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Diophantine inequalities for the non-Archimedean line $\mathbb{F}_q((1/T))$

by

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1. Introduction. In 1946, Davenport and Heilbronn [3] adapted the Hardy–Littlewood method to prove that if λ_i $(i = 1, \ldots, K)$ are non-zero real numbers, not all of the same sign, and if λ_1/λ_2 is irrational, then the values of

$$\lambda_1 x_1^k + \ldots + \lambda_K x_K^k$$

as x_i's run independently through all natural numbers, are everywhere dense on the real line provided that $K \geq 2^k + 1$. In the case $k = 1$, Baker [1] (see also [11] and [13]) showed that for any positive integer n there exist infinitely many primes p_1, p_2, p_3 satisfying the inequality

$$|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3| < (\ln p)^{-n},$$

where p denotes the maximum of $p_1,\ p_2,\ p_3$. More recently, Harman [5] showed that if α is a real number, then there are infinitely many ordered triples of primes $p_1,\ p_2,\ p_3$ for which

$$|\alpha + \lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3| < (\max_j p_j)^{-1/5+\varepsilon}.$$

In the case $k \geq 2$, Ramachandra [11] (see also [12]) showed that when $K \geq 2^k + 1$ if $1 \leq k \leq 11$ and $K \geq 2[2k^2 \ln k + k^2 \ln\ln k + 2.5k^2] - 1$ if $k \geq 12$, the values of

$$\lambda_1 p_1^k + \ldots + \lambda_K p_K^k$$

as the p_j's run independently through all primes, are everywhere dense on the real line. The key to the Hardy–Littlewood method on the real line is

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the integral

$$\int_{-\infty}^{\infty} \exp(xy)\left(\frac{\sin \pi x}{\pi x}\right)^2 dx = \max\{1-|y|, 0\}. \tag{1}$$

In this paper, we study the Hardy–Littlewood method for the completion $\mathbf{K}_\infty = \mathbb{F}_q((1/T))$ of the rational function field $\mathbf{K} = \mathbb{F}_q(T)$ at the infinite place, where $\mathbb{F}_q$ denotes the finite field with q elements. We have a natural discrete valuation $|\cdot|$ on $\mathbf{K}_\infty$ defined by

$$|f| = q^{\deg f},$$

where $\deg f$ denotes the degree of $f \in \mathbf{K}_\infty$ at T, and set $\deg 0 = -\infty$. Since $\mathbf{K}_\infty$ is complete under the non-Archimedean valuation $|\cdot|$ and the Pontryagin (self) duality $\widehat{\mathbf{K}}_\infty = \mathbf{K}_\infty$ holds (cf. Section 2), we have the following basic analogy:

$$\mathbb{F}_q[T] \sim \mathbb{Z}, \qquad \mathbf{K} \sim \mathbb{Q}, \qquad \mathbf{K}_\infty \sim \mathbb{R}.$$

Let p be the characteristic of $\mathbb{F}_q$, let $\lambda_1, \ldots, \lambda_D$ be non-zero elements in $\mathbf{K}_\infty$ satisfying $\lambda_1/\lambda_2 \notin \mathbf{K}$ and

$$\operatorname{sgn} \lambda_1 + \ldots + \operatorname{sgn} \lambda_D = 0,$$

where $\operatorname{sgn} f \in \mathbb{F}_q$ denotes the leading coefficient of $f \in \mathbf{K}_\infty$. We show that if $p > d \geq 1$ and

$$D \geq \begin{cases} 1 + 2^d & \text{if } 2 \leq d < 11, \\ 2[2d^2 \ln d + d^2 \ln\ln d + 2d^2 - 2d] + 1 & \text{if } d \geq 11, \end{cases}$$

then the values of the sum

$$\lambda_1 P_1^d + \ldots + \lambda_D P_D^d,$$

as the P_i's run independently through all monic irreducible polynomials in $\mathbb{F}_q[T]$, are everywhere dense on the "non-Archimedean" line $\mathbf{K}_\infty$. In fact, we obtain a more explicit inequality in Theorem 2.1. In the proof of Theorem 2.1, the integral (cf. Lemma 2.2)

$$\int_{\mathbf{K}_\infty} E(af)\chi_n(a)\, da = \begin{cases} 1 & \text{if } \deg f < n, \\ 0 & \text{if } \deg f \geq n, \end{cases}$$

plays a role entirely analogous to the integral (1) on the real line.

We studied the case $d = 1,\ D = 3$ in [8]. In the present paper, we attack this problem in the case when $d \geq 2$. In this situation, we need more additive theory of monic irreducible polynomials in $\mathbb{F}_q[T]$ (see, e.g., Theorems 4.3, 4.4, and 2.4).

