Diophantine inequalities for the non-Archimedean line $\mathbb{F}_q((1/T))$

by

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1. Introduction. In 1946, Davenport and Heilbronn [3] adapted the Hardy–Littlewood method to prove that if $\lambda_i$ ($i = 1, \ldots, K$) are non-zero real numbers, not all of the same sign, and if $\lambda_1/\lambda_2$ is irrational, then the values of

$$\lambda_1 x_1^k + \ldots + \lambda_K x_K^k$$

as $x_i$’s run independently through all natural numbers, are everywhere dense on the real line provided that $K \geq 2^k + 1$. In the case $k = 1$, Baker [1] (see also [11] and [13]) showed that for any positive integer $n$ there exist infinitely many primes $p_1, p_2, p_3$, satisfying the inequality

$$|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3| < (\ln p)^{-n},$$

where $p$ denotes the maximum of $p_1, p_2, p_3$. More recently, Harman [5] showed that if $\alpha$ is a real number, then there are infinitely many ordered triples of primes $p_1, p_2, p_3$ for which

$$|\alpha + \lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3| < (\max_j p_j)^{-1/5+\varepsilon}.$$  

In the case $k \geq 2$, Ramachandra [11] (see also [12]) showed that when $K \geq 2^k + 1$ if $1 \leq k \leq 11$ and $K \geq 2[2k^2 \ln k + k^2 \ln \ln k + 2.5k^2] - 1$ if $k \geq 12$, the values of

$$\lambda_1 p_1^k + \ldots + \lambda_K p_K^k$$

as the $p_j$’s run independently through all primes, are everywhere dense on the real line. The key to the Hardy–Littlewood method on the real line is

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the integral
\begin{equation}
\int_{-\infty}^{\infty} \exp(xy) \left( \frac{\sin \pi x}{\pi x} \right)^2 \, dx = \max\{1 - |y|, 0\}.
\end{equation}

In this paper, we study the Hardy–Littlewood method for the completion \( K_\infty = \mathbb{F}_q((1/T)) \) of the rational function field \( K = \mathbb{F}_q(T) \) at the infinite place, where \( \mathbb{F}_q \) denotes the finite field with \( q \) elements. We have a natural discrete valuation \(|\cdot|\) on \( K_\infty \) defined by
\[ |f| = q^{\deg f}, \]
where \( \deg f \) denotes the degree of \( f \in K_\infty \) at \( T \), and set \( \deg 0 = -\infty \). Since \( K_\infty \) is complete under the non-Archimedean valuation \(|\cdot|\) and the Pontryagin (self) duality \( \hat{K}_\infty = K_\infty \) holds (cf. Section 2), we have the following basic analogy:
\[ \mathbb{F}_q[T] \sim \mathbb{Z}, \quad K \sim \mathbb{Q}, \quad K_\infty \sim \mathbb{R}. \]
Let \( p \) be the characteristic of \( \mathbb{F}_q \), let \( \lambda_1, \ldots, \lambda_D \) be non-zero elements in \( K_\infty \) satisfying \( \lambda_1/\lambda_2 \not\in K \) and
\[ \text{sgn} \lambda_1 + \ldots + \text{sgn} \lambda_D = 0, \]
where \( \text{sgn} f \in \mathbb{F}_q \) denotes the leading coefficient of \( f \in K_\infty \). We show that if \( p > d \geq 1 \) and
\[ D \geq \begin{cases} 1 + 2d & \text{if } 2 \leq d < 11, \\ 2[2d^2 \ln d + d^2 \ln \ln d + 2d^2 - 2d] + 1 & \text{if } d \geq 11, \end{cases} \]
then the values of the sum
\[ \lambda_1 P_1^d + \ldots + \lambda_D P_D^d, \]
as the \( P_i \)'s run independently through all monic irreducible polynomials in \( \mathbb{F}_q[T] \), are everywhere dense on the “non-Archimedean” line \( K_\infty \). In fact, we obtain a more explicit inequality in Theorem 2.1. In the proof of Theorem 2.1, the integral (cf. Lemma 2.2)
\[ \int_{K_\infty} E(a f) \chi_n(a) \, da = \begin{cases} 1 & \text{if } \deg f < n, \\ 0 & \text{if } \deg f \geq n, \end{cases} \]
plays a role entirely analogous to the integral (1) on the real line.

We studied the case \( d = 1, \ D = 3 \) in [8]. In the present paper, we attack this problem in the case when \( d \geq 2 \). In this situation, we need more additive theory of monic irreducible polynomials in \( \mathbb{F}_q[T] \) (see, e.g., Theorems 4.3, 4.4, and 2.4).

