An arithmetic criterion for the values of the exponential function

by

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1. Introduction. The motivation for the present work comes from the following conjecture due to S. Schanuel (see [1], Historical notes of Chapter III):

CONJECTURE 1. Let \( l \) be a positive integer and let \( y_1, \ldots, y_l \in \mathbb{C} \) be linearly independent over \( \mathbb{Q} \). Then

\[
\text{tr.deg}_\mathbb{Q} \mathbb{Q}(y_1, \ldots, y_l; e^{y_1}, \ldots, e^{y_l}) \geq l.
\]

This conjecture is known to be true when \( l = 1 \) (Hermite–Lindemann theorem) and when \( y_1, \ldots, y_l \in \overline{\mathbb{Q}} \) (Lindemann–Weierstrass theorem), where \( \overline{\mathbb{Q}} \) denotes the algebraic closure of \( \mathbb{Q} \) in \( \mathbb{C} \). There are other evidences for this conjecture, but the general case is open, including the algebraic independence of \( e \) and \( \pi \) (take \( y_1 = 1 \) and \( y_2 = \pi i \)).

Here, we will show that this conjecture is equivalent to the following algebraic statement where the symbol \( D \) stands for the derivation:

\[
D = \frac{\partial}{\partial X_0} + X_1 \frac{\partial}{\partial X_1}
\]

in the field \( \mathbb{C}(X_0, X_1) \), and where the height of a polynomial \( P \in \mathbb{C}[X_0, X_1] \) is defined as the maximum of the absolute values of its coefficients.

CONJECTURE 2. Let \( l \) be a positive integer, let \( y_1, \ldots, y_l \in \mathbb{C} \) be linearly independent over \( \mathbb{Q} \) and let \( \alpha_1, \ldots, \alpha_l \in \mathbb{C}^\times \). Moreover, let \( s_0, s_1, t_0, t_1, u \) be positive numbers satisfying

(1) \( \max\{1, 2t_0, 2t_1\} < \min\{s_0, 2s_1\}, \quad \max\{s_0, s_1 + t_1\} < u < \frac{1}{2}(1 + t_0 + t_1). \)

Assume that, for any sufficiently large positive integer \( N \), there exists a nonzero polynomial \( P_N \in \mathbb{Z}[X_0, X_1] \) with partial degree \( \leq N^{t_0} \) in \( X_0 \), partial

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degree ≤ N^{t_1} in X_1 and height ≤ e^N which satisfies
\[ \left| (D^k P_N) \left( \sum_{j=1}^{l} m_j y_j \prod_{j=1}^{l} \alpha_j^{m_j} \right) \right| ≤ \exp(-N^u), \]
for any integers \( k, m_1, \ldots, m_l \in \mathbb{N} \) with \( k ≤ N^{s_0} \) and \( \max\{m_1, \ldots, m_l\} ≤ N^{s_1} \). Then \( \text{tr.\deg}_Q Q(y_1, \ldots, y_l, \alpha_1, \ldots, \alpha_l) ≥ l \).

Note that this arithmetic statement is similar to the present criteria of algebraic independence (see for example [2] and [3]). It suggests, we hope, a reasonable approach toward Schanuel’s conjecture. We will show that if it is true for some positive integer \( l \) and some choice of parameters \( s_0, s_1, t_0, t_1, u \) satisfying (1), then Schanuel’s conjecture is true for this value of \( l \). This follows from a general construction of an auxiliary function due to Michel Waldschmidt (Theorem 3.1 of [4]). Conversely, we will show that, if Conjecture 1 is true for some positive integer \( l \), then Conjecture 2 is also true for the same value of \( l \) and for any choice of parameters satisfying (1). In particular, Conjecture 2 is true in the case \( l = 1 \). Moreover, if, for fixed \( l \), Conjecture 2 is true for at least one choice of parameters satisfying (1), then it is true for all of them. We prove the reverse implication as a consequence of the following criterion concerning the values of the exponential function.