2. The main theorem and definition. Let $\mathbb{F}_q$ be the finite field with q elements. Let p be its characteristic and let $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ be the subfield of

$\mathbb{F}_q$ with p elements. Let $\psi_0 : \mathbb{F}_p \to \mathbb{C}^\times$ be the canonical additive character defined by

$$\psi_0([c]) = \exp\left(\frac{2\pi i \cdot c}{p}\right),$$

where $[c]$ denotes the canonical image of c in $\mathbb{F}_p$. Let $\psi : \mathbb{F}_q \to \mathbb{C}^\times$ be the additive character defined by $\psi(x) = \psi_0(\mathrm{Tr}(x))$ for all $x \in \mathbb{F}_q$ where Tr is the trace map from $\mathbb{F}_q$ to $\mathbb{F}_p$. Let $\mathbf{A} = \mathbb{F}_q[T]$ (resp. $\mathbf{K} = \mathbb{F}_q(T)$) be the polynomial ring (resp. rational function field) with coefficients in $\mathbb{F}_q$. Let $\mathbf{A}_+$ denote the subset of $\mathbf{A}$ consisting of all monic polynomials. Let $\mathbf{K}_\infty = \mathbb{F}_q((1/T))$ denote the completion of $\mathbf{K}$ at the infinite place; in other words, for every $a \in \mathbf{K}_\infty$, if $a \neq 0$, then a can be expressed as

$$a = \sum_{i=d}^{-\infty} c_i T^i,$$

where $c_i \in \mathbb{F}_q$ and $c_d \neq 0$. The sign, degree, and absolute value of a are defined by $\operatorname{sgn} a = c_d, \deg a = d$, and $|a| = q^d$. The residue of a at the infinite place is denoted by $\mathrm{Res}_\infty f = c_{-1}$. The exponential map $E : \mathbf{K}_\infty \to \mathbb{C}^\times$ is defined by

$$E(a) = \psi(\mathrm{Res}_\infty a).$$

The exponential map E is a non-trivial additive character from $\mathbf{K}_\infty$ to $\mathbb{C}^\times$ and the Pontryagin (self) duality $\widehat{\mathbf{K}}_\infty = \mathbf{K}_\infty$ is deduced by the bilinear map

$$\mathbf{K}_\infty \times \mathbf{K}_\infty \to \mathbb{C}^\times, \quad (a, f) \mapsto E(a \cdot f).$$

In this paper, the Haar integral for $\mathbf{K}_\infty$ is defined to satisfy

$$\int\limits_{\deg a \le -1} 1\, da = 1.$$

This implies that

$$\int\limits_{\mathbf{K}_\infty} f(a)\, d(ba) = |b| \int\limits_{\mathbf{K}_\infty} f(a)\, da$$

for all $b \in \mathbf{K}_\infty$ and continuous functions f (with compact support). With these properties, we have the following basic analogy:

$$\mathbf{A} \sim \mathbb{Z}, \quad \mathbf{K} \sim \mathbb{Q}, \quad \mathbf{K}_\infty \sim \mathbb{R}, \quad E \sim \exp.$$

The main theorem of this paper is

THEOREM 2.1. *Suppose that d, D, m are positive integers and $\lambda, \lambda_1, \ldots$ $\ldots, \lambda_D$ are non-zero elements in $\mathbf{K}_\infty$ satisfying $\lambda_1/\lambda_2 \notin \mathbf{K}, 2 \le d < p$,*

$$\deg \lambda_1 = \ldots = \deg \lambda_D = 0, \tag{2}$$

and

$$\operatorname{sgn} \lambda_1 + \ldots + \operatorname{sgn} \lambda_D = 0.$$

Then if

$$D \geq \begin{cases} 1 + 2^d & \textit{if } 2 \leq d < 11, \\ 2[2d^2 \ln d + d^2 \ln\ln d + 2d^2 - 2d] + 1 & \textit{if } d \geq 11, \end{cases}$$

then there exist infinitely many positive integers N for which there are

$$\gg \frac{q^{(D-d)N}}{N^{D+m}}$$

D-tuples $(P_1, \ldots, P_D)$ of monic irreducible polynomials with $\deg(\lambda_i P_i) = N$ *and*

$$\deg(\lambda + \lambda_1 P_1^d + \ldots + \lambda_D P_D^d) < -m \ln N + 1,$$

where the implied constant depends only on $\mathbf{A}$, λ, λ_i, d, D, *and* m, *but not on* N.

Remark 1. The complete proof of Theorem 2.1 is given in Section 5. In fact, if we define the value of $I_j(a)$ in (3) to be

$$I_j(a) = I(aT^{\Lambda - \deg \lambda_j} \lambda_j),$$

where

$$\Lambda = \max_{1 \leq j \leq D} \{\deg \lambda_j\},$$

then without the condition (2), the statement of Theorem 2.1 is also true.

2. The choice of N depends on $\lambda_1/\lambda_2 \in \mathbf{K}_\infty/\mathbf{K}$ and this condition is used only in Lemmas 4.2 and 4.6. Combining this theorem and [8], Theorem 1.2, we obtain

Consequence 1. *Under the hypothesis of Theorem* 2.1, *suppose* $p > d \geq 1$ *and*

$$D \geq \begin{cases} 1 + 2^d & \textit{if } d < 11, \\ 2[2d^2 \ln d + d^2 \ln\ln d + 2d^2 - 2d] + 1 & \textit{if } d \geq 11. \end{cases}$$

Then the values of the sum

$$\lambda_1 P_1^d + \ldots + \lambda_D P_D^d,$$

as the P_i's run independently through all monic irreducible polynomials in $\mathbb{F}_q[T]$, *are everywhere dense on the non-Archimedean line* $\mathbb{F}_q((1/T))$.