2. The main theorem and definition. Let \( \mathbb{F}_q \) be the finite field with \( q \) elements. Let \( p \) be its characteristic and let \( \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \) be the subfield of
\[ \mathbb{F}_q \text{ with } p \text{ elements. Let } \psi_0 : \mathbb{F}_p \rightarrow \mathbb{C}^\times \text{ be the canonical additive character defined by} \]
\[ \psi_0([c]) = \exp \left( \frac{2\pi i \cdot c}{p} \right), \]
where \([c]\) denotes the canonical image of \(c\) in \(\mathbb{F}_p\). Let \(\psi : \mathbb{F}_q \rightarrow \mathbb{C}^\times\) be the additive character defined by \(\psi(x) = \psi_0(\text{Tr}(x))\) for all \(x \in \mathbb{F}_q\) where \(\text{Tr}\) is the trace map from \(\mathbb{F}_q\) to \(\mathbb{F}_p\). Let \(A = \mathbb{F}_q[T]\) (resp. \(K = \mathbb{F}_q(T)\)) be the polynomial ring (resp. rational function field) with coefficients in \(\mathbb{F}_q\). Let \(A_+\) denote the subset of \(A\) consisting of all monic polynomials. Let \(K_\infty = \mathbb{F}_q((1/T))\) denote the completion of \(K\) at the infinite place; in other words, for every \(a \in K_\infty\), if \(a \neq 0\), then \(a\) can be expressed as
\[ a = \sum_{i=d}^{-\infty} c_i T^i, \]
where \(c_i \in \mathbb{F}_q\) and \(c_d \neq 0\). The sign, degree, and absolute value of \(a\) are defined by \(\text{sgn} \ a = c_d\), \(\deg a = d\), and \(|a| = q^d\). The residue of \(a\) at the infinite place is denoted by \(\text{Res}_\infty f = c_{-1}\). The exponential map \(E : K_\infty \rightarrow \mathbb{C}^\times\) is defined by
\[ E(a) = \psi(\text{Res}_\infty a). \]
The exponential map \(E\) is a non-trivial additive character from \(K_\infty\) to \(\mathbb{C}^\times\) and the Pontryagin (self) duality \(\hat{K}_\infty = K_\infty\) is deduced by the bilinear map
\[ K_\infty \times K_\infty \rightarrow \mathbb{C}^\times, \quad (a, f) \mapsto E(a \cdot f). \]
In this paper, the Haar integral for \(K_\infty\) is defined to satisfy
\[ \int_{\deg a \leq -1} 1 \ da = 1. \]
This implies that
\[ \int_{K_\infty} f(a) d(ba) = |b| \int_{K_\infty} f(a) \ da \]
for all \(b \in K_\infty\) and continuous functions \(f\) (with compact support). With these properties, we have the following basic analogy:
\[ A \sim \mathbb{Z}, \quad K \sim \mathbb{Q}, \quad K_\infty \sim \mathbb{R}, \quad E \sim \exp. \]
The main theorem of this paper is

**Theorem 2.1.** Suppose that \(d, D, m\) are positive integers and \(\lambda, \lambda_1, \ldots, \lambda_D\) are non-zero elements in \(K_\infty\) satisfying \(\lambda_1/\lambda_2 \notin \mathbb{K}, 2 \leq d < p, \)
\[ \deg \lambda_1 = \ldots = \deg \lambda_D = 0, \]
and
\[ \text{sgn} \lambda_1 + \ldots + \text{sgn} \lambda_D = 0. \]
Then if
\[ D \geq \begin{cases} 
1 + 2^d & \text{if } 2 \leq d < 11, \\
2[2d^2 \ln d + d^2 \ln \ln d + 2d^2 - 2d] + 1 & \text{if } d \geq 11,
\end{cases} \]
then there exist infinitely many positive integers \( N \) for which there are
\[
\gg \frac{q^{(D-d)N}}{N^{D+m}}
\]
\( D \)-tuples \((P_1, \ldots, P_D)\) of monic irreducible polynomials with \( \deg(\lambda_i P_i) = N \) and
\[
\deg(\lambda + \lambda_1 P_1^d + \ldots + \lambda_D P_D^d) < -m \ln N + 1,
\]
where the implied constant depends only on \( A, \lambda, \lambda_i, d, D, \) and \( m \), but not on \( N \).

Remark 1. The complete proof of Theorem 2.1 is given in Section 5. In fact, if we define the value of \( I_j(a) \) in (3) to be
\[
I_j(a) = I(aT^{A-\deg \lambda_j} \lambda_j),
\]
where
\[
A = \max_{1 \leq j \leq D} \{\deg \lambda_j\},
\]
then without the condition (2), the statement of Theorem 2.1 is also true.

2. The choice of \( N \) depends on \( \lambda_1/\lambda_2 \in \mathbf{K}_\infty/\mathbf{K} \) and this condition is used only in Lemmas 4.2 and 4.6. Combining this theorem and [8], Theorem 1.2, we obtain

Consequence 1. Under the hypothesis of Theorem 2.1, suppose \( p > d \geq 1 \) and
\[
D \geq \begin{cases} 
1 + 2^d & \text{if } d < 11, \\
2[2d^2 \ln d + d^2 \ln \ln d + 2d^2 - 2d] + 1 & \text{if } d \geq 11.
\end{cases} \]
Then the values of the sum
\[
\lambda_1 P_1^d + \ldots + \lambda_D P_D^d,
\]
as the \( P_i \)'s run independently through all monic irreducible polynomials in \( \mathbb{F}_q[T] \), are everywhere dense on the non-Archimedean line \( \mathbb{F}_q((1/T)) \).

