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On the stationary points of Hardy's function $Z(t)$

by

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1. Introduction. Hardy's function, sometimes referred to as the signed modulus, is defined by the equation

$$Z(t) = e^{i\theta(t)}\zeta\left(\frac{1}{2}+it\right) = \left\{\pi^{-it}\,\frac{\Gamma\left(\frac{1}{4}+\frac{1}{2}it\right)}{\Gamma\left(\frac{1}{4}-\frac{1}{2}it\right)}\right\}^{1/2}\zeta\left(\frac{1}{2}+it\right). \tag{1}$$

It is a consequence of the functional equation that $Z(t)$ is an even function of t which is real when t is real. Furthermore t is a real zero of $Z(t)$ if and only if $1/2+it$ is a zero of the zeta-function on the critical line $\operatorname{Re} s = 1/2$. We denote the sequence of zeros of $Z(t)$ in $\mathbb{R}^+$, counted according to multiplicity and arranged in non-decreasing order, by $\{t_n\}$; a recent result [5] is that

$$\Lambda := \limsup \frac{t_{n+1}-t_n}{2\pi/\log t_n} \geq \sqrt{11/2} = 2.345207\ldots \tag{2}$$

independently of any unproved hypothesis. By Rolle's theorem if $t_{n+1} > t_n$, then $Z'(t)$ must vanish at least once in (t_n, t_{n+1}); we denote the non-decreasing sequence of real positive zeros (again counted according to multiplicity) of $Z'(t)$ by $\{u_n\}$, with $t_1 = 14.13\ldots < u_1 < t_2 = 21.02\ldots$ We have $Z(0) < 0$, and we shall see later that $Z''(0) > 0$. We find that $Z(t)$ has two stationary points $u_{-1}, u_0 \in (0, t_1)$: a maximum approximately equal to $-.52\ldots$ near $t = 2.4$ and a minimum approximately equal to $-1.55\ldots$ near 10.4.

We define $N_0^*(T) := \operatorname{card}\{n : 0 < u_n \leq T\}$ and we see that

$$N_0^*(T) \geq N_0(T) - \varepsilon(T) + 2 \tag{3}$$

where as usual $N_0(T) := \{n : t_n \leq T\}$ and $\varepsilon(T) = 0$ or 1. This inequality is true whether or not $Z(t)$ has multiple zeros, moreover $\varepsilon(T) = 0$ if

$$\max\{t : t \leq T,\ Z'(t) = 0\} > \max\{t : t \leq T,\ Z(t) = 0\}. \tag{4}$$

Thus $\varepsilon(T) = 0$ for some arbitrarily large values of T (when we have just passed a zero of $Z'(t)$).

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In view of (3), it seems worthwhile to see what can be said about the possibly complex zeros of the function $Z'(w)$. We show that there are infinitely many purely imaginary zeros, which we call trivial and which we are able to describe fairly precisely (Theorem 5), and that the remaining, non-trivial, zeros lie in a strip $\{w : |\mathrm{Im}\, w| < B\}$ (Theorem 1); moreover, on the Riemann Hypothesis all these non-trivial zeros are real (Theorem 2). We show unconditionally that there are some relatively large spaces between the real zeros: this result (Theorem 4) is analogous to (2). The question arises whether similar statements about $Z''(w)$ and the higher derivatives are valid. This is not touched on here and we draw it to the reader's attention.

There are a number of results in the literature concerning the zeros of $\zeta^{(k)}(s)$ and $\xi^{(k)}(s)$: we mention Speiser [10], Berndt [1], Spira [11]–[14], Levinson [8] and Levinson and Montgomery [9]. As far as we are aware, $Z'(w)$ has not been studied directly before now, however it appears in a slightly disguised form in Conrey and Ghosh [3] where these authors prove the following formula (conditional on the Riemann Hypothesis):

$$\sum_{t_n \le T} \max\left\{\left|\zeta\left(\frac{1}{2}+it\right)\right|^2 : t_n < t < t_{n+1}\right\} \sim \frac{e^2-5}{4\pi}\, T\log^2 T. \tag{5}$$

It is known that on this hypothesis, the function $|\zeta(1/2+it)|$ has exactly one maximum in each interval between successive zeros (except for $(-t_1, t_1)$), so that the sum in (5) is

$$\sum_{t_n \le T} Z(u_n)^2. \tag{6}$$

In order to study the stationary points of $Z(T)$ we replace t in (1) by the complex variable w and we put $s = 1/2 + iw$ so that (1) becomes

$$Z(w) = \mathcal{Z}(s) := \{\chi(1-s)\}^{1/2}\zeta(s), \qquad \mathcal{Z}(s) = \mathcal{Z}(1-s), \tag{7}$$

in which

$$\chi(1-s) = 2^{1-s}\pi^{-s}\cos\tfrac{1}{2}s\pi\Gamma(s). \tag{8}$$

It is easier to work in the familiar s-plane where the terms critical line etc. have their usual meaning: we have only to note the correspondence between the half-planes $\{w : \mathrm{Im}\, w \le 0\}$, $\{s : \mathrm{Re}\, s \ge 1/2\}$ and their boundaries together with the relation $Z'(w) = i\mathcal{Z}'(s)$. We see from (7) and (8) that $\mathcal{Z}(s)$ has algebraic singularities at the points $\ldots, -6, -4, -2, 0, 1, 3, 5, \ldots$, and is analytic in any simply connected domain not including these singularities. This function shares the complex zeros of $\zeta(s)$ in the critical strip; there are no poles or trivial zeros. We are interested in the zeros of $\mathcal{Z}'(s)$ and we refer to the real zeros of this function as trivial, and define $N^*(T)$ to be the number of zeros (counted according to multiplicity) such that $0 < \mathrm{Im}\, s \le T$;

we see that $N_0^*(T)$, defined above, is the number of these zeros on the critical line. The trivial zeros of $Z'(w)$ lie on the imaginary axis. We deduce from [3] (Lemma, part (iii)) that in our notation, $N^*(T) = N(T) + O(\log T)$.