Let $\mathfrak{M}$ be the subring of $\mathbf{K}_\infty$ consisting of $a \in \mathbf{K}_\infty$ with $\deg a \leq -1$ and let χ_0 be the characteristic function of $\mathfrak{M}$; in other words, $\chi_0 : \mathbf{K}_\infty \to \mathbb{R}$ satisfies

$$\chi_0(a) = \begin{cases} 1 & \text{if } a \in \mathfrak{M}, \\ 0 & \text{otherwise.} \end{cases}$$

Given any integer n, the function $\chi_n : \mathbf{K}_\infty \to \mathbb{R}$ is defined by

$$\chi_n(a) = q^n \chi_0(aT^n) \quad \text{for } a \in \mathbf{K}_\infty.$$

LEMMA 2.2. *We have*

$$\int_{\mathbf{K}_\infty} E(af)\chi_n(a)\,da = \begin{cases} 1 & \text{if } \deg f < n,\\ 0 & \text{if } \deg f \ge n.\end{cases}$$

Proof. See [6], Theorem 3.5. ■

Let $p > d \ge 2,\ N$ be fixed positive integers. We define functions

$$(3)\qquad \begin{gathered} S(a) = {\sum_{\deg P = N}}' E(aP^d), \qquad I(a) = \frac{1}{N}\int_{y\in T^N + T^N\mathfrak{M}} E(ay^d)\,dy,\\ S_j(a) = S(a\lambda_j), \qquad I_j(a) = I(a\lambda_j), \qquad j = 1,\ldots,D,\\ F(a) = \prod_{j=1}^{D} S_j(a), \qquad H(a) = \prod_{j=1}^{D} I_j(a), \end{gathered}$$

where $\sum'$ denotes the sum over monic irreducible polynomials in $\mathbf{A}$. Let π_N denote the number of monic irreducible polynomials in $\mathbf{A}$ of degree N. The prime number theorem for $\mathbf{A}$ is

$$(4)\qquad q^N/N - q^{N/2} < \pi_N \le q^N/N.$$

As $\deg\lambda_j = 0$, by the definition of E we have

$$(5)\qquad I_j(a) = \begin{cases} q^N/N & \text{if } \deg a < -dN-1,\\ (q^N/N)\psi(\operatorname{sgn}(a\lambda_j)) & \text{if } \deg a = -dN-1,\end{cases}$$

and

$$(6)\qquad S_j(a) = \begin{cases} \pi_N & \text{if } \deg a < -dN-1,\\ \pi_N\psi(\operatorname{sgn}(a\lambda_j)) & \text{if } \deg a = -dN-1.\end{cases}$$

LEMMA 2.3. *If* $\deg a \ge -dN$, *then* $I_j(a) = 0$.

Proof. Since $\deg\lambda_j = 0$, it suffices to show that $I(a) = 0$ for $\deg a \ge -dN$. Let $\deg a = -dN + l$ for some integer $l \ge 0$ and let

$$a = a_l T^{-dN+l} + \ldots + a_{-1}T^{-dN-1} + a' \in \mathbf{K}_\infty,$$

where $a_j \in \mathbb{F}_q,\ a_l \ne 0$, and $\deg a' \le -dN-2$. Let

$$y = T^N + \sum_{j=1}^{\infty} b_{-j}T^{N-j} \in T^N + T^N\mathfrak{M},$$

where $b_{-j} \in \mathbb{F}_q$. Then we have

$$(7)\qquad y^d = T^{dN} + \sum_{j=1}^{\infty}(db_{-j} + c_{-j}(b_{-1},\ldots,b_{-(j-1)}))T^{dN-j},$$

for some $c_{-j}(x_1,\dots,x_{j-1}) \in \mathbb{F}_q[x_1,\dots,x_{j-1}]$ and $c_{-1}=0$. Since $E(a'y^d) = 1$, we have

$$E(ay^d) = E((a-a')y^d) = \psi\Big(a_{-1} + \sum_{j=0}^{l} a_j b'_{-(j+1)}\Big), \tag{8}$$

where

$$b'_{-j} = db_{-j} + c_{-j}(b_{-1},\dots,b_{-(j-1)}).$$

By (7), since $2 \le d < p$, we know that the dth power mapping

$$F : T^N + T^N\mathfrak{M} \to T^{dN} + T^{dN}\mathfrak{M}, \qquad y \mapsto y^d$$

is bijective and satisfies

$$F(y + T^{N-(l+1)}\mathfrak{M}) = y^{dN} + T^{dN-(l+1)}\mathfrak{M}. \tag{9}$$

By (8), (9), since ψ is a non-trivial additive character of $\mathbb{F}_q$, and $a_l \ne 0$, we obtain

$$\begin{aligned} I(a) &= \frac{1}{N} \int\limits_{y\in T^N + T^N\mathfrak{M}} E(ay^d)\,dy = \frac{1}{N} \int\limits_{y\in T^N + T^N\mathfrak{M}} E((a-a')y^d)\,dy \\ &= \frac{1}{N} \int\limits_{z\in T^{N-(l+1)}\mathfrak{M}} \sum_{c\in\mathbb{F}_q} q^l \cdot \psi(c)\,dz = 0. \ \blacksquare \end{aligned}$$

