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# Infinite sums as linear combinations of polygamma functions

by

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*Dedicated to Professor Yu. V. Nesterenko*
*on the occasion of his 60th birthday*

**1. Introduction. 1.** We begin with some notations and definitions. Let $d$ be a positive square-free integer. We denote by $\mathbb{Z}$, $\mathbb{Q}$, $\overline{\mathbb{Q}}$, and $\mathbb{Q}(i\sqrt{d})$ the set of integers, the field of rational numbers, the field of algebraic numbers, and an imaginary quadratic field, respectively.

We will use the *polygamma function*

$$\psi^{(k)}(z) = \frac{d^k}{dz^k}\psi(z) = \frac{d^{k+1}}{dz^{k+1}}\log\Gamma(z), \qquad k = 1, 2, \ldots,$$

which has the following series expansion (see [2, §1.16]):

$$\psi^{(k)}(z) = (-1)^{k+1}k!\sum_{n=0}^{\infty}\frac{1}{(n+z)^{k+1}}, \qquad z \neq 0, -1, -2, \ldots, \tag{1}$$

and the logarithmic derivative of $\Gamma(z)$,

$$\psi(z) = \frac{d}{dz}\log\Gamma(z) = -\gamma + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+z}\right), \qquad z \neq 0, -1, -2, \ldots,$$

called the *digamma function*. Obviously, $\psi(1) = -\gamma$, where $\gamma$ is Euler's constant. The function $\psi^{(k)}(z)$, $k = 0, 1, 2, \ldots$, is single-valued and analytic in the whole complex plane except for the points $z = -m$, $m = 0, 1, 2, \ldots$,

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where it has poles of order $k+1$. The polygamma function satisfies many functional relations [2, §1.16] such as

- "recurrence formula":

$$\psi^{(k)}(z+1) = \psi^{(k)}(z) + \frac{(-1)^k k!}{z^{k+1}}, \tag{2}$$

- "reflection formula":

$$\psi^{(k)}(1-z) + (-1)^{k+1}\psi^{(k)}(z) = (-1)^k \pi \frac{d^k}{dz^k} \cot \pi z, \tag{3}$$

- "multiplication formula":

$$\psi^{(k)}(mz) = \delta \log m + \frac{1}{m^{k+1}} \sum_{r=0}^{m-1} \psi^{(k)}\left(z + \frac{r}{m}\right),$$

where $\delta = 1$ if $k = 0$ and $\delta = 0$ if $k > 0$.

We also introduce its alternating analog (see [2, §1.16])

$$g^{(k)}(z) = (-1)^k k! \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+z)^{k+1}} = \frac{1}{2^{k+1}}\left(\psi^{(k)}\left(\frac{z+1}{2}\right) - \psi^{(k)}\left(\frac{z}{2}\right)\right), \tag{4}$$

which satisfies the similar functional relations

$$g^{(k)}(z+1) = \frac{(-1)^k k!}{z^{k+1}} - g^{(k)}(z), \qquad k = 1, 2, \ldots, \tag{5}$$

$$g^{(k)}(z) + (-1)^k g^{(k)}(1-z) = \pi \frac{d^k}{dz^k}\left(\frac{1}{\sin \pi z}\right). \tag{6}$$

Obviously by (1) and (4), the numbers $\psi^{(k)}(1)/\zeta(k+1)$, $g^{(k)}(1)/\zeta(k+1)$, $\psi^{(k)}(1/2)/\zeta(k+1)$ are rational (here $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$ is the Riemann zeta function) and therefore from (2), (5) we get the following inclusions:

$$\psi^{(2k-1)}(m), g^{(2k-1)}(m), \psi^{(2k-1)}(m+1/2) \in \mathbb{Q}^{\times} \cdot \pi^{2k} + \mathbb{Q}, \qquad m \in \mathbb{N}. \tag{7}$$

**2.** In this paper, we consider the values of the series

$$S = \sum_{n=0}^{\infty} \frac{P(n)}{Q(n)}, \qquad T = \sum_{n=0}^{\infty} \frac{P(n)}{Q(n)}(-1)^n, \qquad U = \sum_{n=0}^{\infty} \frac{P(n)}{Q(n)} f(n), \tag{8}$$

where $P(x), Q(x) \in \overline{\mathbb{Q}}[x]$ and $f$ is a periodic number-theoretic function, and express them as linear combinations of values of the polygamma functions (see Lemmas 1–2 below). Such a representation allows one to give simple sufficient conditions for the numbers $S, T$ to be algebraic or transcendental, which is done in Section 2. Further, we assume that all the zeros of $Q(x)$ are in the imaginary quadratic field $\mathbb{Q}(i\sqrt{d})$ and the polynomials $P(x), Q(x)$ have some symmetry properties. By formulas (3), (6), summing the series

$S, T, U$ explicitly and applying Nesterenko's famous result [7] on algebraic independence of the numbers $\pi, e^{\pi\sqrt{d}}$ we show that the infinite sums (8) either have a computable algebraic value or are transcendental. (By a *computable value*, we mean a number which can be explicitly determined in terms of its defining parameters.) Actually, we describe a mixed approach for computation of infinite sums (8) combining linear combinations of values of the polygamma functions and contour integration. The latter can be applied to the trigonometric series

$$V = \sum_{n=-\infty}^{\infty} \frac{P_1(n)e^{i\beta_1 n} + \cdots + P_s(n)e^{i\beta_s n}}{Q(n)}, \qquad \beta_1, \ldots, \beta_s \in \mathbb{Q},$$

and enables us to prove that under certain conditions on the polynomials $P_1, \ldots, P_s, Q$, the sum $V$ is either zero or transcendental. As a consequence, we establish the transcendence of some Fourier series (see Section 4). In Section 5 we extend these results to a more general set of roots of the polynomial $Q(x)$ provided that the Schanuel conjecture holds. This generalizes the well-known result of P. Bundschuh on the series $\sum_{n=2}^{\infty} 1/(n^{2k} - 1)$, $k \geq 2$ (see [3], [12, Section 3.2]).