THEOREM 1. *The non-trivial zeros of the function* $\mathcal{Z}'(s)$ *all lie in the strip* $|\mathrm{Re}\, s - 1/2| < 15/2$ (*and so* $Z'(w)$ *has no non-trivial zeros such that* $|\mathrm{Im}\, w| \ge 15/2$). *Suppose that* T *is not the ordinate of a zero of the zeta function or of the function*

$$(9)\qquad H(s) := \frac{\mathcal{Z}'(s)}{\mathcal{Z}(s)} = \frac{\zeta'(s)}{\zeta(s)} - \frac{1}{2}\frac{\chi'(1-s)}{\chi(1-s)} = \frac{\zeta'(s)}{\zeta(s)} + \frac{1}{2}\frac{\Gamma'(s)}{\Gamma(s)} - \frac{\pi}{4}\tan\frac{\pi s}{2} - \frac{1}{2}\log 2\pi,$$

and put

$$(10)\qquad A(T) := \frac{1}{\pi}\arg H\left(\frac{1}{2} + iT\right),$$

defined by continuous variation along the line segments $[8, 8 + iT]$, $[8 + iT, 1/2 + iT]$. ($\arg H(8) := 0$.) *Then we have*, *for sufficiently large* T,

$$(11)\qquad N^*(T) = N(T) + A(T) + 3/2.$$

From Conrey and Ghosh's result noted above, we see that

$$(12)\qquad A(T) \ll \log T.$$

In the next two theorems we assume the truth of the Riemann Hypothesis. Theorems 1, 4 and 5 are unconditional: they do not depend on any unproved hypothesis.

THEOREM 2. *On the Riemann Hypothesis*, *we have*

$$(13)\qquad N^*(T) = N(T) - \frac{1}{2}\,\mathrm{sgn}\,\frac{Z'(T)}{Z(T)} + \frac{3}{2},$$

provided T *is not the ordinate of a zero of the zeta-function or of the function* $H(s)$ *defined in* (9). *A corollary is that for large* T, $N^*(T) = N_0^*(T)$ *identically*, *that is*, *all the non-trivial zeros of* $\mathcal{Z}'(s)$ *lie on the critical line*, *equivalently the non-trivial stationary points of Hardy's function are all real.*

It is interesting to compare the case of $\mathcal{Z}'(s)$ with that of $\xi'(s)$ (Titchmarsh [15, eqs. (2.1.9), (2.12.5)]):

$$(14)\qquad \xi(s) = \frac{1}{2}\, s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{2}\, e^{as}\prod\{(1 - s/\varrho)e^{s/\varrho}\},$$

in which the product runs over the complex zeros ϱ and

$$(15)\qquad a = \frac{1}{2}\log 4\pi - 1 - \frac{1}{2}\,\overline{\gamma} = -\sum \mathrm{Re}\,\frac{1}{\varrho}.$$

(In this paper, $\overline{\gamma}$ denotes Euler's constant, and $\varrho = \beta + i\gamma$ is a typical complex zero of $\zeta(s)$.) It is well known and easy to prove that on the Riemann Hypothesis, all critical points of $\xi(s)$ lie on the critical line. Indeed we have

$$\text{(16)} \quad \begin{aligned} \operatorname{Re}\frac{\xi'(s)}{\xi(s)} &= \sum_{\varrho} \frac{\sigma-\beta}{(\sigma-\beta)^2+(t-\gamma)^2} \\ &= \left(\sigma-\frac{1}{2}\right)\sum_{\varrho} \frac{(\sigma-\beta)(\sigma-1+\beta)+(t-\gamma)^2}{\{(\sigma-\beta)^2+(t-\gamma)^2\}\{(\sigma-1+\beta)^2+(t-\gamma)^2\}} \\ &=: \left(\sigma-\frac{1}{2}\right)g(s) \end{aligned}$$

where we have symmetrized by averaging the terms involving β and $1-\beta$. On the Riemann Hypothesis,

$$\text{(17)} \qquad g(s) = \sum_{\varrho} \frac{1}{(\sigma-1/2)^2+(t-\gamma)^2} > 0,$$

and so $\sigma \neq 1/2$ implies $\xi'(s) \neq 0$. A formula connecting $\mathcal{Z}'(s)$ with $\xi'(s)$ is

$$\text{(18)} \qquad \frac{\mathcal{Z}'(s)}{\mathcal{Z}(s)} = \frac{\xi'(s)}{\xi(s)} - \frac{1}{4}\left\{\frac{\Gamma'(s/2)}{\Gamma(s/2)} - \frac{\Gamma'((1-s)/2)}{\Gamma((1-s)/2)}\right\} - \frac{2s-1}{s(s-1)}$$

and we remark that it should be possible to derive the corollary to Theorem 2 directly from (17) and (18). This would involve some numerical computations which I have not checked, perhaps making the proposed proof a bit clumsy. I sketch out this method just after the proof of Theorem 2 given below.

This conclusion, that on RH the non-trivial zeros of $Z'(w)$ are real, is not one which has been taken for granted. (See for example [3, p. 196, l. 1]; Conrey and Ghosh proved a result equivalent to the following: if the Riemann Hypothesis is true then the zeros of $\mathcal{Z}'(s)$ satisfy $|\operatorname{Re} s - 1/2| < 1/9$.) Actually $\zeta'(s)$ does have zeros to the right of the critical line (even to the right of the line $\sigma = 2$: see Titchmarsh [15, Theorem 11.5C]). Formulae involving these zeros with $\sigma > 1/2$ appear in [9].

