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On the Diophantine equation $(x^2 \pm C)(y^2 \pm D) = z^4$

by

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1. Introduction. Let $L > 0$ and M be rational integers such that $L - 4M > 0$ and $(L, M) = 1$. Let α and β be the two roots of the trinomial $x^2 - \sqrt{L}\,x + M$. For a non-negative integer n, the nth term in the Lehmer sequence $\{P_n\}$ and the associated Lehmer sequence $\{Q_n\}$ (see [11]) are defined by

$$P_n := P_n(\alpha,\beta) = \begin{cases} \dfrac{\alpha^n - \beta^n}{\alpha - \beta} & \text{for } n \text{ odd},\\[2mm] \dfrac{\alpha^n - \beta^n}{\alpha^2 - \beta^2} & \text{for } n \text{ even},\end{cases}$$

and

$$Q_n := Q_n(\alpha,\beta) = \begin{cases} \dfrac{\alpha^n + \beta^n}{\alpha + \beta} & \text{for } n \text{ odd},\\[2mm] \alpha^n + \beta^n & \text{for } n \text{ even}.\end{cases}$$

Lehmer sequences have many interesting properties and often arise in the study of Diophantine equations. The arithmetic properties of the numbers P_n can be found in [11, 25].

Let a, b be positive integers such that ab is not a square. Diophantine equations of the form

$$aX^4 - bY^2 = c, \tag{1.1}$$

where $c \in \{\pm 1, \pm 2, \pm 4\}$, have received considerable interest, as we see from the references [2, 7, 8, 17, 19, 22, 23]. The study of these equations goes back to the classical work of Ljunggren [12, 13, 15, 16], who was able to prove many sharp results on (1.1). The following cases have been considered: Ljunggren [15] ($c = -1$), [16] ($c = 4$), Luca and Walsh [17] ($c = -2$), Luo and Yuan [18] ($c = \pm 4$), Akhtari [1] ($c = 1$) and Yuan and Li [28] ($c = 2$).

As an application of some results on (1.1), Luca and Walsh [17] proved the following theorem.

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Theorem LW1 (Theorem 3 in [17]).

1. *The equation*
$$(X^2+1)(Y^2+1)=Z^4$$
has no positive integer solutions.
2. *The only positive integer solutions of the equation*
$$(X^2+1)(Y^2-1)=Z^4$$
are $(X,Y,Z)=(1,3,2),(239,3,26)$.
3. *The equation*
$$(X^2-1)(Y^2-1)=Z^4$$
has no positive integer solutions.

In this paper, we will investigate the positive integer solutions (x,y,z) of the Diophantine equations of the type

$$(x^2\pm C)(y^2\pm D)=z^4, \tag{1.2}$$

where $C,D\in\{1,2,4\}$. The main purpose is try to completely solve the remaining eighteen equations of the type (1.2). The main results of the present paper are as follows. Throughout, $\square$ stands for a square, and $\left(\frac{A}{B}\right)$ for the Jacobi symbol of A with respect to B, where A and B are coprime integers.

Theorem 1.1. *Let* $A>1$ *be a positive integer. Then the Diophantine equation*

$$(AX^2+1)(AY^2+1)=Z^4 \tag{1.3}$$

has no positive integer solutions (X,Y,Z) *with* $X\neq Y$.

Theorem 1.2.

(1) *The only positive integer solutions of the equation*
$$(X^2+4)(Y^2+4)=Z^4 \tag{1.4}$$
are $(X,Y,Z)=(1,11,5),(11,1,5)$.

(2) *The equation*
$$(X^2-4)(Y^2-4)=Z^4 \tag{1.5}$$
has no positive integer solutions.

(3) *The equation*
$$(X^2-2)(Y^2-2)=Z^4 \tag{1.6}$$
has no positive integer solutions.

(4) *The only positive integer solutions of the equation*
$$(X^2+2)(Y^2+2)=Z^4 \tag{1.7}$$
are $(X,Y,Z)=(1,5,3),(5,1,3)$.

(5) *The equation*

$$(X^2+2)(Y^2-2)=Z^4 \tag{1.8}$$

has no positive integer solutions.

(6) *The equation*

$$(X^2+2)(Y^2+1)=Z^4 \tag{1.9}$$

has no positive integer solutions.

(7) *The equation*

$$(X^2-2)(Y^2+1)=Z^4 \tag{1.10}$$

has no positive integer solutions.

(8) *The equation*

$$(X^2+2)(Y^2-4)=Z^4 \tag{1.11}$$

has no positive integer solutions.

(9) *The equation*

$$(X^2+2)(Y^2+4)=Z^4 \tag{1.12}$$

has no positive integer solutions.

(10) *The only positive integer solution to the equation*

$$(X^2+2)(Y^2-1)=Z^4 \tag{1.13}$$

is $(X,Y,Z)=(5,2,3)$.

(11) *The only positive integer solutions to the equation*

$$(X^2+4)(Y^2+1)=Z^4 \tag{1.14}$$

are $(X,Y,Z)=(11,2,5),(2,239,26),\ (478,1,26)$.

(12) *The only positive integer solutions of the equation*

$$(X^2+4)(Y^2-4)=Z^4 \tag{1.15}$$

are $(X,Y,Z)=(2,6,4),\ (478,6,52)$.

(13) *The equation*

$$(X^2+4)(Y^2-1)=Z^4 \tag{1.16}$$

has no positive integer solutions.

(14) *The equation*

$$(X^2-4)(Y^2+1)=Z^4 \tag{1.17}$$

has no positive integer solutions.

(15) *The equation*

$$(X^2-4)(Y^2-1)=Z^4 \tag{1.18}$$

has only infinitely many trivial positive solutions $(X,Y,Z) = (2Y,Y,2S)$, *where* Y,S *are positive integers with* $Y^2-2S^2=1$.

(16) *The only positive integer solutions to the equation*

$$(X^2-2)(Y^2+4)=Z^4 \tag{1.19}$$

are $(X,Y,Z)=(2,2,2),\ (2,478,26)$.

(17) *The equation*

$$(X^2-4)(Y^2-1)=4Z^4, \quad 2\nmid X, \tag{1.20}$$

has no positive integer solutions.

However, we have not been able to solve the following two equations:

$$(X^2-2)(Y^2-4)=Z^4, \quad 2\,|\,XY, \tag{1.21}$$

$$(X^2-2)(Y^2-1)=Z^4, \quad 2\,|\,X. \tag{1.22}$$

We leave this as an open question.

2. The results on the equation $ax^2-by^4=c$**.** In this section, we will list all the related results on equations $ax^2-by^4=\pm 2,\pm 4$, which will be used later.

Let a and b be odd positive integers such that the equation

$$aX^2-bY^2=2 \tag{2.1}$$

is solvable in positive integers (X,Y). Let (a_1,b_1) be the minimal positive solution to (2.1), and define

$$\alpha=\frac{a_1\sqrt{a}+b_1\sqrt{b}}{\sqrt{2}}. \tag{2.2}$$

Furthermore, for k odd, define

$$\alpha^k=\frac{a_k\sqrt{a}+b_k\sqrt{b}}{\sqrt{2}}, \tag{2.3}$$

where $(a_k,\ b_k)$ are positive integers. It is well known that all positive integer solutions (X,Y) of (2.1) are of the form $(a_k,\ b_k)$.

By investigating the occurrence of squares and certain square classes in some sets of Lehmer sequences, Luca and Walsh [17] completely solved the Diophantine equations of the type

$$ax^2-by^4=2. \tag{2.4}$$

Theorem LW2 (Theorem 2 in [17]).

1. *If* b_1 *is not a square, then equation* (2.4) *has no solutions.*
2. *If* b_1 *is a square and* b_3 *is not a square, then* $(X,Y)=(a_1,\sqrt{b_1})$ *is the only solution of* (2.4).
3. *If* b_1 *and* b_3 *are both squares, then* $(X,Y)=(a_1,\sqrt{b_1}),\ (a_3,\sqrt{b_3})$ *are the only solutions of* (2.4).

Recently, by the method similar to that in Luca and Walsh [17], Yuan and Li [28] confirmed a conjecture of Akhtari, Togbe and Walsh [3] by proving the following result.

Theorem YL ([28]). *For any positive odd integers a, b, the equation $aX^4 - bY^2 = 2$ has at most one solution in positive integers, and such a solution must arise from the minimal solution to the quadratic equation $aX^2 - bY^2 = 2$.*

Let A and B be odd positive integers such that the Diophantine equation

$$Ax^2 - By^2 = 4 \tag{2.5}$$

has solutions in odd, positive integers x, y. Let a_1, b_1 be the minimal positive integer solution. Define

$$\frac{a_n\sqrt{A} + b_n\sqrt{B}}{2} = \left(\frac{a_1\sqrt{A} + b_1\sqrt{B}}{2}\right)^n. \tag{2.6}$$

With these assumptions, Ljunggren [16] showed the following two results by computing some Jacobi's symbols of the related Lehmer sequences.