Let \( \mathfrak{M} \) be the subring of \( \mathbf{K}_\infty \) consisting of \( a \in \mathbf{K}_\infty \) with \( \deg a \leq -1 \) and let \( \chi_0 \) be the characteristic function of \( \mathfrak{M} \); in other words, \( \chi_0 : \mathbf{K}_\infty \to \mathbb{R} \) satisfies
\[
\chi_0(a) = \begin{cases} 
1 & \text{if } a \in \mathfrak{M}, \\
0 & \text{otherwise}.
\end{cases}
\]
Given any integer \( n \), the function \( \chi_n : \mathbf{K}_\infty \to \mathbb{R} \) is defined by
\[
\chi_n(a) = q^n \chi_0(aT^n) \quad \text{for } a \in \mathbf{K}_\infty.
\]
Lemma 2.2. We have
\[ E(af) \chi_n(a) \, da = \begin{cases} 
1 & \text{if } \deg f < n, \\
0 & \text{if } \deg f \geq n.
\end{cases} \]

Proof. See [6], Theorem 3.5. \( \blacksquare \)

Let \( p > d \geq 2 \), \( N \) be fixed positive integers. We define functions
\[
S(a) = \sum_{\deg P = N} E(aP^d), \quad I(a) = \frac{1}{N} \int_{y \in T^N + T^N \mathfrak{M}} E(ay^d) \, dy,
\]
\[
S_j(a) = S(a\lambda_j), \quad I_j(a) = I(a\lambda_j), \quad j = 1, \ldots, D,
\]
\[
F(a) = \prod_{j=1}^{D} S_j(a), \quad H(a) = \prod_{j=1}^{D} I_j(a),
\]
where \( \sum' \) denotes the sum over monic irreducible polynomials in \( \mathfrak{A} \). Let \( \pi_N \) denote the number of monic irreducible polynomials in \( \mathfrak{A} \) of degree \( N \). The prime number theorem for \( \mathfrak{A} \) is
\[
q^N/N - q^{N/2} < \pi_N \leq q^N/N.
\]
As \( \deg \lambda_j = 0 \), by the definition of \( E \) we have
\[
I_j(a) = \begin{cases} 
q^N/N & \text{if } \deg a < -dN - 1, \\
(q^N/N)\psi(\text{sgn}(a\lambda_j)) & \text{if } \deg a = -dN - 1,
\end{cases}
\]
and
\[
S_j(a) = \begin{cases} 
\pi_N & \text{if } \deg a < -dN - 1, \\
\pi_N\psi(\text{sgn}(a\lambda_j)) & \text{if } \deg a = -dN - 1.
\end{cases}
\]

Lemma 2.3. If \( \deg a \geq -dN \), then \( I_j(a) = 0 \).

Proof. Since \( \deg \lambda_j = 0 \), it suffices to show that \( I(a) = 0 \) for \( \deg a \geq -dN \). Let \( \deg a = -dN + l \) for some integer \( l \geq 0 \) and let
\[
a = a_l T^{-dN+l} + \ldots + a_{-1} T^{-dN-1} + a' \in \mathbb{K}_\infty,
\]
where \( a_j \in \mathbb{F}_q \), \( a_l \neq 0 \), and \( \deg a' \leq -dN - 2 \). Let
\[
y = T^N + \sum_{j=1}^{\infty} b_{-j} T^{N-j} \in T^N + T^N \mathfrak{M},
\]
where \( b_{-j} \in \mathbb{F}_q \). Then we have
\[
y^d = T^{dN} + \sum_{j=1}^{\infty} (db_{-j} + c_{-j}(b_{-1}, \ldots, b_{-(j-1)})) T^{dN-j},
\]
for some $c_{-j}(x_1, \ldots, x_{j-1}) \in \mathbb{F}_q[x_1, \ldots, x_{j-1}]$ and $c_{-1} = 0$. Since $E(a'y^d) = 1$, we have

$$E(ay^d) = E((a-a')y^d) = \psi \left( a_{-1} + \sum_{j=0}^{l} a_{j}b'_{-(j+1)} \right),$$

where

$$b'_{-j} = db_{-j} + c_{-j}(b_{-1}, \ldots, b_{-(j-1)}).$$

By (7), since $2 \leq d < p$, we know that the $d$th power mapping

$$F : T^N + T^N \mathfrak{M} \to T^{dN} + T^{dN} \mathfrak{M}, \quad y \mapsto y^d$$

is bijective and satisfies

$$F(y + T^{N-(l+1)} \mathfrak{M}) = y^{dN} + T^{dN-(l+1)} \mathfrak{M}.$$ 

By (8), (9), since $\psi$ is a non-trivial additive character of $\mathbb{F}_q$, and $a_l \neq 0$, we obtain

$$I(a) = \frac{1}{N} \int_{y \in T^{N} + T^{N} \mathfrak{M}} E(ay^d) \, dy = \frac{1}{N} \int_{y \in T^{N} + T^{N} \mathfrak{M}} E((a-a')y^d) \, dy$$

$$= \frac{1}{N} \int_{z \in T^{N-(l+1)} \mathfrak{M}} \sum_{c \in \mathbb{F}_q} q^l \cdot \psi(c) \, dz = 0. \blacksquare$$