We see from (16) that there exists $V : \mathbb{R} \to \mathbb{R}$ such that

$$\text{(19)} \qquad \frac{\xi'(1/2+it)}{\xi(1/2+it)} = \frac{1}{i}V(t);$$

moreover from (18) we have

$$\text{(20)} \qquad \frac{Z'(t)}{Z(t)} = V(t) + \frac{1}{2}\operatorname{Im}\frac{\Gamma'\left(\frac{1}{4}+\frac{1}{2}it\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}it\right)} - \frac{2t}{1/4+t^2} = V(t) + \frac{\pi}{4} + O(1/t).$$

On the Riemann Hypothesis we have

$$V'(t) = -\sum_{\varrho} \frac{1}{(t-\gamma)^2}, \qquad \zeta(1/2+it) \neq 0, \tag{21}$$

and V has a unique zero $v_n \in (t_n, t_{n+1})$. The following result was drawn to my attention by the referee of an earlier version of this paper.

THEOREM 3. *On the Riemann Hypothesis, we have*

$$0 < u_n - v_n \ll \frac{1}{\log u_n \log\log u_n}. \tag{22}$$

One expects that, unconditionally, the roots of $\mathcal{Z}'(s)$ and $\xi'(s)$ are close together in the sense that, if ω is a root of either of these functions, then the disc $\{z : |z-\omega| < \psi(|\omega|)/\log|\omega|\}$ contains a root of the other; here ψ is some function which tends to 0 as $|\omega| \to \infty$. I do not have a proof of this.

Conrey and Ghosh also had a result about the spacing of the u_n. They state that, on the Riemann Hypothesis, there exists $u_n \in (T, 2T]$ such that

$$(u_{n+1} - u_n)\,\frac{\log(T/2\pi)}{2\pi} > 1.4 \tag{23}$$

whereas it is clear that, on this hypothesis, the average value of the left-hand side is equal to 1.

THEOREM 4. *Let $\varepsilon(T) \to 0$ in such a way that $\varepsilon(T)\log T \to \infty$. Then for sufficiently large T, there exists an interval contained in $[T, (1+\varepsilon(T))T]$ which is free of zeros of $Z'(t)$ and has length at least*

$$\sqrt{\frac{7723}{3230}} \cdot \left\{1 + O\left(\frac{1}{\varepsilon(T)\log T}\right)\right\} \frac{2\pi}{\log T}. \tag{24}$$

Thus

$$\Lambda^* := \limsup \frac{u_{n+1} - u_n}{2\pi/\log u_n} \geq \sqrt{\frac{7723}{3230}} = 1.546292\ldots \tag{25}$$

This result depends on the method developed in [4], [5] (which rather suggests that in some of these questions about gaps between zeros, the Riemann Hypothesis is irrelevant, at least for the large gaps). A consequence of (25) is that, infinitely often, the stationary points of Hardy's function are some distance from the Gram points. These are points g_m $(m = -1, 0, 1, 2, 3, \ldots)$ such that

$$\theta(g_m) = m\pi \tag{26}$$

(where as usual $\theta(t) = \arg \Gamma\big(\frac{1}{4} + \frac{1}{2}it\big) - \frac{1}{2}t\log\pi\big)$; at one time it was believed that $g_{n-2} < t_n < g_{n-1}$ always, but this was disproved by Hutchinson [7]. In this too simple model one would expect $Z(t)$ to have a maximum or minimum between t_n and t_{n+1} close to g_{n-1}. It is not difficult to show that both $g_m \sim 2\pi m/\log m$ and $g_m - g_{m-1} \sim 2\pi/\log g_m$ so that this rule, sometimes referred to as Gram's Law (but I am not aware that Gram stated it), certainly fails infinitely often, for example by (2). Different kinds of these failures are called Lehmer's or Rosser's phenomena.

We require information about the zeros and poles of an auxiliary function $F(s)$. We explain this next. We differentiate (8) logarithmically to obtain

$$\mathcal{Z}'(s) = \{\chi(1-s)\}^{1/2}F(s) := \{\chi(1-s)\}^{1/2}H(s)\zeta(s) \tag{27}$$

with $H(s)$ as in (9). Since $\mathcal{Z}'(1-s) = -\mathcal{Z}'(s)$ we deduce immediately that

$$F(1-s) = -2^{1-s}\pi^{-s}\cos\frac{s\pi}{2}\Gamma(s)F(s), \qquad H(1-s) = -H(s). \tag{28}$$

(A formula equivalent to this one appears in [3, Lemma, part (ii)].) Notice that (27) is equivalent to

$$F(s) = \zeta'(s) + \left\{\frac{1}{2}\frac{\Gamma'(s)}{\Gamma(s)} - \frac{\pi}{4}\tan\frac{s\pi}{2} - \frac{1}{2}\log 2\pi\right\}\zeta(s). \tag{29}$$

In this investigation, we are primarily interested in the zeros of $F(s)$ in $\mathbb{H} := \{s = \sigma + it : t > 0\}$. However, in our application of the Principle of the Argument it will be convenient to consider all the zeros and poles of $F(s)$ in $\mathbb{C}$. We see that $F(s)$ is a meromorphic function with a double pole at $s = 1$ and simple poles at $s = 0$ and $3, 5, 7, \ldots$, these latter arising from the tangent in (29). There are no other singularities for $\sigma > 0$ by (27) and therefore no singularities on $\mathbb{R}^-$: we can see this from (28). An immediate consequence of (28) is that $F(s)$ has a zero of odd order at $s = 1/2$, moreover the non-real zeros are positioned symmetrically with respect to both $\mathbb{R}$ and the line $\operatorname{Re} s = 1/2$. We consider the other real zeros. From (29), F jumps from $-\infty$ to ∞ as s increases through the values $3, 5, 7, \ldots$; hence F has an odd number of zeros, say n_4 in $(3,5)$, n_6 in $(5,7)$ and so on. Also from (29), $F(s)$ is negative near the double pole at $s = 1$ and so the number of zeros n_2 in $(1,3)$ is even. There are an odd number of zeros $n_{1/2}$ in $(0,1)$. The real zeros are positioned symmetrically with respect to the critical line: there are no trivial zeros on $-2\mathbb{N}$. The next result is really a lemma; I state it as a theorem because it is basic to our investigation. It concerns the location of the trivial zeros.