**Theorem 1.** Let \((y, \alpha) \in \mathbb{C} \times \mathbb{C}^\times\), and let \( s_0, s_1, t_0, t_1, u \) be positive numbers satisfying the inequalities (1). Then the following conditions are equivalent:

(a) there exists an integer \( d ≥ 1 \) such that \( \alpha^d = e^{dy} \);

(b) for any sufficiently large positive integer \( N \), there exists a nonzero polynomial \( Q_N \in \mathbb{Z}[X_0, X_1] \) with partial degree \( ≤ N^{t_0} \) in \( X_0 \), partial degree \( ≤ N^{t_1} \) in \( X_1 \) and height \( ≤ e^N \) such that
\[ \left| (D^k Q_N) (m y, \alpha^m) \right| ≤ \exp(-N^u) \]
for any \( k, m \in \mathbb{N} \) with \( k ≤ N^{s_0} \) and \( m ≤ N^{s_1} \).

Again the proof that (a) implies (b) follows from Waldschmidt’s construction. To prove the reverse implication, we establish a new interpolation lemma for holomorphic functions \( F(z_1, z_2) \) of two complex variables. This interpolation lemma takes into account not only the values of \( F \) on a subgroup of \( \mathbb{C}^2 \) of rank 2, but also the values of its derivatives in the direction of a nonzero point \( w = (w_1, w_2) \) of \( \mathbb{C}^2 \). The corresponding derivation is denoted by
\[ D_w = w_1 \frac{\partial}{\partial z_1} + w_2 \frac{\partial}{\partial z_2}. \]

To state this result, we need to fix additional notation. We define
\[ B(0, R) = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| ≤ R, |z_2| ≤ R\} \]
for any $R > 0$. For a continuous function $F : B(0, R) \to \mathbb{C}$, we put

$$|F|_R = \sup\{|F(z_1, z_2)| : |z_1| = |z_2| = R\}.$$ 

By the maximum modulus principle, when $F$ is holomorphic in the interior of $B(0, R)$, this coincides with the supremum of $|F|$ on $B(0, R)$.

**Theorem 2.** Let $\{u, w\}$ be a basis of $\mathbb{C}^2$, let $v \in \mathbb{C}^2$ and let $a$ be the complex number for which $v - au \in \mathbb{C}w$. Then there exists a constant $c \geq 1$ which depends only on $u$, $v$ and $w$, and which satisfies $u, v \in B(0, c)$ and the following property: For any integer $N \geq 1$ with

$$\min\{|m + na| : m, n \in \mathbb{Z}, 0 < \max\{|m|, |n|\} < N\} \geq 2^{-N},$$

for any pair of real numbers $r, R$ with $R \geq 2r$ and $r \geq cN$, and for any continuous function $F : B(0, R) \to \mathbb{C}$ which is holomorphic inside $B(0, R)$, we have

$$|F|_r \leq \left(\frac{cr}{N}\right)^{N^2} \max_{0 \leq k < N^2} \left\{ \frac{1}{k!} |D^k_F(mu + n\nu)|N^k \right\} + \left(\frac{cr}{R}\right)^{N^2} |F|_R.$$ 

The condition (2) is satisfied for infinitely many values of $N$ if $a \notin \mathbb{Q}$ (see Lemma 4 below). It can be shown that such a condition is necessary in the above result. However, for a function $F$ satisfying $D^k_F(mu + n\nu) = 0$ for $0 \leq k < N^2$ and $0 \leq m, n < N$, Theorem 2 gives

$$|F|_r \leq \left(\frac{cr}{R}\right)^{N^2} |F|_R,$$

and it is not clear that a Diophantine condition like (2) is needed any more. We refer the reader to Chapter 7 of [5] for related conjectures and results concerning the growth of holomorphic functions vanishing at points of finitely generated subgroups of $\mathbb{C}^n$.

The organization of this paper is as follows. In Section 2 below, we establish a first interpolation formula. The proof of Theorem 2 follows in Section 3, using this formula. Finally, the proof of Theorem 1 and the equivalence between the two conjectures are established in Sections 4 and 5 respectively.

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2. A first interpolation formula. We fix a point $(a, b) \in \mathbb{C}^2$ and a positive integer $N$. We put $L = N^2$ and we assume that $a$ satisfies the condition (2) in the statement of Theorem 2, that is, $|m + na| \geq 2^{-N}$ for any pair $(m, n) \in \mathbb{Z}^2$ with $0 < \max\{|m|, |n|\} < N$. 