Let a positive integer $l$ satisfy $l \leq N/2$ and let $y$ be a monic element in $K_\infty$ of degree $N$. Let $\pi_N(y, l)$ denote the number of monic irreducible polynomials $P \in A_+$ of degree $N$ with $\deg(P-y) < N-l$. In [7], Corollary 2.6, or [2], Theorem 1.4, we have

$$\pi_N(y, l) = \frac{q^{N-l}}{N} + O(q^{N/2}),$$

where the implied constant depends only on $A$. Given

$$x = \sum_{i=-dN+l-1}^{-\infty} a_iT^i \in K_\infty, \quad f = T^{dN} + \sum_{j=dN-1}^{0} f_jT^j \in A,$$

where $a_i, f_j \in \mathbb{F}_q$, $a_{-dN+l-1} \neq 0$, and setting $f_{dN} = 1$, we have

$$\text{Res}_\infty(xf) = \sum_{k=0}^{l} a_{-dN+k-1}f_{dN-k}.$$ 

Let $\pi_{N,d}(f, l)$ be the number of monic irreducible polynomials $P$ of degree $N$ with $\deg(P^d - f) < dN - l$. By (7), since $2 \leq d < p$, there exists a monic element $y \in K_\infty$ of degree $N$ satisfying $\deg(y^d - f) < dN - l$ and $\pi_{N,d}(f, l) = \pi_{N,d}(y^d, l) = \pi_N(y, l)$. Thus by (10) we get

$$\pi_{N,d}(f, l) = \frac{q^{N-l}}{N} + O(q^{N/2}),$$

where

$$\psi(a_0y^d) = 1,$$
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where the implied constant depends only on \( A \). If \( P \) is a monic irreducible polynomial of degree \( N \) satisfying \( \deg(P^d - f) < dN - l \), then by (11), \( \text{Res}_\infty(xP^d) = \text{Res}_\infty(xf) \). Hence combining (12), (11), and \( a_{-dN+l-1} \neq 0 \), we get

\[
\#\{\text{monic irreducibles } P \mid \deg P = N, \ \text{Res}_\infty(xP^d) = c\} = q^{l-1}
\left(\frac{q^{N-l}}{N} + O(q^{N/2})\right) = \frac{q^{N-1}}{N} + O(q^{l+N/2})
\]

for any \( c \in \mathbb{F}_q \). Since

\[
E(xP^d) = \exp\left(\frac{2\pi i \text{Tr}(\text{Res}_\infty(xP^d))}{p}\right),
\]

and \( \text{Tr} \) is a surjective \( \mathbb{F}_p \)-linear mapping from \( \mathbb{F}_q \) onto \( \mathbb{F}_p \), by (13) we obtain

\[
|S(x)| = \left| \sum'_{\deg P = N} E(xP^d) \right| = O(q^{l+N/2}).
\]

Therefore we have

**Theorem 2.4.** Let \( m \) be an integer satisfying \( 0 \leq m < N/2 \). Then

\[
|S(x)| = O(q^{m+N/2})
\]

for all \( x \in K_\infty \) with \( \deg x = -dN + m \), where the implied constant depends only on \( A \).

**Remark.** If \( m \geq N/2 \), then the result of Theorem 2.4 is trivial.

**3. The major arcs**

**Lemma 3.1.** Let \( n \) be a positive integer and let \( -dN \leq m \leq -dN + N/4 \). Then

\[
\int_{\deg a \leq m} |F(a) - H(a)|\chi_{-n}(a) \, da = o\left(\frac{q^{(D-d)N-N/2-n}}{ND}\right),
\]

as \( N \to \infty \).

**Proof.** Using (5), (6), \( \text{sgn } \lambda_1 + \ldots + \text{sgn } \lambda_D = 0 \), \( \chi_{-n}(a) \leq q^{-n} \), Lemma 2.3, and (4), we obtain

\[
\int_{\deg a \leq m} |F(a) - H(a)|\chi_{-n}(a) \, da
\]

\[
\leq q^{-n} \int_{\deg a \leq -dN-1} |\pi_D^N - q^{DN}/ND| \, da + q^{-n} \sum_{i=-dN}^{m} \int_{\deg a = i} |F(a)| \, da
\]

\[
\leq O\left(\frac{q^{(D-d)N-N/2-n}}{ND-1}\right) + q^{-n} \sum_{i=-dN}^{m} \int_{\deg a = i} |F(a)| \, da,
\]
where the implied constant depends only on $\mathbf{A}$. Since $S_j(a) = S(a\lambda_j)$ and $\deg(a\lambda_j) = \deg a$, by Theorem 2.4, we obtain
\[
|S_j(a)| = O(q^{dN+m+N/2})
\]
for $-dN \leq \deg a \leq m$, where the implied constant depends only on $\mathbf{A}$. As $d \geq 2$ and $D \geq 1 + 2^d \geq 5$, we obtain
\[
q^{-n} \sum_{i=-dN}^{m} \int_{\deg a = i} |F(a)| \, da = O\left(q^{-n} \sum_{i=-dN}^{m} \int_{\deg a = i} q^{dN+DN/2+Dm} \, da \right)
\]
\[
= O(q^{-n} \cdot q^{m} \cdot q^{dN+DN/2+Dm})
\]
\[
= O(q^{(dD+D/2)N+(D+1)m-n})
\]
\[
= O\left(\frac{q^{(D-d)N-N/2-n}}{N^D-1}\right),
\]
where the implied constant depends only on $\mathbf{A}$. ■

**Lemma 3.2.** Let $n$ be a positive integer and let $\lambda \in K_\infty$. Then if $m \geq -dN$ and $dN > \deg \lambda$, we have
\[
\int_{\deg a \leq m} H(a)E(a\lambda)\chi_{-n}(a) \, da = \frac{q^{(D-d)N-N/2-n}}{N^D}.
\]