Special cases of the infinite sums (8) were considered by P. Bundschuh in [3]. Using Baker's theory on linear forms in logarithms, he proved that the value of the series

$$F(z) = z \sum_{m=1}^{\infty} \frac{a_m}{m(m-z)},$$

where $\{a_m\}_{m=1}^{\infty}$ is a periodic sequence of algebraic numbers and $z \in \mathbb{Q} \cap (0, 1)$, is either zero or transcendental. In particular, this yields the transcendence of the numbers $\psi(z) + \gamma$, $\psi(z) - \psi(z/2)$ for any $z \in \mathbb{Q} \setminus \mathbb{Z}$, and of the series $\sum_{n=2}^{\infty} \zeta(n) z^n$, $\sum_{n=2}^{\infty} \beta(n) z^n$ for any rational $z$ with $0 < |z| < 1$, where $\beta(s) = \sum_{k=0}^{\infty} (-1)^k/(2k+1)^s$ is the Dirichlet beta function.

The case when all the roots $\alpha_1, \ldots, \alpha_m$ of $Q(x)$ are distinct rational numbers was considered in [1], where by Baker's theory it was proved that each of the numbers (8) is either a computable algebraic number or is transcendental. In particular, if $Q(x)$ is a reduced polynomial, i.e., if $\alpha_1, \ldots, \alpha_m$ are distinct rational numbers from $[-1, 0)$, then $S, T, U$ and the series

$$\sum_{n=0}^{\infty} \frac{P_1(n)\beta_1^n + \cdots + P_s(n)\beta_s^n}{Q(n)}, \qquad \beta_1, \ldots, \beta_s \in \overline{\mathbb{Q}},$$

are either zero or transcendental.

Notice that from [1] it follows that for any rational numbers $\alpha_1, \ldots, \alpha_m$ distinct from nonnegative integers and such that $\alpha_k - \alpha_l \notin \mathbb{Z}$, $1 \leq k \neq l \leq m$, all the values

$$\psi(\alpha_1), \ldots, \psi(\alpha_m) \tag{9}$$

are transcendental except for at most one value of $\alpha_k$ (compare this with [6, Theorem 3]). In fact, taking into account (2) we can assume without loss of generality that $\alpha_1, \ldots, \alpha_m$ are distinct numbers from $(0, 1]$ and then by [1, Theorem 3] we have, for $k \neq l$,

$$\psi(\alpha_l) - \psi(\alpha_k) = \sum_{n=0}^{\infty}\left(\frac{1}{n+\alpha_k} - \frac{1}{n+\alpha_l}\right) = \sum_{n=0}^{\infty} \frac{\alpha_l - \alpha_k}{(n+\alpha_k)(n+\alpha_l)} \notin \overline{\mathbb{Q}}.$$

Therefore the set (9) cannot contain two algebraic numbers.

In 2001, G. Molteni [5] considered the generating power series for the sequence $\{\zeta(2k+1)\}_{k=1}^{\infty}$, which can also be written as a linear combination of values of the digamma function,

$$F(z) = \sum_{k=1}^{\infty} \zeta(2k+1) z^{2k} = -\frac{1}{2}\,\psi(1+z) - \frac{1}{2}\,\psi(1-z) + \psi(1),$$

and proved that the numbers $1, F(\alpha_1), \ldots, F(\alpha_m)$ are linearly independent over $\overline{\mathbb{Q}}$ if all $\alpha_k = a_k/b_k$ are distinct rational numbers from the interval $(0, 1)$ such that $(a_k, b_k) = 1$ and for any $k$ there exists an odd prime $p_k$ dividing $b_k$ and $p_k \nmid b_j$ when $j \neq k$. An obvious corollary is that $F(\alpha)$ is transcendental for all $\alpha = a/b \in (0, 1)$ with $b$ not a power of 2. Actually, this restriction can be removed and $F(\alpha)$ is transcendental for any rational $\alpha$ with $0 < |\alpha| < 1$ by [1, Theorem 3], since

$$F(\alpha) = \sum_{n=0}^{\infty} \frac{\alpha^2}{(n+1)(n+1+\alpha)(n+1-\alpha)}$$

and the last series does not vanish.

## 2. Sums $S, T, U$ as linear combinations of polygamma functions

LEMMA 1. *Let $f : \mathbb{Z} \to \overline{\mathbb{Q}}$ be periodic with period $q \in \mathbb{N}$. Suppose that $P(x), Q(x) \in \overline{\mathbb{Q}}[x]$, $\deg P(x) \le \deg Q(x) - 1$, and $Q(x) = (x+\alpha_1)^{l_1} \ldots \ldots (x+\alpha_m)^{l_m}$, where $l_1, \ldots, l_m \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_m$ are distinct, and distinct from non-negative integers. If $\deg P(x) = \deg Q(x) - 1$, suppose also that $\sum_{t=0}^{q-1} f(t) = 0$ (convergence condition). Then the series*

$$U = \sum_{n=0}^{\infty} \frac{P(n)}{Q(n)}\, f(n)$$

*converges and we have the following representation:*

$$U = \sum_{t=0}^{q-1} f(t) \sum_{k=1}^{m} \sum_{l=1}^{l_k} \frac{(-1)^l}{(l-1)!}\, \frac{A_{k,l}}{q^l}\, \psi^{(l-1)}\left(\frac{t+\alpha_k}{q}\right) \tag{10}$$

*with*

$$A_{k,l} = \frac{1}{(l_k - l)!} \frac{d^{l_k - l}}{dx^{l_k - l}} \left( \frac{P(x)}{Q(x)} (x + \alpha_k)^{l_k} \right) \Big|_{x = -\alpha_k} \in \overline{\mathbb{Q}}. \tag{11}$$