For each triple of integers \((m, n, k)\) with \(0 \leq m, n < N\) and \(0 \leq k < L\), we define
\[
g_{m,n,k}(z, w) = (w - nb)^k \prod_{(m', n') \neq (m,n)} \frac{z - m' - n'a}{m + na - m' - n'a}
\]
where the product on the right hand side is over all pairs of integers \((m', n')\) with \(0 \leq m', n' < N\) and \((m', n') \neq (m, n)\). By construction, these polynomials have the following interpolation property:
\[
\frac{\partial}{\partial w} k' g_{m,n,k}(m' + n'a, n'b) = \begin{cases} 
k! & \text{if } (m', n', k') = (m, n, k), \\
0 & \text{otherwise.}
\end{cases}
\]

They also satisfy:

**Lemma 1.** For each triple \((m, n, k)\) as above, we have
\[
|g_{m,n,k}|_N \leq c_1^L ((1 + |b|)N)^k
\]
where \(c_1 = 8e(2 + |a|)\).

**Proof.** Fix an integer \(s\) with \(0 \leq s < N\) and consider the set
\[
I_s = \{m + na - m' - sa : m' = 0, 1, \ldots, N - 1\}.
\]
The elements of \(I_s\) are \(N\) complex numbers which differ from one another by an integer. Let \(x_0\) be an element of \(I_s\) whose real part has minimal absolute value. It is possible to order the remaining elements \(x_1, \ldots, x_{N-1}\) of \(I_s\) so that the absolute values of their real parts are respectively bounded from below by \(1/2, 2/2, \ldots, (N-1)/2\). Since \(|x_0| \geq 2^{-N}\) if \(x_0 \neq 0\), this implies
\[
\prod_{x \in I_s \setminus \{0\}} |x| \geq 2^{-N} \frac{(N-1)!}{2^{N-1}} \geq 2^{-N} e^{-N} \left( \frac{N}{2} \right)^{N-1} \geq \left( \frac{N}{8e} \right)^N.
\]
So, for the denominator of \(g_{m,n,k}\), we get
\[
\prod_{(m', n') \neq (m,n)} |m + na - m' - n'a| = \prod_{s=0}^{N-1} \prod_{x \in I_s \setminus \{0\}} |x| \geq \left( \frac{N}{8e} \right)^L.
\]
Using this lower bound, we deduce
\[
|g_{m,n,k}|_N \leq (N + n|b|)^k (2N + N|a|)^{L-1} \left( \frac{8e}{N} \right)^L \leq (8e(2 + |a|))^L ((1 + |b|)N)^k.
\]

**Lemma 2.** For each \((r, s) \in \mathbb{N}^2\), define
\[
f_{r,s}(z, w) = \sum_{0 \leq m, n < N} \sum_{0 \leq k < L} g_{m,n,k}(z, w)(m + na)^r \binom{s}{k} (nb)^{s-k}.
\]
Then \( f_{r,s}(z,w) = z^r w^s \) whenever \( \max\{r,s\} < L \). Moreover, for any \((r,s) \in \mathbb{N}^2\),

\[
|f_{r,s}|_N \leq (2c_1)^L ((1 + |a| + 2|b|)N)^{r+s}.
\]

**Proof.** For the first assertion, consider the vector subspace \( V \) of \( \mathbb{C}[z,w] \) consisting of all polynomials of partial degree \(< L\) in \( z \) and partial degree \(< L\) in \( w \). By virtue of (3), the \( L^2 \) functions \( g_{m,n,k} \) form a basis of \( V \) with the dual basis given by the linear functionals

\[
\varphi_{m,n,k}(g) = (1/k!)(\partial/\partial w)^k g(m + na, nb).
\]

When \( \max\{r,s\} < L \), the polynomial \( z^r w^s \) belongs to \( V \) and its image under \( \varphi_{m,n,k} \) is the same as that of \( f_{r,s} \) for \( 0 \leq m, n < N \) and \( 0 \leq k < L \). So, the two polynomials must be equal. For the second assertion, we use Lemma 1. It gives

\[
|f_{r,s}|_N \leq \sum_{0 \leq m,n < N \atop 0 \leq k < L} |g_{m,n,k}|_N (m + n|a|)^r \left( \frac{s}{k} \right) (n|b|)^{s-k}
\]

\[
\leq c_1^L \sum_{0 \leq m,n < N} (m + n|a|)^r ((1 + |b|)N + n|b|)^s
\]

\[
\leq c_1^L N^2 ((1 + |a|)N)^r (1 + 2|b|)N)^s.
\]

The conclusion follows if we use \( N^2 = L \leq 2L \).