**Proof.** By Lemma 2.3 and the definition of $H$, we have $H(a) = 0$ if $\deg a \geq m \geq -dN$. Thus
\[
\int_{\deg a \geq m} H(a)E(a\lambda)\chi_{-n}(a) \, da = 0.
\]
By the definitions of $H$, $E$ and Lemma 2.3, we have
\[
\int_{K_\infty} H(a)E(a\lambda)\chi_{-n}(a) \, da
\]
\[
= \frac{1}{N^D} \int_{\deg a < -dN} \int_{T^N + T^N \mathbb{M}} \ldots \int_{T^N + T^N \mathbb{M}} E\left(a\left(\lambda + \sum_{j=1}^{D} \lambda_j y_j^d\right)\right) 
\]
\[
\times \chi_{-n}(a) \, dy_1 \ldots dy_D \, da
\]
\[
= \frac{1}{N^D} \int_{T^N + T^N \mathbb{M}} \ldots \int_{T^N + T^N \mathbb{M}} \int_{\deg a < -dN} E\left(a\left(\lambda + \sum_{j=1}^{D} \lambda_j y_j^d\right)\right) 
\]
\[
\times \chi_{-n}(a) \, da \, dy_1 \ldots dy_D.
\]
By the definition of $\chi_n(a)$ and since $\deg a < -dN$, the above is

$$\frac{q^{-n}}{N^D} \int_{T^N + T^N \mathfrak{M}} \ldots \int_{T^N + T^N \mathfrak{M}} E\left(\lambda + \sum_{j=1}^{D} \lambda_j y_j^d\right) da dy_1 \ldots dy_D.$$ 

Given any $y_1, \ldots, y_D \in T^N + T^N \mathfrak{M}$, set

$$f = \lambda + \lambda_1 y_1^d + \ldots + \lambda_D y_D^d.$$ 

Since $dN > \deg \lambda$, $\deg \lambda_j = 0$, and $\text{sgn} \lambda_1 + \ldots + \text{sgn} \lambda_D = 0$, we have $\deg f < dN$. This implies

$$\int_{\deg a < -dN} E(a f) \, da = \int_{\deg a < -dN} 1 \, da = q^{-dN}.$$ 

Therefore

$$\int_{\mathbf{K}_\infty} H(a) E(a \lambda) \chi_n(a) \, da = \frac{q^{-n}}{N^D} \cdot q^{DN} \cdot q^{-dN} = \frac{q^{(D-d)N-n}}{N^D}.$$ 

Combining these with (14), we complete the proof. ■

4. The minor arcs. We recall Dirichlet’s theorem for $\mathbf{A}$ in $A$.

THEOREM 4.1. Given any $\alpha \in \mathbf{K}_\infty$ and a positive integer $N$, there exists a unique monic polynomial $Q$ and a polynomial $a$ in $\mathbf{A}$ satisfying $(Q, a) = 1$, $\deg Q \leq N$, and $\deg(\alpha - a/Q) \leq -(\deg Q + N + 1)$.

Proof. See [6]. ■

For any $x \in \mathbf{K}_\infty$, define

$$V(x) = \min\{|S_1(x)|, |S_2(x)|\}.$$ 

LEMMA 4.2. Suppose $p > d \geq 2$ and that positive numbers $\varepsilon, D_1,$ and $\sigma_0$ satisfy $d - 6\varepsilon < 2D_1 < d$. Then there exist infinitely many positive integers $N$ such that

$$V(x) \ll q^N / N^{\sigma_0} \quad \text{for all } x \in \mathbf{K}_\infty \text{ with } -(d - \varepsilon)N \leq \deg x \leq D_1 N,$$ 

where the implied constant depends only on $d, \varepsilon, D_1$ and $\sigma_0$.

Proof. Since $\lambda_1/\lambda_2 \in \mathbf{K}_\infty \setminus \mathbf{K}$, by Theorem 4.1 there exist infinitely many monic polynomials $Q$ and polynomials $a$ in $\mathbf{A}$ such that $(Q, a) = 1$ and

$$\deg(\lambda_1/\lambda_2 - a/Q) < -2\deg Q.$$ 

For a fixed pair $(Q, a)$, let $N$ be the least integer satisfying $2\deg Q \leq dN$ and write

$$\frac{\lambda_1}{\lambda_2} = \frac{a}{Q} + f \quad \text{for some } f \in \mathbf{K}_\infty \text{ with } \deg f < -2\deg Q.$$
Throughout the proof of this lemma, assume that \((d - 2D_1)N \geq 6d\). Given any \(x \in \mathbf{K}_\infty\) satisfying \(-(d - \varepsilon)N \leq \deg x \leq D_1N\), let \(m\) denote the least integer satisfying \((5d + 2D_1)N/6 \leq m\). For any \(j = 1, 2\), again by Theorem 4.1 there exist monic polynomials \(Q_1, Q_2\) and polynomials \(a_1, a_2\) such that
\[
\text{(17)} \quad \deg(x\lambda_j - a_j/Q_j) < -\deg Q_j - m, \quad j = 1, 2,
\]
where \((Q_j, a_j) = 1\) and \(\deg Q_j \leq m\). Since \(\deg \lambda_j = 0\), \(\deg(x\lambda_j) = \deg x \geq -(d - \varepsilon)N\). Combining this with (17) and \(m > (d - \varepsilon)N\) because \(d - 6\varepsilon < 2D_1\), we have \(a_j \neq 0\) and we can write
\[
x\lambda_j = \frac{a_j}{Q_j} + \frac{f_j}{Q_j} = \frac{a_j}{Q_j} \left(1 + \frac{f_j}{a_j}\right)
\]
for some \(f_j \in \mathbf{K}_\infty\) with \(\deg f_j < -m\).