THEOREM 5. *We have* (*in the above notation*) $n_2 = 0$, $n_4 = n_6 = n_8 = \ldots = 1$, *and* $n_{1/2} = 1$. *Moreover for* $k \ge 4$, *the zero in* $(2k-1, 2k+1)$ *is in* $(2k, 2k+1)$.

We shall see in the course of the proof that $H'(1/2) > 0$ so that as claimed above, $Z''(0) > 0$ and $Z(0)$ is a minimum.

2. Proofs of the theorems. It is logical to start with the last theorem, which is required for the proof of Theorems 1 and 2. The proof is technical, albeit entirely elementary, and the reader might prefer to take it on trust at a first reading.

Proof of Theorem 5. We begin by showing that $H(s)$ is negative throughout $(1,3)$ and decreasing in each of the intervals $(3,5)$, $(5,7)$ &c., thereby establishing the first two parts of our assertion. We note that on $(1,3)$,

$$\frac{\Gamma'(s)}{\Gamma(s)} < \frac{\Gamma'(3)}{\Gamma(3)} = \frac{3}{2} - \gamma < \log 2\pi \tag{30}$$

whence our claim holds on $[2,3)$ because ζ decreases and the tangent is positive. It will therefore be sufficient to prove that for $0 < x < 1$ we have

$$-\frac{\zeta'(1+x)}{\zeta(1+x)} > \frac{\pi}{4}\cot\frac{\pi x}{2}. \tag{31}$$

This is not quite straightforward and we indicate to the reader the main steps.

For $1 < \sigma < 2$ we have

$$\begin{aligned} \zeta(\sigma) &= \frac{1}{\sigma-1} + \frac{1}{2} + \frac{\sigma(\sigma+1)}{2}\int_1^\infty \frac{\{u\}(1-\{u\})}{u^{\sigma+2}}\,du \\ &< \frac{1}{\sigma-1} + \frac{1}{2} + \frac{\sigma}{8} < \frac{1}{\sigma-1} + \frac{3}{4}, \end{aligned} \tag{32}$$

the $\{u\}$ denoting fractional part. From (32),

$$\begin{aligned} -\zeta'(\sigma) &= \frac{1}{(\sigma-1)^2} + \frac{\sigma(\sigma+1)}{2}\int_1^\infty \frac{\{u\}(1-\{u\})}{u^{\sigma+2}}\log u\,du \\ &\quad - \frac{2\sigma+1}{2}\int_1^\infty \frac{\{u\}(1-\{u\})}{u^{\sigma+2}}\,du \\ &> \frac{1}{(\sigma-1)^2} - \frac{2\sigma+1}{8(\sigma+1)} > \frac{1}{(\sigma-1)^2} - \frac{5}{24}. \end{aligned} \tag{33}$$

We also have $\tan\theta > \theta + \frac{1}{3}\theta^3$ for $0 < \theta < \pi/2$ (because tangent has non-negative Maclaurin coefficients) and so our assertion will follow from

$$\left(1 + \frac{\pi^2}{12}x^2\right)\frac{1 - \frac{5}{24}x^2}{1 + \frac{3}{4}x} > \frac{1}{2} \qquad (0 < x < 1), \tag{34}$$

that is,

$$\frac{1}{2} - \frac{3}{8}x + \left(\frac{\pi^2}{12} - \frac{5}{24}\right)x^2 - \frac{5\pi^2}{288}x^4 > 0. \tag{35}$$

The first term exceeds the second and the third exceeds the fourth whence (35) is true and so is (31). Hence $F(s) < 0$ on $(1,3)$ as required.

For the next part it will be sufficient to show that for $s > 3$ we have

$$\frac{\pi^2}{8}\sec^2\frac{s\pi}{2} > \frac{d}{ds}\left\{\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{2}\frac{\Gamma'(s)}{\Gamma(s)}\right\} = \sum_{n=1}^{\infty}\frac{\Lambda(n)\log n}{n^s} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{(n+s)^2}, \tag{36}$$

so that $H(s)$ decreases between the poles. The first sum on the right is less than

$$\sum_{n=2}^{\infty}\frac{\log^2 n}{n^3} < \frac{\log^2 2}{8} + \int_2^{\infty}\frac{\log^2 u}{u^3}\,du < .27$$

while the second sum is less than $\zeta(2) - 5/4 < .4$, so that the right-hand side of (36) is less than .47 and the inequality is clear. Also we may check that if $k \ge 4$ then $H(2k) > 0$ so that the zero in $(2k-1, 2k+1)$ lies in $(2k, 2k+1)$.

The more difficult third part is $n_{1/2} = 1$, which will follow from the proposition

$$H(\sigma) \gg \sigma - 1/2 \qquad (1/2 < \sigma < 1), \tag{37}$$

because $\zeta(\sigma) < 0$ throughout $(-2, 1)$. We have

$$\frac{\Gamma'((1-s)/2)}{\Gamma((1-s)/2)} = \frac{\Gamma'((1+s)/2)}{\Gamma((1+s)/2)} - \pi\tan\frac{s\pi}{2} \tag{38}$$

and we insert this into (18) to obtain

$$H(s) = \frac{\xi'(s)}{\xi(s)} + \frac{1}{4}\left\{\frac{\Gamma'((1+s)/2)}{\Gamma((1+s)/2)} - \frac{\Gamma'(s/2)}{\Gamma(s/2)}\right\} - \frac{\pi}{4}\tan\frac{s\pi}{2} - \frac{2s-1}{s(s-1)}. \tag{39}$$