Let a positive integer l satisfy $l \le N/2$ and let y be a monic element in $\mathbf{K}_\infty$ of degree N. Let $\pi_N(y,l)$ denote the number of monic irreducible polynomials $P \in \mathbf{A}_+$ of degree N with $\deg(P-y) < N-l$. In [7], Corollary 2.6, or [2], Theorem 1.4, we have

$$\pi_N(y,l) = \frac{q^{N-l}}{N} + O(q^{N/2}), \tag{10}$$

where the implied constant depends only on $\mathbf{A}$. Given

$$x = \sum_{i=-dN+l-1}^{-\infty} a_i T^i \in \mathbf{K}_\infty, \qquad f = T^{dN} + \sum_{j=dN-1}^{0} f_j T^j \in \mathbf{A},$$

where $a_i, f_j \in \mathbb{F}_q$, $a_{-dN+l-1} \ne 0$, and setting $f_{dN} = 1$, we have

$$\mathrm{Res}_\infty(xf) = \sum_{k=0}^{l} a_{-dN+k-1} f_{dN-k}. \tag{11}$$

Let $\pi_{N,d}(f,l)$ be the number of monic irreducible polynomials P of degree N with $\deg(P^d - f) < dN - l$. By (7), since $2 \le d < p$, there exists a monic element $y \in \mathbf{K}_\infty$ of degree N satisfying $\deg(y^d - f) < dN - l$ and $\pi_{N,d}(f,l) = \pi_{N,d}(y^d,l) = \pi_N(y,l)$. Thus by (10) we get

$$\pi_{N,d}(f,l) = \frac{q^{N-l}}{N} + O(q^{N/2}), \tag{12}$$

where the implied constant depends only on $\mathbf{A}$. If P is a monic irreducible polynomial of degree N satisfying $\deg(P^d - f) < dN - l$, then by (11), $\mathrm{Res}_\infty(xP^d) = \mathrm{Res}_\infty(xf)$. Hence combining (12), (11), and $a_{-dN+l-1} \neq 0$, we get

$$\#\{\text{monic irreducibles } P \mid \deg P = N,\ \mathrm{Res}_\infty(xP^d) = c\} \\ = q^{l-1}\bigg(\frac{q^{N-l}}{N} + O(q^{N/2})\bigg) = \frac{q^{N-1}}{N} + O(q^{l+N/2}) \tag{13}$$

for any $c \in \mathbb{F}_q$. Since

$$E(xP^d) = \exp\bigg(\frac{2\pi i\,\mathrm{Tr}(\mathrm{Res}_\infty(xP^d))}{p}\bigg),$$

and Tr is a surjective $\mathbb{F}_p$-linear mapping from $\mathbb{F}_q$ onto $\mathbb{F}_p$, by (13) we obtain

$$|S(x)| = \Big|\sideset{}{'}\sum_{\deg P=N} E(xP^d)\Big| = O(q^{l+N/2}).$$

Therefore we have

Theorem 2.4. *Let m be an integer satisfying $0 \le m < N/2$. Then*

$$|S(x)| = O(q^{m+N/2})$$

for all $x \in \mathbf{K}_\infty$ with $\deg x = -dN + m$, where the implied constant depends only on $\mathbf{A}$.

Remark. If $m \ge N/2$, then the result of Theorem 2.4 is trivial.

3. The major arcs

Lemma 3.1. *Let n be a positive integer and let $-dN \le m \le -dN+N/4$. Then*

$$\int\limits_{\deg a \le m} |F(a) - H(a)|\chi_{-n}(a)\,da = o\bigg(\frac{q^{(D-d)N-N/2-n}}{N^D}\bigg),$$

as $N \to \infty$.

Proof. Using (5), (6), $\mathrm{sgn}\,\lambda_1 + \ldots + \mathrm{sgn}\,\lambda_D = 0$, $\chi_{-n}(a) \le q^{-n}$, Lemma 2.3, and (4), we obtain

$$\begin{aligned} \int\limits_{\deg a \le m} &|F(a) - H(a)|\chi_{-n}(a)\,da \\ &\le q^{-n} \int\limits_{\deg a \le -dN-1} |\pi_N^D - q^{DN}/N^D|\,da + q^{-n} \sum_{i=-dN}^{m} \int\limits_{\deg a = i} |F(a)|\,da \\ &\le O\bigg(\frac{q^{(D-d)N-N/2-n}}{N^{D-1}}\bigg) + q^{-n} \sum_{i=-dN}^{m} \int\limits_{\deg a = i} |F(a)|\,da, \end{aligned}$$

where the implied constant depends only on $\mathbf{A}$. Since $S_j(a) = S(a\lambda_j)$ and $\deg(a\lambda_j) = \deg a$, by Theorem 2.4, we obtain

$$|S_j(a)| = O(q^{dN+m+N/2})$$

for $-dN \le \deg a \le m$, where the implied constant depends only on $\mathbf{A}$. As $d \ge 2$ and $D \ge 1 + 2^d \ge 5$, we obtain

$$\begin{aligned} q^{-n} \sum_{i=-dN}^{m} \int_{\deg a = i} |F(a)|\, da &= O\Big(q^{-n} \sum_{i=-dN}^{m} \int_{\deg a=i} q^{dDN+DN/2+Dm}\, da\Big) \\ &= O(q^{-n} \cdot q^m \cdot q^{dDN+DN/2+Dm}) \\ &= O(q^{(dD+D/2)N+(D+1)m-n}) \\ &= O\bigg(\frac{q^{(D-d)N-N/2-n}}{N^{D-1}}\bigg), \end{aligned}$$

where the implied constant depends only on $\mathbf{A}$. ■

LEMMA 3.2. *Let n be a positive integer and let $\lambda \in \mathbf{K}_\infty$. Then if $m \ge -dN$ and $dN > \deg \lambda$, we have*

$$\int_{\deg a \le m} H(a)E(a\lambda)\chi_{-n}(a)\, da = \frac{q^{(D-d)N-n}}{N^D}.$$