Theorem Lj. *The Diophantine equation $Ax^4 - By^2 = 4$ has at most two solutions in positive integers x, y.*

1. *If $a_1 = h^2$ and $Aa_1^2 - 3 = k^2$, there are only two solutions, namely, $x = \sqrt{a_1} = h$ and $x = \sqrt{a_3} = hk$.*
2. *If $a_1 = h^2$ and $Aa_1^2 - 3 \neq k^2$, then $x = \sqrt{a_1} = h$ is the only solution.*
3. *If $a_1 = 5h^2$ and $A^2a_1^4 - 5Aa_1^2 + 5 = 5k^2$, then the only solution is $x = \sqrt{a_5} = 5hk$.*

Otherwise there are no solutions.

By computing more Jacobi's symbols of the related Lehmer sequences, Luo and Yuan [18] proved the following result.

Theorem LY ([18]).

1. *If b_1 is not a square, then the equation*

$$Ax^2 - By^4 = 4 \tag{2.7}$$

   has no positive integer solutions except in the case $b_1 = 3h^2$ and $Bb_1^2 + 3 = 3k^2$, when $y = \sqrt{b_3}$ is the only solution of (2.7).
2. *If b_1 is a square, then (2.7) has at most one positive integer solution other than $y = \sqrt{b_1}$, which is given by either $y = \sqrt{b_3}$ or $y = \sqrt{b_2}$, the latter occurring if and only if a_1 and b_1 are both squares and $A = 1$ and $B \neq 5$.*

3. Other lemmas. In this section, we present some other lemmas that will be used later.

LEMMA 3.1 ([27]). *Let $D \neq 2$ be a positive non-square integer with $8 \nmid D$.*

(i) *If $2 \mid D$, then one and only one of the Diophantine equations*

$$kx^2 - ly^2 = 1 \tag{3.1}$$

has integer solutions, where (k, l) ranges over all pairs of integers such that $k > 1$, $kl = D$.

(ii) *If $2 \nmid D$, then one and only one of the Diophantine equations*

$$kx^2 - ly^2 = 1, \qquad kx^2 - ly^2 = 2 \tag{3.2}$$

has integer solutions, where (k, l) in the former equation ranges over all pairs of integers such that $k > 1, kl = D$, and (k, l) in the latter equation ranges over all pairs of integers such that $k > 0$, $kl = D$.

(iii) *If $2 \nmid D$ and the Diophantine equation $x^2 - Dy^2 = 4$ has solutions in odd integers x and y, then one and only one of the Diophantine equations*

$$kx^2 - ly^2 = 4 \tag{3.3}$$

has integer solutions, where (k, l) ranges over all pairs of integers such that $k > 1$, $kl = D$.

The following lemma will be used in the proofs.

LEMMA 3.2.

(i) *Let $k > 1$ and l be odd positive integers such that $kx^2 - ly^2 = 4$, $2 \nmid xy$, has positive integer solutions. Then $kx^2 - ly^2 = 1$ has positive integer solutions.*

(ii) *Let D be a positive integer such that $x^2 - Dy^2 = 4$, $2 \nmid xy$, is solvable. Then one and only one of the Diophantine equations*

$$kx^2 - ly^2 = 1$$

has integer solutions, where (k, l) ranges over all pairs of integers such that $k > 1$, $kl = D$.

Proof. Obvious from Lemma 3.1(iii). ■

We also need the following ten known results.

LEMMA 3.3 ([19]). *Let p be an odd prime. If $(L, M) \equiv (0, 3) \pmod 4$ and $\left(\frac{L}{M}\right) = 1$, then the equation $P_p = px^2$ with x an integer has no solutions.*

LEMMA 3.4 ([18]). *Let L and M be coprime positive odd integers with $L - 4M > 0$. If $Q_n = ku^2$, $k \mid n$, then $n = 1, 3, 5$. If $Q_n = 2ku^2$, $k \mid n$, then $n = 3$.*

LEMMA 3.5 ([28]). *Let p be an odd prime. If $(L, M) \equiv (2, 3) \pmod 4$ and $\left(\frac{L}{M}\right) = 1$, then the equation $P_p = px^2$ with x an integer has no integer solutions provided that $p > 3$, and the equation $P_p = x^2$ has no integer solutions.*

LEMMA 3.6 ([17]). *Let p be an odd prime. If $(L, M) \equiv (2, 1) \pmod 4$ and $\left(\frac{L}{M}\right) = 1$, then the equation $P_p = x^2$ with x an integer has no integer solutions provided that $p > 3$, and the equation $P_p = px^2$ has no integer solutions.*

LEMMA 3.7 ([14]). *The only positive integer solutions to the equation*

$$x^2 - 2y^4 = -1$$

are $(x, y) = (1, 1), (239, 13)$.

LEMMA 3.8 ([6], [26]). *Let $d > 3$ be a non-square such that the Pell equation*

$$X^2 - dY^2 = -1$$

is solvable in positive integers, and let $\tau = v + u\sqrt{d}$ denote its minimal positive integer solution. Then the only positive integer solution to the equation

$$X^2 - dY^4 = -1$$

is $(X, Y) = (v, \sqrt{u})$.

LEMMA 3.9 ([21]).

(i) *Let a and b be positive integers, with a non-square, such that the equation $aX^2 - bY^2 = 1$ is solvable in positive integers. Let (v, w) be the solution with v minimal, and put $\tau = v\sqrt{a} + w\sqrt{b}$. Let $w = n^2 l$ with l odd and square-free. Then the Diophantine equation*

$$ax^2 - by^4 = 1 \tag{3.4}$$

has at most one solution in positive integers. If a solution (x, y) to (3.4) exists, then $x\sqrt{a} + y^2\sqrt{b} = \tau^l$.

(ii) *Let $D > 0$ be a non-square integer. Define*

$$T_n + U_n\sqrt{D} = (T_1 + U_1\sqrt{D})^n,$$

where $T_1 + U_1\sqrt{D}$ is the fundamental solution of the Pell equation

$$X^2 - DY^2 = 1. \tag{3.5}$$

Then there are at most two positive integer solutions (X, Y) to the equation

$$X^2 - DY^4 = 1. \tag{3.6}$$

1. *If two solutions with $Y_1 < Y_2$ exist, then $Y_1^2 = U_1$ and $Y_2^2 = U_2$, except when $D = 1785$ or $D = 16 \cdot 1785$, in which case $Y_1^2 = U_1$ and $Y_2^2 = U_4$.*

2. *If only one positive integer solution* (X,Y) *to equation* (3.6) *exists, then* $Y^2 = U_l$ *where* $U_1 = lv^2$ *for some square-free integer* l*, and either* $l = 1$, $l = 2$ *or* $l = p$ *for some prime* $p \equiv 3 \pmod 4$.

LEMMA 3.10 ([20], [9]). *Let the fundamental solution of the equation* $v^2 - du^2 = 1$ *be* $a + b\sqrt{d}$. *Then the only possible solutions to the equation* $X^4 - dY^2 = 1$ *are given by* $X^2 = a$ *and* $X^2 = 2a^2 - 1$*; both solutions occur in the following cases:* $d = 1785, 7140, 28560$.

LEMMA 3.11 ([5]). *Let* s, d *be positive integers with* $s > 1$. *Then the Diophantine equation*

$$s^2X^4 - dY^2 = 1$$

has at most one positive integer solution (X,Y), *which can be given by* $X^2s + \sqrt{d}Y = as + b\sqrt{d}$, *where* $as + b\sqrt{d}$ *is the minimal positive integer solution of the equation* $s^2T^2 - dU^2 = 1$.

Let $A > 1$ and B be positive integers with AB non-square, and let $v\sqrt{A}+w\sqrt{B}$ be the minimal positive integer solution to the equation $Ax^2 - By^2 = 1$. By the result of the first author [29], Bennett, Togbe and Walsh [4] and Akhtari [1], we have the following lemma.

LEMMA 3.12 ([4], [1]). *The Diophantine equation*

$$Ax^4 - By^2 = 1 \tag{3.7}$$

has at most two positive integer solutions. Moreover, (3.7) *is solvable if and only if* v *is a square; and if* $x^2\sqrt{A} + y\sqrt{B} = (v\sqrt{A} + w\sqrt{B})^k$, *then* $k = 1$ *or* $k = p \equiv 3 \pmod 4$ *is a prime.*

The following lemma is a generalization of an old result (Theorem 7.4.8 in [29]) of the first author.

LEMMA 3.13. *Suppose the equation*

$$A(ru^2)^2 - By^2 = 1,$$

where $A > 1$, AB *is not a square, and* $r \mid A$, *has a solution. Let* $a_1\sqrt{A}+b_1\sqrt{B}$ *be its minimal positive integer solution. Then* $a_1 = rv^2$ *for some positive integer* v.