On taking real parts and applying (16) this becomes

$$\operatorname{Re} H(s) = \left(\sigma - \frac{1}{2}\right)g(s) + \frac{1}{4}\operatorname{Re}\left\{\frac{\Gamma'((1+s)/2)}{\Gamma((1+s)/2)} - \frac{\Gamma'(s/2)}{\Gamma(s/2)}\right\} - \frac{\pi}{4}\operatorname{Re}\tan\frac{s\pi}{2} - \operatorname{Re}\left\{\frac{1}{s} + \frac{1}{s-1}\right\}; \tag{40}$$

in particular

$$H(\sigma) = \left(\sigma - \frac{1}{2}\right)g(\sigma) + \frac{1}{4}h(\sigma) - \frac{\pi}{4}\tan\frac{\sigma\pi}{2} + \frac{1}{1-\sigma} - \frac{1}{\sigma} \tag{41}$$

where

$$g(\sigma) = \sum_{\varrho}\frac{(\sigma-\beta)(\sigma-1+\beta)+\gamma^2}{\{(\sigma-\beta)^2+\gamma^2\}\{(\sigma-1+\beta)^2+\gamma^2\}} > 0 \tag{42}$$

(because $|\gamma| > 14$), and

$$h(s) := \frac{\Gamma'\left(\frac{1}{2}s+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}s+\frac{1}{2}\right)} - \frac{\Gamma'\left(\frac{1}{2}s\right)}{\Gamma\left(\frac{1}{2}s\right)} = 2\left\{\frac{1}{s} - \frac{1}{s+1} + \frac{1}{s+2} - \frac{1}{s+3} + \ldots\right\}. \tag{43}$$

We notice that $h(1/2) = \pi$ (equivalently, $H(1/2) = 0$, as in (28)). Put $\sigma = 1-\delta$, $0 < \delta \le 1/3$. Then

$$\frac{\pi}{4}\tan\frac{\sigma\pi}{2} < \frac{1}{2\delta} \tag{44}$$

and since $h(\sigma) > 2/\sigma(\sigma+1)$ directly from (43), we have

$$H(\sigma) > \frac{1}{2\delta} - \frac{1}{1-\delta} + \frac{1}{2(1-\delta)(2-\delta)} > \frac{1}{2(1-\delta)(2-\delta)} > \frac{1}{2}\left(\frac{1}{2}-\delta\right), \tag{45}$$

which establishes (37) in this case.

On the range $1/2 < \sigma \le 2/3$ we have, by convexity,

$$\tan\frac{\sigma\pi}{2} \le 1 + 6(\sqrt{3}-1)(\sigma - 1/2) \tag{46}$$

and so (37) will follow from (41), (42) and (46) if we can show that for $1/2 \le \sigma \le 2/3$,

$$\frac{2\sigma-1}{\sigma(1-\sigma)} + \frac{1}{4}\,(h(\sigma)-\pi) \ge B(\sigma - 1/2), \qquad B > \frac{3}{2}\pi(\sqrt{3}-1) = 3.4497\ldots \tag{47}$$

From (43), with a little algebra,

$$\begin{aligned} h\left(\tfrac{1}{2}\right) - h(\sigma) &= 4(\sigma - 1/2)\left\{\frac{1}{\sigma} - \frac{1}{3(\sigma+1)} + \frac{1}{5(\sigma+2)} - \frac{1}{7(\sigma+3)} + \ldots\right\} \\ &< \frac{4}{\sigma}(\sigma - 1/2) \end{aligned} \tag{48}$$

so that the left-hand side of (47) exceeds

$$\frac{1+\sigma}{\sigma(1-\sigma)}\,(\sigma - 1/2) \ge 6(\sigma - 1/2). \tag{49}$$

This is all we need, and (37) follows. Thus $n_{1/2} = 1$ as required. Moreover we have $H'(1/2) = -Z''(0)/Z(0) > 0$, and since $Z(0)$ is negative, we see that $Z''(0) > 0$.

Proof of Theorem 1. Since N, in any of its guises, is right continuous we may assume that T is not the ordinate of a zero of $\zeta(s)$ or $H(s)$, that is, $F(s)$ has no zero with $\operatorname{Im} s = T$. We may re-write (28) in the form

$$\pi^{(1-s)/2}\Gamma\left(\tfrac{1}{2} - \tfrac{1}{2}s\right)F(1-s) = -\pi^{-s/2}\Gamma\left(\tfrac{1}{2}s\right)F(s), \tag{50}$$

which is equivalent to the assertion that the function

$$G(z) := \pi^{-(1/2+z)/2}\Gamma\left(\tfrac{1}{4} + \tfrac{1}{2}z\right)F\left(\tfrac{1}{2}+z\right)/z \tag{51}$$

is an even function of z with a removable singularity at $z = 0$, $G(0) \ne 0$. Furthermore $G(z)$ is real for real z and it is analytic except for poles at $\pm 1/2, \pm 5/2, \pm 9/2, \ldots$, the first pair only being double. There is a simple zero in each interval between two simple poles: for $h \ge 4$ the zero in $(2h - 3/2, 2h+1/2)$ lies in $(2h-1/2, 2h+1/2)$. Let R be the rectangle with corners

$\pm a \pm iT$ in which $a = 2h - 1/2$ for some $h \in \mathbb{N}$, $h \geq 4$, and T is not the ordinate of a zero of the zeta-function, or of a zero of $G(z)$. We shall see later that on the line $\operatorname{Re} s = 2h\, (h \geq 4)$ we have $\operatorname{Re} H(s) > 0$, so $G(z)$ does not vanish on $\sigma = a$. From Theorem 5, $G(z)$ has $2h-4$ zeros in $(-a, a)$ and there are $2h+2$ poles. If C is that part of R in the first quadrant then

$$\frac{2}{\pi i}\int_C \frac{G'(z)}{G(z)}\,dz = N - P = 2N_h^+ - 6 \tag{52}$$

where N and P denote the number of zeros and poles of $G(z)$ inside R and N_h^+ denotes the number of such zeros in the region $\{z : |x| < a,\ 0 < y < T\}$. The integral is

$$\frac{2}{\pi}\Delta \arg\left\{\pi^{-s/2}\Gamma\left(\tfrac{1}{2}s\right)F(s)/(s-1/2)\right\} = \frac{2}{\pi}\Delta\left\{\frac{H(s)\xi(s)}{s(s-1/2)(s-1)}\right\} \tag{53}$$

where $\Delta \arg$ is the change in the argument as s moves from $a+1/2$ to $a + 1/2 + iT$ and thence to $1/2 + iT$. We have employed (14) and (27) on the right-hand side of (53).