We are now ready to prove:

**PROPOSITION 1.** Let \((a,b), N\) and \( L\) be as above. Let \( R \geq 2(1 + |a| + 2|b|)N \), and let \( F(z,w) \) be a complex-valued function which is continuous on \( B(0,R) \) and holomorphic inside. Put

\[
A = \max \left\{ \frac{1}{k!} \left| \frac{\partial^k F}{\partial w^k}(m + na, nb) \right| N^k : 0 \leq m, n < N \text{ and } 0 \leq k < L \right\}.
\]

Then

\[
|F|_N \leq c_2^L A + (c_2 N/R)^L |F|_R \quad \text{with } c_2 = 8(1 + 2c_1)(1 + |a| + 2|b|).
\]

**Proof.** Define \( \mathbf{T} = \{(\xi, \zeta) \in \mathbb{C}^2 : |\xi| = |\zeta| = R\} \). For any continuous function \( G: \mathbf{T} \to \mathbb{C} \), we put

\[
\langle F, G \rangle = \frac{1}{(2\pi i)^2} \int_{\mathbf{T}} F(\xi, \zeta)G(\xi, \zeta) \, d\xi \, d\zeta.
\]

This integral satisfies \(|\langle F, G \rangle| \leq R^2 |F|_R |G|_R\) where \(|G|_R\) denotes the supremum of \(|G|\) on the torus \( \mathbf{T} \). On the other hand, Cauchy’s integral formulas give

\[
F(z, w) = \left\langle F, \frac{1}{(\xi - z)(\zeta - w)} \right\rangle
\]
for any point \((z, w)\) in the interior of \(B(0, R)\). For a triple \((m, n, k)\) of integers with \(0 \leq m, n < N\) and \(0 \leq k < L\), they also give

\[
\frac{1}{k!} \cdot \frac{\partial^k F}{\partial w^k} (m + na, nb) = \langle F, D_{m,n,k} \rangle
\]

where

\[
D_{m,n,k}(\xi, \zeta) := \frac{1}{(\zeta - m - na)(\xi - nb)^{k+1}}
\]

since \((m + na, nb)\) belongs to the interior of \(B(0, R)\). We claim that

\[
(4) \quad \frac{1}{(\xi - z)(\zeta - w)} = \sum_{0 \leq m, n < N} \sum_{0 \leq k < L} g_{m,n,k}(z, w) \cdot D_{m,n,k}(\xi, \zeta) + U(z, w, \xi, \zeta)
\]

where the remainder \(U\) satisfies \(|U(z, w, \xi, \zeta)| \leq (c_2 N^L R^{-L-2}\) for any \((z, w, \xi, \zeta) \in B(0, N) \times T\). If we take this for granted, then, multiplying both sides of (4) by \(F(\xi, \zeta)\) and integrating over \(T\), we get, by linearity of the integral,

\[
|F|_N \leq \sum_{0 \leq m, n < N} \sum_{0 \leq k < L} \left| g_{m,n,k} \right|_N \left| \frac{1}{k!} \cdot \frac{\partial^k F}{\partial w^k} (m + na, nb) \right| + \left( \frac{c_2 N}{R} \right)^L |F|_R.
\]

Using Lemma 1, we deduce

\[
|F|_N \leq c_1^L \sum_{0 \leq m, n < N} \sum_{0 \leq k < L} (1 + |b|)^k \cdot A + \left( \frac{c_2 N}{R} \right)^L |F|_R \leq c_2^L \cdot A + \left( \frac{c_2 N}{R} \right)^L |F|_R
\]

and the proposition is proved.

To prove the claim, we use the developments

\[
\frac{1}{(\xi - z)(\zeta - w)} = \sum_{r,s \geq 0} \frac{z^r w^s}{\xi^{r+1} \zeta^{s+1}}
\]

and

\[
D_{m,n,k}(\xi, \zeta) = \sum_{r,s \geq 0} \frac{(m + na)^r (nb)^{s-k}}{\xi^{r+1} \zeta^{s+1}}
\]

which converge absolutely and represent these functions whenever \((z, w) \in B(0, N)\) and \((\xi, \zeta) \in T\). Using Lemma 2, we deduce that the function \(U\) defined by (4) is given by

\[
U(z, w, \xi, \zeta) = \sum_{\max\{r,s\} \geq L} \frac{z^r w^s - f_{r,s}(z, w)}{\xi^{r+1} \zeta^{s+1}}
\]
for \((z, w, \xi, \zeta) \in B(0, N) \times T\). For those values of \((z, w, \xi, \zeta)\), we get