Proof. Let (a_k, b_k) be positive integers such that

$$a_k\sqrt{A} + b_k\sqrt{B} = (a_1\sqrt{A} + b_1\sqrt{B})^k. \tag{3.8}$$

We have $a_k = a_1 \cdot \frac{a_k}{a_1} = ru^2$ and $\gcd(a_1, a_k/a_1) \mid k$, $r \mid k$. Hence

$$P_k = a_k/a_1 = r_1l\square, \qquad a_1 = r_2l\square,\ r = r_1r_2,\ r_1l \mid k.$$

Now we show that $r_1l = 1$. Assume that this is not so and let $p > 2$ be a prime divisor of r_1l. Then

$$P_k/P_{k/p} = pv^2 \tag{3.9}$$

for some positive integer v. This sequence satisfies the hypothesis of Lemma 3.3, therefore (3.9) is impossible, so $r_1 l = 1$, as desired. Hence $a_1 = r\Box$. ∎

LEMMA 3.14. *Let a and b be odd positive integers such that the equation* (2.1) *is solvable in positive integers (X, Y). Let (a_1, b_1) and (a_k, b_k) be defined by* (2.2) *and* (2.3), *respectively.*

(i) *If $a_k = r\Box$, $r \mid aa_1k$, r square-free, then $k = 1$ or 3.*
(ii) *If $b_k = s\Box$, $s \mid bb_1k$, r square-free, then $k = 1$ or 3.*

Proof. First we prove (ii). Since $b_k = b_1 \cdot (b_k/b_1) = r\Box$, $s \mid bb_1k$ and $\gcd(b_1, b_k/b_1) \mid k$, we have

$$P_k = b_k/b_1 = s_1 l\Box, \qquad b_1 = s_2 l\Box,\ s = s_1 s_2,\ s_1 l \mid k.$$

Let p be the largest prime divisor of k. Since

$$P_k = \frac{P_k}{P_{k/p}} \cdot P_{k/p} = s_1 l\Box, \quad \gcd(P_k/P_{k/p}, P_{k/p}) \mid p,$$

we have $P_k/P_{k/p} = \Box$ or $p\Box$. Applying Lemma 3.6 to

$$\frac{P_k}{P_{k/p}} = P'_p = \frac{\alpha^k - \overline{\alpha}^k}{\alpha^{k/p} - \overline{\alpha}^{k/p}}$$

we find that $p = 3$. Hence $k = 3^m$ for some non-negative integer m. If $m > 1$, then the above argument and Lemma 3.6 show that $P_9 = \Box$ and $P_3 = \Box$, which implies that the equation $ax^2 - bb_1^2 y^4 = 2$ has three positive integer solutions (x, y) with $y = 1, \sqrt{P_3}$ and $\sqrt{P_9}$, which contradicts Theorem LW2. Therefore $k = 1$ or 3.

Next we prove (i). By Lemma 3.5, we get $k = 3^m$ for some non-negative integer m. If $m > 1$, then a similar argument and Lemma 3.5 show that $P_9 = 3P_3\Box$ and $P_3 = 3\Box$, which implies that the equation $aa_1x^4 - by^2 = 2$ has two positive integer solutions (x, y) with $x = 1$ and $\sqrt{P_9}$, contradicting Theorem YL. Therefore $k = 1$ or 3. ∎

We also need the following two lemmas.

LEMMA 3.15.

(i) *The equation*

$$5x^4 + 5x^2 + 1 = y^2$$

has no positive integer solutions.

(ii) *The only positive integer solutions of the equation*

$$5x^4 - 5x^2 + 1 = y^2$$

are $(x, y) = (1, 1), (3, 19)$.

Proof. We obtain the results by MAGMA computations. ∎

LEMMA 3.16. *The only positive integer solution to the system*

$$\begin{cases} 3x^2 - y^2 = 2, \\ 2x^2 - z^2 = 1, \end{cases}$$

is $(x, y, z) = (1, 1, 1)$.

Proof. We have $x^2 + y^2 = 2z^2$ and $2 \nmid xyz$. Hence there are integers u, v such that

$$z = u^2 + v^2, \quad x = u^2 - v^2 + 2uv.$$

Substituting this into $2x^2 - z^2 = 1$ we get

$$u^4 + 8u^3v + 2u^2v^2 - 8uv^3 + v^4 = 1.$$

By a MAGMA computation, we obtain $uv=0$, and thus $(x, y, z)=(1, 1, 1)$. ∎

4. Proof of Theorem 1.1. Define

$$R = \{p \mid (AX^2 + 1);\ \mathrm{ord}_p(AX^2 + 1) \equiv 1 \pmod 4\},$$
$$S = \{p \mid (AX^2 + 1);\ \mathrm{ord}_p(AX^2 + 1) \equiv 2 \pmod 4\},$$
$$Q = \{p \mid (AX^2 + 1);\ \mathrm{ord}_p(AX^2 + 1) \equiv 3 \pmod 4\},$$

and

$$r = \prod_{p \in R} p, \quad s = \prod_{p \in S} p, \quad t = \prod_{p \in Q} p.$$

By the above notation and (1.3) we have

$$rts^2(tu_1^2)^2 - AX^2 = 1, \quad rts^2(ru_2^2)^2 - AY^2 = 1 \tag{4.1}$$

for some positive integers u_1 and u_2. Suppose $rts^2 > 1$, and denote by $\varepsilon = T_1\sqrt{rts^2} + U_1\sqrt{2}$ the minimal positive solution of the equation

$$rts^2T^2 - AU^2 = 1. \tag{4.2}$$

Then, by Lemma 3.13 and (4.1), we obtain

$$T_1 = t\square = r\square,$$

and so $rt = 1$ since $\gcd(r, t) = 1$. Hence (4.1) becomes

$$s^2u_1^4 - AX^2 = 1, \quad s^2u_2^4 - AY^2 = 1, \tag{4.3}$$

which has no positive integer solutions with $X \neq Y$ by Lemma 3.11. Therefore (1.3) has no positive integer solutions with $X \neq Y$. ∎

5. Proof of Theorem 1.2

(1) *The equation* $(X^2 + 4)(Y^2 + 4) = Z^4$. We first consider the solution (X, Y, Z) of (1.4) with $2 \nmid XY$. Define

$$R = \{p \,|\, (X^2+4);\ \mathrm{ord}_p(X^2+4) \equiv 1 \pmod 4\},$$
$$S = \{p \,|\, (X^2+4);\ \mathrm{ord}_p(X^2+4) \equiv 2 \pmod 4\},$$
$$Q = \{p \,|\, (X^2+4);\ \mathrm{ord}_p(X^2+4) \equiv 3 \pmod 4\},$$

and

$$r = \prod_{p\in R} p, \qquad s = \prod_{p\in S} p, \qquad t = \prod_{p\in Q} p.$$

Then

$$X^2+4 = rts^2(tu_1^2)^2, \quad Y^2+4 = rts^2(ru_2^2)^2, \quad Z = rstu_1u_2. \tag{5.1}$$

We denote by (T_1, U_1) the minimal positive solution of the equation

$$rts^2T^2 - U^2 = 4 \tag{5.2}$$

and let

$$\alpha = \frac{T_1\sqrt{rts^2} + U_1}{2}.$$

For a positive integer $k \geq 1$, we define (T_k, U_k) to be positive integers such that

$$\frac{T_k\sqrt{rts^2} + U_k}{2} = \alpha^k.$$

It is well known that all odd positive solutions of (5.2) are of the form $(T, U) = (T_k, U_k)$ for some positive integer k with $3 \nmid k$. With the above notations, for any positive integer solution (X, Y, Z) to $(X^2+4)(Y^2+4) = Z^4$ with $2 \nmid XY$, we have $X = U_k$ and $Y = U_l$ for some integers k and l and $3 \nmid kl$, and that

$$T_k = tu_1^2, \qquad T_l = ru_2^2 \tag{5.3}$$

for some odd positive integers u_1 and u_2.

Let $d = \gcd(k, l)$, $k = dk_1$, $l = dl_1$. Then $2 \nmid kl$. Noting that every prime divisor of $\gcd(T_k/T_d, rtT_d)$ divides k_1, we have

$$T_k/T_d = k_2\square, \quad k_2 \,|\, k_1.$$

Now we apply Lemma 3.4 to

$$Q_{k_1} = \frac{T_k}{T_d} = \frac{\alpha^{k_1d} + \overline{\alpha}^{k_1d}}{\alpha^d + \overline{\alpha}^d},$$

to deduce that $k_1 \in \{1, 5\}$. Similarly, $l_1 \in \{1, 5\}$.

Since $k \neq l$, we may assume that $k_1 = 1$ and $l_1 = 5$. Hence

$$T_d = tu_1^2, \qquad T_{5d} = ru_2^2.$$

If $t > 1$, then $t \,|\, T_{5d}/T_d$ since rt square-free, $\gcd(T_{5d}/T_d, rt) \,|\, 5$, so $t = 5$ and $T_{5d} = 5u_1^2$. Similarly, if $r > 1$, then $r = 5$. Therefore $rt = 5$.

If $r=1$ and $t=5$, then $T_d=5u_1^2$ and $T_{5d}=u_2^2$. By a direct computation we get $5s^4T_d^4-5s^2T_d^2+1=(u_2/5u_1)^2$, so $sT_d=1$ or 3 by Lemma 3.15, which is impossible since $5\mid T_d$.

