On the Diophantine equation \((x^2 \pm C)(y^2 \pm D) = z^4\)

by

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1. Introduction. Let \(L > 0\) and \(M\) be rational integers such that \(L - 4M > 0\) and \((L, M) = 1\). Let \(\alpha\) and \(\beta\) be the two roots of the trinomial \(x^2 - \sqrt{L}x + M\). For a non-negative integer \(n\), the \(n\)th term in the Lehmer sequence \(\{P_n\}\) and the associated Lehmer sequence \(\{Q_n\}\) (see [11]) are defined by

\[
P_n := P_n(\alpha, \beta) = \begin{cases} 
\frac{\alpha^n - \beta^n}{\alpha - \beta} & \text{for } n \text{ odd,} \\
\frac{\alpha^n - \beta^n}{\alpha^2 - \beta^2} & \text{for } n \text{ even,}
\end{cases}
\]

and

\[
Q_n := Q_n(\alpha, \beta) = \begin{cases} 
\frac{\alpha^n + \beta^n}{\alpha + \beta} & \text{for } n \text{ odd,} \\
\alpha^n + \beta^n & \text{for } n \text{ even.}
\end{cases}
\]

Lehmer sequences have many interesting properties and often arise in the study of Diophantine equations. The arithmetic properties of the numbers \(P_n\) can be found in [11, 25].

Let \(a, b\) be positive integers such that \(ab\) is not a square. Diophantine equations of the form

\[(1.1) \quad aX^4 - bY^2 = c,\]

where \(c \in \{\pm 1, \pm 2, \pm 4\}\), have received considerable interest, as we see from the references [2, 7, 8, 17, 19, 22, 23]. The study of these equations goes back to the classical work of Ljunggren [12, 13, 15, 16], who was able to prove many sharp results on (1.1). The following cases have been considered: Ljunggren [15] \((c = -1)\), [16] \((c = 4)\), Luca and Walsh [17] \((c = -2)\), Luo and Yuan [18] \((c = \pm 4)\), Akhtari [1] \((c = 1)\) and Yuan and Li [28] \((c = 2)\).

As an application of some results on (1.1), Luca and Walsh [17] proved the following theorem.

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The equation 
\[(X^2 + 1)(Y^2 + 1) = Z^4\]
has no positive integer solutions.

2. The only positive integer solutions of the equation
\[(X^2 + 1)(Y^2 - 1) = Z^4\]
are \((X, Y, Z) = (1, 3, 2), (239, 3, 26)\).

3. The equation
\[(X^2 - 1)(Y^2 - 1) = Z^4\]
has no positive integer solutions.

In this paper, we will investigate the positive integer solutions \((x, y, z)\) of the Diophantine equations of the type
\[(1.2)\quad (x^2 \pm C)(y^2 \pm D) = z^4,\]
where \(C, D \in \{1, 2, 4\}\). The main purpose is try to completely solve the remaining eighteen equations of the type \([1.2]\). The main results of the present paper are as follows. Throughout, \(\Box\) stands for a square, and \((A/B)\) for the Jacobi symbol of \(A\) with respect to \(B\), where \(A\) and \(B\) are coprime integers.

**Theorem 1.1.** Let \(A > 1\) be a positive integer. Then the Diophantine equation
\[(1.3)\quad (AX^2 + 1)(AY^2 + 1) = Z^4\]
has no positive integer solutions \((X, Y, Z)\) with \(X \neq Y\).

**Theorem 1.2.**

1. The only positive integer solutions of the equation
\[(1.4)\quad (X^2 + 4)(Y^2 + 4) = Z^4\]
are \((X, Y, Z) = (1, 11, 5), (11, 1, 5)\).

2. The equation
\[(1.5)\quad (X^2 - 4)(Y^2 - 4) = Z^4\]
has no positive integer solutions.

3. The equation
\[(1.6)\quad (X^2 - 2)(Y^2 - 2) = Z^4\]
has no positive integer solutions.

4. The only positive integer solutions of the equation
\[(1.7)\quad (X^2 + 2)(Y^2 + 2) = Z^4\]
are \((X, Y, Z) = (1, 5, 3), (5, 1, 3)\).
On the Diophantine equation \((x^2 \pm C)(y^2 \pm D) = z^4\)

(5) The equation

\[(X^2 + 2)(Y^2 - 2) = Z^4\]

has no positive integer solutions.

(6) The equation

\[(X^2 + 2)(Y^2 + 1) = Z^4\]

has no positive integer solutions.

(7) The equation

\[(X^2 - 2)(Y^2 + 1) = Z^4\]

has no positive integer solutions.

(8) The equation

\[(X^2 + 2)(Y^2 - 4) = Z^4\]

has no positive integer solutions.

(9) The equation

\[(X^2 + 2)(Y^2 + 4) = Z^4\]

has no positive integer solutions.

(10) The only positive integer solution to the equation

\[(X^2 + 2)(Y^2 - 1) = Z^4\]

is \((X, Y, Z) = (5, 2, 3)\).

(11) The only positive integer solutions to the equation

\[(X^2 + 4)(Y^2 + 1) = Z^4\]

are \((X, Y, Z) = (11, 2, 5), (2, 239, 26), (478, 1, 26)\).

(12) The only positive integer solutions of the equation

\[(X^2 + 4)(Y^2 - 4) = Z^4\]

are \((X, Y, Z) = (2, 6, 4), (478, 6, 52)\).

(13) The equation

\[(X^2 + 4)(Y^2 - 1) = Z^4\]

has no positive integer solutions.

(14) The equation

\[(X^2 - 4)(Y^2 + 1) = Z^4\]

has no positive integer solutions.

(15) The equation

\[(X^2 - 4)(Y^2 - 1) = Z^4\]

has only infinitely many trivial positive solutions \((X, Y, Z) = (2Y, Y, 2S)\), where \(Y, S\) are positive integers with \(Y^2 - 2S^2 = 1\).
The only positive integer solutions to the equation
\[(X^2 - 2)(Y^2 + 4) = Z^4\]
are \((X, Y, Z) = (2, 2, 2), (2, 478, 26)\).

The equation
\[(X^2 - 4)(Y^2 - 1) = 4Z^4, \quad 2 \nmid X,\]
has no positive integer solutions.

However, we have not been able to solve the following two equations:
\[(X^2 - 2)(Y^2 - 4) = Z^4, \quad 2 \mid XY,\]
\[(X^2 - 2)(Y^2 - 1) = Z^4, \quad 2 \mid X.\]

We leave this as an open question.

2. The results on the equation \(ax^2 - by^4 = c\). In this section, we will list all the related results on equations \(ax^2 - by^4 = \pm 2, \pm 4\), which will be used later.

Let \(a\) and \(b\) be odd positive integers such that the equation
\[aX^2 - bY^2 = 2\]
is solvable in positive integers \((X, Y)\). Let \((a_1, b_1)\) be the minimal positive solution to (2.1), and define
\[\alpha = \frac{a_1\sqrt{a} + b_1\sqrt{b}}{\sqrt{2}}.\]
Furthermore, for \(k\) odd, define
\[\alpha^k = \frac{a_k\sqrt{a} + b_k\sqrt{b}}{\sqrt{2}},\]
where \((a_k, b_k)\) are positive integers. It is well known that all positive integer solutions \((X, Y)\) of (2.1) are of the form \((a_k, b_k)\).

By investigating the occurrence of squares and certain square classes in some sets of Lehmer sequences, Luca and Walsh \[17\] completely solved the Diophantine equations of the type
\[ax^2 - by^4 = 2.\]

Theorem LW2 (Theorem 2 in \[17\]).
1. If \(b_1\) is not a square, then equation (2.4) has no solutions.
2. If \(b_1\) is a square and \(b_3\) is not a square, then \((X, Y) = (a_1, \sqrt{b_1})\) is the only solution of (2.4).
3. If \(b_1\) and \(b_3\) are both squares, then \((X, Y) = (a_1, \sqrt{b_1}), (a_3, \sqrt{b_3})\) are the only solutions of (2.4).
Recently, by the method similar to that in Luca and Walsh [17], Yuan and Li [28] confirmed a conjecture of Akhtari, Togbe and Walsh [3] by proving the following result.

**Theorem YL (28).** For any positive odd integers \( a, b \), the equation \( aX^4 - bY^2 = 2 \) has at most one solution in positive integers, and such a solution must arise from the minimal solution to the quadratic equation \( aX^2 - bY^2 = 2 \).

Let \( A \) and \( B \) be odd positive integers such that the Diophantine equation (2.5)
\[
Ax^2 - By^2 = 4
\]
has solutions in odd, positive integers \( x, y \). Let \( a_1, b_1 \) be the minimal positive integer solution. Define
\[
a_n\sqrt{A} + b_n\sqrt{B} = \left( \frac{a_1\sqrt{A} + b_1\sqrt{B}}{2} \right)^n.
\]
With these assumptions, Ljunggren [16] showed the following two results by computing some Jacobi’s symbols of the related Lehmer sequences.

**Theorem Lj.** The Diophantine equation \( Ax^4 - By^2 = 4 \) has at most two solutions in positive integers \( x, y \).

1. If \( a_1 = h^2 \) and \( Aa_1^2 - 3 = k^2 \), there are only two solutions, namely, \( x = \sqrt{a_1} = h \) and \( x = \sqrt{a_3} = hk \).
2. If \( a_1 = h^2 \) and \( Aa_1^2 - 3 \neq k^2 \), then \( x = \sqrt{a_1} = h \) is the only solution.
3. If \( a_1 = 5h^2 \) and \( A^2a_1^4 - 5Aa_1^2 + 5 = 5k^2 \), then the only solution is \( x = \sqrt{a_5} = 5hk \).