Thus
\[
\frac{\lambda_1}{\lambda_2} = \frac{x\lambda_1}{x\lambda_2} = \frac{Q_2a_1}{Q_1a_2} \left(1 + \frac{f_1}{a_1}\right) \left(1 + \frac{f_2}{a_2}\right)^{-1}.
\]

Since \(\deg \lambda_1 = \deg \lambda_2 = 0\), we have \(\deg(Q_2a_1) = \deg(Q_1a_2)\). We may write
\[
\frac{\lambda_1}{\lambda_2} = \frac{Q_2a_1}{Q_1a_2} + f_3
\]
for some \(f_3 \in \mathbf{K}_\infty\) with \(\deg f_3 < -m\).

By (16), and since \(0 < m \leq d(N - 1)\) because \((d - 2D_1)N \geq 6d\), we have
\[
\deg \left(\frac{a}{Q} - \frac{Q_2a_1}{Q_1a_2}\right) < -m.
\]

This implies
\[
\deg(a_2Q_1a - Q_2a_1Q) < dN/2 - m + \deg(Q_1a_2).
\]

If \(a_2Q_1a - Q_2a_1Q \neq 0\), then \(\deg(Q_1a_2) > -dN/2 + m\). If \(a_2Q_1a - Q_2a_1Q = 0\), then
\[
\frac{a}{Q} = \frac{Q_2a_1}{Q_1a_2}.
\]

Since \((Q, a) = 1\) and \(d \geq 2\), \(\deg(Q_1a_2) \geq \deg Q > d(N - 1)/2 \geq -dN/2 + m\) because \((d - 2D_1)N \geq 6d\). Thus we always have \(\deg(Q_1a_2) > -dN/2 + m\).

Since \(\deg(x\lambda_2 - a_2/Q_2) < -\deg Q_2 - m\), \(-(d - \varepsilon)N \leq \deg x \leq D_1N\), \(\deg \lambda_2 = 0\), and \(m > (d - \varepsilon)N\), we have \(D_1N \geq \deg x = \deg x\lambda_2 = \deg(a_2/Q_2)\).

Combining these, we have
\[
\deg(Q_1Q_2) = \deg(Q_1a_2) + \deg(Q_2/a_2) > -dN/2 + m - D_1N.
\]

This implies that \(\max\{\deg Q_1, \deg Q_2\} > -dN/4 + (m - D_1N)/2\). Without loss of generality, assume that \(\deg Q_1 > -dN/4 + (m - D_1N)/2\). By the definition of \(m\), we have
\[
\deg Q_1 + m > -\frac{dN}{4} + \frac{m - D_1N}{2} + m \geq dN.
\]
Combining this with (17), we obtain
\[ S_1(x) = S(x\lambda_1) = S(a_1/Q_1). \]
Set \( \sigma = (dN - m)/\ln N \). Since \( m - 1 < (5d + 2D_1)N/6 \) and \( d > 2D_1 \), we have
\[ \sigma > \frac{(d - 2D_1)N - 6}{6\ln N} \geq d2^{6d}(\sigma_0 + 1) \]
for large \( N \). Since
\[ \sigma \ln N = dN - m < \deg Q_1 \leq m = dN - \sigma \ln N, \]
by Theorem 4.3 below, we obtain
\[ |S_1(x)| = |S(x\lambda_1)| = |S(a_1/Q_1)| \ll q^N/N^{\sigma_0} \]
for large \( N \). Thus there exist infinitely many positive integers \( N \) such that
\[ V(x) \ll q^N/N^{\sigma_0} \quad \text{for all } - (d - \varepsilon)N \leq \deg x \leq D_1N. \]

Now we recall three theorems proved in [9]. They are used in the proof of Lemma 4.6 and in the proofs of polynomial Waring and polynomial Waring–Goldbach problems (cf. [4] and [9]).

**Theorem 4.3.** Let \( 2 \leq d < p \) and let \( \sigma_0 \geq 0 \). Suppose that \( (Q, a) = 1 \), \( \sigma \ln N \leq \deg Q \leq dN - \sigma \ln N \). Then, if \( \sigma \geq d2^{6d}(\sigma_0 + 1) \), we have
\[ |S(a/Q)| \ll q^N/N^{\sigma_0}, \]
where the implied constant depends only on \( d, \sigma_0, \) and \( q \).

**Proof.** See [9], Theorem 11.8. □

**Theorem 4.4 (Hua’s lemma).** Suppose that \( 1 \leq d < p \). Then
\[ \int \mathbb{M} \left| \sum_{x \in \mathbf{A}_+, \deg x = N} E(x^d a) \right|^{2^d} da \ll N^C q^{N(2^d - d)} \]
for some \( C \), where the implied constant and the constant \( C \) depend on \( d \) and \( \mathbf{A} \), but not on \( N \). In other words, the number of solutions of
\[ x_1^d + \ldots + x_{2^d-1}^d = y_1^d + \ldots + y_{2^d-1}^d \]
with \( x_i, y_i \in \mathbf{A}_+ \) and \( \deg x_i = \deg y_i = N \) is \( \ll N^C q^{N(2^d - d)} \).