If $r=5$ and $t=1$, then $T_d=u_1^2$ and $T_{5d}=5u_2^2$. Similarly, we have $5s^4T_d^4-5s^2T_d^2+1=(u_2/u_1)^2$, thus $sT_d=1$ or 3. If $sT_d=3$, then $s=3$, $T_d=1$ and $45-4=U_d^2$, which is impossible. If $sT_d=1$, then $U_d=1, u_2=5$ and $(X,Y,Z)=(1,11,5)$.

Next we consider the solution (X,Y,Z) of (1.4) with $2\mid XY$. Then $2\mid X$ and $2\mid Y$, say $X=2X_1, Y=2Y_1, Z=2Z_1$, and we obtain

$$(X_1^2+1)(Y_1^2+1)=Z_1^4.$$

By item 1 of Theorem LW1, the above equation has no positive integer solutions. Therefore, the only positive integer solutions to (1.4) are $(X,Y,Z)=(1,11,5),(11,1,5)$.

(2) *The equation* $(X^2-4)(Y^2-4)=Z^4$. We first consider the solution (X,Y,Z) of (1.5) with $2\nmid XY$. We retain the definitions of r,s, and t as given at the beginning of the proof of Theorem 1.2(1), but define them to be square-free numbers built up from prime divisors of X^2-4 instead of X^2+4. We denote by (T_1,U_1) the minimal positive solution of the equation

$$T^2-rts^2U^2=4 \tag{5.4}$$

and let

$$\alpha=\frac{T_1+U_1\sqrt{rts^2}}{2}.$$

For a positive integer $k\ge 1$, we define (T_k,U_k) to be positive integers such that

$$\frac{T_k+U_k\sqrt{rts^2}}{2}=\alpha^k.$$

Proceeding as before, it follows that there are integers k and l such that $X=T_k$ and $Y=T_l$,

$$U_k=tu_1^2,\qquad U_l=ru_2^2 \tag{5.5}$$

for some odd positive integers u_1 and u_2.

We may assume that $d=\gcd(k,l)$, $k=dk_1$, $l=2^ul_1d, 2\nmid k_1l_1$, $u\ge 0$. Then $U_d=\gcd(U_k,U_l)=c\square$ with $c\mid rt$ since $\gcd(r,t)=1$. Since

$$U_l=\frac{U_l}{U_{l_1d}}\cdot U_{l_1d},\qquad \gcd(U_l/U_{l_1d},rU_{l_1d})=1,$$

we have

$$U_{l_1d}=r\square.$$

Since every prime divisor of $\gcd(U_{k_1d}/U_d,rtU_d)$ divides k_1, we obtain

$$U_{k_1d}/U_d=n\square,\qquad n\mid k_1.$$

Applying Lemma 3.4 to

$$Q_{k_1} = \frac{U_{k_1 d}}{U_d} = \frac{(\alpha^d)^{k_1} + (-\overline{\alpha}^d)^{k_1}}{(\alpha^d) + (-\overline{\alpha}^d)},$$

we have $k \in \{1, 5\}$. Similarly, $l \in \{1, 5\}$. Since $2 \nmid U_k U_l$, $k \neq l$, we may assume that $k_1 = 1$ and $l_1 = 5$. Hence

$$U_d = tu_1^2, \qquad U_{5d} = ru_2^2.$$

If $t > 1$, then $t \mid U_5/U_1$ since rt is square-free, $\gcd(T_5/T_1, rt) \mid 5$, so $t = 5$ and $U_1 = 5u_1^2$. Similarly, if $r > 1$, then $r = 5$. Thus $rt = 5$ when $rt > 1$.

If $r = 1$ and $t = 5$, then $U_d = 5u_1^2$ and $U_{5d} = u_2^2$. It follows that $5s^4U_d^4 + 5s^2U_d^2 + 1 = (u_2/5u_1)^2$. This yields $sU_d = 0$ by Lemma 3.15, which is impossible. If $r = 5$ and $t = 1$, then $U_d = u_1^2$ and $U_{5d} = 5u_2^2$, and $5s^4U_d^4 + 5s^2U_d^2 + 1 = (u_2/u_1)^2$. This yields $sU_d = 0$ by Lemma 3.15 again, which is also impossible.

Next we consider the solution (X, Y, Z) of (1.5) with $X \neq Y$ and $2 \mid XY$. If $2 \mid X$ and $2 \mid Y$, then $X = 2X_1$, $Y = 2Y_1$, $Z = 2Z_1$, and we obtain

$$(X_1^2 - 1)(Y_1^2 - 1) = Z_1^4.$$

By item 3 of Theorem LW1 the above equation has no positive integer solutions. If $2 \nmid X$ and $2 \mid Y$ (the case that $2 \nmid Y$ and $2 \mid X$ is similar), say $Y = 2Y_1$, $Z = 2Z_1$, then we obtain

$$(X^2 - 4)(Y_1^2 - 1) = 4Z_1^2, \qquad 2 \nmid X,$$

which has no positive integer solutions by Theorem 1.2(17). Hence (1.5) has no positive integer solutions.

(3) *The equation* $(X^2 - 2)(Y^2 - 2) = Z^4$. It is obvious that for any solution (X, Y, Z) of the equation, we have $X \neq Y$ and $2 \nmid XYZ$. We retain the definitions for r, s, and t as given at the beginning of the proof of Theorem 1.2(1), but define them to be square-free numbers built up from prime divisors of $X^2 - 2$ instead of $X^2 + 4$.

From (1.6) we have

$$X^2 - rt(tsu_1^2)^2 = 2, \quad Y^2 - rt(rsu_2^2)^2 = 2, \quad Z = rstu_1u_2 \tag{5.6}$$

for some positive integers u_1 and u_2. We denote by (T_1, U_1) the minimal positive solution of the equation

$$T^2 - rts^2U^2 = 2 \tag{5.7}$$

and for a positive integer $k \geq 1$, we define (T_k, U_k) to be positive integers such that

$$\frac{T_k + U_k\sqrt{rts^2}}{\sqrt{2}} = \left(\frac{T_1 + U_1\sqrt{rts^2}}{\sqrt{2}}\right)^k.$$

Proceeding as before, it follows that there are integers k and l such that $X = T_k$ and $Y = T_l$ for some odd integers k and l, and

$$U_k = tu_1^2, \quad U_l = ru_2^2 \tag{5.8}$$

for some positive integers u_1 and u_2. By Lemma 3.14, we have $k, l \in \{1, 3\}$. Since $2 \nmid U_k U_l$, $k \neq l$, we may assume that $k = 1$ and $l = 3$. Hence

$$U_1 = tu_1^2, \quad U_3 = ru_2^2.$$

If $t > 1$, then $t \mid U_3/U_1$ since rt is square-free, $\gcd(U_3/U_1, rt) \mid 3$, so $t = 3$ and $U_1 = 3u_1^2$. Similarly, if $r > 1$, then $r = 3$. Thus $rt = 3$ since $\gcd(r, t) = 1$.

If $r = 1$ and $t = 3$, then $U_1 = 3u_1^2$ and $U_3 = u_2^2$. It follows that

$$18s^2u_1^4 + 1 = \left(\frac{u_2}{3u_1}\right)^2,$$

which is also impossible since $2 \nmid su_1^2$. If $r = 3$ and $t = 1$, then $U_1 = u_1^2$ and $U_3 = 3u_2^2$. It follows that

$$2s^2u_1^4 + 1 = \left(\frac{u_2}{u_1}\right)^2,$$

which is impossible since $2 \nmid su_1^2$. Hence (1.6) has no positive integer solutions.

(4) *The equation* $(X^2 + 2)(Y^2 + 2) = Z^4$. It is obvious that for any solution (X, Y, Z) of (1.7), we have $X \neq Y$ and $2 \nmid XYZ$. We retain the definitions for r, s, and t as given at the beginning of the proof of Theorem 1.2(1), but define them to be square-free numbers built up from prime divisors of $X^2 + 2$ instead of $X^2 + 4$.

From (1.7) we have

$$rt(tsu_1^2)^2 - X^2 = 2, \quad rt(rsu_2^2)^2 - Y^2 = 2 \tag{5.9}$$

for some positive integers u_1 and u_2. We denote by (T_1, U_1) the minimal positive solution of the equation

$$rts^2T^2 - U^2 = 2 \tag{5.10}$$

and for a positive integer $k \geq 1$, we define (T_k, U_k) to be positive integers such that

$$\frac{T_k\sqrt{rts^2} + U_k}{\sqrt{2}} = \left(\frac{T_1\sqrt{rts^2} + U_1}{\sqrt{2}}\right)^k.$$

Proceeding as before, we have $X = U_k$ and $T = U_l$ for some odd integers k and l, and

$$T_k = tu_1^2, \quad T_l = ru_2^2 \tag{5.11}$$

for some positive integers u_1 and u_2. Moreover, $rt = 3$.