Otherwise there are no solutions.

By computing more Jacobi’s symbols of the related Lehmer sequences, Luo and Yuan [18] proved the following result.

**Theorem LY (18).**

1. If \( b_1 \) is not a square, then the equation (2.7)
\[
Ax^2 - By^4 = 4
\]
has no positive integer solutions except in the case \( b_1 = 3h^2 \) and \( Bb_1^2 + 3 = 3k^2 \), when \( y = \sqrt{b_3} \) is the only solution of (2.7).
2. If \( b_1 \) is a square, then (2.7) has at most one positive integer solution other than \( y = \sqrt{b_1} \), which is given by either \( y = \sqrt{b_3} \) or \( y = \sqrt{b_2} \), the latter occurring if and only if \( a_1 \) and \( b_1 \) are both squares and \( A = 1 \) and \( B \neq 5 \).
3. Other lemmas. In this section, we present some other lemmas that will be used later.

**Lemma 3.1** ([27]). Let $D \neq 2$ be a positive non-square integer with $8 \nmid D$.

(i) If $2 \mid D$, then one and only one of the Diophantine equations
\begin{equation}
\begin{aligned}
kx^2 - ly^2 &= 1 \\
&\text{has integer solutions, where } (k, l) \text{ ranges over all pairs of integers such that } k > 1, kl = D.
\end{aligned}
\end{equation}

(ii) If $2 \nmid D$, then one and only one of the Diophantine equations
\begin{equation}
\begin{aligned}
kx^2 - ly^2 &= 1, \\
kx^2 - ly^2 &= 2
\end{aligned}
\end{equation}
has integer solutions, where $(k, l)$ in the former equation ranges over all pairs of integers such that $k > 1, kl = D$, and $(k, l)$ in the latter equation ranges over all pairs of integers such that $k > 0, kl = D$.

(iii) If $2 \nmid D$ and the Diophantine equation $x^2 - Dy^2 = 4$ has solutions in odd integers $x$ and $y$, then one and only one of the Diophantine equations
\begin{equation}
kx^2 - ly^2 = 4
\end{equation}
has integer solutions, where $(k, l)$ ranges over all pairs of integers such that $k > 1, kl = D$.

The following lemma will be used in the proofs.

**Lemma 3.2.**

(i) Let $k > 1$ and $l$ be odd positive integers such that $kx^2 - ly^2 = 4$, $2 \nmid xy$, has positive integer solutions. Then $kx^2 - ly^2 = 1$ has positive integer solutions.

(ii) Let $D$ be a positive integer such that $x^2 - Dy^2 = 4$, $2 \nmid xy$, is solvable. Then one and only one of the Diophantine equations
\begin{equation}
kx^2 - ly^2 = 1
\end{equation}
has integer solutions, where $(k, l)$ ranges over all pairs of integers such that $k > 1, kl = D$.

*Proof.* Obvious from Lemma 3.1(iii).

We also need the following ten known results.

**Lemma 3.3** ([19]). Let $p$ be an odd prime. If $(L, M) \equiv (0, 3) \pmod{4}$ and $(\frac{L}{M}) = 1$, then the equation $P_p = px^2$ with $x$ an integer has no solutions.

**Lemma 3.4** ([18]). Let $L$ and $M$ be coprime positive odd integers with $L - 4M > 0$. If $Q_n = ku^2$, $k \mid n$, then $n = 1, 3, 5$. If $Q_n = 2ku^2$, $k \mid n$, then $n = 3$. 

Lemma 3.5 ([28]). Let \( p \) be an odd prime. If \((L, M) \equiv (2, 3) \pmod{4}\) and \(\left( \frac{L}{M} \right) = 1\), then the equation \( P_p = px^2 \) with \( x \) an integer has no integer solutions provided that \( p > 3 \), and the equation \( P_p = x^2 \) has no integer solutions.

Lemma 3.6 ([17]). Let \( p \) be an odd prime. If \((L, M) \equiv (2, 1) \pmod{4}\) and \(\left( \frac{L}{M} \right) = 1\), then the equation \( P_p = x^2 \) with \( x \) an integer has no integer solutions provided that \( p > 3 \), and the equation \( P_p = px^2 \) has no integer solutions.

Lemma 3.7 ([14]). The only positive integer solutions to the equation
\[
x^2 - 2y^4 = -1
\]
are \((x, y) = (1, 1), (239, 13)\).

Lemma 3.8 ([6], [26]). Let \( d > 3 \) be a non-square such that the Pell equation
\[
X^2 - dY^2 = -1
\]
is solvable in positive integers, and let \( \tau = v + u\sqrt{d} \) denote its minimal positive integer solution. Then the only positive integer solution to the equation
\[
X^2 - dY^4 = -1
\]
is \((X, Y) = (v, \sqrt{u})\).

Lemma 3.9 ([21]).

(i) Let \( a \) and \( b \) be positive integers, with a non-square, such that the equation \( aX^2 - bY^2 = 1 \) is solvable in positive integers. Let \((v, w)\) be the solution with \( v \) minimal, and put \( \tau = v\sqrt{a} + w\sqrt{b} \). Let \( w = n^2l \) with \( l \) odd and square-free. Then the Diophantine equation
\[
(3.4) \quad ax^2 - by^4 = 1
\]
has at most one solution in positive integers. If a solution \((x, y)\) to \( (3.4) \) exists, then \( x\sqrt{a} + y^2\sqrt{b} = \tau^l \).

(ii) Let \( D > 0 \) be a non-square integer. Define
\[
T_n + U_n\sqrt{D} = (T_1 + U_1\sqrt{D})^n,
\]
where \( T_1 + U_1\sqrt{D} \) is the fundamental solution of the Pell equation
\[
(3.5) \quad X^2 - DY^2 = 1.
\]
Then there are at most two positive integer solutions \((X, Y)\) to the equation
\[
(3.6) \quad X^2 - DY^4 = 1.
\]
1. If two solutions with \( Y_1 < Y_2 \) exist, then \( Y_1^2 = U_1 \) and \( Y_2^2 = U_2 \), except when \( D = 1785 \) or \( D = 16 \cdot 1785 \), in which case \( Y_1^2 = U_1 \) and \( Y_2^2 = U_4 \).
2. If only one positive integer solution \((X,Y)\) to equation (3.6) exists, then \(Y^2 = U_1\) where \(U_1 = lv^2\) for some square-free integer \(l\), and either \(l = 1\), \(l = 2\) or \(l = p\) for some prime \(p \equiv 3 \pmod{4}\).

**Lemma 3.10** ([20], [9]). Let the fundamental solution of the equation \(v^2 - du^2 = 1\) be \(a + b\sqrt{d}\). Then the only possible solutions to the equation \(X^4 - dY^2 = 1\) are given by \(X^2 = a\) and \(X^2 = 2a^2 - 1\); both solutions occur in the following cases: \(d = 1785, 7140, 28560\).

**Lemma 3.11** ([5]). Let \(s, d\) be positive integers with \(s > 1\). Then the Diophantine equation \(s^2 X^4 - dY^2 = 1\) has at most one positive integer solution \((X,Y)\), which can be given by \(X^2 s + \sqrt{d}Y = as + b\sqrt{d}\), where \(as + b\sqrt{d}\) is the minimal positive integer solution of the equation \(s^2 T^2 - dU^2 = 1\).

Let \(A > 1\) and \(B\) be positive integers with \(AB\) non-square, and let \(v\sqrt{A}+w\sqrt{B}\) be the minimal positive integer solution to the equation \(Ax^2 - By^2 = 1\). By the result of the first author [29], Bennett, Togbe and Walsh [4] and Akhtari [1], we have the following lemma.

**Lemma 3.12** ([4], [1]). The Diophantine equation (3.7) \(Ax^4 - By^2 = 1\) has at most two positive integer solutions. Moreover, (3.7) is solvable if and only if \(v\) is a square; and if \(x^2\sqrt{A} + y\sqrt{B} = (v\sqrt{A} + w\sqrt{B})^k\), then \(k = 1\) or \(k = p \equiv 3 \pmod{4}\) is a prime.

The following lemma is a generalization of an old result (Theorem 7.4.8 in [29]) of the first author.

**Lemma 3.13.** Suppose the equation \(A(r u^2)^2 - By^2 = 1\), where \(A > 1\), \(AB\) is not a square, and \(r \mid A\), has a solution. Let \(a_1 \sqrt{A} + b_1 \sqrt{B}\) be its minimal positive integer solution. Then \(a_1 = rv^2\) for some positive integer \(v\).

**Proof.** Let \((a_k, b_k)\) be positive integers such that (3.8) \(a_k \sqrt{A} + b_k \sqrt{B} = (a_1 \sqrt{A} + b_1 \sqrt{B})^k\). We have \(a_k = a_1 \cdot \frac{a_k}{a_1} = rv^2\) and \(\gcd(a_1, a_k/a_1) \mid k, r \mid k\). Hence \(P_k = a_k/a_1 = r_1 l\square, \quad a_1 = r_2 l\square, \quad r = r_1 r_2, \quad r_1 l \mid k\).