**Proof.** See [9], Theorem 4.2. □

**Remark.** In [4], Theorem 8.13, the right-hand side of (18) is \( q^{N(2^d - d + \varepsilon)} \). Following Hua’s idea (cf. [10], Theorem 4), we improve this to the form of Theorem 4.4.
Theorem 4.5. Suppose $d \geq 9$ and $s \geq 2d^2 \ln d + d^2 \ln \ln d + 2d^2 - 2d$. Then

$$\sum_{x \in \mathbb{A}_+, \deg x = N} E(x^d a)^{2s} \, da \ll q^{N(2s-d)},$$

where the implied constant depends only on $d$, $s$, and $q$. In other words, the number of solutions of

$$x_1^d + \ldots + x_s^d = y_1^d + \ldots + y_s^d$$

with $x_i, y_i \in \mathbb{A}_+$ and $\deg x_i = \deg y_i = N$ is $\ll q^{N(2s-d)}$.

Proof. See [9], Theorem 7.5. □

Lemma 4.6. Let $D, n$ be positive integers and let $d, \varepsilon, D_1$ and $N$ be as in Lemma 4.2. Then, if $D_1 N \geq n$ and

$$D \geq \begin{cases} 1 + 2^d & \text{if } 2 \leq d < 11, \\ 2[2d^2 \ln d + d^2 \ln \ln d + 2d^2 - 2d] + 1 & \text{if } d \geq 11, \end{cases}$$

we have

$$\int_{-(d-\varepsilon)N \leq \deg a} |F(a)| \chi_n(a) \, da \ll q^{(D-d)N}/N^{\sigma_0}$$

for any positive number $\sigma_0$, where the implied constant depends only on $D$, $d$, $\varepsilon$, $D_1$, $\sigma_0$, and the constant $C$ of Theorem 4.4.

Proof. By the definition of $\chi_n$, we know that $\chi_n(a) = 0$ if $\deg a \geq n$. Thus $\chi_n(a) = 0$ if $\deg a \geq D_1 N$. Thus

$$\int_{-(d-\varepsilon)N \leq \deg a} |F(a)| \chi_n(a) \, da = \int_{-(d-\varepsilon)N \leq \deg a \leq D_1 N} |F(a)| \chi_n(a) \, da.$$

If $V(a) = \min\{|S_1(a)|, |S_2(a)|\}$, then

$$|F(a)| \leq V(a) \left( |S_1(a)\prod_{j=3}^D S_j(a)| + |S_2(a)\prod_{j=3}^D S_j(a)| \right).$$

This implies

$$|F(a)| \leq V(a) \left( \sum_{j=1}^D |S_j(a)|^{D-1} \right).$$

Since

$$D \geq \begin{cases} 1 + 2^d & \text{if } 2 \leq d < 11, \\ 2[2d^2 \ln d + d^2 \ln \ln d + 2d^2 - 2d] + 1 & \text{if } d \geq 11, \end{cases}$$

and $\deg \lambda_j = 0$, $|S_j(a)| \leq q^N$, we have
where $s = [2d^2 \ln d + d^2 \ln \ln d + 2d^2 - 2d]$. By Lemma 2.2, the last integral is equal to the number of monic irreducible $2s$-tuples $(P_1, \ldots, P_{2s})$ such that \(\deg P_i = N\) and

$$\deg \left( \sum_{i=1}^{s} (P_i^d - P_{s+i}^d) \right) < -n.$$ 

Since $n > 0$, this integral is equal to the number of monic irreducible $2s$-tuples $(P_1, \ldots, P_{2s})$ such that \(\deg P_i = N\) and

$$\sum_{i=1}^{s} (P_i^d - P_{s+i}^d) = 0.$$ 

Using (19) and Theorems 4.4 and 4.5, we obtain

$$\int_{K_{\infty}} |S_j(a)|^{D-1} \chi_{-n}(a) da \ll \begin{cases} \quad N^C q^{N(D-d-1)} & \text{if } 2 \leq d < 11, \\ \quad q^{N(D-d-1)} & \text{if } d \geq 11, \end{cases}$$

Combining these with Lemma 4.2 (substitute \(\sigma_0 + C\) for \(\sigma_0\)), we obtain

$$\int_{-\sigmaN \leq \deg a} |F(a)| \chi_{-n}(a) da \leq \int_{-\sigmaN \leq \deg a \leq D_1 N} V(a) \sum_{j=1}^{D} |S_j(a)|^{D-1} \chi_{-n}(a) da \ll \frac{q^N}{N^{\sigma_0+C}} \cdot N^C q^{N(D-d-1)} = \frac{q^{(D-d)N}}{N^{\sigma_0}}. \quad \blacksquare$$

**5. Completion of the proof of the main theorem.** We conclude the proof of Theorem 2.1 by collecting the above results. First of all, Lemma 3.2 with \(\varepsilon > 0\) and a positive integer $n$ gives

$$\int_{\deg a \leq -(d-\varepsilon)N} H(a) E(a\lambda) \chi_{-n}(a) da = q^{(D-d)N-n}/N^D,$$

as $dN > \deg \lambda$. Combining this with Lemma 3.1, when $0 < \varepsilon < 1/4$ and $n = \lceil m \ln N \rceil$, we have

$$\int_{\deg a \leq -(d-\varepsilon)N} F(a) E(a\lambda) \chi_{-[m \ln N]}(a) da \gg q^{(D-d)N}/N^{D+m},$$

and

$$\int_{\deg a \leq -(d-\varepsilon)N} H(a) E(a\lambda) \chi_{-[m \ln N]}(a) da = q^{(D-d)N-n}/N^D.$$