*Proof.* Writing $n$ in the form $n = q\tau + t$, $\tau, t \in \mathbb{Z}$, $0 \le t \le q - 1$, $\tau \ge 0$, we get

$$U = \sum_{\tau=0}^{\infty} \sum_{t=0}^{q-1} f(q\tau + t) \frac{P(q\tau + t)}{Q(q\tau + t)} = \sum_{\tau=0}^{\infty} \sum_{t=0}^{q-1} f(t) \frac{P(q\tau + t)}{Q(q\tau + t)}. \tag{12}$$

Decomposing $P(x)/Q(x)$ into partial fractions, we have

$$\frac{P(x)}{Q(x)} = \sum_{k=1}^{m} \sum_{l=1}^{l_k} \frac{A_{k,l}}{(x + \alpha_k)^l},$$

where the coefficients $A_{k,l}$ are defined in (11) and $\sum_{k=1}^{m} A_{k,1} = 0$ if $\deg P(x) \le \deg Q(x) - 2$.

To prove (10), we first suppose that $\deg P(x) \le \deg Q(x) - 2$. Then from (12) we have

$$U = \sum_{t=0}^{q-1} f(t) \sum_{\tau=0}^{\infty} \frac{P(q\tau + t)}{Q(q\tau + t)},$$

where

$$\begin{aligned} \frac{P(q\tau + t)}{Q(q\tau + t)} &= \sum_{k=1}^{m} \sum_{l=1}^{l_k} \frac{A_{k,l}}{(q\tau + t + \alpha_k)^l} \\ &= \sum_{k=1}^{m} \frac{A_{k,1}}{q\tau + t + \alpha_k} + \sum_{k=1}^{m} \sum_{l=2}^{l_k} \frac{A_{k,l}}{(q\tau + t + \alpha_k)^l} \\ &= \frac{1}{q} \sum_{k=2}^{m} A_{k,1} \left( \frac{1}{\tau + \frac{t+\alpha_k}{q}} - \frac{1}{\tau + \frac{t+\alpha_1}{q}} \right) + \sum_{k=1}^{m} \sum_{l=2}^{l_k} \frac{A_{k,l}}{(q\tau + t + \alpha_k)^l}. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{\tau=0}^{\infty} \frac{P(q\tau + t)}{Q(q\tau + t)} &= \frac{1}{q} \sum_{k=2}^{m} A_{k,1} \left( \psi\left( \frac{t + \alpha_1}{q} \right) - \psi\left( \frac{t + \alpha_k}{q} \right) \right) \\ &\quad + \sum_{k=1}^{m} \sum_{l=2}^{l_k} \frac{(-1)^l}{(l-1)!} \frac{A_{k,l}}{q^l} \psi^{(l-1)}\left( \frac{t + \alpha_k}{q} \right) \\ &= \sum_{k=1}^{m} \sum_{l=1}^{l_k} \frac{(-1)^l}{(l-1)!} \frac{A_{k,l}}{q^l} \psi^{(l-1)}\left( \frac{t + \alpha_k}{q} \right), \end{aligned}$$

which yields (10). If $\deg P(x) = \deg Q(x) - 1$, then we find

$$\sum_{t=0}^{q-1} \frac{P(q\tau+t)}{Q(q\tau+t)}\, f(t) = \sum_{t=0}^{q-1} f(t) \sum_{k=1}^{m}\sum_{l=1}^{l_k} \frac{A_{k,l}}{(q\tau+t+\alpha_k)^l}$$
$$= \sum_{t=0}^{q-1} f(t) \sum_{k=1}^{m} \frac{A_{k,1}}{q\tau+t+\alpha_k} + \sum_{t=0}^{q-1} f(t) \sum_{k=1}^{m}\sum_{l=2}^{l_k} \frac{A_{k,l}}{(q\tau+t+\alpha_k)^l}$$
$$= \sum_{k=1}^{m} \frac{A_{k,1}}{q} \sum_{t=1}^{q-1} f(t) \left( \frac{1}{\tau+\frac{t+\alpha_k}{q}} - \frac{1}{\tau+\frac{\alpha_k}{q}} \right) + \sum_{t=0}^{q-1} f(t) \sum_{k=1}^{m}\sum_{l=2}^{l_k} \frac{A_{k,l}}{(q\tau+t+\alpha_k)^l}.$$

Hence, by (12), we get

$$U = \sum_{k=1}^{m} \frac{A_{k,1}}{q} \sum_{t=1}^{q-1} f(t) \left( \psi\left(\frac{\alpha_k}{q}\right) - \psi\left(\frac{t+\alpha_k}{q}\right) \right) + \sum_{t=0}^{q-1} f(t) \sum_{k=1}^{m}\sum_{l=2}^{l_k} \frac{(-1)^l}{(l-1)!}\, \frac{A_{k,l}}{q^l}$$
$$\times\, \psi^{(l-1)}\left(\frac{t+\alpha_k}{q}\right) = \sum_{t=0}^{q-1} f(t) \sum_{k=1}^{m}\sum_{l=1}^{l_k} \frac{(-1)^l}{(l-1)!}\, \frac{A_{k,l}}{q^l}\, \psi^{(l-1)}\left(\frac{t+\alpha_k}{q}\right),$$

as required. ∎

Let us mention two particular cases $q=1$, $f \equiv 1$ and $q=2$, $f(n) = (-1)^n$ of Lemma 1.