In Titchmarsh's proof ([15, 9.3]) of Backlund's Theorem (it is inconsequential that he has a different oblong) he shows that

$$\frac{2}{\pi}\Delta \arg \xi(s) = 2N(T). \tag{54}$$

We assemble (52) and (54) to obtain

$$\begin{aligned} N - P &= 2N(T) + \frac{2}{\pi}\Delta \arg\{H(s)/s(s-1/2)(s-1)\} \\ &= 2N(T) + \frac{2}{\pi}\Delta \arg H(s) - 3, \end{aligned} \tag{55}$$

whence

$$N_h^+ = N(T) + \frac{1}{\pi}\Delta \arg H(s) + \frac{3}{2} = N(T) + A_h(T) + \frac{3}{2} \tag{56}$$

where $\pi A_h(T)$ is the change in the argument of $H(s)$ as s moves from $2h$ to $1/2 + iT$ along the two perpendicular line segments. We claim that provided $h \geq 4$ both sides of (56) are independent of h, that is, $G(z)$ has no zeros in $\{z : |x| \geq 7.5,\ y > 0\}$, $F(s)$ and $H(s)$ have all their non-real zeros in the strip $\{s : -7 < \sigma < 8\}$ and all the stationary points of Hardy's function not on the imaginary axis have $|\operatorname{Im} w| < 7.5$ as stated in the theorem. Moreover $A_h(T) = A_4(T) = A(T)$ by definition and $N_h^+ = N^*(T)$. To prove this it will be sufficient to prove that $\operatorname{Re} H(s) > 0$ on the line segment $\{\sigma + iT : \sigma \geq 8\}$ (for large T), and on all the line segments $\{2h + it : 0 < t \leq T,\ h \in \mathbb{N},\ h \geq 4\}$.

We need some information about the Gamma function, and we quote a formula of Binet from Whittaker and Watson [16, Ch. 12]. This is, for

$\operatorname{Re} s > 0$,

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log s - \frac{1}{2s} - \int_0^\infty \left\{\frac{1}{2}\coth\frac{x}{2} - \frac{1}{x}\right\} e^{-sx}\,dx. \tag{57}$$

The kernel on the right is positive, so we may deduce that

$$\operatorname{Re}\frac{\Gamma'(s)}{\Gamma(s)} \ge \log|s| - \frac{\sigma}{2(\sigma^2+t^2)} + \left\{\frac{\Gamma'(\sigma)}{\Gamma(\sigma)} - \log\sigma + \frac{1}{2\sigma}\right\}. \tag{58}$$

Now for $x \ge 2$ (as $\log \Gamma$ is convex),

$$\frac{\Gamma'(x)}{\Gamma(x)} > \int_{x-1}^{x} \frac{\Gamma'(u)}{\Gamma(u)}\,du = \log(x-1) \tag{59}$$

so (58) yields

$$\operatorname{Re}\frac{\Gamma'(s)}{\Gamma(s)} \ge \log|s| - \log\frac{\sigma}{\sigma-1}. \tag{60}$$

We recall that

$$H(s) = \frac{\zeta'(s)}{\zeta(s)} + \frac{1}{2}\frac{\Gamma'(s)}{\Gamma(s)} - \frac{\pi}{4}\tan\frac{s\pi}{2} - \frac{1}{2}\log 2\pi, \tag{61}$$

and if we note that $\operatorname{Re}\tan(s\pi/2) = 0$ on $\operatorname{Re} s = 2h$ we see that on this line we have

$$\operatorname{Re} H(s) > \frac{\zeta'(8)}{\zeta(8)} + \frac{1}{2}\log\frac{7}{2\pi} > 0. \tag{62}$$

We turn our attention to the horizontal line segment, and we show that in fact $\operatorname{Re} H(s) > 0$ on $\{\sigma + iT : \sigma \ge 8\}$, for large T. By (60) we have

$$\frac{1}{2}\left\{\operatorname{Re}\frac{\Gamma'(s)}{\Gamma(s)} - \log 2\pi\right\} \ge \frac{1}{2}\log\frac{7}{2\pi} = .054\ldots \tag{63}$$

whereas

$$\operatorname{Re}\frac{\zeta'(s)}{\zeta(s)} \ge \frac{\zeta'(8)}{\zeta(8)} > -\frac{1}{100}. \tag{64}$$

We also have

$$\operatorname{Re}\tan(x+iy) = \frac{\sin x\cos x}{\cos^2 x\cosh^2 y + \sin^2 x\sinh^2 y} \ll e^{-2y}. \tag{65}$$

This is all we need and completes the proof of Theorem 1.