If $r = 1$ and $t = 3$, then $T_1 = 3u_1^2$ and $T_3 = u_2^2$. It follows that

$$18s^2u_1^4 - 1 = \left(\frac{u_2}{3u_1}\right)^2,$$

which has no solutions (s, u_1, u_2).

If $r = 3$ and $t = 1$, then $T_1 = u_1^2$ and $T_3 = 3u_2^2$. It follows that

$$2s^2u_1^4 - 1 = \left(\frac{u_2}{u_1}\right)^2.$$

Combining this with the first equation of (5.9) we obtain

$$3s^2u_1^4 - X^2 = 2, \quad 2s^2u_1^4 - m^2 = 1,$$

which has only the positive integer solution $(s, u_1, X, m) = (1, 1, 1, 1)$ by Lemma 3.16. Hence all positive integer solutions of (1.7) are $(X, Y, Z) = (1, 5, 3), (5, 1, 3)$.

For the proofs of Theorem 1.2(5)–(7), we note that the equations in (5)–(7) have no solutions (X, Y, Z) with $2 \mid Z$, so we only consider the solutions (X, Y, Z) with $2 \nmid Z$.

(5) *The equation* $(X^2 + 2)(Y^2 - 2) = Z^2, 2 \nmid XY$. From the equation we have

$$X^2 + 2 = du_1^2, \quad Y^2 - du_2^2 = 2, \qquad Z = du_1u_2,$$

which is impossible by Lemma 3.1 since both equations $x^2 - dy^2 = 2$ and $dx^2 - y^2 = 2$ would then have solutions.

(6) *The equation* $(X^2 + 2)(Y^2 + 1) = Z^2$, $2 \nmid X$. From the equation we have

$$X^2 + 2 = du_1^2, \quad du_2^2 - Y^2 = 1, \quad Z = du_1u_2,$$

which is impossible by Lemma 3.1 since both equations $dx^2 - y^2 = 2$ and $dx^2 - y^2 = 1$ would then have solutions.

(7) *The equation* $(X^2 - 2)(Y^2 + 1) = Z^2$, $2 \nmid X$. From the equation we have

$$X^2 - 2 = du_1^2, \quad du_2^2 - Y^2 = 1, \quad Z = du_1u_2,$$

which is impossible by Lemma 3.1 since both equations $x^2 - dy^2 = 2$ and $dx^2 - y^2 = 1$ would then have solutions.

(8) *The equation* $(X^2 + 2)(Y^2 - 4) = Z^4$. We divide the proof into two cases.

CASE 1: $2 \nmid XY$. We consider the following more general equation:

$$(X^2 + 2)(Y^2 - 4) = Z^2, \qquad 2 \nmid XY.$$

From the above equation we have

$$X^2 + 2 = du_1^2, \quad Y^2 - du_2^2 = 4, \quad Z = du_1u_2. \tag{5.12}$$

It follows from Lemma 3.2(ii) and the second equation of (5.12) that one of the equations $d_1x^2-d_2y^2=1$ with $d_1>1$ and $d_1d_2=d$ has a solution, which is impossible by Lemma 3.1 since both equations $d_1x^2-d_2y^2=1, d_1>1$ and $dx^2-y^2=2$ would then have solutions.

Case 2: $2\mid XY$. It is easy to see that (1.11) has no integer solutions when $2\mid X$ and $2\nmid Y$ by taking the equation modulo 4.

We first consider the subcase $2\mid X$ and $2\mid Y$. Write $X=2X_1$, $Y=2Y_1$, $Z=2Z_1$. Then (1.11) becomes

$$(2X_1^2+1)(Y_1^2-1)=2Z_1^4. \tag{5.13}$$

We retain the definitions for r, s, and t but define them to be square-free numbers built up from prime divisors of $2X^2+1$ instead of AX^2+1, as given at the beginning of the proof of Theorem 1.1. From (5.13) we have

$$rts^2(tu_1^2)^2-2X_1^2=1, \qquad Y_1^2-2rts^2(ru_2^2)^2=1 \tag{5.14}$$

for some positive integers u_1 and u_2. From the second equation of (5.14) and Lemma 3.1, we eventually get

$$rts^2(rm^2)^2-2n^4=1$$

as we did in the proof of Theorem 1.2(4). Hence $(2X_1^2+1)(2n^4+1)=Z_2^4$, which has no positive integer solutions by Theorem 1.1.

Next we deal with the subcase $2\nmid X$ and $2\mid Y$. Write $Y=2Y_1$, $Z=2Z_1$. We obtain the equation

$$(X^2+2)(Y_1^2-1)=4Z_1^4. \tag{5.15}$$

From (5.15), we have

$$rts^2(tu_1^2)^2-X^2=2, \qquad Y_1^2-4rts^2(ru_2^2)^2=1 \tag{5.16}$$

for some positive integers u_1 and u_2. Similarly, from the second equation of (5.16) and Lemma 3.1, we finally obtain

$$rts^2(rm^2)^2-n^4=2, \qquad 2\nmid n.$$

Hence $(X^2+2)(n^4+2)=Z_2^4$, $2\nmid Xn$, which has only the positive integer solution $(X,n,Z_1)=(5,1,3)$ by Theorem 1.2(4), and thus $r=1$, $t=3$, $s=1$. Now the second equation of (5.16) becomes $Y_1^2-12u_2^4=1$, which is easily seen to have no positive integer solutions by Lemma 3.9.

(9) *The equation* $(X^2+2)(Y^2+4)=Z^4$. We divide the proof into two cases.

Case 1: $2\nmid XY$. We consider the more general equation

$$(X^2+2)(Y^2+4)=Z^2, \qquad 2\nmid XY.$$

From the above equation we have

$$X^2+2=du_1^2, \qquad du_2^2-Y^2=4, \qquad Z=du_1u_2. \tag{5.17}$$

It follows from the second equation of (5.17) that the equation $dx^2 - y^2 = 1$ has a solution, which is impossible by Lemma 3.1 since both equations $dx^2 - y^2 = 2$ and $dx^2 - y^2 = 1$ would then have solutions.

CASE 2: $2 \mid XY$. It is easy to see that (1.12) has no integer solutions when $2 \nmid X$ or $2 \nmid Y$ by taking the equation modulo 16. Hence it suffices to consider the case $2 \mid X$ and $2 \mid Y$. Write $X = 2X_1$, $Y = 2Y_1$, $Z = 2Z_1$. Then (1.12) becomes

$$(2X_1^2 + 1)(Y^2 + 1) = 2Z_1^4. \tag{5.18}$$

As before, it follows from (5.18) that

$$rts^2(tu_1^2)^2 - 2X_1^2 = 1, \quad 2rts^2(ru_2^2)^2 - Y_1^2 = 1 \tag{5.19}$$

for some positive integers u_1 and u_2. This contradicts Lemma 3.1 when $rts > 1$. If $rst = 1$, then the first equation of (5.19) becomes $u_1^4 - 2X_1^2 = 1$, which has no positive integer solutions by Lemma 3.10.

(10) *The equation* $(X^2 + 2)(Y^2 - 1) = Z^4$. We divide the proof into two cases.

CASE 1: $2 \nmid X$. We retain the definitions r, s, and t as given at the beginning of the proof of Theorem 1.2(4). From (1.13) we have

$$rts^2(tu_1^2)^2 - X^2 = 2, \quad Y^2 - rts^2(ru_2^2)^2 = 1 \tag{5.20}$$

for some positive integers u_1 and u_2. It is easy to see that $rts^2 \neq 1$. From the second equation of (5.20), we have the following two subcases.

SUBCASE 1: $2 \mid u_2$. Then

$$Y + 1 = 2ar_1^2u_3^4, \quad Y - 1 = 2br_2^2u_4^4, \quad r_1r_2 = 2r, \quad 2u_3u_4 = u_2,$$

and thus $ar_1^2u_3^4 - br_2^2u_4^4 = 1$. If $a > 1$, then both equations $rts^2x^2 - y^2 = 2$ and $ax^2 - by^2 = 1$ have solutions, which contradicts Lemma 3.1. Hence $a = 1$ and $r \mid r_2$. Continuing the above process for the equation $r_1^2u_3^4 - rts^2r_2^2u_4^4 = 1$, we finally get

$$rts^2(rm^2)^2 - n^4 = 2.$$

SUBCASE 2: $2 \nmid u_2$. Then

$$Y + 1 = ar_1^2u_3^4, \quad Y - 1 = br_2^2u_4^4, \quad r_1r_2 = r,\ u_3u_4 = u_2,$$

and thus $ar_1^2u_3^4 - br_2^2u_4^4 = 2$. If $b > 1$, then both equations $rts^2x^2 - y^2 = 2$ and $ax^2 - by^2 = 2$ have solutions, which contradicts Lemma 3.1. Hence $b = 1, a = rts^2$ and $r = r_1, r_2 = 1$, and we also get the equation

$$rts^2(rm^2)^2 - n^4 = 2.$$

It follows that $(X^2 + 2)(n^4 + 2) = Z_1^4$. From the proof of the equation $(X^2 + 2)(Y^2 + 2) = Z^4$ we have $X = 5$, $n = 1$, hence $X = 5$, $Y = 2$, $Z = 3$.