Lemma 2. *Let $P(x), Q(x) \in \overline{\mathbb{Q}}[x]$, $Q(x) = (x+\alpha_1)^{l_1} \dots (x+\alpha_m)^{l_m}$, where $l_1, \dots, l_m \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_m$ are distinct, and distinct from non-negative integers. Suppose that the series*

$$S = \sum_{n=0}^{\infty} \frac{P(n)}{Q(n)}, \qquad T = \sum_{n=0}^{\infty} \frac{P(n)}{Q(n)}\, (-1)^n$$

*converge. Then the following representations are valid:*

$$S = \sum_{k=1}^{m}\sum_{l=1}^{l_k} \frac{(-1)^l}{(l-1)!}\, A_{k,l}\psi^{(l-1)}(\alpha_k), \qquad T = \sum_{k=1}^{m}\sum_{l=1}^{l_k} \frac{(-1)^{l-1}}{(l-1)!}\, A_{k,l} g^{(l-1)}(\alpha_k),$$

*where the coefficients $A_{k,l}$ are defined in* (11).

If $Q(x)$ has only simple zeros, then Lemma 2 enables us to give simple sufficient conditions for $S, T$ to be algebraic or transcendental.

Corollary 1. *Let $P(x), Q(x) \in \overline{\mathbb{Q}}[x]$, $Q(x) = (x+\alpha_1)\dots(x+\alpha_m)$, where $\alpha_1, \dots, \alpha_m$ are distinct, and distinct from non-negative integers, and $\deg P(x) \le \deg Q(x) - 2$. If there is a subset $L$ of $\{1, \dots, m\}$ with $\#L \ge 2$, with $j, k \in L \Rightarrow \alpha_j - \alpha_k \in \mathbb{Z}$, and with $P(-\alpha_l) = 0$ for $l \notin L$, then*

$$S = \sum_{n=0}^{\infty} \frac{P(n)}{Q(n)}$$

*is algebraic.*

*Proof.* This statement easily follows from Lemma 2 and formula (2). ■

REMARK 0.1. In the case $m = 3$ and $\alpha_1, \ldots, \alpha_m \in \mathbb{Q}$, $P(x), Q(x) \in \mathbb{Q}[x]$ the conditions of Corollary 1 are necessary and sufficient for $S$ to be rational (see [9, Theorem 2]).

COROLLARY 2. *Let* $P(x), Q(x) \in \overline{\mathbb{Q}}[x]$, $Q(x) = (x + \alpha_1) \ldots (x + \alpha_m)$, *where* $\alpha_1, \ldots, \alpha_m$ *are distinct, and distinct from non-negative integers, and* $\deg P(x) \le \deg Q(x) - 1$. *If* $\alpha_k - \alpha_1 =: n_k \in \mathbb{Z}$ *for all* $1 \le k \le m$ *and*

$$\sum_{k=1}^{m} (-1)^{n_k} \frac{P(-\alpha_k)}{Q'(-\alpha_k)} = 0, \tag{13}$$

*then*

$$T = \sum_{n=0}^{\infty} \frac{P(n)}{Q(n)} (-1)^n$$

*is algebraic.* (*In particular, if all* $n_k$ *are even and* $\deg P(x) \le \deg Q(x) - 2$, *then condition* (13) *holds automatically.*)

*Proof.* This statement easily follows from Lemma 2 and formula (5). ■

REMARK 0.2. In the case $m = 2$ and $\alpha_1, \ldots, \alpha_m \in \mathbb{Q}$, $P(x), Q(x) \in \mathbb{Q}[x]$ the conditions of Corollary 2 are necessary and sufficient for $T$ to be rational (see [9, Theorem 1] and [10, Theorem 3]).

COROLLARY 3. *Let* $P(x) \in \overline{\mathbb{Q}}[x]$, $Q(x) = (x + \alpha_1) \ldots (x + \alpha_m)$, *where* $\alpha_1, \ldots, \alpha_m$ *are distinct rational numbers, distinct from non-negative integers, and* $\deg P(x) = m-1$. *If* $\alpha_k - \alpha_l \in 2\mathbb{Z}$ *for all* $1 \le k, l \le m$, *then the sum*

$$T = \sum_{n=0}^{\infty} \frac{P(n)}{Q(n)} (-1)^n$$

*is transcendental.*

*Proof.* By Lemma 2 and formula (5) it follows that

$$T = A + ag(\alpha_1) = B \pm ag(\alpha) = B \pm a \sum_{n=0}^{\infty} \left( \frac{1}{2n + \alpha} - \frac{1}{2n + \alpha + 1} \right),$$

where $A, B \in \overline{\mathbb{Q}}$, $a \ne 0$ is the leading coefficient of the polynomial $P(x)$ and $\alpha \equiv \alpha_1 \pmod 1$, $\alpha \in (0, 1]$. Since the infinite sum in the latter expression of $T$ does not vanish, by [1, Theorem 3] we conclude that $T$ is transcendental. ■

LEMMA 3. *For the* $k$*th derivatives we have*

$$\text{(a)}\ (\cot \pi z)^{(k)} = \pi^k p_k(\cot \pi z), \qquad \text{(b)}\ \left( \frac{1}{\sin \pi z} \right)^{(k)} = \pi^k \frac{q_k(\cos \pi z)}{\sin^{k+1} \pi z},$$

*where* $p_k(z), q_k(z) \in \mathbb{Z}[z]$, $\deg(p_k(z) - (-1)^k k! z^{k+1}) \le k$, $\deg(q_k(z) - (-z)^k) \le k - 1$.