Remark. It is not the case that $\operatorname{Re} H(s) > 0$ on the line segment $\{\sigma + iT : \sigma \ge 1/2\}$ even if we assume the Riemann Hypothesis. For this would imply $|A(T)| < 1/2$ which by (11) is impossible: N and N^* are integers. Therefore $A(T)$ is half an odd integer whenever it is defined, whence

$H(1/2+iT)$ is purely imaginary and by continuity we have $\operatorname{Re} H(1/2+it) = 0$ identically. This may be seen directly from (9) since

$$H(s) = -i\,\frac{Z'(w)}{Z(w)}, \qquad H(1/2+it) = -i\,\frac{Z'(t)}{Z(t)}. \tag{66}$$

Proof of Theorem 2. We assume the Riemann Hypothesis in this proof and that of Theorem 3 only, and we show that for $\sigma > 1/2$ and $t \ge t_0$ we have $\operatorname{Re} H(\sigma+it) > 0$. We have already proved this unconditionally for $\sigma \ge 8$, so we may assume $\sigma < 8$. First we notice that for $t \ge 1$ we have, using standard properties of the Gamma function,

$$\begin{aligned} \operatorname{Re} \tan \frac{s\pi}{2} &= \frac{1}{\cosh t\pi} + O\biggl(\frac{\sigma-1/2}{\cosh t\pi}\biggr),\\ \operatorname{Re} h(s) &= \frac{\pi}{\cosh t\pi} + O\biggl(\frac{\sigma-1/2}{t^2}\biggr). \end{aligned} \tag{67}$$

Now recall from (40) and (17) that

$$\begin{aligned} \operatorname{Re} H(s) = {} & -\frac{\sigma-1}{(\sigma-1)^2+t^2} - \frac{\sigma}{\sigma^2+t^2}\\ & + \frac{1}{4}\operatorname{Re} h(s) - \frac{\pi}{4}\operatorname{Re}\tan\frac{s\pi}{2} + (\sigma-1/2)g(s), \end{aligned} \tag{68}$$

where

$$g(s) = \sum_{\varrho} \frac{1}{(\sigma-1/2)^2+(t-\gamma)^2}, \tag{69}$$

whence by (65), there exists an absolute constant K such that

$$\operatorname{Re} H(s) \ge (\sigma-1/2)\biggl\{-\frac{K}{t^2} + \sum_{\varrho}\frac{1}{(\sigma-1/2)^2+(t-\gamma)^2}\biggr\} \qquad (\sigma \ge 1/2,\ t \ge 1). \tag{70}$$

For large t, there is such a zero that $|t-\gamma| \le 1$ whence the summand in (70) exceeds $1/60$, and we ensure that $t^2 > 60K$. This is all we need.

Now we consider $A(T)$. There are two cases according to the sign of $Z'(T)/Z(T)$. If this is positive, then as s moves from 8 towards $1/2+iT$, $H(s)$ stays in the right-hand half-plane and converges to a point on the negative imaginary axis; that is, $\Delta \arg H(s) = -\pi/2$ and $A(T) = -1/2$. Similarly if $Z'(T)/Z(T) < 0$ then $A(T) = 1/2$. We conclude that

$$A(T) = -\frac{1}{2}\operatorname{sgn}\frac{Z'(T)}{Z(T)}, \tag{71}$$

whence our result follows by Theorem 1. This completes the proof of Theorem 2.

We proceed to the corollary. This involves counting the zeros of $Z(t)$ and $Z'(t)$ on $\mathbb{R}^+$. Let us suppose that $T \in (t_N, t_{N+1})$ so that $\operatorname{sgn} Z(T) = (-1)^{N+1}$ because $Z(0) < 0$. On the Riemann Hypothesis, $Z'(t)$ has just one zero in this interval, namely s_N. We have $N(T) = N$ and $N_0^*(T) = N+1$ or $N+2$ according as $T < s_N$ or $T > s_N$. Furthermore $\operatorname{sgn} Z'(T)/Z(T) = 1$ if $T < s_N$ and $= -1$ if $T > s_N$. To fix our ideas, let us suppose that $T \in (t_N, s_N)$, so that we have $N_0^*(T) - N(T) = 1$ and $A(T) = -1/2$, by (71). Theorem 1 implies that

$$N^*(T) - N(T) = 1, \tag{72}$$

and we deduce that

$$N^*(T) - N_0^*(T) = 0. \tag{73}$$

It is easy to check that if $T \in (s_N, t_{N+1})$ then (73) still holds. This completes the proof.

We sketch the alternative proof of the corollary employing (17) and (18). We know unconditionally that

$$\min\{|t-\gamma| : \gamma = \operatorname{Im} \varrho\} = O(1) \qquad (t \in \mathbb{R}), \tag{74}$$

indeed rather more ([15, 9.2]; for an update of this topic see also [6]). Therefore RH implies

$$\operatorname{Re} \frac{\xi'(s)}{\xi(s)} \gg \sigma - \frac{1}{2} \qquad (\sigma > 1/2). \tag{75}$$

Next,

$$\begin{aligned} \operatorname{Re}\left\{\frac{\Gamma'(s/2)}{\Gamma(s/2)} - \frac{\Gamma'((1-s)/2)}{\Gamma((1-s)/2)}\right\} \\ = \operatorname{Re}\left\{\frac{\Gamma'((\sigma+it)/2)}{\Gamma((\sigma+it)/2)} - \frac{\Gamma'((1-\sigma+it)/2)}{\Gamma((1-\sigma+it)/2)}\right\} \ll \frac{\sigma-1/2}{t^2}. \end{aligned} \tag{76}$$

Notice that we conjugate the term involving $1-s$ and then write the difference as an integral; we have

$$\operatorname{Re}\left\{\frac{d}{ds}\frac{\Gamma'(s)}{\Gamma(s)}\right\} \ll \frac{1}{t^2}. \tag{77}$$

We deduce from (18), (75) and (77) that on the Riemann Hypothesis,

$$\operatorname{Re} \frac{\mathcal{Z}'(s)}{\mathcal{Z}(s)} \gg \sigma - \frac{1}{2} \qquad (\sigma > 1/2,\ T > T_0). \tag{78}$$

To complete the proof we should require a numerical value for T_0 and a check on the zeros up to this point.