Proof. By Lemma 2.3 and the definition of H, we have $H(a) = 0$ if $\deg a \ge m \ge -dN$. Thus

$$\int_{\deg a \ge m} H(a)E(a\lambda)\chi_{-n}(a)\, da = 0. \tag{14}$$

By the definitions of H, E and Lemma 2.3, we have

$$\begin{aligned} &\int_{\mathbf{K}_\infty} H(a)E(a\lambda)\chi_{-n}(a)\, da \\ &\quad = \frac{1}{N^D} \int_{\deg a < -dN} \int_{T^N + T^N\mathfrak{M}} \cdots \int_{T^N+T^N\mathfrak{M}} E\Big(a\Big(\lambda + \sum_{j=1}^{D} \lambda_j y_j^d\Big)\Big) \\ &\qquad\qquad \times \chi_{-n}(a)\, dy_1 \ldots dy_D\, da \\ &\quad = \frac{1}{N^D} \int_{T^N+T^N\mathfrak{M}} \cdots \int_{T^N+T^N\mathfrak{M}} \int_{\deg a < -dN} E\Big(a\Big(\lambda + \sum_{j=1}^{D} \lambda_j y_j^d\Big)\Big) \\ &\qquad\qquad \times \chi_{-n}(a)\, da\, dy_1 \ldots dy_D. \end{aligned}$$

By the definition of $\chi_{-n}(a)$ and since $\deg a < -dN$, the above is

$$\frac{q^{-n}}{N^D} \int\limits_{T^N+T^N\mathfrak{M}} \cdots \int\limits_{T^N+T^N\mathfrak{M}} \int\limits_{\deg a<-dN} E\Big(a\Big(\lambda + \sum_{j=1}^{D} \lambda_j y_j^d\Big)\Big)\, da\, dy_1 \ldots dy_D.$$

Given any $y_1, \ldots, y_D \in T^N + T^N\mathfrak{M}$, set

$$f = \lambda + \lambda_1 y_1^d + \ldots + \lambda_D y_D^d.$$

Since $dN > \deg\lambda$, $\deg\lambda_j = 0$, and $\operatorname{sgn}\lambda_1 + \ldots + \operatorname{sgn}\lambda_D = 0$, we have $\deg f < dN$. This implies

$$\int\limits_{\deg a<-dN} E(af)\, da = \int\limits_{\deg a<-dN} 1\, da = q^{-dN}.$$

Therefore

$$\int\limits_{\mathbf{K}_\infty} H(a)E(a\lambda)\chi_{-n}(a)\, da = \frac{q^{-n}}{N^D}\cdot q^{DN}\cdot q^{-dN} = \frac{q^{(D-d)N-n}}{N^D}.$$

Combining these with (14), we complete the proof. ■

4. The minor arcs. We recall Dirichlet's theorem for $\mathbf{A}$ in

THEOREM 4.1. *Given any $\alpha \in \mathbf{K}_\infty$ and a positive integer N, there exists a unique monic polynomial Q and a polynomial a in $\mathbf{A}$ satisfying $(Q, a) = 1, \deg Q \le N$, and $\deg(\alpha - a/Q) \le -(\deg Q + N + 1)$.*

Proof. See [6]. ■

For any $x \in \mathbf{K}_\infty$, define

$$V(x) = \min\{|S_1(x)|, |S_2(x)|\}.$$

LEMMA 4.2. *Suppose $p > d \ge 2$ and that positive numbers ε, D_1, and σ_0 satisfy $d - 6\varepsilon < 2D_1 < d$. Then there exist infinitely many positive integers N such that*

$$V(x) \ll q^N/N^{\sigma_0} \quad \text{for all } x \in \mathbf{K}_\infty \text{ with } -(d-\varepsilon)N \le \deg x \le D_1 N,$$

where the implied constant depends only on d, ε, D_1 and σ_0.

Proof. Since $\lambda_1/\lambda_2 \in \mathbf{K}_\infty \setminus \mathbf{K}$, by Theorem 4.1 there exist infinitely many monic polynomials Q and polynomials a in $\mathbf{A}$ such that $(Q, a) = 1$ and

$$\deg(\lambda_1/\lambda_2 - a/Q) < -2\deg Q. \tag{15}$$

For a fixed pair (Q, a), let N be the least integer satisfying $2\deg Q \le dN$ and write

$$\frac{\lambda_1}{\lambda_2} = \frac{a}{Q} + f \quad \text{for some } f \in \mathbf{K}_\infty \text{ with } \deg f < -2\deg Q. \tag{16}$$