\[
|U(z, w, \xi, \zeta)| \leq \sum_{\max\{r, s\} \geq L} \frac{N^{r+s} + |f_{r,s}|N}{R^{r+s+2}} \leq \left(1 + \frac{(2c_1)^L}{R^2}\right) \sum_{\max\{r, s\} \geq L} \left(\frac{(1 + |a| + 2|b|)N}{R}\right)^{r+s} \leq 8\left(1 + \frac{(2c_1)^L}{R^2}\right) \left(\frac{(1 + |a| + 2|b|)N}{R}\right)^L,
\]

since \((1 + |a| + 2|b|)N/R \leq 1/2\). This proves the claim and thus completes the proof of the proposition.

3. Proof of Theorem 2. We first prove:

**Lemma 3.** Let \(L\) be a positive integer, let \(r_0, r\) and \(R\) be positive numbers with \(r \geq r_0\) and \(R \geq 2r\), and let \(F(z, w)\) be a complex-valued function which is continuous on \(B(0, R)\) and holomorphic inside. Then

\[
|F|_r \leq \left(\frac{L+1}{2}\right)\left(\frac{r}{r_0}\right)^L |F|_{r_0} + (2L + 4)\left(\frac{r}{R}\right)^L |F|_R.
\]

**Proof.** Since \(r < R\), the Taylor expansion of \(F\) around \((0, 0)\) converges normally in \(B(0, r)\) and we get

\[
|F|_r \leq \sum_{j,k \geq 0} \frac{1}{j!k!} \left|\frac{\partial^{j+k} F}{\partial z^j \partial w^k}(0, 0)\right| r^{j+k}.
\]

Using Cauchy’s inequalities, we deduce

\[
|F|_r \leq \sum_{j+k < L} \left(\frac{r}{r_0}\right)^{j+k} |F|_{r_0} + \sum_{j+k \geq L} \left(\frac{r}{R}\right)^{j+k} |F|_R.
\]

The conclusion follows using \(\sum_{j+k \geq L} 2^{L-j-k} = 2L + 4\). Note that a sharper inequality with the factor \(2L + 4\) replaced by \(\sqrt{L+1}\) follows from Lemma 3.4 of [4].

**Proof of Theorem 2.** Let \(T : \mathbb{C}^2 \to \mathbb{C}^2\) be the linear map for which \(T(1, 0) = u\) and \(T(0, 1) = w\), and let \((a, b)\) be the point of \(\mathbb{C}^2\) for which \(T(a, b) = v\). Let \(c_3, c_4\) be positive constants such that \(c_3 \|z\| \leq \|T(z)\| \leq c_4 \|z\|\) for any \(z \in \mathbb{C}^2\), where \(\| \|\) denotes the maximum norm. Assume that \(N\) is a positive integer satisfying the condition (2) in the statement of Theorem 2. Choose real numbers \(r\) and \(R\) with

\[
R \geq 2r \quad \text{and} \quad r \geq \max\{c_3, (1 + |a| + 2|b|)c_4\} N.
\]
Choose also a continuous function $F : B(0, R) \to \mathbb{C}$ which is holomorphic in the interior of $B(0, R)$. Put $L = N^2$ and $G = F \circ T$. Then $G$ is defined and continuous on $B(0, R/c_4)$. It is holomorphic in the interior of this ball and satisfies

\[ |F|_{c_3 N} \leq |G|_N \quad \text{and} \quad |G|_{R/c_4} \leq |F|_R. \]

On the other hand, $T(m + na, nb) = mu + n\nu$ for any $(m, n) \in \mathbb{Z}^2$. Since $T(0,1) = w$, this implies

\[ \frac{\partial^k G}{\partial w^k}(m + na, nb) = D_w F(mu + n\nu) \quad \text{for any integer } k \geq 0. \]

Let $B$ be the maximum of the numbers $|D_w^k F(mu + n\nu)|N^k/k!$ with $0 \leq k < L$ and $0 \leq m, n < N$. For any choice of integers $k, m, n$ in the same intervals, the relation (6) implies

\[ \frac{1}{k!} \left| \frac{\partial^k G}{\partial w^k}(m + na, nb) \right| N^k \leq B. \]

By Proposition 1, we deduce

\[ |G|_N \leq c_2^L B + \left( \frac{c_2 c_4 N}{R} \right)^L |G|_{R/c_4} \]

where $c_2$ depends only on $|a|$ and $|b|$. Combining this with (5) and applying Lemma 3 with $r_0 = c_3 N$, we deduce

\[ |F|_r \leq \left( \frac{L + 1}{2} \right) \left( \frac{r}{c_3 N} \right)^L \left[ c_2^L B + \left( \frac{c_2 c_4 N}{R} \right)^L |F|_R \right] + (2L + 4) \left( \frac{r}{R} \right)^L |F|_R, \]

which proves Theorem 2 for a suitable constant $c$ depending only on $c_2, c_3, c_4$.