Now we show that \(r_1 l = 1\). Assume that this is not so and let \(p > 2\) be a prime divisor of \(r_1 l\). Then (3.9) \(P_k/P_{k/p} = pv^2\)
for some positive integer \(v\). This sequence satisfies the hypothesis of Lemma 3.3, therefore (3.9) is impossible, so \(r_1 l = 1\), as desired. Hence \(a_1 = r\). □

**Lemma 3.14.** Let \(a\) and \(b\) be odd positive integers such that the equation (2.1) is solvable in positive integers \((X, Y)\). Let \((a_1, b_1)\) and \((a_k, b_k)\) be defined by (2.2) and (2.3), respectively.

(i) If \(a_k = r\), \(r | aa_1 k\), \(r\) square-free, then \(k = 1\) or \(3\).

(ii) If \(b_k = s\), \(s | bb_1 k\), \(r\) square-free, then \(k = 1\) or \(3\).

**Proof.** First we prove (ii). Since \(b_k = b_1 \cdot \left(b_k/b_1\right) = r\), \(s | bb_1 k\) and \(\gcd(b_1, b_k/b_1) | k\), we have

\[ P_k = b_k/b_1 = s_1 l\square, \quad b_1 = s_2 l\square, \quad s = s_1 s_2, \quad s_1 l | k. \]

Let \(p\) be the largest prime divisor of \(k\). Since

\[ P_k = \frac{P_k}{P_{k/p}} \cdot P_{k/p} = s_1 l\square, \quad \gcd(P_k/P_{k/p}, P_{k/p}) | p, \]

we have \(P_k/P_{k/p} = \square\) or \(p\). Applying Lemma 3.6 to

\[ \frac{P_k}{P_{k/p}} = \frac{P'}{P_{k/p}} = \frac{\alpha^k - \overline{\alpha}^k}{\alpha^{k/p} - \overline{\alpha}^{k/p}} \]

we find that \(p = 3\). Hence \(k = 3^m\) for some non-negative integer \(m\). If \(m > 1\), then the above argument and Lemma 3.6 show that \(P_9 = \square\) and \(P_3 = \square\), which implies that the equation \(ax^2 - b^2 y^4 = 2\) has three positive integer solutions \((x, y)\) with \(y = 1, \sqrt{P_3}\) and \(\sqrt{P_9}\), which contradicts Theorem LW2. Therefore \(k = 1\) or \(3\).

Next we prove (i). By Lemma 3.5, we get \(k = 3^m\) for some non-negative integer \(m\). If \(m > 1\), then a similar argument and Lemma 3.5 show that \(P_9 = 3P_3\square\) and \(P_3 = 3\square\), which implies that the equation \(aa_1 x^4 - by^2 = 2\) has two positive integer solutions \((x, y)\) with \(x = 1\) and \(\sqrt{P_9}\), contradicting Theorem YL. Therefore \(k = 1\) or \(3\). □

We also need the following two lemmas.

**Lemma 3.15.**

(i) The equation

\[ 5x^4 + 5x^2 + 1 = y^2 \]

has no positive integer solutions.

(ii) The only positive integer solutions of the equation

\[ 5x^4 - 5x^2 + 1 = y^2 \]

are \((x, y) = (1, 1), (3, 19)\).

**Proof.** We obtain the results by MAGMA computations. □
**Lemma 3.16.** The only positive integer solution to the system

\[
\begin{align*}
3x^2 - y^2 &= 2, \\
2x^2 - z^2 &= 1,
\end{align*}
\]

is \((x, y, z) = (1, 1, 1)\).

**Proof.** We have \(x^2 + y^2 = 2z^2\) and \(2 \nmid xyz\). Hence there are integers \(u, v\) such that

\[z = u^2 + v^2, \quad x = u^2 - v^2 + 2uv.\]

Substituting this into \(2x^2 - z^2 = 1\) we get

\[u^4 + 8u^3v + 2u^2v^2 - 8uv^3 + v^4 = 1.\]

By a MAGMA computation, we obtain \(uv = 0\), and thus \((x, y, z) = (1, 1, 1)\).

**4. Proof of Theorem 1.1.** Define

\[R = \{p \mid (AX^2 + 1); \text{ord}_p(AX^2 + 1) \equiv 1 \pmod{4}\},\]

\[S = \{p \mid (AX^2 + 1); \text{ord}_p(AX^2 + 1) \equiv 2 \pmod{4}\},\]

\[Q = \{p \mid (AX^2 + 1); \text{ord}_p(AX^2 + 1) \equiv 3 \pmod{4}\},\]

and

\[r = \prod_{p \in R} p, \quad s = \prod_{p \in S} p, \quad t = \prod_{p \in Q} p.\]

By the above notation and (1.3) we have

\[(4.1) \quad rts^2(tu_1^2)^2 - AX^2 = 1, \quad rts^2(ru_2^2)^2 - AY^2 = 1\]

for some positive integers \(u_1\) and \(u_2\). Suppose \(rts^2 > 1\), and denote by \(\varepsilon = T_1\sqrt{rts^2} + U_1\sqrt{2}\) the minimal positive solution of the equation

\[(4.2) \quad rts^2T^2 - AU^2 = 1.\]

Then, by Lemma 3.13 and (4.1), we obtain

\[T_1 = t\Box = r\Box,\]

and so \(rt = 1\) since \(\gcd(r, t) = 1\). Hence (4.1) becomes

\[(4.3) \quad s^2u_1^4 - AX^2 = 1, \quad s^2u_2^4 - AY^2 = 1,\]

which has no positive integer solutions with \(X \neq Y\) by Lemma 3.11. Therefore (1.3) has no positive integer solutions with \(X \neq Y\).

**5. Proof of Theorem 1.2**

(1) The equation \((X^2 + 4)(Y^2 + 4) = Z^4\). We first consider the solution \((X, Y, Z)\) of (1.4) with \(2 \nmid XY\). Define
On the Diophantine equation \((x^2 \pm C)(y^2 \pm D) = z^4\)

\[ R = \{ p \mid (X^2 + 4); \text{ord}_p(X^2 + 4) \equiv 1 \pmod{4} \}, \]
\[ S = \{ p \mid (X^2 + 4); \text{ord}_p(X^2 + 4) \equiv 2 \pmod{4} \}, \]
\[ Q = \{ p \mid (X^2 + 4); \text{ord}_p(X^2 + 4) \equiv 3 \pmod{4} \}, \]

and

\[ r = \prod_{p \in R} p, \quad s = \prod_{p \in S} p, \quad t = \prod_{p \in Q} p. \]

Then

\[ X^2 + 4 = rts^2(tu_1^2)^2, \quad Y^2 + 4 = rts^2(ru_2^2)^2, \quad Z = rstu_1u_2. \]

We denote by \((T_1, U_1)\) the minimal positive solution of the equation

\[ rts^2T^2 - U^2 = 4 \]

and let

\[ \alpha = \frac{T_1 \sqrt{rts^2} + U_1}{2}. \]

For a positive integer \(k \geq 1\), we define \((T_k, U_k)\) to be positive integers such that

\[ \frac{T_k \sqrt{rts^2} + U_k}{2} = \alpha^k. \]

It is well known that all odd positive solutions of (5.2) are of the form \((T, U) = (T_k, U_k)\) for some positive integer \(k\) with \(3 \nmid k\). With the above notations, for any positive integer solution \((X, Y, Z)\) to \((X^2 + 4)(Y^2 + 4) = Z^4\) with \(2 \nmid X Y\), we have \(X = U_k\) and \(Y = U_l\) for some integers \(k\) and \(l\) and \(3 \nmid kl\), and that

\[ T_k = tu_1^2, \quad T_l = ru_2^2. \]

for some odd positive integers \(u_1\) and \(u_2\).

Let \(d = \gcd(k, l)\), \(k = d k_1, l = d l_1\). Then \(2 \nmid kl\). Noting that every prime divisor of \(\gcd(T_k/T_d, rtT_d)\) divides \(k_1\), we have

\[ T_k/T_d = k_2 \square, \quad k_2 \mid k_1. \]

Now we apply Lemma 3.4 to

\[ Q_{k_1} = \frac{T_k}{T_d} = \frac{\alpha^{k_1d} + \overline{\alpha}^{k_1d}}{\alpha^d + \overline{\alpha}^d}, \]

to deduce that \(k_1 \in \{1, 5\}\). Similarly, \(l_1 \in \{1, 5\}\).

Since \(k \neq l\), we may assume that \(k_1 = 1\) and \(l_1 = 5\). Hence

\[ T_d = tu_1^2, \quad T_{5d} = ru_2^2. \]

If \(t > 1\), then \(t \mid T_{5d}/T_d\) since \(rt\) square-free, \(\gcd(T_{5d}/T_d, rt)\mid 5\), so \(t = 5\) and \(T_{5d} = 5u_1^2\). Similarly, if \(r > 1\), then \(r = 5\). Therefore \(rt = 5\).
If \( r = 1 \) and \( t = 5 \), then \( T_d = 5u_1^2 \) and \( T_{5d} = u_2^2 \). By a direct computation we get \( 5s^4T_d^4 - 5s^2T_{5d}^2 + 1 = (u_2/5u_1)^2 \), so \( sT_d = 1 \) or 3 by Lemma 3.15, which is impossible since \( 5 \nmid T_d \).