*Proof.* The proof is by induction on $k$. Obviously, for $k = 0$ formulas (a), (b) are valid with $p_0(z) = z$ and $q_0(z) = 1$. Assuming (a), (b) to hold for $k$, we will prove them for $k+1$. We have

$$(\cot \pi z)^{(k+1)} = \pi^k (p_k(\cot \pi z))' = \pi^{k+1} p_{k+1}(\cot \pi z),$$

where $p_{k+1}(z) = -p_k'(z)(z^2+1) = (-1)^{k+1}(k+1)!z^{k+2} + c_{k+1}z^{k+1} + \cdots \in \mathbb{Z}[z]$, and

$$\left(\frac{1}{\sin \pi z}\right)^{(k+1)} = \pi^k \left(\frac{q_k(\cos \pi z)}{\sin^{k+1} \pi z}\right)' = \pi^{k+1} \frac{q_{k+1}(\cos \pi z)}{\sin^{k+2} \pi z}$$

with $q_{k+1}(z) = q_k'(z)(z^2-1) - (k+1)zq_k(z) = (-1)^{k+1}z^{k+1} + d_k z^k + \cdots \in \mathbb{Z}[z]$. ■

## 3. Main results

THEOREM 1. *Let* $P_1, \ldots, P_s, Q_1, \ldots, Q_s \in \overline{\mathbb{Q}}[x]$, $m_1, \ldots, m_s \in \mathbb{N}$, $r_1, \ldots, r_s \in \mathbb{Z}$ *satisfy the following conditions*: *for any* $1 \le j \le s$, $\deg P_j \le \deg Q_j - 2$,

$$\frac{P_j(-x)}{Q_j(-x)} = \frac{P_j(r_j+x)}{Q_j(r_j+x)}, \tag{14}$$

$Q_j(x) = \prod_{k=1}^{2m_j}(x-\alpha_{j,k})^{l_{j,k}}$, *where* $\alpha_{j,k} = a_{j,k} + ib_{j,k}\sqrt{d} \in \mathbb{Q}(i\sqrt{d}) \setminus \mathbb{N}_0$, $k = 1, \ldots, 2m_j$, *are distinct and such that* $\alpha_{j,m_j+k} = r_j - \alpha_{j,k}$, $b_{j,k} \ge 0$, $l_{j,m_j+k} = l_{j,k} \in \mathbb{N}$, $k = 1, \ldots, m_j$. *Then the sum*

$$S = \sum_{n=0}^{\infty}\left(\frac{P_1(n)}{Q_1(n)} + \cdots + \frac{P_s(n)}{Q_s(n)}\right)$$

*is either a computable algebraic number or transcendental. Moreover*, $S$ *is transcendental if at least one of the following conditions holds*:

(i) $\alpha_{j,k} \notin \mathbb{Q} \setminus \mathbb{Z}$, $j = 1, \ldots, s$, $k = 1, \ldots, 2m_j$, *and*

$$\sum_{j=1}^{s}\sum_{\substack{k=1 \\ \alpha_{j,k}\notin\mathbb{Z}}}^{m_j} \operatorname*{res}_{z=\alpha_{j,k}} \frac{P_j(z)}{Q_j(z)} \ne 0,$$

(ii) $b_{j_0,k_0} := \min\{b_{j,k} : b_{j,k} > 0\}$ *is a unique minimum of the positive numbers* $b_{j,k}$ *and* $\operatorname{res}_{z=\alpha_{j_0,k_0}} P_{j_0}(z)/Q_{j_0}(z) \ne 0$,

(iii) *there exists a unique maximum* $l_{j_0,k_0}$ *of the sequence* $l_{j,k}$, $1 \le j \le s$, $1 \le k \le m_j$, *and* $b_{j_0,k_0} > 0$, $P_{j_0}(\alpha_{j_0,k_0}) \ne 0$.

*Proof.* By Lemma 2, we have

$$S = \sum_{j=1}^{s}\sum_{n=0}^{\infty}\frac{P_j(n)}{Q_j(n)} = \sum_{j=1}^{s}\sum_{k=1}^{2m_j}\sum_{l=1}^{l_{j,k}} \frac{(-1)^l}{(l-1)!} A_{j,k,l}\psi^{(l-1)}(-\alpha_{j,k}),$$

where

$$(15)\qquad A_{j,k,l} = \frac{1}{(l_{j,k}-l)!}\left(\frac{d}{dx}\right)^{l_{j,k}-l}\left(\frac{P_j(x)}{Q_j(x)}\,(x-\alpha_{j,k})^{l_{j,k}}\right)\bigg|_{x=\alpha_{j,k}} \in \overline{\mathbb{Q}}.$$

From (14), (15) for $1 \le k \le m_j$ it follows that

$$\begin{aligned} A_{j,m_j+k,l} &= \frac{1}{(l_{j,k}-l)!}\left(\frac{d}{dx}\right)^{l_{j,k}-l}\left(\frac{P_j(r_j-x)}{Q_j(r_j-x)}\,(x-r_j+\alpha_{j,k})^{l_{j,k}}\right)\bigg|_{x=r_j-\alpha_{j,k}} \\ &= \frac{(-1)^l}{(l_{j,k}-l)!}\left(\frac{d}{dy}\right)^{l_{j,k}-l}\left(\frac{P_j(y)}{Q_j(y)}\,(y-\alpha_{j,k})^{l_{j,k}}\right)\bigg|_{y=\alpha_{j,k}} = (-1)^l A_{j,k,l} \end{aligned}$$

with $y = r_j - x$. Therefore,

$$S = \sum_{j=1}^{s}\sum_{k=1}^{m_j}\sum_{l=1}^{l_{j,k}} \frac{(-1)^l}{(l-1)!}\,A_{j,k,l}(\psi^{(l-1)}(-\alpha_{j,k}) + (-1)^l\psi^{(l-1)}(\alpha_{j,k}-r_j)).$$

