integers k, m, n in the same intervals, the relation (6) implies

$$\frac{1}{k!}\left|\frac{\partial^k G}{\partial w^k}(m+na, nb)\right| N^k \le B.$$

By Proposition 1, we deduce

$$|G|_N \le c_2^L B + \left(\frac{c_2 c_4 N}{R}\right)^L |G|_{R/c_4}$$

where c_2 depends only on $|a|$ and $|b|$. Combining this with (5) and applying Lemma 3 with $r_0 = c_3 N$, we deduce

$$|F|_r \le \binom{L+1}{2}\left(\frac{r}{c_3 N}\right)^L \left[c_2^L B + \left(\frac{c_2 c_4 N}{R}\right)^L |F|_R\right] + (2L+4)\left(\frac{r}{R}\right)^L |F|_R,$$

which proves Theorem 2 for a suitable constant c depending only on c_2, c_3, c_4.

4. Proof of Theorem 1. We will need the following special case of Theorem 3.1 of [4]:

THEOREM 3 (M. Waldschmidt). *Let Δ, r, T_0, T_1, U be positive numbers. Assume $U \ge 3$,*

$$\log\left((T_0+1)(T_1+1)\right) + \Delta + T_0 \log(er) + erT_1 \le U$$

and $(8U)^2 \le \Delta T_0 T_1$. Then there exists a nonzero polynomial $P \in \mathbb{Z}[X_0, X_1]$ with partial degree $\le T_0$ in X_0, partial degree $\le T_1$ in X_1 and height $\le e^{\Delta}$ such that the function $f(z) = P(z, e^z)$ satisfies $|f|_r \le e^{-U}$.

We divide the proof of Theorem 1 into two propositions. Each proves one implication but assumes a weaker condition than (1) on the parameters s_0, s_1, t_0, t_1 and u.

PROPOSITION 2. *Let* $(y,\alpha)\in\mathbb{C}\times\mathbb{C}^{\times}$ *and let* s_0, s_1, t_0, t_1, u *be positive numbers satisfying*

$$\max\{1,s_0,t_0,s_1+t_1\}<u<\tfrac{1}{2}(1+t_0+t_1).$$

Assume that αe^{-y} *is a root of unity. Then, the condition* (b) *of Theorem* 1 *holds for the pair* (y,α).

Proof. Write $\alpha=\zeta e^{y}$ with $\zeta\in\mathbb{C}^{\times}$. By hypothesis, $\zeta^{d}=1$ for some integer $d\ge 1$. Choose ε with

$$0<\varepsilon<\min\bigl\{1,t_0,t_1,\tfrac{1}{5}(1+t_0+t_1-2u)\bigr\}.$$

Then, for any sufficiently large integer N, the conditions of Theorem 3 are satisfied with $\Delta=N^{1-\varepsilon}$, $r=N^{s_1+\varepsilon}$, $T_0=N^{t_0-\varepsilon}$, $T_1=N^{t_1-\varepsilon}$ and $U=N^{u+\varepsilon}$. Fix such an integer N and choose a nonzero polynomial $P_N\in\mathbb{Z}[X_0,X_1]$ with the properties corresponding to this choice of parameters. Then

$$Q_N(X_0,X_1)=\prod_{k=0}^{d-1}P_N(X_0,\zeta^{k}X_1)$$

is also a nonzero polynomial with integral coefficients. If N is sufficiently large, its partial degree in X_j is $\le dT_j\le N^{t_j}$ for $j=0,1$, and its height is $\le ((T_0+1)(T_1+1)e^{\Delta})^{d}\le e^{N}$. We define entire functions $f_{N,k}(z)=P_N(z,\zeta^{k}e^{z})$ for $k=0,\dots,d-1$ and

$$g_N(z)=Q_N(z,e^{z})=\prod_{k=0}^{d-1}f_{N,k}(z).$$

By construction, $|f_{N,0}|_r\le e^{-U}$, while, for $k=1,\dots,d-1$, a direct estimate gives

$$|f_{N,k}|_r\le (T_0+1)(T_1+1)\exp(\Delta+T_0\log(r)+rT_1)\le \exp(N^{u})$$

provided that N is large enough. From these inequalities we deduce, if N is sufficiently large,

$$|g_N|_r\le \exp(-N^{u+\varepsilon}+(d-1)N^{u})\le \exp(-2N^{u}).$$

On the other hand, $g_N(z)=Q_N(z,\zeta^{m}e^{z})$ for any $m\in\mathbb{Z}$ and any $z\in\mathbb{C}$. For fixed m, we deduce