If \( r = 5 \) and \( t = 1 \), then \( T_d = u_1^2 \) and \( T_{5d} = 5u_2^2 \). Similarly, we have \( 5s^4T_d^4 - 5s^2T_{5d}^2 + 1 = (u_2/u_1)^2 \), thus \( sT_d = 1 \) or 3. If \( sT_d = 3 \), then \( s = 3, T_d = 1 \) and \( 45 - 4 = U_2^2 \), which is impossible. If \( sT_d = 1 \), then \( U_d = 1, u_2 = 5 \) and \((X, Y, Z) = (1, 11, 5)\).

Next we consider the solution \((X, Y, Z)\) of (1.4) with \( 2 \nmid XY \). Then \( 2 \mid X \) and \( 2 \mid Y \), say \( X = 2X_1, Y = 2Y_1, Z = 2Z_1 \), and we obtain
\[
(X_1^2 + 1)(Y_1^2 + 1) = Z_1^4.
\]
By item 1 of Theorem LW1, the above equation has no positive integer solutions. Therefore, the only positive integer solutions to (1.4) are \((X, Y, Z) = (1, 11, 5), (11, 1, 5)\).

(2) The equation \((X^2 - 4)(Y^2 - 4) = Z^4\). We first consider the solution \((X, Y, Z)\) of (1.5) with \( 2 \nmid XY \). We retain the definitions of \( r, s, \) and \( t \) as given at the beginning of the proof of Theorem 1.2(1), but define them to be square-free numbers built up from prime divisors of \( X^2 + 4 \) instead of \( X^2 + 4 \). We denote by \((T_1, U_1)\) the minimal positive solution of the equation \((5.4)\)
\[
T^2 - rts^2U^2 = 4
\]
and let
\[
\alpha = \frac{T_1 + U_1\sqrt{rts^2}}{2}.
\]
For a positive integer \( k \geq 1 \), we define \((T_k, U_k)\) to be positive integers such that
\[
T_k + U_k\sqrt{rts^2} = \alpha^k.
\]
Proceeding as before, it follows that there are integers \( k \) and \( l \) such that \( X = T_k \) and \( Y = T_l \),
\[
(U_k = tu_1^2, \quad U_l = ru_2^2)
\]
for some odd positive integers \( u_1 \) and \( u_2 \).

We may assume that \( d = \gcd(k, l), k = dk_1, l = 2u_1l_1, 2 \nmid k_1l_1, u \geq 0 \). Then \( U_d = \gcd(U_k, U_l) = c\Box \) with \( c \mid rt \) since \( \gcd(r, t) = 1 \). Since
\[
U_l = \frac{U_l}{U_{1_l}} \cdot U_{1_l}, \quad \gcd(U_l/U_{1_l}, ru_{1_l}d) = 1,
\]
we have
\[
U_{1_l} = r\Box.
\]
Since every prime divisor of \( \gcd(U_{k_1l_1}/U_d, rtU_d) \) divides \( k_1 \), we obtain
\[
U_{k_1l_1}/U_d = n\Box, \quad n \mid k_1.
\]
Applying Lemma 3.4 to
\[ Q_{k_1} = \frac{U_{k_1}d}{U_d} = \frac{(\alpha d)^{k_1} + (-\bar{\alpha} d)^{k_1}}{(\alpha d) + (-\bar{\alpha} d)}, \]
we have \( k \in \{1, 5\} \). Similarly, \( l \in \{1, 5\} \). Since \( 2 \nmid U_kU_l \), \( k \neq l \), we may assume that \( k_1 = 1 \) and \( l_1 = 5 \). Hence
\[ U_d = tu_1^2, \quad U_{5d} = ru_2^2. \]
If \( t > 1 \), then \( t \mid U_5/U_1 \) since \( rt \) is square-free, \( \gcd(T_5/T_1, rt) \mid 5 \), so \( t = 5 \) and \( U_1 = 5u_1^2 \). Similarly, if \( r > 1 \), then \( r = 5 \). Thus \( rt = 5 \) when \( rt > 1 \).

If \( r = 1 \) and \( t = 5 \), then \( U_d = 5u_1^2 \) and \( U_{5d} = u_2^2 \). It follows that
\[ 5s^4U_d^4 + 5s^2U_d^2 + 1 = (u_2/5u_1)^2. \]
This yields \( sU_d = 0 \) by Lemma 3.15, which is impossible. If \( r = 5 \) and \( t = 1 \), then \( U_d = u_1^2 \) and \( U_{5d} = 5u_2^2 \), and
\[ 5s^4U_d^4 + 5s^2U_d^2 + 1 = (u_2/5u_1)^2. \]
This yields \( sU_d = 0 \) by Lemma 3.15 again, which is also impossible.

Next we consider the solution \((X, Y, Z)\) of (1.5) with \( X \neq Y \) and \( 2 \nmid XY \).
If \( 2 \mid X \) and \( 2 \mid Y \), then \( X = 2X_1, \ Y = 2Y_1, \ Z = 2Z_1 \), and we obtain
\[ (X_1^2 - 1)(Y_1^2 - 1) = Z_1^4. \]
By item 3 of Theorem LW1 the above equation has no positive integer solutions. If \( 2 \nmid X \) and \( 2 \mid Y \) (the case that \( 2 \nmid Y \) and \( 2 \mid X \) is similar), say \( Y = 2Y_1, \ Z = 2Z_1 \), then we obtain
\[ (X^2 - 4)(Y_1^2 - 1) = 4Z_1^2, \quad 2 \nmid X, \]
which has no positive integer solutions by Theorem 1.2(17). Hence (1.5) has no positive integer solutions.

(3) The equation \((X^2 - 2)(Y^2 - 2) = Z^4\). It is obvious that for any solution \((X, Y, Z)\) of the equation, we have \( X \neq Y \) and \( 2 \nmid XYZ \). We retain the definitions for \( r, s \), and \( t \) as given at the beginning of the proof of Theorem 1.2(1), but define them to be square-free numbers built up from prime divisors of \( X^2 - 2 \) instead of \( X^2 + 4 \).

From (1.6) we have
\[ X^2 - rt(tsu_1^2)^2 = 2, \quad Y^2 - rt(rsu_2^2)^2 = 2, \quad Z = rstu_1u_2 \]
for some positive integers \( u_1 \) and \( u_2 \). We denote by \((T_1, U_1)\) the minimal positive solution of the equation
\[ T^2 - rts^2U^2 = 2 \]
and for a positive integer \( k \geq 1 \), we define \((T_k, U_k)\) to be positive integers such that
\[ \frac{T_k + U_k\sqrt{rts^2}}{\sqrt{2}} = \left( \frac{T_1 + U_1\sqrt{rts^2}}{\sqrt{2}} \right)^k. \]
Proceeding as before, it follows that there are integers $k$ and $l$ such that $X = T_k$ and $Y = T_l$ for some odd integers $k$ and $l$, and

(5.8)\[ U_k = tu_1^2, \quad U_l = ru_2^2 \]

for some positive integers $u_1$ and $u_2$. By Lemma 3.14, we have $k, l \in \{1, 3\}$. Since $2 \nmid U_k U_l$, $k \neq l$, we may assume that $k = 1$ and $l = 3$. Hence

$$U_1 = tu_1^2, \quad U_3 = ru_2^2.$$ 

If $t > 1$, then $t \mid U_3/U_1$ since $rt$ is square-free, $\gcd(U_3/U_1, rt) \mid 3$, so $t = 3$ and $U_1 = 3u_1^2$. Similarly, if $r > 1$, then $r = 3$. Thus $rt = 3$ since $\gcd(r, t) = 1$.

If $r = 1$ and $t = 3$, then $U_1 = 3u_1^2$ and $U_3 = u_2^2$. It follows that

$$18s^2u_1^4 + 1 = \left( \frac{u_2}{3u_1} \right)^2,$$

which is also impossible since $2 \nmid su_1^2$. If $r = 3$ and $t = 1$, then $U_1 = u_1^2$ and $U_3 = 3u_2^2$. It follows that

$$2s^2u_1^4 + 1 = \left( \frac{u_2}{u_1} \right)^2,$$

which is impossible since $2 \nmid su_1^2$. Hence (1.6) has no positive integer solutions.

(4) The equation $(X^2 + 2)(Y^2 + 2) = Z^4$. It is obvious that for any solution $(X, Y, Z)$ of (1.7), we have $X \neq Y$ and $2 \nmid XYZ$. We retain the definitions for $r, s, t$ as given at the beginning of the proof of Theorem 1.2(1), but define them to be square-free numbers built up from prime divisors of $X^2 + 2$ instead of $X^2 + 4$.

From (1.7) we have

(5.9)\[ rt(ts_1^2)^2 - X^2 = 2, \quad rt(rs_2^2)^2 - Y^2 = 2 \]

for some positive integers $u_1$ and $u_2$. We denote by $(T_1, U_1)$ the minimal positive solution of the equation

(5.10)\[ rts^2T^2 - U^2 = 2 \]

and for a positive integer $k \geq 1$, we define $(T_k, U_k)$ to be positive integers such that

$$T_k \sqrt{rts^2} + U_k \sqrt{2} = \left( T_1 \sqrt{rts^2} + U_1 \sqrt{2} \right)^k.$$

Proceeding as before, we have $X = U_k$ and $T = U_l$ for some odd integers $k$ and $l$, and

(5.11)\[ T_k = tu_1^2, \quad T_l = ru_2^2 \]

for some positive integers $u_1$ and $u_2$. Moreover, $rt = 3$. 
On the Diophantine equation \((x^2 \pm C)(y^2 \pm D) = z^4\)

If \(r = 1\) and \(t = 3\), then \(T_1 = 3u_1^2\) and \(T_3 = u_2^2\). It follows that

\[18s^2u_1^4 - 1 = \left( \frac{u_2}{3u_1} \right)^2,\]

which has no solutions \((s, u_1, u_2)\).