Proof of Theorem 3. We put $t = u_n$ in (18) and we have

$$V(v_n) = 0 = V(u_n) + \pi/4 + O(1/u_n) \tag{79}$$

whence there exists $w_n \in (t_n, t_{n+1})$ such that

$$(v_n - u_n)V'(w_n) = V(v_n) - V(u_n) = \pi/4 + O(1/u_n), \tag{80}$$

and it is a matter of showing that $-V'(w_n)$ is large. On the Riemann Hypothesis we have $S(t) := \arg\zeta(1/2+it) \ll \log t/\log\log t$ and we deduce from Backlund's formula that there exists a fixed c such that for sufficiently large t,

$$N\left(t + \frac{c}{\log\log t}\right) - N(t) \geq \frac{c\log t}{7\log\log t}. \tag{81}$$

We apply (81), in each of the ranges

$$\frac{kc}{\log\log w_n} < \gamma - w_n \leq \frac{(k+1)c}{\log\log w_n} \qquad (k \in \mathbb{N}), \tag{82}$$

to the sum in (21) to obtain

$$-V'(w_n) \gg \log w_n \log\log w_n \tag{83}$$

whereby (22) follows from (80).

Proof of Theorem 4. We follow the argument employed in [5] except that we find that we can streamline this a little: also we require

LEMMA 1. *We have, as $T \to \infty$,*

$$\int_0^T Z'(t)^4\,dt = \frac{1}{1120\pi^2}T\log^8 T + O(T\log^7 T), \tag{84}$$

$$\int_0^T |Z'(t)Z''(t)|^2\,dt = \frac{19}{604800\pi^2}T\log^{10} T + O(T\log^9 T), \tag{85}$$

$$\int_0^T Z''(t)^4\,dt = \frac{17}{1774080\pi^2}T\log^{12} T + O(T\log^{11} T). \tag{86}$$

Formulae (84) and (85) are to be found in [4] and (86) is proved by a similar method. These calculations are becoming prohibitive and I hope to design a Maple (or similar) programme to perform this algebra in future.

LEMMA 2. *Suppose that $y = y(x) \in C^2[0,\pi]$ and $y(0) = y(\pi) = 0$, also that $\nu \geq 0$. Then we have*

$$\int_0^\pi \{y'(x)^4 + 6\nu y(x)^2 y'(x)^2\}\,dx \geq 3\lambda_0(\nu)\int_0^\pi y(x)^4\,dx, \tag{87}$$

where

$$\lambda_0(\nu) = \frac{1}{8}\{1 + 4\nu + \sqrt{1+8\nu}\}. \tag{88}$$

The constant is sharp for every ν.

This result was proved in [5] by the Calculus of Variations. By a suitable linear transformation we find that if $y = y(x) \in C^2[a, b]$ and $y(a) = y(b) = 0$ then

$$\int_a^b \left\{ \left(\frac{b-a}{\pi} \right)^4 y'(x)^4 + 6\nu \left(\frac{b-a}{\pi} \right)^2 y(x)^2 y'(x)^2 - 3\lambda_0(\nu) y(x)^4 \right\} dx \geq 0. \tag{89}$$

Suppose that $\varepsilon(T)$ is as in the statement of the theorem and that u_l is the first zero of $Z'(t)$ not less than T and u_m is the last zero not exceeding $(1 + \varepsilon(T))T$. Suppose further that for $l \leq n < m$ we have

$$u_{n+1} - u_n \leq \frac{2\pi\kappa}{\log T}. \tag{90}$$

We deduce from (89) and (90) that

$$\int_{u_n}^{u_{n+1}} \left\{ \left(\frac{2\kappa}{\log T} \right)^4 Z''(t)^4 + 6\nu \left(\frac{2\kappa}{\log T} \right)^2 Z'(t)^2 Z''(t)^2 - 3\lambda_0(\nu) Z'(t)^4 \right\} dt \geq 0. \tag{91}$$

We sum this inequality for $l \leq n < m$: as in [5] we find that, with negligible error, we may replace the limits of integration by T and $(1 + \varepsilon(T))T$; we then apply Lemma 1 to obtain

$$\left\{ \frac{17\kappa^4}{110880\pi^2} + \frac{19\nu\kappa^2}{25200\pi^2} - \frac{3\lambda_0(\nu)}{1120\pi^2} \right\} T \log^8 T + O(T \log^7 T) \geq 0 \tag{92}$$

or

$$\kappa^4 + \frac{418}{85}\, \nu\kappa^2 - \frac{297}{17}\, \lambda_0(\nu) \geq -\frac{C}{\varepsilon(T) \log T}. \tag{93}$$

We deduce from (93) that

$$\kappa^2 \geq -b\nu + \sqrt{b^2\nu^2 + c\lambda_0(\nu)} + O\left(\frac{1}{\varepsilon(T) \log T} \right), \qquad b = \frac{209}{85},\ c = \frac{297}{17}. \tag{94}$$

We maximize the function of ν on the right-hand side of (94). It is easier to work with λ as independent variable and employ the relation $\nu = 2\lambda - \sqrt{\lambda}$, which is equivalent to (88). Notice that $\lambda \geq 1/4$ and ν is increasing. Denoting this function by $K(\lambda)$ we find that $K'(\lambda) > 0$ if

$$4b^2\nu \frac{d\nu}{d\lambda} + c - 4b^2\lambda \left(\frac{d\nu}{d\lambda} \right)^2 > 0, \tag{95}$$

which reduces to

$$4b^2\sqrt{\lambda} < c + b^2 \tag{96}$$

so that we require $c \geq b^2$. This condition is satisfied in the application: in [5] we had $b = 7,\ c = 105$. We put $\sqrt{\lambda} = (1 + c/b^2)/4$ and this implies that

$$\nu = \frac{1}{8}\left(\frac{c^2}{b^4} - 1 \right), \qquad \max K(\lambda) = \frac{1}{4}\left(b + \frac{c}{b} \right). \tag{97}$$

We insert the values of b and c to obtain our result. I have worked out the general form of this inequality as it may be useful in the future. In [5] I used an ad hoc method to compute the maximum and I was surprised at the time that it was rational.

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