$$\frac{d^{k}g_N}{dz^{k}}(z)=(\mathcal{D}^{k}Q_N)(z,\zeta^{m}e^{z})\quad\text{and so}\quad \frac{d^{k}g_N}{dz^{k}}(my)=(\mathcal{D}^{k}Q_N)(my,\alpha^{m})$$

for any integer $k\ge 0$. Suppose that N is large enough so that $N^{s_1}|y|+1\le r$ and $N^{s_0}\log(N^{s_0})\le N^{u}$. Then, for any pair of integers (k,m) with $0\le k\le N^{s_0}$ and $0\le m\le N^{s_1}$, Cauchy's inequalities give the estimate

$$|(\mathcal{D}^{k}Q_N)(my,\alpha^{m})|=\left|\frac{d^{k}g_N}{dz^{k}}(my)\right|\le k!|g_N|_{|my|+1}\le \exp(N^{u})|g_N|_r.$$

Since $|g_N|_r \le \exp(-2N^u)$ when N is large enough, the sequence of polynomials $(Q_N)_{N\ge N_0}$ has the required properties for a suitable choice of N_0.

For the next proposition, we will need the following fact:

LEMMA 4. *Let a be an irrational complex number. Then there are infinitely many positive integers N such that*

$$\min\{|m+na| : m,n\in\mathbb{Z},\ 0<\max\{|m|,|n|\}<N\} \ge 1/(2N). \tag{7}$$

Proof. Assume on the contrary that, for any integer N larger than some constant N_0, there are integers $m(N)$ and $n(N)$ such that

$$0<\max\{|m(N)|,|n(N)|\}<N \quad\text{and}\quad |m(N)+n(N)a|<1/(2N).$$

For $N>N_0$, these conditions imply $n(N)\ne 0$ and we find

$$\begin{aligned}&|m(N)n(N+1)-m(N+1)n(N)|\\ &\quad\le |m(N)+n(N)a|\cdot|n(N+1)|+|m(N+1)+n(N+1)a|\cdot|n(N)|<1,\end{aligned}$$

and so the integer $m(N)n(N+1)-m(N+1)n(N)$ is zero. This shows that the ratio $m(N)/n(N)$ is a constant $r\in\mathbb{Q}$. Since

$$|r+a| = |m(N)+n(N)a|/|n(N)| < 1/(2N)$$

for any $N>N_0$, we deduce that $a=-r$ in contradiction with the hypothesis $a\notin\mathbb{Q}$.

PROPOSITION 3. *Let $(y,\alpha)\in\mathbb{C}\times\mathbb{C}^\times$, and let s_0, s_1, t_0, t_1, u be positive numbers such that*

$$\max\{1,t_0,2t_1\}<\min\{s_0,2s_1\}<u.$$

Suppose that αe^{-y} is not a root of unity. Then the condition (b) *of Theorem* 1 *does not hold for the pair (y,α).*

Proof. Choose $\lambda\in\mathbb{C}$ such that $e^\lambda=\alpha$. The ratio $a=(\lambda-y)/(2\pi i)$ is by hypothesis an irrational number. Therefore there exist infinitely many positive integers N which satisfy the condition (7) of Lemma 4. Fix such an integer N. Put $s=\min\{s_0/2,s_1\}$, and let M denote the smallest positive integer for which $N\le M^s$. Choose also a nonzero polynomial $Q\in\mathbb{Z}[X_0,X_1]$ with partial degree $\le M^{t_0}$ in X_0, partial degree $\le M^{t_1}$ in X_1 and height $\le e^M$. We will show that, if N is sufficiently large, the number

$$A=\max_{\substack{0\le k\le M^{s_0}\\ 0\le n\le M^{s_1}}} |(\mathcal{D}^kQ)(ny,\alpha^n)|$$

satisfies $A>\exp(-M^u)$. This will prove the proposition.