If \(r = 3\) and \(t = 1\), then \(T_1 = u_1^2\) and \(T_3 = 3u_2^2\). It follows that

\[2s^2u_1^4 - 1 = \left( \frac{u_2}{u_1} \right)^2.\]

Combining this with the first equation of \((5.9)\) we obtain

\[3s^2u_1^4 - X^2 = 2, \quad 2s^2u_1^4 - m^2 = 1,\]

which has only the positive integer solution \((s, u_1, X, m) = (1, 1, 1, 1)\) by Lemma 3.16. Hence all positive integer solutions of \((1.7)\) are \((X, Y, Z) = (1, 5, 3), (5, 1, 3)\).

For the proofs of Theorem 1.2(5)–(7), we note that the equations in (5)–(7) have no solutions \((X, Y, Z)\) with \(2 \mid Z\), so we only consider the solutions \((X, Y, Z)\) with \(2 \nmid Z\).

(5) The equation \((X^2 + 2)(Y^2 - 2) = Z^2, 2 \nmid XY\). From the equation we have

\[X^2 + 2 = du_1^2, \quad Y^2 - du_2^2 = 2, \quad Z = du_1u_2,\]

which is impossible by Lemma 3.1 since both equations \(x^2 - dy^2 = 2\) and \(dx^2 - y^2 = 2\) would then have solutions.

(6) The equation \((X^2 + 2)(Y^2 + 1) = Z^2, 2 \nmid X\). From the equation we have

\[X^2 + 2 = du_1^2, \quad du_2^2 - Y^2 = 1, \quad Z = du_1u_2,\]

which is impossible by Lemma 3.1 since both equations \(dx^2 - y^2 = 2\) and \(dx^2 - y^2 = 1\) would then have solutions.

(7) The equation \((X^2 - 2)(Y^2 + 1) = Z^2, 2 \nmid X\). From the equation we have

\[X^2 - 2 = du_1^2, \quad du_2^2 - Y^2 = 1, \quad Z = du_1u_2,\]

which is impossible by Lemma 3.1 since both equations \(x^2 - dy^2 = 2\) and \(dx^2 - y^2 = 1\) would then have solutions.

(8) The equation \((X^2 + 2)(Y^2 - 4) = Z^4\). We divide the proof into two cases.

Case 1: \(2 \nmid XY\). We consider the following more general equation:

\[(X^2 + 2)(Y^2 - 4) = Z^2, \quad 2 \nmid XY.\]

From the above equation we have

\[(5.12) \quad X^2 + 2 = du_1^2, \quad Y^2 - du_2^2 = 4, \quad Z = du_1u_2.\]
It follows from Lemma 3.2(ii) and the second equation of (5.12) that one of the equations $d_1x^2 - dy^2 = 1$ with $d_1 > 1$ and $d_1d_2 = d$ has a solution, which is impossible by Lemma 3.1 since both equations $d_1x^2 - dy^2 = 1$, $d_1 > 1$ and $dx^2 - y^2 = 2$ would then have solutions.

Case 2: $2 | XY$. It is easy to see that (1.11) has no integer solutions when $2 | X$ and $2 \nmid Y$ by taking the equation modulo 4.

We first consider the subcase $2 | X$ and $2 | Y$. Write $X = 2X_1$, $Y = 2Y_1$, $Z = 2Z_1$. Then (1.11) becomes

$$(2X_1^2 + 1)(Y_1^2 - 1) = 2Z_1^4.$$  

We retain the definitions for $r, s$, and $t$ but define them to be square-free numbers built up from prime divisors of $2X^2 + 1$ instead of $AX^2 + 1$, as given at the beginning of the proof of Theorem 1.1. From (5.13) we have

$$rts^2(tu_1^2)^2 - 2X_1^2 = 1, \quad Y_1^2 - 2rts^2(ru_2^2)^2 = 1$$

for some positive integers $u_1$ and $u_2$. From the second equation of (5.14) and Lemma 3.1, we eventually get

$$rts^2(rm^2)^2 - 2n^4 = 1$$

as we did in the proof of Theorem 1.2(4). Hence $(2X_1^2 + 1)(2n^4 + 1) = Z_1^4$, which has no positive integer solutions by Theorem 1.1.

Next we deal with the subcase $2 \nmid X$ and $2 | Y$. Write $Y = 2Y_1$, $Z = 2Z_1$. We obtain the equation

$$(X^2 + 2)(Y_1^2 - 1) = 4Z_1^4.$$  

From (5.15), we have

$$(X^2 + 2)(Y_1^2 - 1) = 4Z_1^4.$$  

From (5.15), we have

$$rts^2(tu_1^2)^2 - 2X_1^2 = 2, \quad Y_1^2 - 4rts^2(ru_2^2)^2 = 1$$

for some positive integers $u_1$ and $u_2$. Similarly, from the second equation of (5.16) and Lemma 3.1, we finally obtain

$$rts^2(rm^2)^2 - n^4 = 2, \quad 2 \nmid n.$$  

Hence $(X^2 + 2)(n^4 + 2) = Z_1^4$, $2 \nmid Xn$, which has only the positive integer solution $(X, n, Z_1) = (5, 1, 3)$ by Theorem 1.2(4), and thus $r = 1$, $t = 3$, $s = 1$. Now the second equation of (5.16) becomes $Y_1^2 - 12u_2^4 = 1$, which is easily seen to have no positive integer solutions by Lemma 3.9.

(9) The equation $(X^2 + 2)(Y^2 + 4) = Z^4$. We divide the proof into two cases.

Case 1: $2 \nmid XY$. We consider the more general equation

$$(X^2 + 2)(Y^2 + 4) = Z^2, \quad 2 \nmid XY.$$  

From the above equation we have

$$(X^2 + 2) = du_1^2, \quad du_2^2 - Y^2 = 4, \quad Z = du_1u_2.$$
On the Diophantine equation \((x^2 \pm C)(y^2 \pm D) = z^4\)

It follows from the second equation of (5.17) that the equation \(dx^2 - y^2 = 1\) has a solution, which is impossible by Lemma 3.1 since both equations \(dx^2 - y^2 = 2\) and \(dx^2 - y^2 = 1\) would then have solutions.

**Case 2**: \(2 \mid XY\). It is easy to see that (1.12) has no integer solutions when \(2 \nmid X\) or \(2 \nmid Y\) by taking the equation modulo 16. Hence it suffices to consider the case \(2 \mid X\) and \(2 \mid Y\). Write \(X = 2X_1, Y = 2Y_1, Z = 2Z_1\). Then (1.12) becomes

\[(5.18) \quad (2X_1^2 + 1)(Y_1^2 + 1) = 2Z_1^4.\]

As before, it follows from (5.18) that

\[(5.19) \quad rts^2(tu_1^2)^2 - 2X_1^2 = 1, \quad 2rts^2(ru_2^2)^2 - Y_1^2 = 1\]

for some positive integers \(u_1\) and \(u_2\). This contradicts Lemma 3.1 when \(rst > 1\). If \(rst = 1\), then the first equation of (5.19) becomes \(u_1^4 - 2X_1^2 = 1\), which has no positive integer solutions by Lemma 3.10.

**Case 1**: \(2 \nmid X\). We retain the definitions \(r, s, t\) as given at the beginning of the proof of Theorem 1.2(4). From (1.13) we have

\[(5.20) \quad rts^2(tu_1^2)^2 - X^2 = 2, \quad Y^2 - rts^2(ru_2^2)^2 = 1\]

for some positive integers \(u_1\) and \(u_2\). It is easy to see that \(rts^2 \neq 1\). From the second equation of (5.20), we have the following two subcases.

**Subcase 1**: \(2 \mid u_2\). Then

\[Y + 1 = 2ar_1^2u_3^4, \quad Y - 1 = 2br_2^2u_4^4, \quad r_1r_2 = 2r, \quad 2u_3u_4 = u_2,\]

and thus \(ar_1^2u_3^4 - br_2^2u_4^4 = 1\). If \(a > 1\), then both equations \(rts^2x^2 - y^2 = 2\) and \(ax^2 - by^2 = 1\) have solutions, which contradicts Lemma 3.1. Hence \(a = 1\) and \(r \mid r_2\). Continuing the above process for the equation \(r_1^2u_3^4 - rts^2r_2^2u_4^4 = 1\), we finally get

\[rts^2(rm^2)^2 - n^4 = 2.\]

**Subcase 2**: \(2 \nmid u_2\). Then

\[Y + 1 = ar_1^2u_3^4, \quad Y - 1 = br_2^2u_4^4, \quad r_1r_2 = r, \quad u_3u_4 = u_2,\]

and thus \(ar_1^2u_3^4 - br_2^2u_4^4 = 2\). If \(b > 1\), then both equations \(rts^2x^2 - y^2 = 2\) and \(ax^2 - by^2 = 2\) have solutions, which contradicts Lemma 3.1. Hence \(b = 1, a = rts^2\) and \(r = r_1, r_2 = 1\), and we also get the equation

\[rts^2(rm^2)^2 - n^4 = 2.\]

It follows that \((X^2 + 2)(n^4 + 2) = Z_1^4\). From the proof of the equation \((X^2 + 2)(Y^2 + 2) = Z^4\) we have \(X = 5, n = 1\), hence \(X = 5, Y = 2, Z = 3\).
Therefore the equation \((X^2 + 2)(Y^2 - 1) = Z^4\) has only the positive integer solution \((X, Y, Z) = (5, 2, 3)\) with \(2 \nmid X\).