4. Proof of Theorem 1. We will need the following special case of Theorem 3.1 of [4]:

**Theorem 3 (M. Waldschmidt).** Let $\Delta, r, T_0, T_1, U$ be positive numbers. Assume $U \geq 3$,

\[ \log ((T_0 + 1)(T_1 + 1)) + \Delta + T_0 \log (er) + erT_1 \leq U \]

and $(8U)^2 \leq \Delta T_0 T_1$. Then there exists a nonzero polynomial $P \in \mathbb{Z}[X_0, X_1]$ with partial degree $\leq T_0$ in $X_0$, partial degree $\leq T_1$ in $X_1$ and height $\leq e^\Delta$ such that the function $f(z) = P(z, e^z)$ satisfies $|f|_r \leq e^{-U}$.

We divide the proof of Theorem 1 into two propositions. Each proves one implication but assumes a weaker condition than (1) on the parameters $s_0, s_1, t_0, t_1$ and $u$. 
Proposition 2. Let \((y, \alpha) \in \mathbb{C} \times \mathbb{C}^\times\) and let \(s_0, s_1, t_0, t_1, u\) be positive numbers satisfying
\[
\max\{1, s_0, t_0, s_1 + t_1\} < u < \frac{1}{2}(1 + t_0 + t_1).
\]
Assume that \(\alpha e^{-y}\) is a root of unity. Then, the condition (b) of Theorem 1 holds for the pair \((y, \alpha)\).

Proof. Write \(\alpha = \zeta e^y\) with \(\zeta \in \mathbb{C}^\times\). By hypothesis, \(\zeta^d = 1\) for some integer \(d \geq 1\). Choose \(\varepsilon\) with
\[
0 < \varepsilon < \min\{1, t_0, t_1, \frac{1}{5}(1 + t_0 + t_1 - 2u)\}.
\]
Then, for any sufficiently large integer \(N\), the conditions of Theorem 3 are satisfied with \(\Delta = N^{1-\varepsilon}, r = N^{s_1+\varepsilon}, T_0 = N^{t_0-\varepsilon}, T_1 = N^{t_1-\varepsilon}\) and \(U = N^{u+\varepsilon}\). Fix such an integer \(N\) and choose a nonzero polynomial \(P_N \in \mathbb{Z}[X_0, X_1]\) with the properties corresponding to this choice of parameters. Then
\[
Q_N(X_0, X_1) = \prod_{k=0}^{d-1} P_N(X_0, \zeta^k X_1)
\]
is also a nonzero polynomial with integral coefficients. If \(N\) is sufficiently large, its partial degree in \(X_j\) is \(\leq dT_j \leq N^{t_j}\) for \(j = 0, 1\), and its height is \(\leq ((T_0 + 1)(T_1 + 1)e^{d})^d \leq e^{N}\). We define entire functions \(f_{N,k}(z) = P_N(z, \zeta^k e^z)\) for \(k = 0, \ldots, d - 1\) and
\[
g_N(z) = Q_N(z, e^z) = \prod_{k=0}^{d-1} f_{N,k}(z).
\]
By construction, \(|f_{N,0}|_r \leq e^{-U}\), while, for \(k = 1, \ldots, d - 1\), a direct estimate gives
\[
|f_{N,k}|_r \leq (T_0 + 1)(T_1 + 1) \exp(\Delta + T_0 \log(r) + rT_1) \leq \exp(N^u)
\]
provided that \(N\) is large enough. From these inequalities we deduce, if \(N\) is sufficiently large,
\[
|g_N|_r \leq \exp(-N^{u+\varepsilon} + (d - 1)N^u) \leq \exp(-2N^u).
\]
On the other hand, \(g_N(z) = Q_N(z, \zeta^m e^z)\) for any \(m \in \mathbb{Z}\) and any \(z \in \mathbb{C}\). For fixed \(m\), we deduce
\[
\frac{d^k g_N}{dz^k}(z) = (D^k Q_N)(z, \zeta^m e^z) \quad \text{and so} \quad \frac{d^k g_N}{dz^k}(my) = (D^k Q_N)(my, \alpha^m)
\]
for any integer \(k \geq 0\). Suppose that \(N\) is large enough so that \(N^{s_1}|y| + 1 \leq r\) and \(N^{s_0} \log(N^{s_0}) \leq N^u\). Then, for any pair of integers \((k, m)\) with \(0 \leq k \leq N^{s_0}\) and \(0 \leq m \leq N^{s_1}\), Cauchy’s inequalities give the estimate
\[
|(D^k Q_N)(my, \alpha^m)| = \left| \frac{d^k g_N}{dz^k}(my) \right| \leq k!|g_N|_{my+1} \leq \exp(N^u)|g_N|_r.
\]
Since $|g_N|_r \leq \exp(-2N^u)$ when $N$ is large enough, the sequence of polynomials $(Q_N)_{N \geq N_0}$ has the required properties for a suitable choice of $N_0$.