To this end, we consider the entire function $F:\mathbb{C}^2\to\mathbb{C}$ given by $F(z,w)=Q(z,e^w)$, and the vectors

$$\mathbf{u}=(0,2\pi i),\qquad \mathbf{v}=(y,\lambda),\qquad \mathbf{w}=(1,1).$$

The differential operator $D_{\mathbf{w}} = \partial/\partial z + \partial/\partial w$ satisfies $(D_{\mathbf{w}}^k F)(z,w) = (\mathcal{D}^k Q)(z, e^w)$ for any integer $k \geq 0$ and any $(z,w) \in \mathbb{C}^2$. In particular, we get

$$(D_{\mathbf{w}}^k F)(m\mathbf{u} + n\mathbf{v}) = (\mathcal{D}^k Q)(ny, \alpha^n)$$

for any $k \in \mathbb{N}$ and any $(m,n) \in \mathbb{Z}^2$. Since $N^2 \leq M^{s_0}$ and $N \leq M^{s_1}$, this implies

$$\max_{\substack{0 \leq k < N^2 \\ 0 \leq m,n < N}} \left\{ \frac{1}{k!} |D_{\mathbf{w}}^k F(m\mathbf{u} + n\mathbf{v})| N^k \right\} \leq A \sum_{k=0}^{\infty} \frac{N^k}{k!} = A e^N.$$

Let c be the constant of Theorem 2 associated with the present choice of $\mathbf{u}, \mathbf{v}, \mathbf{w}$. Because of the choice of N, the condition (2) of this theorem is satisfied. Thus, if we put $r = cN$ and $R = ecr$, Theorem 2 gives

$$|F|_r \leq c^{2N^2} e^N A + e^{-N^2} |F|_R.$$

Since $\max\{1, t_0, s + t_1\} < 2s$, we find

$$|F|_R \leq (M^{t_0} + 1)(M^{t_1} + 1) \exp(M + M^{t_0} \log(R) + RM^{t_1}) \leq e^{N^2}/2$$

provided that N is large enough. On the other hand, since Q is a nonzero polynomial with *integral* coefficients, we have

$$1 \leq H(Q) \leq |Q|_1 \leq |F|_\pi \leq |F|_r$$

if $r \geq \pi$. Since $2s < u$, we conclude that when N is sufficiently large we have

$$A \geq \tfrac{1}{2} c^{-2N^2} e^{-N} > \exp(-M^u),$$

as required.

5. Equivalence of the two conjectures

1° Under the hypotheses of Conjecture 2, Theorem 1 shows that there exists an integer $d \geq 1$ such that $\alpha_j^d = e^{dy_j}$ for $j = 1, \ldots, l$. Since $dy_1, \ldots, dy_l$ are linearly independent over $\mathbb{Q}$, Schanuel's conjecture, if it is true, implies

$$\operatorname{tr.deg}_{\mathbb{Q}} \mathbb{Q}(dy_1, \ldots, dy_l, \alpha_1^d, \ldots, \alpha_l^d) \geq l.$$

Thus Conjecture 1 implies Conjecture 2.

2° Conversely, let l and $y_1, \ldots, y_l$ be as in Conjecture 1. Put $\alpha_j = e^{y_j}$ for $j = 1, \ldots, l$ and choose real numbers s_0, s_1, t_0, t_1, u satisfying the condition (1) from Conjecture 2. We apply Theorem 3 with $\Delta = N$, $r = 1 + cN^{s_1}$, $T_0 = N^{t_0}$, $T_1 = N^{t_1}$ and $U = 2N^u$ where $c = |y_1| + \ldots + |y_l|$. For sufficiently large N, this theorem ensures the existence of a nonzero polynomial $P_N \in \mathbb{Z}[X_0, X_1]$ with partial degree $\leq T_j$ in X_j for $j = 0, 1$ and height $\leq e^N$ such that the function $f_N(z) = P_N(z, e^z)$ satisfies $|f_N|_r \leq e^{-U}$. For any

$k, m_1, \ldots, m_l \in \mathbb{N}$ with $k \le N^{s_0}$ and $\max\{m_1, \ldots, m_l\} \le N^{s_1}$, we find

$$\begin{aligned}\Big|(\mathcal{D}^k P_N)\Big(\sum_{j=1}^{l} m_j y_j, \prod_{j=1}^{l} \alpha_j^{m_j}\Big)\Big| &= \Big|\frac{d^k f_N}{dz^k}\Big(\sum_{j=1}^{l} m_j y_j\Big)\Big| \\ &\le k!|f_N|_r \le \exp(-N^u)\end{aligned}$$

if N is sufficiently large. Assuming that Conjecture 2 is true, this implies

$$\mathrm{tr.deg}_{\mathbb{Q}}\mathbb{Q}(y_1, \ldots, y_l, e^{y_1}, \ldots, e^{y_l}) \ge l.$$

Thus Conjecture 2 implies Conjecture 1.

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