Combining these results with (19) and Theorems 4.4 and 4.5, we obtain

$$\int_{K_{\infty}} |S_j(a)|^{D-1} \chi_{-n}(a) da \ll \begin{cases} \quad N^C q^{N(D-d-1)} & \text{if } 2 \leq d < 11, \\ \quad q^{N(D-d-1)} & \text{if } d \geq 11, \end{cases}$$

where $s = [2d^2 \ln d + d^2 \ln \ln d + 2d^2 - 2d]$. By Lemma 2.2, the last integral is equal to the number of monic irreducible $2s$-tuples $(P_1, \ldots, P_{2s})$ such that \(\deg P_i = N\) and

$$\deg \left( \sum_{i=1}^{s} (P_i^d - P_{s+i}^d) \right) < -n.$$ 

Since $n > 0$, this integral is equal to the number of monic irreducible $2s$-tuples $(P_1, \ldots, P_{2s})$ such that \(\deg P_i = N\) and

$$\sum_{i=1}^{s} (P_i^d - P_{s+i}^d) = 0.$$ 

Using (19) and Theorems 4.4 and 4.5, we obtain

$$\int_{K_{\infty}} |S_j(a)|^{D-1} \chi_{-n}(a) da \ll \begin{cases} \quad N^C q^{N(D-d-1)} & \text{if } 2 \leq d < 11, \\ \quad q^{N(D-d-1)} & \text{if } d \geq 11. \end{cases}$$

Combining these with Lemma 4.2 (substitute \(\sigma_0 + C\) for \(\sigma_0\)), we obtain

$$\int_{-\sigmaN \leq \deg a} |F(a)| \chi_{-n}(a) da \leq \int_{-\sigmaN \leq \deg a \leq D_1 N} V(a) \sum_{j=1}^{D} |S_j(a)|^{D-1} \chi_{-n}(a) da \ll \frac{q^N}{N^{\sigma_0+C}} \cdot N^C q^{N(D-d-1)} = \frac{q^{(D-d)N}}{N^{\sigma_0}}. \quad \blacksquare$$

**5. Completion of the proof of the main theorem.** We conclude the proof of Theorem 2.1 by collecting the above results. First of all, Lemma 3.2 with \(\varepsilon > 0\) and a positive integer $n$ gives

$$\int_{\deg a \leq -(d-\varepsilon)N} H(a) E(a\lambda) \chi_{-n}(a) da = q^{(D-d)N-n}/N^D,$$

as $dN > \deg \lambda$. Combining this with Lemma 3.1, when $0 < \varepsilon < 1/4$ and $n = \lceil m \ln N \rceil$, we have

$$\int_{\deg a \leq -(d-\varepsilon)N} F(a) E(a\lambda) \chi_{-[m \ln N]}(a) da \gg q^{(D-d)N}/N^{D+m},$$

and

$$\int_{\deg a \leq -(d-\varepsilon)N} H(a) E(a\lambda) \chi_{-[m \ln N]}(a) da = q^{(D-d)N-n}/N^D.$$
as $N \to \infty$. In Lemmas 4.2 and 4.6, if $\varepsilon, D_1, d, D, n$ and $\sigma_0$ satisfy $d - 6\varepsilon < 2D_1 < d$, $2 \leq d < p$, $\sigma_0 = D + m + 1$, $n = \lfloor m \ln N \rfloor$, $D_1 N \geq n$, and

$$D \geq \begin{cases} 1 + 2^d & \text{if } 2 \leq d < 11, \\ 2[2d^2 \ln d + d^2 \ln n + 2d^2 - 2d] + 1 & \text{if } d \geq 11, \end{cases}$$

then there are infinitely many positive integers $N$ (note that these $N$ come from $\lambda_1/\lambda_2 \in K_\infty/K$) such that

$$\chi_{\lfloor m \ln N \rfloor}(a) da \ll q^{(D-d)N}/N^{D+m+1}. \quad (21)$$

Therefore, taking $\varepsilon = 1/6$, $D_1 = d/2 - 1/4$ and combining (20) and (21), we see that for any positive integer $m$,

$$\int_{K_\infty} \sum_{\deg P_1 = N} \cdots \sum_{\deg P_D = N} E\left( a \left( \lambda + \sum_{i=1}^{D} \lambda_i P_i^d \right) \right) \chi_{\lfloor m \ln N \rfloor}(a) da$$

$$= \int_{K_\infty} F(a) E(a\lambda) \chi_{\lfloor m \ln N \rfloor}(a) da \gg q^{(D-d)N}/N^{D+m}.$$ 

It follows from Lemma 2.2 that there exist infinitely many positive integers $N$ for which there are $\gg q^{(D-d)N}/N^{D+m}$ $D$-tuples $(P_1, \ldots, P_D)$ of monic irreducible polynomials with $\deg P_i = N$ and

$$\deg(\lambda + \lambda_1 P_1^d + \cdots + \lambda_D P_D^d) < -m \ln N + 1.$$ 

This completes the proof of Theorem 2.1. $\blacksquare$

References

Hardy–Littlewood method


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