For the next proposition, we will need the following fact:

**Lemma 4.** Let $a$ be an irrational complex number. Then there are infinitely many positive integers $N$ such that

\[(7) \quad \min\{|m + na| : m, n \in \mathbb{Z}, 0 < \max\{|m|, |n|\} < N\} \geq 1/(2N).\]

**Proof.** Assume on the contrary that, for any integer $N$ larger than some constant $N_0$, there are integers $m(N)$ and $n(N)$ such that

$$0 < \max\{|m(N)|, |n(N)|\} < N \quad \text{and} \quad |m(N) + n(N)a| < 1/(2N).$$

For $N > N_0$, these conditions imply $n(N) \neq 0$ and we find

$$|m(N)n(N + 1) - m(N + 1)n(N)| \leq |m(N) + n(N)a| \cdot |n(N + 1)| + |m(N + 1) + n(N + 1)a| \cdot |n(N)| < 1,$$

and so the integer $m(N)n(N + 1) - m(N + 1)n(N)$ is zero. This shows that the ratio $m(N)/n(N)$ is a constant $r \in \mathbb{Q}$. Since

$$|r + a| = |m(N) + n(N)a|/|n(N)| < 1/(2N) \quad \text{for any } N > N_0,$$

we deduce that $a = -r$ in contradiction with the hypothesis $a \notin \mathbb{Q}$.

**Proposition 3.** Let $(y, \alpha) \in \mathbb{C} \times \mathbb{C}^\times$, and let $s_0, s_1, t_0, t_1, u$ be positive numbers such that

$$\max\{1, t_0, 2t_1\} < \min\{s_0, 2s_1\} < u.$$

Suppose that $\alpha e^{-y}$ is not a root of unity. Then the condition (b) of Theorem 1 does not hold for the pair $(y, \alpha)$.

**Proof.** Choose $\lambda \in \mathbb{C}$ such that $e^\lambda = \alpha$. The ratio $a = (\lambda - y)/(2\pi i)$ is by hypothesis an irrational number. Therefore there exist infinitely many positive integers $N$ which satisfy the condition $(7)$ of Lemma 4. Fix such an integer $N$. Put $s = \min\{s_0/2, s_1\}$, and let $M$ denote the smallest positive integer for which $N \leq M^s$. Choose also a nonzero polynomial $Q \in \mathbb{Z}[X_0, X_1]$ with partial degree $\leq M^{t_0}$ in $X_0$, partial degree $\leq M^{t_1}$ in $X_1$ and height $\leq e^M$. We will show that, if $N$ is sufficiently large, the number

$$A = \max_{0 \leq k \leq M^{t_0}} \left|((D^k Q)(ny, \alpha^n))\right|_{0 \leq n \leq M^{t_1}}$$

satisfies $A > \exp(-M^u)$. This will prove the proposition.