**Case 2:** \(2 \mid X\). Write \(X = 2X_1, Z = 2Z_1\). Then \((1.13)\) becomes

\[
(2X_1^2 + 1)(Y^2 - 1) = 8Z_1^4.
\]

The remaining proof is similar to the proof of Case 1 of Theorem 1.2(8). Thus the Diophantine equation \((X^2 + 2)(Y^2 - 1) = Z^4\) with \(2 \mid X\) has no positive integer solutions.

Therefore the equation \((X^2 + 2)(Y^2 - 1) = Z^4\) has only the positive integer solution \((X, Y, Z) = (5, 2, 3)\).

\[(11) \text{ The equation } (X^2 + 4)(Y^2 + 1) = Z^4. \text{ We divide the proof into two cases.}\]

**Case 1:** \(2 \nmid X\). An argument similar to the one employed for \((1.4)\) shows that there exist odd integers \(k\) and \(l\) such that \(3 \mid l\) and \(X = U_k\) and \(Y = U_l\) and

\[
T_k = tu_1^2, \quad T_l = 2ru_2^2
\]

for some positive integers \(u_1\) and \(u_2\).

Let \(d = \gcd(k, l), k = dk_1, l = dl_1\). Then \(2 \nmid k_1l_1\). By a similar method to the proof of Theorem 1.2(1) and by Lemma 3.4, we have \(k_1 \in \{1, 5\}\) and \(l_1 = 3\). We first consider the case \(k_1 = 1\). Then

\[
T_d = t\Box, \quad T_{3d} = 2r\Box.
\]

Since \(\gcd(T_{3d}/T_d, rt) \mid 3\), \(rt \mid T_{3d}/T_d\) and \(3 \nmid rts^2\), we have \(rt = 1\), which is impossible. Hence

\[
k_1 = 5, \quad T_{3d} = 2r\Box, \quad T_{5d} = t\Box.
\]

Since \(\gcd(T_{3d}/T_d, rt) \mid 3\) and \(3 \nmid rt\), we have \(r = 1\). Similarly, \(t = 5\). Now from \(T_d = \gcd(T_{3d}, T_{5d}) = \Box, T_5 = 5\Box, \) and \(r = 1, t = 5\), we derive that

\[
5s^4T_d^4 - 5s^2T_d^2 + 1 = \Box, \quad \text{and so } sT_d = 1 \text{ or } 3 \text{ by Lemma 3.15. If } sT_d = 1, \text{ then } s = 1, T_d = 1, U_d = 1, T_{3d} = 2, T_{5d} = 5, \text{ and thus } (1.14) \text{ has a solution } (X, Y, Z) = (11, 2, 5). \text{ If } sT_d = 3, \text{ then } s = 3, T_d = 1, \text{ which is impossible since } 3 \nmid Z. \text{ Therefore } (1.14) \text{ has only one positive integer solution } (X, Y, Z) = (11, 2, 5).
\]

**Case 2:** \(2 \mid X\). Write \(X = 2X_1, Z = 2Z_1\). As before we obtain the equation

\[
(X_1^2 + 1)(Y_1^2 + 1) = 4Z_1^4,
\]

and from \((5.23)\) we have

\[
2rts^2(tu_1^2)^2 - X_1^2 = 1, \quad 2rts^2(ru_2^2)^2 - Y^2 = 1.
\]
Similarly, by Lemma 3.13, we have \( rt = 1 \), and so

\[
(5.25) \quad 2s^2u_1^4 - X_1^2 = 1, \quad 2s^2u_2^4 - Y^2 = 1,
\]

which implies that \( s = 1 \) by Lemma 3.8. Thus

\[
(5.26) \quad X_1^2 - 2u_1^4 = -1, \quad Y^2 - 2u_2^4 = -1.
\]

It follows from Lemma 3.8 that \((X_1, Y, u_1, u_2) = (1, 239, 1, 13)\), \((239, 1, 13, 1)\).

Therefore the only positive integer solutions to the Diophantine equation

\[
(X^2 + 4)(Y^2 + 1) = Z^4
\]

are \((X,Y,Z) = (11, 2, 5), (239, 2, 6), (478, 1, 26)\).

The equation \((X^2 + 4)(Y^2 - 4) = Z^4\). We divide the proof into two cases.

CASE 1: \( 2 \nmid XY \). We define \( r, s, t \) as at the beginning of the proof of Theorem 1.2(1). We only consider the solution \((X, Y, Z)\) of \((1.15)\) with \(2 \nmid XY\). From \((1.15)\) we have

\[
(5.27) \quad rts^2(tu_1^2)^2 - X^2 = 4, \quad Y^2 - rts^2(ru_2^2)^2 = 4, \quad Z = rstu_1u_2, \quad 2 \nmid Z.
\]

From the second equation of \((5.27)\) there are positive integers \(a, b, r_1, r_2, u_3, u_4\) such that

\[
Y + 2 = ar_1^2u_3^4, \quad Y - 2 = br_2^2u_4^4, \quad ab = rts^2, \quad r = r_1r_2, \quad u_2 = u_3u_4,
\]

hence

\[
(5.28) \quad ar_1^2u_3^4 - br_2^2u_4^4 = 4, \quad 2 \nmid abr_1r_2u_3u_4.
\]

If \(a, b > 1\), then both equations \(rts^2x^2 - y^2 = 1\) and \(ax^2 - by^2 = 1\) with \(ab = rts^2, a, b > 1\) have integer solutions, contradicting Lemma 3.1.

If \(a > 1\) and \(b = 1\), then \(r_1 = r, r_2 = 1\), and so

\[
(5.29) \quad rts^2(ru_3^2)^2 - u_4^4 = 4, \quad u_4 \mid u_2.
\]

If \(a = 1\) and \(b = rts^2\), then repeating the above process for the equation \(u_3^4 - rts^2(ru_4^2)^2 = 4\) we eventually obtain

\[
(5.30) \quad rts^2(rm^2)^2 - n^4 = 4, \quad n \mid u_2.
\]

Combining \((5.30)\) or \((5.29)\) and the first equation of \((5.27)\) we get

\[
(5.31) \quad (n^4 + 4)(X^2 + 4) = Z^4_1, \quad 2 \nmid Xn.
\]

By Theorem 1.2(1), equation \((5.31)\) has no positive integer solutions. Therefore, \((1.15)\) has no positive integer solutions with \(2 \nmid XY\).

CASE 2: \( 2 \mid XY \). It is easy to see that the equation \((X^2+4)(Y^2-4) = Z^4\) has no integer solutions when \(2 \mid X\) and \(2 \nmid Y\) by taking the equation modulo 16.

Assume \(2 \mid X\) and \(2 \mid Y\). Write \(X = 2X_1, Y = 2Y_1, Z = 2Z_1\). Then, from \((1.15)\), we obtain

\[
(5.32) \quad (X_1^2 + 1)(Y_1^2 - 1) = Z_1^4.
\]
By Theorem LW1, the above equation has only the positive integer solutions
\((X_1, Y_1, Z_1) = (1, 3, 2), (239, 3, 26)\).

Next we consider the case \(2 \nmid X\) and \(2 \mid Y\). Write \(Y = 2Y_1, Z = 2Z_1\). Then
\[ (5.33) \quad (X^2 + 4)(Y^2_1 - 1) = 4Z_1^4. \]
From (5.33) we have
\[ (5.34) \quad rts^2(tu_1^2)^2 - X^2 = 4, \quad Y^2_1 - 4rts^2(ru_2^2)^2 = 1, \]
\[ Z_1 = rstu_1u_2, \quad 2 \nmid X. \]
Similarly, from the second equation of (5.34) and Lemma 3.1, we eventually obtain
\[ (5.35) \quad rts^2(rm^2)^2 - 4n_4^4 = 1 \quad \text{or} \quad rts^2(rm^2)^2 - n_4^4 = 1. \]
Combining (5.35) and the first equation of (5.34) we get
\[ (5.36) \quad (4n_4^4 + 1)(X^2 + 4) = Z_2^4 \quad \text{or} \quad (n_4^4 + 1)(X^2 + 4) = Z_2^4, \quad 2 \nmid X. \]
By the proof of Theorem 1.2(11), only the first equation in (5.36) has the positive integer solution
\((X, n, Z_2) = (11, 1, 5)\).

Therefore, (1.15) has only the positive integer solutions \((X, Y, Z) = (2, 6, 4), (478, 6, 52)\).

(13) The equation \((X^2 + 4)(Y^2 - 1) = Z^4\). We first consider the solution \((X, Y, Z)\) of (1.16) with \(2 \nmid X\). We retain the definitions for \(r, s, \) and \(t\) as given at the beginning of the proof of Theorem 1.2(1). Then from (1.16) we have
\[ (5.37) \quad rts^2(tu_1^2)^2 - X^2 = 4, \quad Y^2 - rts^2(ru_2^2)^2 = 1, \quad Z = rstu_1u_2. \]
If \(2 \nmid u_2\), then from the second equation of (5.37) there are positive integers \(a, b, r_1, r_2, u_3, u_4\) such that
\[ Y + 1 = ar_1^2u_3^4, \quad Y - 1 = br_2^2u_4^4, \quad ab = rts^2, \quad r = r_1r_2, \quad u_2 = u_3u_4, \]

hence
\[ (5.38) \quad ar_1^2u_3^4 - br_2^2u_4^4 = 2, \quad 2 \nmid abr_1r_2u_3u_4. \]
It follows that both equations \(rts^2x^2 - y^2 = 1\) and \(ax^2 - by^2 = 2, ab = rts^2, 2 \nmid xy\) have integer solutions, contradicting Lemma 3.1.