Throughout the proof of this lemma, assume that $(d-2D_1)N \geq 6d$. Given any $x \in \mathbf{K}_\infty$ satisfying $-(d-\varepsilon)N \leq \deg x \leq D_1 N$, let m denote the least integer satisfying $(5d+2D_1)N/6 \leq m$. For any $j = 1, 2$, again by Theorem 4.1 there exist monic polynomials $Q_1,\ Q_2$ and polynomials $a_1,\ a_2$ such that

$$\deg(x\lambda_j - a_j/Q_j) < -\deg Q_j - m, \qquad j = 1, 2, \tag{17}$$

where $(Q_j, a_j) = 1$ and $\deg Q_j \leq m$. Since $\deg \lambda_j = 0$, $\deg(x\lambda_j) = \deg x \geq -(d-\varepsilon)N$. Combining this with (17) and $m > (d-\varepsilon)N$ because $d-6\varepsilon < 2D_1$, we have $a_j \neq 0$ and we can write

$$x\lambda_j = \frac{a_j}{Q_j} + \frac{f_j}{Q_j} = \frac{a_j}{Q_j}\left(1 + \frac{f_j}{a_j}\right) \quad \text{for some } f_j \in \mathbf{K}_\infty \text{ with } \deg f_j < -m.$$

Thus

$$\frac{\lambda_1}{\lambda_2} = \frac{x\lambda_1}{x\lambda_2} = \frac{Q_2 a_1}{Q_1 a_2}\left(1 + \frac{f_1}{a_1}\right)\left(1 + \frac{f_2}{a_2}\right)^{-1}.$$

Since $\deg \lambda_1 = \deg \lambda_2 = 0$, we have $\deg(Q_2 a_1) = \deg(Q_1 a_2)$. We may write

$$\frac{\lambda_1}{\lambda_2} = \frac{Q_2 a_1}{Q_1 a_2} + f_3 \quad \text{for some } f_3 \in \mathbf{K}_\infty \text{ with } \deg f_3 < -m.$$

By (16), and since $0 < m \leq d(N-1)$ because $(d-2D_1)N \geq 6d$, we have

$$\deg\left(\frac{a}{Q} - \frac{Q_2 a_1}{Q_1 a_2}\right) < -m.$$

This implies

$$\deg(a_2 Q_1 a - Q_2 a_1 Q) < dN/2 - m + \deg(Q_1 a_2).$$

If $a_2 Q_1 a - Q_2 a_1 Q \neq 0$, then $\deg(Q_1 a_2) > -dN/2 + m$. If $a_2 Q_1 a - Q_2 a_1 Q = 0$, then

$$\frac{a}{Q} = \frac{Q_2 a_1}{Q_1 a_2}.$$

Since $(Q, a) = 1$ and $d \geq 2$, $\deg(Q_1 a_2) \geq \deg Q > d(N-1)/2 \geq -dN/2 + m$ because $(d-2D_1)N \geq 6d$. Thus we always have $\deg(Q_1 a_2) > -dN/2 + m$. Since $\deg(x\lambda_2 - a_2/Q_2) < -\deg Q_2 - m$, $-(d-\varepsilon)N \leq \deg x \leq D_1 N$, $\deg \lambda_2 = 0$, and $m > (d-\varepsilon)N$, we have $D_1 N \geq \deg x = \deg x\lambda_2 = \deg(a_2/Q_2)$. Combining these, we have

$$\deg(Q_1 Q_2) = \deg(Q_1 a_2) + \deg(Q_2/a_2) > -dN/2 + m - D_1 N.$$

This implies that $\max\{\deg Q_1, \deg Q_2\} > -dN/4 + (m - D_1 N)/2$. Without loss of generality, assume that $\deg Q_1 > -dN/4 + (m - D_1 N)/2$. By the definition of m, we have

$$\deg Q_1 + m > -\frac{dN}{4} + \frac{m - D_1 N}{2} + m \geq dN.$$

Combining this with (17), we obtain

$$S_1(x) = S(x\lambda_1) = S(a_1/Q_1).$$

Set $\sigma = (dN - m)/\ln N$. Since $m - 1 < (5d + 2D_1)N/6$ and $d > 2D_1$, we have

$$\sigma > \frac{(d - 2D_1)N - 6}{6 \ln N} \ge d2^{6d}(\sigma_0 + 1)$$

for large N. Since

$$\sigma \ln N = dN - m < \deg Q_1 \le m = dN - \sigma \ln N,$$

by Theorem 4.3 below, we obtain

$$|S_1(x)| = |S(x\lambda_1)| = |S(a_1/Q_1)| \ll q^N/N^{\sigma_0}$$

for large N. Thus there exist infinitely many positive integers N such that

$$V(x) \ll q^N/N^{\sigma_0} \quad \text{for all } -(d - \varepsilon)N \le \deg x \le D_1 N. \ \blacksquare$$

Now we recall three theorems proved in [9]. They are used in the proof of Lemma 4.6 and in the proofs of polynomial Waring and polynomial Waring–Goldbach problems (cf. [4] and [9]).

THEOREM 4.3. *Let $2 \le d < p$ and let $\sigma_0 \ge 0$. Suppose that $(Q, a) = 1$, $\sigma \ln N \le \deg Q \le dN - \sigma \ln N$. Then, if $\sigma \ge d2^{6d}(\sigma_0 + 1)$, we have*

$$|S(a/Q)| \ll q^N/N^{\sigma_0},$$

where the implied constant depends only on d, σ_0, and q.