Therefore the equation $(X^2+2)(Y^2-1)=Z^4$ has only the positive integer solution $(X,Y,Z)=(5,2,3)$ with $2\nmid X$.

CASE 2: $2\,|\,X$. Write $X=2X_1$, $Z=2Z_1$. Then (1.13) becomes

$$(2X_1^2+1)(Y^2-1)=8Z_1^4. \tag{5.21}$$

The remaining proof is similar to the proof of Case 1 of Theorem 1.2(8). Thus the Diophantine equation $(X^2+2)(Y^2-1)=Z^4$ with $2\,|\,X$ has no positive integer solutions.

Therefore the equation $(X^2+2)(Y^2-1)=Z^4$ has only the positive integer solution $(X,Y,Z)=(5,2,3)$.

(11) *The equation* $(X^2+4)(Y^2+1)=Z^4$. We divide the proof into two cases.

CASE 1: $2\nmid X$. An argument similar to the one employed for (1.4) shows that there exist odd integers k and l such that $3\,|\,l$ and $X=U_k$ and $Y=U_l$ and

$$T_k=tu_1^2, \qquad T_l=2ru_2^2 \tag{5.22}$$

for some positive integers u_1 and u_2.

Let $d=\gcd(k,l)$, $k=dk_1$, $l=dl_1$. Then $2\nmid k_1l_1$. By a similar method to the proof of Theorem 1.2(1) and by Lemma 3.4, we have $k_1\in\{1,5\}$ and $l_1=3$. We first consider the case $k_1=1$. Then

$$T_d=t\Box, \quad T_{3d}=2r\Box.$$

Since $\gcd(T_{3d}/T_d,rt)\,|\,3$, $rt\,|\,T_{3d}/T_d$ and $3\nmid rts^2$, we have $rt=1$, which is impossible. Hence

$$k_1=5, \qquad T_{3d}=2r\Box, \qquad T_{5d}=t\Box.$$

Since $\gcd(T_{3d}/T_d,rt)\,|\,3$ and $3\nmid rt$, we have $r=1$. Similarly, $t=5$. Now from $T_d=\gcd(T_{3d},T_{5d})=\Box$, $T_5=5\Box$, and $r=1$, $t=5$, we derive that $5s^4T_d^4-5s^2T_d^2+1=\Box$, and so $sT_d=1$ or 3 by Lemma 3.15. If $sT_d=1$, then $s=1$, $T_d=1$, $U_d=1$, $T_{3d}=2$, $T_{5d}=5$, and thus (1.14) has a solution $(X,Y,Z)=(11,2,5)$. If $sT_d=3$, then $s=3, T_d=1$, which is impossible since $3\nmid Z$. Therefore (1.14) has only one positive integer solution $(X,Y,Z)=(11,2,5)$.

CASE 2: $2\,|\,X$. Write $X=2X_1$, $Z=2Z_1$. As before we obtain the equation

$$(X_1^2+1)(Y_1^2+1)=4Z_1^4, \tag{5.23}$$

and from (5.23) we have

$$2rts^2(tu_1^2)^2-X_1^2=1, \qquad 2rts^2(ru_2^2)^2-Y^2=1. \tag{5.24}$$

Similarly, by Lemma 3.13, we have $rt = 1$, and so

$$(5.25) \qquad 2s^2u_1^4 - X_1^2 = 1, \quad 2s^2u_2^4 - Y^2 = 1,$$

which implies that $s = 1$ by Lemma 3.8. Thus

$$(5.26) \qquad X_1^2 - 2u_1^4 = -1, \quad Y^2 - 2u_2^4 = -1.$$

It follows from Lemma 3.8 that $(X_1, Y, u_1, u_2) = (1, 239, 1, 13), (239, 1, 13, 1)$.

Therefore the only positive integer solutions to the Diophantine equation $(X^2 + 4)(Y^2 + 1) = Z^4$ are $(X, Y, Z) = (11, 2, 5), (2, 239, 26), (478, 1, 26)$.

(12) *The equation* $(X^2 + 4)(Y^2 - 4) = Z^4$. We divide the proof into two cases.

Case 1: $2 \nmid XY$. We define r, s, and t as at the beginning of the proof of Theorem 1.2(1). We only consider the solution (X, Y, Z) of (1.15) with $2 \nmid XY$. From (1.15) we have

$$(5.27) \quad rts^2(tu_1^2)^2 - X^2 = 4, \quad Y^2 - rts^2(ru_2^2)^2 = 4, \quad Z = rstu_1u_2, \quad 2 \nmid Z.$$

From the second equation of (5.27) there are positive integers a, b, r_1, r_2, u_3, u_4 such that

$$Y + 2 = ar_1^2u_3^4, \quad Y - 2 = br_2^2u_4^4, \quad ab = rts^2,\ r = r_1r_2,\ u_2 = u_3u_4,$$

hence

$$(5.28) \qquad ar_1^2u_3^4 - br_2^2u_4^4 = 4, \quad 2 \nmid abr_1r_2u_3u_4.$$

If $a, b > 1$, then both equations $rts^2x^2 - y^2 = 1$ and $ax^2 - by^2 = 1$ with $ab = rts^2, a, b > 1$ have integer solutions, contradicting Lemma 3.1.

If $a > 1$ and $b = 1$, then $r_1 = r$, $r_2 = 1$, and so

$$(5.29) \qquad rts^2(ru_3^2)^2 - u_4^4 = 4, \quad u_4 \mid u_2.$$

If $a = 1$ and $b = rts^2$, then repeating the above process for the equation $u_3^4 - rts^2(ru_4^2)^2 = 4$ we eventually obtain

$$(5.30) \qquad rts^2(rm^2)^2 - n^4 = 4, \quad n \mid u_2.$$

Combining (5.30) or (5.29) and the first equation of (5.27) we get

$$(5.31) \qquad (n^4 + 4)(X^2 + 4) = Z_1^4, \quad 2 \nmid Xn.$$

By Theorem 1.2(1), equation (5.31) has no positive integer solutions. Therefore, (1.15) has no positive integer solutions with $2 \nmid XY$.

Case 2: $2 \mid XY$. It is easy to see that the equation $(X^2+4)(Y^2-4) = Z^4$ has no integer solutions when $2 \mid X$ and $2 \nmid Y$ by taking the equation modulo 16.

Assume $2 \mid X$ and $2 \mid Y$. Write $X = 2X_1$, $Y = 2Y_1$, $Z = 2Z_1$. Then, from (1.15), we obtain

$$(5.32) \qquad (X_1^2 + 1)(Y_1^2 - 1) = Z_1^4.$$

By Theorem LW1, the above equation has only the positive integer solutions $(X_1, Y_1, Z_1) = (1, 3, 2)$, $(239, 3, 26)$.

Next we consider the case $2 \nmid X$ and $2 \mid Y$. Write $Y = 2Y_1$, $Z = 2Z_1$. Then

$$(X^2 + 4)(Y_1^2 - 1) = 4Z_1^4. \tag{5.33}$$

From (5.33) we have

$$\begin{gathered} rts^2(tu_1^2)^2 - X^2 = 4, \quad Y_1^2 - 4rts^2(ru_2^2)^2 = 1, \\ Z_1 = rstu_1u_2, \quad 2 \nmid X. \end{gathered} \tag{5.34}$$

Similarly, from the second equation of (5.34) and Lemma 3.1, we eventually obtain

$$rts^2(rm^2)^2 - 4n^4 = 1 \quad \text{or} \quad rts^2(rm^2)^2 - n^4 = 1. \tag{5.35}$$

Combining (5.35) and the first equation of (5.34) we get

$$(4n^4 + 1)(X^2 + 4) = Z_2^4 \quad \text{or} \quad (n^4 + 1)(X^2 + 4) = Z_2^4, \quad 2 \nmid X. \tag{5.36}$$

By the proof of Theorem 1.2(11), only the first equation in (5.36) has the positive integer solution $(X, n, Z_2) = (11, 1, 5)$.

Therefore, (1.15) has only the positive integer solutions $(X, Y, Z) = (2, 6, 4)$, $(478, 6, 52)$.

(13) *The equation* $(X^2 + 4)(Y^2 - 1) = Z^4$. We first consider the solution (X, Y, Z) of (1.16) with $2 \nmid X$. We retain the definitions for r, s, and t as given at the beginning of the proof of Theorem 1.2(1). Then from (1.16) we have

$$rts^2(tu_1^2)^2 - X^2 = 4, \quad Y^2 - rts^2(ru_2^2)^2 = 1, \quad Z = rstu_1u_2. \tag{5.37}$$

If $2 \nmid u_2$, then from the second equation of (5.37) there are positive integers a, b, r_1, r_2, u_3, u_4 such that

$$Y + 1 = ar_1^2u_3^4, \quad Y - 1 = br_2^2u_4^4, \quad ab = rts^2,\ r = r_1r_2,\ u_2 = u_3u_4,$$

hence

$$ar_1^2u_3^4 - br_2^2u_4^4 = 2, \quad 2 \nmid abr_1r_2u_3u_4. \tag{5.38}$$

It follows that both equations $rts^2x^2 - y^2 = 1$ and $ax^2 - by^2 = 2, ab = rts^2, 2 \nmid xy$ have integer solutions, contradicting Lemma 3.1.