If \(2 \mid u_2\), then from the second equation of (5.38) there are positive integers \(a, b, r_1, r_2, u_3, u_4\) such that
\[ Y + 1 = 2ar_1^2u_3^4, \quad Y - 1 = 2br_2^2u_4^4, \quad ab = rts^2, \quad 2r = r_1r_2, \quad u_2 = u_3u_4, \]

hence
\[ (5.39) \quad ar_1^2u_3^4 - br_2^2u_4^4 = 1. \]
On the Diophantine equation \((x^2 \pm C)(y^2 \pm D) = z^4\)

If \(a, b > 1\), then both equations \(rts^2x^2 - y^2 = 1\) and \(ax^2 - by^2 = 1\), \(ab = rts^2\), \(a, b > 1\) have integer solutions, contradicting Lemma 3.1. If \(a > 1\) and \(b = 1\), then \(r_1 = r, r_2 = 1\), and so

\[
(5.40) \quad rts^2(ru_3^2)^2 - 4u_4^4 = 1.
\]

If \(a = 1\) and \(b = rts^2\), then repeating the above process for the equation \(u_4^4 - rts^2(ru_3^2)^2 = 1\) we eventually obtain

\[
(5.41) \quad rts^2(rm^2)^2 - 4n^4 = 1.
\]

Combining (5.41) or (5.40) and the first equation of (5.37) we get

\[
(5.42) \quad (4n^4 + 1)(X^2 + 4) = Z_1^4, \quad 2 \nmid X.
\]

By the proof of Theorem 1.2(11), equation (5.42) has only the positive integer solution \((X, n, Z_1) = (11, 1, 5)\). Therefore, (1.16) has no positive integer solutions with \(2 \nmid X\).

Next we consider the case \(2 \parallel X\). Write \(X = 2X_1, Z = 2Z_1\) with \(X_1\) odd. From (1.16) we obtain

\[
(5.43) \quad X_1^2 + 1 = 2rts^2(tu_4^2)^2, \quad Y^2 - 1 = 2rts^2(ru_2^2)^2.
\]

Similarly, from the second equation of (5.43) and Lemma 2.1, we obtain

\[
(5.44) \quad 2rts^2(2ru_3^2)^2 = u_4^4 + 1, \quad u_3, u_4 \in \mathbb{N}.
\]

Combining the first equation of (5.43) and equation (5.44) leads to

\[
(u_4^4 + 1)(X_1^2 + 1) = Z_2^4,
\]

which is impossible by Theorem LW1.

Now we consider the case \(4 \mid X\). Write \(X = 2X_1, Z = 2Z_1\) with \(X_1\) even. We obtain

\[
(5.45) \quad X_1^2 + 1 = rts^2(tu_1^2)^2, \quad Y^2 - 1 = rts^2(2ru_2^2)^2.
\]

Similarly, from the second equation of (5.45) and Lemma 3.1, we obtain

\[
(5.46) \quad rts^2(ru_3^2)^2 = (2u_4^2)^2 + 1, \quad u_3, u_4 \in \mathbb{N}.
\]

Combining the first equation of (5.45) and equation (5.46), we derive

\[
((2u_4^2)^2 + 1)(X_1^2 + 1) = Z_2^4,
\]

which is impossible by Theorem LW1. Thus the Diophantine equation \((X^2 + 4)(Y^2 - 1) = Z^4\) has no positive integer solutions.

(14) The equation \((X^2 - 4)(Y^2 + 1) = Z^4\). We consider the solution \((X, Y, Z)\) of (1.17) with \(2 \nmid X\). We retain the definitions for \(r, s,\) and \(t\) as given at the beginning of the proof of Theorem 1.2(2). Then

\[
(5.47) \quad X^2 - 4 = rts^2(tu_1^2)^2, \quad Y^2 + 1 = rts^2(ru_2^2)^2, \quad Z = rstu_1u_2.
\]
From the first equation of (5.47) there are positive integers $a, b, t_1, t_2, u_3, u_4$ such that
\[ X + 2 = at_1^2u_3^4, \quad X - 2 = bt_2^2u_4^4, \quad ab = rts^2, \quad t = t_1t_2, \quad u = u_3u_4, \]

hence
\[ (5.48) \quad at_1^2u_3^4 - bt_2^2u_4^4 = 4, \quad 2 \not| abr_2u_3u_4. \]

If $a, b > 1$, then both equations $rts^2x^2 - y^2 = 1$ and $ax^2 - by^2 = 1, ab = rts^2$, have integer solutions, contradicting Lemma 3.1.

If $a > 1$ and $b = 1$, then $r_1 = r, r_2 = 1$, and so
\[ (5.49) \quad rts^2(ru_2^2)^2 - u_4^4 = 4, \quad u_4 | u_2. \]

If $a = 1$ and $b = rts^2$, then repeating the above process for the equation $u_3^4 - rts^2(ru_2^2)^2 = 4$ we finally obtain
\[ (5.50) \quad rts^2(rm^2)^2 - n^4 = 4, \quad n | u_2. \]

Combining (5.50) or (5.49) and the second equation of (5.47) we get
\[ (5.51) \quad (n^4 + 4)(Y^2 + 1) = Z_1^4, \quad 2 \not| n. \]

By Theorem 1.2(11), equation (5.51) has no positive integer solutions. Therefore, (1.17) has no positive integer solutions with $2 \not| X$.

We now consider the case $2 \parallel X$. Write $X = 2X_1, Z = 2Z_1$ with $X_1$ odd. We obtain
\[ (5.52) \quad X_1^2 - 1 = 2rts^2(tu_1^2)^2, \quad Y^2 + 1 = 2rts^2(ru_2^2)^2. \]

From the first equality of (5.52) and Lemma 3.1 we get
\[ (5.53) \quad X_1 + 1 = 4rts^2(2ru_2^2)^2, \quad X_1 - 1 = 2u_4^4. \]

Thus
\[ (5.54) \quad 2rts^2(2ru_2^2)^2 = u_4^4 + 1, \]

which is impossible by taking the equation modulo 4.

Now we assume that $4 \mid X$. Write $X = 2X_1, Z = 2Z_1$ with $X_1$ even. We obtain
\[ (5.55) \quad X_1^2 - 1 = rts^2(tu_1^2)^2, \quad Y^2 + 1 = rts^2(2ru_2^2)^2; \]

however, the second equation of (5.55) is impossible by taking it modulo 4. Thus the Diophantine equation $(X^2 - 4)(Y^2 + 1) = Z^4$ has no positive integer solutions.

(15) The equation $(X^2 - 4)(Y^2 - 1) = Z^4$. We first consider the case $2 \not| X$. An argument similar to the one employed in the solution of (1.18) shows that there exist positive integers $k$ and $l$ such that $3 \mid l$ and $X = T_k$ and $Y = T_l$ and
we have (5.56)  
\[ U_k = tu_1^2, \quad U_l = 2ru_2^2 \]
for some positive integers \( u_1 \) and \( u_2 \).

We may assume that \( d = \gcd(k,l) \), \( k = dk_1 \), \( l = 2u_1u_2 \), \( 2 \nmid k_1l_1 \), \( u \geq 0 \). Then \( U_d = \gcd(U_k, U_l) = c\square \) with \( c \mid rt \) since \( \gcd(r,t) = 1 \). Since

\[
U_l = \frac{U_l}{U_{l_1d}} \cdot U_{l_1d}, \quad \gcd(U_l/U_{l_1d}, rU_{l_1d}) = 1,
\]
we have

\[
U_{l_1d} = 2r\square.
\]
Since every prime divisor of \( \gcd(U_{l_1d}/U_d, rtU_d) \) divides \( l_1 \), we obtain

\[
U_{l_1d}/U_d = m\square, \quad m \mid l_1.
\]
Applying Lemma 3.4 to

\[
Ql_1 = \frac{U_{l_1d}}{U_d} = \frac{(\alpha^d)^{l_1} + (-\alpha^d)^{l_1}}{(\alpha^d) + (-\alpha^d)}
\]
we have \( l_1 = 3 \). Similarly, \( k_1 \in \{1, 5\} \). We first consider the case \( k_1 = 1 \). Then

\[
U_d = t\square, \quad U_{3d} = 2r\square.
\]
Since every prime divisor of \( \gcd(T_{3d}/T_d, rt) \) divides 3, and \( rt \mid T_{3d}/T_d \) (as \( \gcd(r,t) = 1 \)), we have \( rt = 3 \), which is impossible since \( T_d^2 - 3s^2U_d^2 = 4 \) and \( 2 \nmid T_d \). Hence

\[
k_1 = 5, \quad T_{3d} = 2r\square, \quad T_{5d} = t\square.
\]
Since every prime divisor of \( \gcd(T_{3d}/T_d, rt) \) divides 3, we have \( r \mid 3 \); similarly, \( t \nmid 5 \).