Now if for some pair $(j,k)$ we have $-\alpha_{j,k}$ and $\alpha_{j,k}-r_j \in \mathbb{N}$, then by (2), (7), we get

$$\begin{aligned} S = C_0 &+ \sum_{j=1}^{s}\sum_{\substack{k=1\\ \alpha_{j,k}\in\mathbb{Z}}}^{m_j}\sum_{\substack{l=1\\ l\ \text{even}}}^{l_{j,k}} C_{j,k,l}\pi^l \\ &+ \sum_{j=1}^{s}\sum_{\substack{k=1\\ \alpha_{j,k}\notin\mathbb{Z}}}^{m_j}\sum_{l=1}^{l_{j,k}} \frac{A_{j,k,l}}{(l-1)!}\,(\psi^{(l-1)}(\alpha_{j,k}+1) + (-1)^l\psi^{(l-1)}(-\alpha_{j,k})), \end{aligned}$$

where $C_0, C_{j,k,l} \in \overline{\mathbb{Q}}$. Combining this with (3) and Lemma 3 we conclude that

$$\begin{aligned} (16)\qquad S = C_0 &+ \sum_{j=1}^{s}\sum_{\substack{k=1\\ \alpha_{j,k}\in\mathbb{Z}}}^{m_j}\sum_{\substack{l=1\\ l\ \text{even}}}^{l_{j,k}} C_{j,k,l}\pi^l \\ &+ \sum_{j=1}^{s}\sum_{\substack{k=1\\ \alpha_{j,k}\notin\mathbb{Z}}}^{m_j}\sum_{l=1}^{l_{j,k}} \frac{(-1)^{l-1}A_{j,k,l}}{(l-1)!}\,\pi^l p_{l-1}(-\cot\pi\alpha_{j,k}). \end{aligned}$$

According to the formula

$$\cot\pi\alpha_{j,k} = i\,\frac{e^{2\pi i a_{j,k}} + e^{2\pi b_{j,k}\sqrt{d}}}{e^{2\pi i a_{j,k}} - e^{2\pi b_{j,k}\sqrt{d}}} = -i - \frac{2ie^{2\pi i a_{j,k}}}{e^{2\pi b_{j,k}\sqrt{d}} - e^{2\pi i a_{j,k}}}$$

we see that $S - C_0 \in \overline{\mathbb{Q}}(\pi, e^{\pi\sqrt{d}/B})$, where $B \in \mathbb{N}$ is the least common denominator of the numbers $b_{j,k}$, and therefore $S - C_0$ is either zero or transcendental in view of the algebraic independence of $\pi$ and $e^{\pi\sqrt{d}}$ [7].

If we suppose that $S$ is algebraic and condition (i) holds, then considering the summands in (16) involving $\pi$ to the first power we get

$$-\pi\sum_{\substack{j=1\\ \alpha_{j,k}\notin\mathbb{Z}}}^{s}\sum_{k=1}^{m_j}A_{j,k,1}\cot\pi\alpha_{j,k}+\pi^2(\ldots)=0$$

or

$$\pi i\sum_{\substack{j=1\\ b_{j,k}>0}}^{s}\sum_{k=1}^{m_j}A_{j,k,1}+2\pi i\sum_{\substack{j=1\\ b_{j,k}>0}}^{s}\sum_{k=1}^{m_j}\frac{A_{j,k,1}e^{2\pi i a_{j,k}}}{e^{2\pi b_{j,k}\sqrt{d}}-e^{2\pi i a_{j,k}}}+\pi^2(\ldots)=0.$$

Now multiplying both sides of the last equality by

$$\prod_{\substack{j=1\\ b_{j,k}>0}}^{s}\prod_{k=1}^{m_j}(e^{2\pi b_{j,k}\sqrt{d}}-e^{2\pi i a_{j,k}})^{l_{j,k}} \tag{17}$$

we get a contradiction with the algebraic independence of $\pi$ and $e^{\pi\sqrt{d}}$.

If (ii) is valid and $S$ is algebraic, then (16) can be rewritten as

$$\pi C_1+2\pi i\sum_{\substack{j=1\\ b_{j,k}>0}}^{s}\sum_{k=1}^{m_j}\frac{A_{j,k,1}e^{2\pi i a_{j,k}}}{e^{2\pi b_{j,k}\sqrt{d}}-e^{2\pi i a_{j,k}}}+\pi^2(\ldots)=0. \tag{18}$$

If $C_1\neq 0$, then this is impossible by the same argument as above. If $C_1=0$, then multiplying both sides of (18) by (17) we get

$$2\pi i A_{j_0,k_0,1}e^{2\pi i a_{j_0,k_0}}e^{2\pi(\beta-b_{j_0,k_0})\sqrt{d}}+\cdots=0,$$

which is impossible, and therefore $S$ is transcendental.

If condition (iii) holds, then the summands with the maximal power of $\pi$ in (16) have the form

$$\pi^{l_{j_0,k_0}}\left(\pm\frac{A_{j_0,k_0,l_{j_0,k_0}}}{(l_{j_0,k_0}-1)!}\,p_{l_{j_0,k_0}-1}(-\cot\pi\alpha_{j_0,k_0})+C_{j_0,k_0,l_{j_0,k_0}}\right), \tag{19}$$

where $A_{j_0,k_0,l_{j_0,k_0}}, C_{j_0,k_0,l_{j_0,k_0}}\in\overline{\mathbb{Q}}$ and $A_{j_0,k_0,l_{j_0,k_0}}$ is not zero by (15). Since $\cot\pi\alpha_{j_0,k_0}$ is transcendental, the term (19) does not vanish in (16), and hence $S$ is transcendental. This completes the proof of the theorem. ∎

REMARK 1.1. If under the assumptions of Theorem 1 we have $r_1=\cdots=r_s=-1$, then $S$ is either zero or transcendental.

COROLLARY 4. *If* $a,b\in\mathbb{Z}$, $4b>a^2$, $m\in\mathbb{N}$, *then the sum*

$$\sum_{n=0}^{\infty}\frac{P(n)}{(n^2+an+b)^m}$$

*is transcendental for any polynomial* $P(x) \in \overline{\mathbb{Q}}[x]$ *such that*

$$\deg P(x) \le 2m-2 \quad \text{and} \quad P(-x) = P(x-a).$$

*In particular, the sum of the series*

$$\sum_{n=0}^{\infty} \frac{(n^2+an+c)^k}{(n^2+an+b)^m}$$

*is transcendental for any* $c, k \in \mathbb{Z}$, $0 \le k < m$.