If $2 \mid u_2$, then from the second equation of (5.38) there are positive integers a, b, r_1, r_2, u_3, u_4 such that

$$Y + 1 = 2ar_1^2u_3^4, \quad Y - 1 = 2br_2^2u_4^4, \quad ab = rts^2,\ 2r = r_1r_2,\ u_2 = u_3u_4,$$

hence

$$ar_1^2u_3^4 - br_2^2u_4^4 = 1. \tag{5.39}$$

If $a, b > 1$, then both equations $rts^2x^2 - y^2 = 1$ and $ax^2 - by^2 = 1$, $ab = rts^2$, $a, b > 1$ have integer solutions, contradicting Lemma 3.1. If $a > 1$ and $b = 1$, then $r_1 = r, r_2 = 1$, and so

$$rts^2(ru_3^2)^2 - 4u_4^4 = 1. \tag{5.40}$$

If $a = 1$ and $b = rts^2$, then repeating the above process for the equation $u_3^4 - rts^2(ru_4^2)^2 = 1$ we eventually obtain

$$rts^2(rm^2)^2 - 4n^4 = 1. \tag{5.41}$$

Combining (5.41) or (5.40) and the first equation of (5.37) we get

$$(4n^4 + 1)(X^2 + 4) = Z_1^4, \quad 2 \nmid X. \tag{5.42}$$

By the proof of Theorem 1.2(11), equation (5.42) has only the positive integer solution $(X, n, Z_1) = (11, 1, 5)$. Therefore, (1.16) has no positive integer solutions with $2 \nmid X$.

Next we consider the case $2 \parallel X$. Write $X = 2X_1$, $Z = 2Z_1$ with X_1 odd. From (1.16) we obtain

$$X_1^2 + 1 = 2rts^2(tu_1^2)^2, \quad Y^2 - 1 = 2rts^2(ru_2^2)^2. \tag{5.43}$$

Similarly, from the second equation of (5.43) and Lemma 2.1, we obtain

$$2rts^2(2ru_3^2)^2 = u_4^4 + 1, \quad u_3, u_4 \in \mathbb{N}. \tag{5.44}$$

Combining the first equation of (5.43) and equation (5.44) leads to

$$(u_4^4 + 1)(X_1^2 + 1) = Z_2^4,$$

which is impossible by Theorem LW1.

Now we consider the case $4 \mid X$. Write $X = 2X_1$, $Z = 2Z_1$ with X_1 even. We obtain

$$X_1^2 + 1 = rts^2(tu_1^2)^2, \quad Y^2 - 1 = rts^2(2ru_2^2)^2. \tag{5.45}$$

Similarly, from the second equation of (5.45) and Lemma 3.1, we obtain

$$rts^2(ru_3^2)^2 = (2u_4^2)^2 + 1, \quad u_3, u_4 \in \mathbb{N}. \tag{5.46}$$

Combining the first equation of (5.45) and equation (5.46), we derive

$$((2u_4^2)^2 + 1)(X_1^2 + 1) = Z_2^4,$$

which is impossible by Theorem LW1. Thus the Diophantine equation $(X^2 + 4)(Y^2 - 1) = Z^4$ has no positive integer solutions.

(14) *The equation* $(X^2 - 4)(Y^2 + 1) = Z^4$. We consider the solution (X, Y, Z) of (1.17) with $2 \nmid X$. We retain the definitions for r, s, and t as given at the beginning of the proof of Theorem 1.2(2). Then

$$X^2 - 4 = rts^2(tu_1^2)^2, \quad Y^2 + 1 = rts^2(ru_2^2)^2, \quad Z = rstu_1u_2. \tag{5.47}$$

From the first equation of (5.47) there are positive integers a, b, t_1, t_2, u_3, u_4 such that

$$X + 2 = at_1^2u_3^4, \quad X - 2 = bt_2^2u_4^4, \quad ab = rts^2,\ t = t_1t_2,\ u_2 = u_3u_4,$$

hence

$$at_1^2u_3^4 - bt_2^2u_4^4 = 4, \quad 2 \nmid abr_1r_2u_3u_4. \tag{5.48}$$

If $a, b > 1$, then both equations $rts^2x^2 - y^2 = 1$ and $ax^2 - by^2 = 1$, $ab = rts^2$, have integer solutions, contradicting Lemma 3.1.

If $a > 1$ and $b = 1$, then $r_1 = r$, $r_2 = 1$, and so

$$rts^2(ru_3^2)^2 - u_4^4 = 4, \quad u_4 \mid u_2. \tag{5.49}$$

If $a = 1$ and $b = rts^2$, then repeating the above process for the equation $u_3^4 - rts^2(ru_4^2)^2 = 4$ we finally obtain

$$rts^2(rm^2)^2 - n^4 = 4, \quad n \mid u_2. \tag{5.50}$$

Combining (5.50) or (5.49) and the second equation of (5.47) we get

$$(n^4 + 4)(Y^2 + 1) = Z_1^4, \quad 2 \nmid n. \tag{5.51}$$

By Theorem 1.2(11), equation (5.51) has no positive integer solutions. Therefore, (1.17) has no positive integer solutions with $2 \nmid X$.

We now consider the case $2 \parallel X$. Write $X = 2X_1$, $Z = 2Z_1$ with X_1 odd. We obtain

$$X_1^2 - 1 = 2rts^2(tu_1^2)^2, \quad Y^2 + 1 = 2rts^2(ru_2^2)^2. \tag{5.52}$$

From the first equality of (5.52) and Lemma 3.1 we get

$$X_1 + 1 = 4rts^2(2ru_3^2)^2, \quad X_1 - 1 = 2u_4^4. \tag{5.53}$$

Thus

$$2rts^2(2ru_3^2)^2 = u_4^4 + 1, \tag{5.54}$$

which is impossible by taking the equation modulo 4.

Now we assume that $4 \mid X$. Write $X = 2X_1$, $Z = 2Z_1$ with X_1 even. We obtain

$$X_1^2 - 1 = rts^2(tu_1^2)^2, \quad Y^2 + 1 = rts^2(2ru_2^2)^2; \tag{5.55}$$

however, the second equation of (5.55) is impossible by taking it modulo 4. Thus the Diophantine equation $(X^2-4)(Y^2+1) = Z^4$ has no positive integer solutions.

(15) *The equation* $(X^2 - 4)(Y^2 - 1) = Z^4$. We first consider the case $2 \nmid X$. An argument similar to the one employed in the solution of (1.18) shows that there exist positive integers k and l such that $3 \mid l$ and $X = T_k$ and $Y = T_l$ and

$$U_k = tu_1^2, \qquad U_l = 2ru_2^2 \tag{5.56}$$

for some positive integers u_1 and u_2.

We may assume that $d = \gcd(k, l)$, $k = dk_1$, $l = 2^u l_1 d$, $2 \nmid k_1 l_1$, $u \geq 0$. Then $U_d = \gcd(U_k, U_l) = c\square$ with $c \mid rt$ since $\gcd(r, t) = 1$. Since

$$U_l = \frac{U_l}{U_{l_1 d}} \cdot U_{l_1 d}, \quad \gcd(U_l/U_{l_1 d}, rU_{l_1 d}) = 1,$$

we have

$$U_{l_1 d} = 2r\square.$$

Since every prime divisor of $\gcd(U_{l_1 d}/U_d, rtU_d)$ divides l_1, we obtain

$$U_{l_1 d}/U_d = m\square, \qquad m \mid l_1.$$

Applying Lemma 3.4 to

$$Q_{l_1} = \frac{U_{l_1 d}}{U_d} = \frac{(\alpha^d)^{l_1} + (-\overline{\alpha}^d)^{l_1}}{(\alpha^d) + (-\overline{\alpha}^d)}$$

we have $l_1 = 3$. Similarly, $k_1 \in \{1, 5\}$. We first consider the case $k_1 = 1$. Then

$$U_d = t\square, \quad U_{3d} = 2r\square.$$

Since every prime divisor of $\gcd(T_{3d}/T_d, rt)$ divides 3, and $rt \mid T_{3d}/T_d$ (as $\gcd(r, t) = 1$), we have $rt = 3$, which is impossible since $T_d^2 - 3s^2U_d^2 = 4$ and $2 \nmid T_d$. Hence

$$k_1 = 5, \quad T_{3d} = 2r\square, \quad T_{5d} = t\square.$$

Since every prime divisor of $\gcd(T_{3d}/T_d, rt)$ divides 3, we have $r \mid 3$; similarly, $t \mid 5$.