Since \( T_d^2 - rts^2U_d^2 = 4, 2 \nmid T_d \), we have \( rt \neq 1, 3, 15 \), so \( r = 1 \) and \( t = 5 \). Now from \( U_d = \gcd(U_{3d}, U_{5d}) = \square, U_{5d} = 5\square, r = 1, t = 5 \), we derive that \( 5s^4U_d^4 + 5s^2U_d^2 + 1 = \square \), and so \( sT_d = 0 \) by Lemma 3.15, which is impossible. Therefore (1.18) has no positive integer solutions with \( 2 \nmid X \).

Now we consider the case \( 2 \mid X \). Write \( X = 2X_1, Z = 2Z_1 \). Then (1.18) becomes

(5.57)  
\[ (X_1^2 - 1)(Y^2 - 1) = 4Z_1^2. \]
We first consider the case \( 2 \mid X_1Y \). We may assume that \( 2 \mid Y \) and \( 2 \nmid X_1 \). From (5.57), there are positive integers \( u_1, u_2 \) such that

(5.58)  
\[ Y^2 - 1 = rts^2(ru_2^2), \quad X_1^2 - 1 = 4rts^2(tu_1^2)^2, \quad 2 \nmid rtsu_2. \]
From the first equation of (5.58), there exist odd integers \( m, n, r_1, r_2, u_3, u_4 \) such that

(5.59)  
\[ m(r_1u_3^2)^2 - n(r_2u_4^2)^2 = 2, \quad mn = rts^2, \quad r_1r_2 = r, \quad u_3u_4 = u_2. \]
From the second equation of (5.58) and Lemma 3.1, there exist positive integers $t_1, t_2, u_5, u_6$ such that

$$X + 1 = 2t_1^2 u_5^4, \quad X - 1 = 2t_2^2 rts^2 u_6^4, \quad 2 \mid u_6, t_1t_2 = t. \quad (5.60)$$

It follows that $t_1 = 1$ and

$$u_5^4 - rts^2 t^2 u_6^4 = 1, \quad 2 \mid u_6. \quad (5.61)$$

From (5.59), (5.61) and Lemma 3.1, we derive

$$u_2^2 + 1 = 2u_7^2,$$

which implies that $u_5 = 239$ and

$$239^2 - 1 = 3 \cdot 5 \cdot 7 \cdot 17 \cdot 25 = 8rts^2 u_8^4,$$

which is impossible.

Finally we consider the case $2 \nmid X_1 Y$. From (5.57), there are positive integers $u_1, u_2$ such that

$$Y^2 - 1 = 2rtes^2 (ru_2)^2, \quad X_1^2 - 1 = 2rtes^2 (tu_6)^2, \quad 2 \nmid rtsu_2. \quad (5.62)$$

From the first equation of (5.62), there exist positive integers $m > 1, n, r_1, r_2, u_3, u_4$ such that

$$m(r_1u_3^2)^2 - n(r_2u_4^2)^2 = 1, \quad mn = 2rtes^2 \text{ or } mn = rtes^2/2, \quad r_1r_2 = r, \quad u_3u_4 = u_2. \quad (5.63)$$

From the second equation of (5.62), (5.63) and Lemma 3.1, there exist positive integers $t_1, t_2, u_5, u_6$ such that

$$mt_1^2 u_5^4 - nt_2^2 u_6^4 = 1, \quad t_1t_2 = t. \quad (5.64)$$

Since $m > 1$, it follows from Lemma 3.13, (5.63) and (5.64) that $r_1t_1 = 1$, and so $rt \mid n$ and $m = 2s_1^2$. Therefore we have the equation

$$2s_1^2 u_3^4 - rts_2^2 (ru_4^2)^2 = 1, \quad 2s_1^2 u_5^4 - rts_2^2 (tu_6)^2 = 1, \quad s_1s_2 = s \text{ or } s_1s_2 = s/2. \quad (5.65)$$

We denote by $(T_1, U_1)$ the minimal positive integer solution of the Pell equation

$$2s_1^2 T^2 - rts_2^2 U^2 = 1 \quad (5.66)$$

and let $\varepsilon = T_1 \sqrt{2s_1^2} + U_1 \sqrt{rts_2^2}$. For a positive integer $k \geq 1$, let $(T_k, U_k)$ be positive integers given by

$$T_k \sqrt{2s_1^2} + U_k \sqrt{rts_2^2} = \varepsilon^k.$$

Assume $rt > 1$. By Lemma 3.12, we assume that $T_1 \sqrt{2s_1^2} + U_1 \sqrt{rts_2^2} = u_3 \sqrt{2s_1^2} + ru_4 \sqrt{rts_2^2}$ and $T_k \sqrt{2s_1^2} + U_k \sqrt{rts_2^2} = u_5 \sqrt{2s_1^2} + tu_6 \sqrt{rts_2^2}$. Then $U_k = tu_6^2 = U_1 \cdot tu_6^2 / (ru_4^2)$. It follows that $rt \mid k$, say $k = rtl$ for some positive
integer \( l \). Observe that \( k = p \equiv 3 \pmod{4} \) and \( rt \geq 1 \); by Lemma 3.12 again, we obtain \( l = 1 \) and the equation
\[
2s_1^2 - ps_2^2U^4 = 1, \quad p \equiv 3 \pmod{4},
\]
which is impossible by taking it modulo 8.

Now we assume that \( rt = 1 \). Then, by Lemma 3.9, the equation \( X^2 - 2s^2U^4 = 1 \) has at most one positive integer solution \((X, U)\), so \( X_1 = Y, u_1 = u_2 \) by (5.58).

Obviously, (5.58) has infinite many trivial solutions \((X_1, Y, S, u_1, u_2) = (Y, Y, S, 1, 1)\), where \( Y^2 - 2S^2 = 1 \).

Therefore the Diophantine equation \( (X^2 - 4)(Y^2 - 1) = Z^4 \) has only the trivial solutions \((X, Y, Z) = (2Y, Y, 2S)\), where \( Y^2 - 2S^2 = 1 \).

\(16\) The equation \( (X^2 - 2)(Y^2 + 4) = Z^4 \). We divide the proof into two cases.

**Case 1:** \( 2 \nmid XY \). We consider the more general equation
\[
(X^2 - 2)(Y^2 + 4) = Z^2, \quad 2 \nmid XY.
\]
From the above equation we have
\[
X^2 + 2 = du_1^2, \quad du_2^2 - Y^2 = 4, \quad Z = du_1u_2.
\]
It follows from the second equation of (5.67) that the equation \( dx^2 - y^2 = 1 \) has a solution, which is impossible by Lemma 3.1 since both equations \( x^2 - dy^2 = 2 \) and \( dx^2 - y^2 = 1 \) would then have solutions.

**Case 2:** \( 2 \mid XY \). It is easy to see that the equation \( (X^2 - 2)(Y^2 + 4) = Z^4 \) has no integer solutions when \( 2 \mid X \) and \( 2 \nmid Y \) by taking the equation modulo 4. We consider two subcases.

**Subcase 1:** \( 2 \mid X \) and \( 2 \mid Y \). Write \( X = 2X_1, Y = 2Y_1, Z = 2Z_1 \). We obtain
\[
(2X_1^2 - 1)(Y_1^2 + 1) = 2Z_1^4.
\]
We retain the definitions for \( r, s \) and \( t \) as given at the beginning of the proof of Theorem 1.1, but define them to be square-free numbers built up from prime divisors of \( 2X_1^2 - 1 \) instead of \( AX_1^2 + 1 \). We obtain
\[
2X_1^2 - rts^2(tu_1^2)^2 = 1, \quad 2rts^2(ru_2^2)^2 - Y_1^2 = 1,
\]
for some positive integers \( u_1 \) and \( u_2 \) with \( Z_1 = rtsu_1u_2 \). It follows from Lemma 3.1 that \( rts^2 = 1 \). Therefore
\[
2X_1^2 - u_1^4 = 1, \quad Y_1^2 - 2u_2^4 = -1.
\]
It follows from (5.71), (5.72) and Lemma 3.7, and a theorem of Ljunggren, that \( X_1 = 1, u_1 = 1, (Y_1, u_2) = (1, 1), (239, 13) \).
Subcase 2: $2 \nmid X$ and $2 \mid Y$. Write $Y = 2Y_1$, $Z = 2Z_1$. We obtain

\[(5.73) \quad (X^2 - 2)(Y_1^2 + 1) = 4Z_1^4.\]

We retain the definitions for $r, s$ and $t$ as given at the beginning of the proof of Theorem 1.2(3). We have

\[(5.74) \quad X^2 - rts^2(tu_1^2)^2 = 2,\]
\[(5.75) \quad rts^2(2ru_2^2)^2 - Y_1^2 = 1,\]

for some positive integers $u_1$ and $u_2$. It follows from Lemma 3.1 that $rts^2 = 1$.

\[(5.76) \quad X^2 - u_1^4 = 2,\]

which is impossible. Thus the only positive integer solutions of the Diophantine equation $(X^2 - 2)(Y^2 + 4) = Z^4$ are $(X, Y, Z) = (2, 2, 2)$ and $(2, 478, 26)$.

\[(17) \quad \text{The equation } (X^2 - 4)(Y^2 - 1) = 4Z^4. \quad \text{The proof is almost the same} \quad \text{as for } (X^2 - 4)(Y^2 - 1) = Z^4, \quad 2 \nmid X; \quad \text{we leave the details to the reader.}\]

This completes the proof of Theorem 1.2. □

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