THEOREM 2. *Let* $P_1, \ldots, P_s$, $Q_1, \ldots, Q_s \in \overline{\mathbb{Q}}[x]$, $m_1, \ldots, m_s \in \mathbb{N}$, $r_1, \ldots, r_s \in \mathbb{Z}$ *satisfy the following conditions*: *for any* $1 \le j \le s$, $\deg P_j \le \deg Q_j - 1$,

$$\frac{P_j(-x)}{Q_j(-x)} = (-1)^{r_j} \frac{P_j(r_j+x)}{Q_j(r_j+x)}, \tag{20}$$

$Q_j(x) = \prod_{k=1}^{2m_j} (x-\alpha_{j,k})^{l_{j,k}}$, *where* $\alpha_{j,k} = a_{j,k} + ib_{j,k}\sqrt{d} \in \mathbb{Q}(i\sqrt{d}) \setminus \mathbb{N}_0$, $k = 1, \ldots, 2m_j$, *are distinct and such that* $\alpha_{j,m_j+k} = r_j - \alpha_{j,k}$, $b_{j,k} \ge 0$, $l_{j,m_j+k} = l_{j,k} \in \mathbb{N}$, $k = 1, \ldots, m_j$. *Then the sum*

$$T = \sum_{n=0}^{\infty} \left( \frac{P_1(n)}{Q_1(n)} + \cdots + \frac{P_s(n)}{Q_s(n)} \right) (-1)^n$$

*is either a computable algebraic number or transcendental. Moreover*, $T$ *is transcendental if at least one of the following conditions holds*:

(i) $b_{j_0,k_0} := \min\{b_{j,k} : b_{j,k} > 0\}$ *is a unique minimum of the positive numbers* $b_{j,k}$ *and* $\operatorname{res}_{z=\alpha_{j_0,k_0}} P_{j_0}(z)/Q_{j_0}(z) \ne 0$,
(ii) *there exists a unique maximum* $l_{j_0,k_0}$ *of the sequence* $l_{j,k}$, $1 \le j \le s$, $1 \le k \le m_j$, *and* $b_{j_0,k_0} > 0$, $P_{j_0}(\alpha_{j_0,k_0}) \ne 0$.

*Proof.* From Lemma 2 it follows that

$$T = \sum_{j=1}^{s} \sum_{n=0}^{\infty} \frac{P_j(n)}{Q_j(n)} (-1)^n = \sum_{j=1}^{s} \sum_{k=1}^{2m_j} \sum_{l=1}^{l_{j,k}} \frac{(-1)^{l-1}}{(l-1)!} A_{j,k,l} g^{(l-1)}(-\alpha_{j,k}),$$

where the coefficients $A_{j,k,l}$ are defined in (15). According to (15) and (20) for $1 \le k \le m_j$ we have $A_{j,m_j+k,l} = (-1)^{r_j+l} A_{j,k,l}$. Then

$$T = \sum_{j=1}^{s} \sum_{k=1}^{m_j} \sum_{l=1}^{l_{j,k}} \frac{(-1)^{l-1}}{(l-1)!} A_{j,k,l} (g^{(l-1)}(-\alpha_{j,k}) + (-1)^{r_j+l} g^{(l-1)}(\alpha_{j,k} - r_j)).$$

Now if for some pair $(j,k)$ we have $-\alpha_{j,k}$ and $\alpha_{j,k} - r_j \in \mathbb{N}$, then by (5), (7), we get

$$T=C_0+\sum_{\substack{j=1\\ \alpha_{j,k}\in\mathbb{Z}}}^{s}\sum_{k=1}^{m_j}\sum_{\substack{l=1\\ l\,\text{even}}}^{l_{j,k}}C_{j,k,l}\pi^l$$
$$+\sum_{\substack{j=1\\ \alpha_{j,k}\notin\mathbb{Z}}}^{s}\sum_{k=1}^{m_j}\sum_{l=1}^{l_{j,k}}\frac{A_{j,k,l}}{(l-1)!}\left((-1)^{l-1}g^{(l-1)}(-\alpha_{j,k})+g^{(l-1)}(\alpha_{j,k}+1)\right),$$

where $C_0, C_{j,k,l}\in\overline{\mathbb{Q}}$. Hence, by (6) and Lemma 3, we have

$$T=C_0+\sum_{\substack{j=1\\ \alpha_{j,k}\in\mathbb{Z}}}^{s}\sum_{k=1}^{m_j}\sum_{\substack{l=1\\ l\,\text{even}}}^{L}C_{j,k,l}\pi^l-\sum_{\substack{j=1\\ \alpha_{j,k}\notin\mathbb{Z}}}^{s}\sum_{k=1}^{m_j}\sum_{l=1}^{l_{j,k}}\frac{A_{j,k,l}}{(l-1)!}\,\pi^l\,\frac{q_{l-1}(\cos\pi\alpha_{j,k})}{\sin^l\pi\alpha_{j,k}} \tag{21}$$

and according to Euler's formulas for cos and sin we conclude that either $T=C_0$ or $T$ is transcendental.

If $T$ is algebraic and condition (i) holds, then we rewrite (21) as

$$\pi C_1+\pi\sum_{\substack{j=1\\ b_{j,k}>0}}^{s}\sum_{k=1}^{m_j}\frac{A_{j,k,1}}{\sin\pi\alpha_{j,k}}+\pi^2(\ldots)=0,$$

from which by the same argument as in the proof of Theorem 1(ii) and formula

$$\frac{1}{\sin\pi\alpha_{j,k}}=-\frac{2ie^{i\pi a_{j,k}}e^{\pi b_{j,k}\sqrt{d}}}{e^{2\pi b_{j,k}\sqrt{d}}-e^{2\pi i a_{j,k}}}$$

we get a contradiction.