Since $T_d^2 - rts^2U_d^2 = 4, 2 \nmid T_d$, we have $rt \neq 1, 3, 15$, so $r = 1$ and $t = 5$. Now from $U_d = \gcd(U_{3d}, U_{5d}) = \square$, $U_{5d} = 5\square$, $r = 1$, $t = 5$, we derive that $5s^4U_d^4 + 5s^2U_d^2 + 1 = \square$, and so $sT_d = 0$ by Lemma 3.15, which is impossible. Therefore (1.18) has no positive integer solutions with $2 \nmid X$.

Now we consider the case $2 \mid X$. Write $X = 2X_1$, $Z = 2Z_1$. Then (1.18) becomes

$$(X_1^2 - 1)(Y^2 - 1) = 4Z_1^2. \tag{5.57}$$

We first consider the case $2 \mid X_1Y$. We may assume that $2 \mid Y$ and $2 \nmid X_1$. From (5.57), there are positive integers u_1, u_2 such that

$$Y^2 - 1 = rts^2(ru_2)^2, \qquad X_1^2 - 1 = 4rts^2(tu_1^2)^2, \quad 2 \nmid rtsu_2. \tag{5.58}$$

From the first equation of (5.58), there exist odd integers m, n, r_1, r_2, u_3, u_4 such that

$$m(r_1u_3^2)^2 - n(r_2u_4^2)^2 = 2, \qquad mn = rts^2,\ r_1r_2 = r,\ u_3u_4 = u_2. \tag{5.59}$$

From the second equation of (5.58) and Lemma 3.1, there exist positive integers t_1, t_2, u_5, u_6 such that

$$X+1 = 2t_1^2u_5^4, \quad X-1 = 2t_2^2rts^2u_6^4, \quad 2\,|\,u_6,\ t_1t_2 = t. \tag{5.60}$$

It follows that $t_1 = 1$ and

$$u_5^4 - rts^2t^2u_6^4 = 1, \quad 2\,|\,u_6. \tag{5.61}$$

From (5.59), (5.61) and Lemma 3.1, we derive

$$u_5^2 + 1 = 2u_7^2,$$

which implies that $u_5 = 239$ and

$$239^2 - 1 = 3\cdot 5\cdot 7\cdot 17\cdot 2^5 = 8rts^2u_8^4,$$

which is impossible.

Finally we consider the case $2 \nmid X_1Y$. From (5.57), there are positive integers u_1, u_2 such that

$$Y^2 - 1 = 2rts^2(ru_2)^2, \quad X_1^2 - 1 = 2rts^2(tu_1^2)^2, \quad 2 \nmid rtsu_2. \tag{5.62}$$

From the first equation of (5.62), there exist positive integers $m > 1$, n, r_1, r_2, u_3, u_4 such that

$$\begin{gathered} m(r_1u_3^2)^2 - n(r_2u_4^2)^2 = 1, \quad mn = 2rts^2 \text{ or } mn = rts^2/2, \\ r_1r_2 = r, \quad u_3u_4 = u_2. \end{gathered} \tag{5.63}$$

From the second equation of (5.62), (5.63) and Lemma 3.1, there exist positive integers t_1, t_2, u_5, u_6 such that

$$mt_1^2u_5^4 - nt_2^2u_6^4 = 1, \quad t_1t_2 = t. \tag{5.64}$$

Since $m > 1$, it follows from Lemma 3.13, (5.63) and (5.64) that $r_1t_1 = 1$, and so $rt\,|\,n$ and $m = 2s_1^2$. Therefore we have the equation

$$\begin{gathered} 2s_1^2u_3^4 - rts_2^2(ru_4^2)^2 = 1, \quad 2s_1^2u_5^2 - rts_2^2(tu_6^2)^2 = 1, \\ s_1s_2 = s \text{ or } s_1s_2 = s/2. \end{gathered} \tag{5.65}$$

We denote by (T_1, U_1) the minimal positive integer solution of the Pell equation

$$2s_1^2T^2 - rts_2^2U^2 = 1 \tag{5.66}$$

and let $\varepsilon = T_1\sqrt{2s_1^2} + U_1\sqrt{rts_2^2}$. For a positive integer $k \geq 1$, let (T_k, U_k) be positive integers given by

$$T_k\sqrt{2s_1^2} + U_k\sqrt{rts_2^2} = \varepsilon^k.$$

Assume $rt > 1$. By Lemma 3.12, we assume that $T_1\sqrt{2s_1^2} + U_1\sqrt{rts_2^2} = u_3^2\sqrt{2s_1^2} + ru_4^2\sqrt{rts_2^2}$ and $T_k\sqrt{2s_1^2} + U_k\sqrt{rts_2^2} = u_5^2\sqrt{2s_1^2} + tu_6^2\sqrt{rts_2^2}$. Then $U_k = tu_6^2 = U_1\cdot tu_6^2/(ru_4^2)$. It follows that $rt\,|\,k$, say $k = rtl$ for some positive integer l. Observe that $k = p \equiv 3 \pmod 4$ and $rt > 1$; by Lemma 3.12 again, we obtain $l = 1$ and the equation

$$2s_1^2 - ps_2^2U^4 = 1, \quad p \equiv 3 \pmod 4,$$

which is impossible by taking it modulo 8.

Now we assume that $rt = 1$. Then, by Lemma 3.9, the equation $X^2 - 2s^2U^4 = 1$ has at most one positive integer solution (X, U), so $X_1 = Y$, $u_1 = u_2$ by (5.58). Obviously, (5.58) has infinite many trivial solutions $(X_1, Y, S, u_1, u_2) = (Y, Y, S, 1, 1)$, where $Y^2 - 2S^2 = 1$.

Therefore the Diophantine equation $(X^2 - 4)(Y^2 - 1) = Z^4$ has only the trivial solutions $(X, Y, Z) = (2Y, Y, 2S)$, where $Y^2 - 2S^2 = 1$.

(16) *The equation* $(X^2 - 2)(Y^2 + 4) = Z^4$. We divide the proof into two cases.

CASE 1: $2 \nmid XY$. We consider the more general equation

$$(X^2 - 2)(Y^2 + 4) = Z^2, \qquad 2 \nmid XY.$$

From the above equation we have

$$X^2 + 2 = du_1^2, \qquad du_2^2 - Y^2 = 4, \qquad Z = du_1u_2. \tag{5.67}$$

It follows from the second equation of (5.67) that the equation $dx^2 - y^2 = 1$ has a solution, which is impossible by Lemma 3.1 since both equations $x^2 - dy^2 = 2$ and $dx^2 - y^2 = 1$ would then have solutions.

CASE 2: $2 \mid XY$. It is easy to see that the equation $(X^2 - 2)(Y^2 + 4) = Z^4$ has no integer solutions when $2 \mid X$ and $2 \nmid Y$ by taking the equation modulo 4. We consider two subcases.

SUBCASE 1: $2 \mid X$ and $2 \mid Y$. Write $X = 2X_1$, $Y = 2Y_1$, $Z = 2Z_1$. We obtain

$$(2X_1^2 - 1)(Y_1^2 + 1) = 2Z_1^4. \tag{5.68}$$

We retain the definitions for r, s and t as given at the beginning of the proof of Theorem 1.1, but define them to be square-free numbers built up from prime divisors of $2X_1^2 - 1$ instead of $AX_1^2 + 1$. We obtain

$$2X_1^2 - rts^2(tu_1^2)^2 = 1, \tag{5.69}$$

$$2rts^2(ru_2^2)^2 - Y_1^2 = 1, \tag{5.70}$$

for some positive integers u_1 and u_2 with $Z_1 = rtsu_1u_2$. It follows from Lemma 3.1 that $rts^2 = 1$. Therefore

$$2X_1^2 - u_1^4 = 1, \tag{5.71}$$

$$Y_1^2 - 2u_2^4 = -1. \tag{5.72}$$

It follows from (5.71), (5.72) and Lemma 3.7, and a theorem of Ljunggren, that $X_1 = 1$, $u_1 = 1$, $(Y_1, u_2) = (1, 1), (239, 13)$.

SUBCASE 2: $2 \nmid X$ and $2 \mid Y$. Write $Y = 2Y_1$, $Z = 2Z_1$. We obtain

$$(X^2 - 2)(Y_1^2 + 1) = 4Z_1^4. \tag{5.73}$$

We retain the definitions for r, s and t as given at the beginning of the proof of Theorem 1.2(3). We have

$$X^2 - rts^2(tu_1^2)^2 = 2, \tag{5.74}$$

$$rts^2(2ru_2^2)^2 - Y_1^2 = 1, \tag{5.75}$$

for some positive integers u_1 and u_2. It follows from Lemma 3.1 that $rts^2 = 1$. Therefore

$$X^2 - u_1^4 = 2, \tag{5.76}$$

which is impossible. Thus the only positive integer solutions of the Diophantine equation $(X^2-2)(Y^2+4) = Z^4$ are $(X, Y, Z) = (2, 2, 2)$ and $(2, 478, 26)$.

(17) *The equation* $(X^2-4)(Y^2-1) = 4Z^4$. The proof is almost the same as for $(X^2 - 4)(Y^2 - 1) = Z^4$, $2 \nmid X$; we leave the details to the reader.

This completes the proof of Theorem 1.2. ∎

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