Proof. See [9], Theorem 11.8. $\blacksquare$

THEOREM 4.4 (Hua's lemma). *Suppose that $1 \le d < p$. Then*

$$\int_{\mathfrak{M}} \Big| \sum_{x \in \mathbf{A}_+,\, \deg x = N} E(x^d a) \Big|^{2^d} da \ll N^C q^{N(2^d - d)} \tag{18}$$

for some C, where the implied constant and the constant C depend on d and $\mathbf{A}$, but not on N. In other words, the number of solutions of

$$x_1^d + \ldots + x_{2^{d-1}}^d = y_1^d + \ldots + y_{2^{d-1}}^d$$

with $x_i, y_i \in \mathbf{A}_+$ and $\deg x_i = \deg y_i = N$ is $\ll N^C q^{N(2^d - d)}$.

Proof. See [9], Theorem 4.2. $\blacksquare$

REMARK. In [4], Theorem 8.13, the right-hand side of (18) is $q^{N(2^d - d + \varepsilon)}$. Following Hua's idea (cf. [10], Theorem 4), we improve this to the form of Theorem 4.4.

THEOREM 4.5. *Suppose $d \geq 9$ and $s \geq 2d^2 \ln d + d^2 \ln\ln d + 2d^2 - 2d$. Then*

$$\int_{\mathfrak{M}} \Big| \sum_{x \in \mathbf{A}_+,\, \deg x = N} E(x^d a) \Big|^{2s} da \ll q^{N(2s-d)},$$

where the implied constant depends only on d, s, and q. In other words, the number of solutions of

$$x_1^d + \ldots + x_s^d = y_1^d + \ldots + y_s^d$$

with $x_i, y_i \in \mathbf{A}_+$ and $\deg x_i = \deg y_i = N$ is $\ll q^{N(2s-d)}$.

Proof. See [9], Theorem 7.5. ∎

LEMMA 4.6. *Let D, n be positive integers and let d, ε, D_1 and N be as in Lemma 4.2. Then, if $D_1 N \geq n$ and*

$$D \geq \begin{cases} 1 + 2^d & \text{if } 2 \leq d < 11, \\ 2[2d^2 \ln d + d^2 \ln\ln d + 2d^2 - 2d] + 1 & \text{if } d \geq 11, \end{cases}$$

we have

$$\int_{-(d-\varepsilon)N \leq \deg a} |F(a)| \chi_{-n}(a)\, da \ll q^{(D-d)N}/N^{\sigma_0}$$

for any positive number σ_0, where the implied constant depends only on D, d, ε, D_1, σ_0, and the constant C of Theorem 4.4.

Proof. By the definition of χ_{-n}, we know that $\chi_{-n}(a) = 0$ if $\deg a \geq n$. Thus $\chi_{-n}(a) = 0$ if $\deg a \geq D_1 N$. Thus

$$\int_{-(d-\varepsilon)N \leq \deg a} |F(a)| \chi_{-n}(a)\, da = \int_{-(d-\varepsilon)N \leq \deg a \leq D_1 N} |F(a)| \chi_{-n}(a)\, da.$$

If $V(a) = \min\{|S_1(a)|, |S_2(a)|\}$, then

$$|F(a)| \leq V(a) \Big(\Big| S_1(a) \prod_{j=3}^{D} S_j(a) \Big| + \Big| S_2(a) \prod_{j=3}^{D} S_j(a) \Big| \Big).$$

This implies

$$|F(a)| \leq V(a) \Big(\sum_{j=1}^{D} |S_j(a)|^{D-1} \Big).$$

Since

$$D \geq \begin{cases} 1 + 2^d & \text{if } 2 \leq d < 11, \\ 2[2d^2 \ln d + d^2 \ln\ln d + 2d^2 - 2d] + 1 & \text{if } d \geq 11, \end{cases}$$

and $\deg \lambda_j = 0$, $|S_j(a)| \leq q^N$, we have

$$(19)\qquad \int\limits_{\mathbf{K}_\infty} |S_j(a)|^{D-1}\chi_{-n}(a)\,da \le \begin{cases} q^{N(D-1-2^d)}\int_{\mathbf{K}_\infty} |S(a)|^{2^d}\chi_{-n}(a)\,da & \text{if } 2\le d<11,\\ q^{N(D-2s-1)}\int_{\mathbf{K}_\infty} |S(a)|^{2s}\chi_{-n}(a)\,da & \text{if } d\ge 11,\end{cases}$$

where $s=[2d^2\ln d+d^2\ln\ln d+2d^2-2d]$. By Lemma 2.2, the last integral is equal to the number of monic irreducible $2s$-tuples $(P_1,\ldots,P_{2s})$ such that $\deg P_i=N$ and

$$\deg\Big(\sum_{i=1}^{s}(P_i^d-P_{s+i}^d)\Big)<-n.$$

Since $n>0$, this integral is equal to the number of monic irreducible $2s$-tuples $(P_1,\ldots,P_{2s})$ such that $\deg P_i=N$ and

$$\sum_{i=1}^{s}(P_i^d-P_{s+i}^d)=0.$$

Using (19) and Theorems 4.4 and 4.5, we obtain

$$\int\limits_{\mathbf{K}_\infty} |S_j(a)|^{D-1}\chi_{-n}(a)\,da \ll \begin{cases} N^C q^{N(D-d-1)} & \text{if } 2\le d<11,\\ q^{N(D-d-1)} & \text{if } d\ge 11.\end{cases}$$