To this end, we consider the entire function $F : \mathbb{C}^2 \to \mathbb{C}$ given by $F(z, w) = Q(z, e^w)$, and the vectors

$$u = (0, 2\pi i), \quad v = (y, \lambda), \quad w = (1, 1).$$
Let satised. Thus, if we put \( u \) the differential operator \( D_w = \partial/\partial z + \partial/\partial w \) satisfies \((D_w^kF)(z,w) = (D^kQ)(z,e^w)\) for any integer \( k \geq 0 \) and any \((z,w) \in \mathbb{C}^2\). In particular, we get
\[
(D_w^kF)(mu + nv) = (D^kQ)(ny, \alpha^n)
\]
for any \( k \in \mathbb{N} \) and any \((m,n) \in \mathbb{Z}^2\). Since \( N^2 \leq M^{s_0} \) and \( N \leq M^{s_1} \), this implies
\[
\max_{0 \leq k < N^2; 0 \leq m,n < N} \left\{ \frac{1}{k!} |D_w^kF(mu + nv)|^N \right\} \leq A \sum_{k=0}^{\infty} \frac{N^k}{k!} = Ae^N.
\]
Let \( c \) be the constant of Theorem 2 associated with the present choice of \( u, v, w \). Because of the choice of \( N \), the condition (2) of this theorem is satisfied. Thus, if we put \( r = cN \) and \( R = ecr \), Theorem 2 gives
\[
|F|_r \leq c^{2N^2} e^N A + e^{-N^2} |F|_R.
\]
Since \( \max\{1, t_0, s + t_1\} < 2s \), we find
\[
|F|_R \leq (M^{t_0} + 1)(M^{t_1} + 1) \exp(M + M^{t_0} \log(R) + RM^{t_1}) \leq e^{N^2/2}
\]
provided that \( N \) is large enough. On the other hand, since \( Q \) is a nonzero polynomial with integral coefficients, we have
\[
1 \leq H(Q) \leq |Q|_1 \leq |F|_\pi \leq |F|_r
\]
if \( r \geq \pi \). Since \( 2s < u \), we conclude that when \( N \) is sufficiently large we have
\[
A \geq \frac{1}{2} e^{-2N^2} e^{-N} > \exp(-M^u),
\]
as required.

5. Equivalence of the two conjectures

1° Under the hypothesis of Conjecture 2, Theorem 1 shows that there exists an integer \( d \geq 1 \) such that \( \alpha_j^d = e^{dy_j} \) for \( j = 1, \ldots, l \). Since \( dy_1, \ldots, dy_l \) are linearly independent over \( \mathbb{Q} \), Schanuel’s conjecture, if it is true, implies
\[
\text{tr.deg}_\mathbb{Q} \mathbb{Q}(dy_1, \ldots, dy_l, \alpha_1^d, \ldots, \alpha_l^d) \geq l.
\]
Thus Conjecture 1 implies Conjecture 2.

2° Conversely, let \( l \) and \( y_1, \ldots, y_l \) be as in Conjecture 1. Put \( \alpha_j = e^{y_j} \) for \( j = 1, \ldots, l \) and choose real numbers \( s_0, s_1, t_0, t_1, u \) satisfying the condition (1) from Conjecture 2. We apply Theorem 3 with \( \Delta = N, r = 1 + cN^{s_1}, T_0 = N^{t_0}, T_1 = N^{t_1} \) and \( U = 2N^u \) where \( c = |y_1| + \ldots + |y_l| \). For sufficiently large \( N \), this theorem ensures the existence of a nonzero polynomial \( P_N \in \mathbb{Z}[X_0, X_1] \) with partial degree \( \leq T_j \) in \( X_j \) for \( j = 0, 1 \) and height \( \leq e^N \) such that the function \( f_N(z) = P_N(z, e^z) \) satisfies \(|f_N|_r \leq e^{-U} \). For any
$k, m_1, \ldots, m_l \in \mathbb{N}$ with $k \leq N^{s_0}$ and $\max\{m_1, \ldots, m_l\} \leq N^{s_1}$, we find
\[
\left| \left( D^k P_N \right) \left( \sum_{j=1}^{l} m_j y_j \prod_{j=1}^{l} \alpha_j^{m_j} \right) \right| = \left| \frac{d^k f_N}{d z^k} \left( \sum_{j=1}^{l} m_j y_j \right) \right|
\leq k! |f_N|_r \leq \exp(-N^u)
\]
if $N$ is sufficiently large. Assuming that Conjecture 2 is true, this implies
\[
\text{tr. deg}_{\mathbb{Q}}(\mathbb{Q}(y_1, \ldots, y_l; e^{y_1}, \ldots, e^{y_l})) \geq l.
\]
Thus Conjecture 2 implies Conjecture 1.

References


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