Combining these with Lemma 4.2 (substitute σ_0+C for σ_0), we obtain

$$\begin{aligned}\int\limits_{-(d-\varepsilon)N\le\deg a} |F(a)|\chi_{-n}(a)\,da &\le \int\limits_{-(d-\varepsilon)N\le\deg a\le D_1N} V(a)\sum_{j=1}^{D}|S_j(a)|^{D-1}\chi_{-n}(a)\,da\\ &\ll \frac{q^N}{N^{\sigma_0+C}}\cdot N^Cq^{N(D-d-1)}=\frac{q^{(D-d)N}}{N^{\sigma_0}}.\ \blacksquare\end{aligned}$$

5. Completion of the proof of the main theorem. We conclude the proof of Theorem 2.1 by collecting the above results. First of all, Lemma 3.2 with $\varepsilon>0$ and a positive integer n gives

$$\int\limits_{\deg a\le -(d-\varepsilon)N} H(a)E(a\lambda)\chi_{-n}(a)\,da = q^{(D-d)N-n}/N^D,$$

as $dN>\deg\lambda$. Combining this with Lemma 3.1, when $0<\varepsilon<1/4$ and $n=[m\ln N]$, we have

$$(20)\qquad \int\limits_{\deg a\le -(d-\varepsilon)N} F(a)E(a\lambda)\chi_{-[m\ln N]}(a)\,da \gg q^{(D-d)N}/N^{D+m},$$

as $N \to \infty$. In Lemmas 4.2 and 4.6, if ε, D_1, d, D, n and σ_0 satisfy $d - 6\varepsilon < 2D_1 < d$, $2 \le d < p$, $\sigma_0 = D + m + 1$, $n = [m \ln N]$, $D_1 N \ge n$, and

$$D \ge \begin{cases} 1 + 2^d & \text{if } 2 \le d < 11, \\ 2[2d^2 \ln d + d^2 \ln\ln d + 2d^2 - 2d] + 1 & \text{if } d \ge 11, \end{cases}$$

then there are infinitely many positive integers N (note that these N come from $\lambda_1/\lambda_2 \in \mathbf{K}_\infty/\mathbf{K}$) such that

$$\int\limits_{-(d-\varepsilon)N \le \deg a} F(a) E(a\lambda) \chi_{-[m \ln N]}(a)\, da \ll q^{(D-d)N}/N^{D+m+1}. \tag{21}$$

Therefore, taking $\varepsilon = 1/6$, $D_1 = d/2 - 1/4$ and combining (20) and (21), we see that for any positive integer m,

$$\int\limits_{\mathbf{K}_\infty} \sideset{}{'}\sum_{\deg P_1 = N} \cdots \sideset{}{'}\sum_{\deg P_D = N} E\Big(a\Big(\lambda + \sum_{i=1}^{D} \lambda_i P_i^d\Big)\Big) \chi_{-[m \ln N]}(a)\, da$$
$$= \int\limits_{\mathbf{K}_\infty} F(a) E(a\lambda) \chi_{-[m \ln N]}(a)\, da \gg q^{(D-d)N}/N^{D+m}.$$

It follows from Lemma 2.2 that there exist infinitely many positive integers N for which there are $\gg q^{(D-d)N}/N^{D+m}$ D-tuples $(P_1, \ldots, P_D)$ of monic irreducible polynomials with $\deg P_i = N$ and

$$\deg(\lambda + \lambda_1 P_1^d + \ldots + \lambda_D P_D^d) < -m \ln N + 1.$$

This completes the proof of Theorem 2.1. ∎

References

[1] A. Baker, *On some diophantine inequalities involving primes*, J. Reine Angew. Math. 228 (1967), 166–181.

[2] M. Car, *Distribution des polynômes irréductibles dans* $\mathbb{F}_q[T]$, Acta Arith. 88 (1999), 141–153.

[3] H. Davenport and H. Heilbronn, *On indefinite quadratic forms in five variables*, J. London Math. Soc. 21 (1946), 185–193.

[4] G. W. Effinger and D. R. Hayes, *Additive Number Theory of Polynomials Over a Finite Field*, Clarendon Press, Oxford, 1991.

[5] G. Harman, *Diophantine approximation by prime numbers*, J. London Math. Soc. (2) 44 (1991), 218–226.

[6] D. R. Hayes, *The expression of a polynomial as a sum of three irreducibles*, Acta Arith. 11 (1966), 461–488.

[7] C. N. Hsu, *The distribution of irreducible polynomials in* $\mathbb{F}_q[t]$, J. Number Theory 61 (1996), 85–96.

[8] —, *Diophantine inequalities for polynomial rings*, ibid. 78 (1999), 46–61.

[9] —, *On polynomial Waring–Goldbach problem*, preprint.

[10] L. K. Hua, *Additive Theory of Prime Numbers*, Amer. Math. Soc., Providence, RI, 1965.

[11] K. Ramachandra, *On the sums* $\sum_{j=1}^{K} \lambda_j f_j(p_j)$, J. Reine Angew. Math. 262–263 (1973), 158–165.
[12] W. Schwarz, *Über die Lösbarkeit gewisser Ungleichungen durch Primzahlen*, ibid. 212 (1963), 150–157.
[13] R. C. Vaughan, *Diophantine approximation by prime numbers II*, Proc. London Math. Soc. (3) 28 (1974), 385–401.

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