If condition (ii) is valid and $T$ is algebraic, then from (21) we have

$$\pi^{l_{j_0,k_0}}\left(C_{j_0,k_0,l_{j_0,k_0}}-\frac{A_{j_0,k_0,l_{j_0,k_0}}}{(l_{j_0,k_0}-1)!}\,\frac{q_{l_{j_0,k_0}-1}(\cos\pi\alpha_{j_0,k_0})}{\sin^{l_{j_0,k_0}}\pi\alpha_{j_0,k_0}}\right)+\cdots=0,$$

where $A_{j_0,k_0,l_{j_0,k_0}}\neq 0$ by (15). Now applying Lemma 3 we easily see that the term containing $\pi$ to the maximal power does not vanish and we get a contradiction with the algebraic independence of $\pi$ and $e^{\pi\sqrt{d}}$. This completes the proof. ∎

Remark 2.1. If under the assumptions of Theorem 2 we have $r_1=\cdots=r_s=-1$, then $T$ is either zero or transcendental.

Remark 2.2. We note that there are alternative proofs of formulas (16), (21) based on application of the residue theorem to the complex integrals

$$\frac{1}{2\pi i}\int_{L_N}\left(\sum_{j=1}^{s}\frac{P_j(z)}{Q_j(z)}\right)(\pi\cot\pi z)\,dz\quad\text{and}\quad\frac{1}{2\pi i}\int_{L_N}\left(\sum_{j=1}^{s}\frac{P_j(z)}{Q_j(z)}\right)\frac{\pi}{\sin\pi z}\,dz,$$

where $L_N$ is a square contour with vertices $(N+1/2)(\pm 1 \pm i)$. (See also [3, Theorem 2].)

COROLLARY 5. *Let $a, b \in \mathbb{Z}$, $4b > a^2$, and $m \in \mathbb{N}$. Then for any polynomial $P(x) \in \overline{\mathbb{Q}}[x]$ such that $\deg P(x) < 2m$, $P(-x) = (-1)^a P(x-a)$, the sum*

$$\sum_{n=0}^{\infty} \frac{(-1)^n P(n)}{(n^2+an+b)^m}$$

*is transcendental. In particular, if $k \in \mathbb{Z}, 0 \le k < 2m$, and the numbers $k, a$ have the same parity, then the sum*

$$\sum_{n=0}^{\infty} \frac{(-1)^n (n+a/2)^k}{(n^2+an+b)^m}$$

*is transcendental.*

THEOREM 3. *Let $f : \mathbb{Z} \to \overline{\mathbb{Q}}$ be periodic with period $q \in \mathbb{N}$. Suppose that $r \in \mathbb{Z}$, $m, l_1, \ldots, l_m \in \mathbb{N}$, $P(x), Q(x) \in \overline{\mathbb{Q}}[x]$,*

$$\frac{P(-x)}{Q(-x)} = \pm \frac{P(x+qr)}{Q(x+qr)}, \tag{22}$$

*$Q(x) = (x-\alpha_0)\prod_{k=1}^{2m}(x-\alpha_k)^{l_k}$, where $\alpha_0 = qr/2$, $\alpha_k = a_k + ib_k\sqrt{d} \in \mathbb{Q}(i\sqrt{d}) \setminus \mathbb{N}$, $k = 1, \ldots, 2m$, are distinct, $\alpha_{m+k} = qr - \alpha_k$, $l_{m+k} = l_k$, $b_k \ge 0$, $k = 1, \ldots, m$, and $f$ is an even or odd function according to whether we have the "plus" or "minus" sign in (22). Suppose further that the series*

$$U = \sum_{n=1}^{\infty} \frac{P(n)}{Q(n)} f(n)$$

*converges. Then $U$ is either a computable algebraic number or transcendental. Moreover, $U$ is transcendental if at least one of the following conditions holds:*

(i) *$P(qr/2) = 0$ and*

$$\sum_{\substack{t=1 \\ t-\alpha_k \notin q\mathbb{Z}}}^{q} \sum_{k=1}^{m} f(t) \operatorname*{res}_{z=\alpha_k} \frac{P(z)}{Q(z)} \ne 0,$$

(ii) *$P(qr/2) = 0$, $b_{k_0} := \min\{b_k > 0\}$ is a unique minimum of the positive numbers $b_k$, $\operatorname{res}_{z=\alpha_{k_0}} P(z)/Q(z) \ne 0$ and $\sum_{t=1}^{q} f(t) e^{-2\pi i t/q} \ne 0$,*

(iii) $\displaystyle\sum_{\substack{t=1 \\ t-\alpha_k \notin q\mathbb{Z}}}^{q-1} \sum_{k=1}^{m} f(t) \operatorname*{res}_{z=\alpha_k} \frac{P(z)}{Q(z)} \ne \frac{i}{2} \frac{P(qr/2)}{Q'(qr/2)} \sum_{\substack{t=1 \\ t \ne q/2}}^{q-1} f(t) \cot\left(\frac{\pi t}{q} + \pi\left\{\frac{r}{2}\right\}\right)$ *and $P(qr/2) \ne 0$, where $\{x\}$ denotes the fractional part of $x$.*

*Proof.* By Lemma 1, using the partial fraction expansion

$$\frac{P(x)}{Q(x)}=\sum_{k=1}^{2m}\sum_{l=1}^{l_k}\frac{A_{k,l}}{(x-\alpha_k)^l}+\frac{A_{0,1}}{x-qr/2},$$

where the coefficients $A_{k,l}$ are defined in (11) with $\alpha_k$ replaced by $-\alpha_k$ and $A_{0,1}=P(qr/2)/Q'(qr/2)$, we have

$$U=\sum_{t=1}^{q}f(t)\sum_{k=1}^{2m}\sum_{l=1}^{l_k}\frac{(-1)^l}{(l-1)!}\,\frac{A_{k,l}}{q^l}\,\psi^{(l-1)}\biggl(\frac{t-\alpha_k}{q}\biggr)-\frac{A_{0,1}}{q}\sum_{t=1}^{q}f(t)\psi\biggl(\frac{t}{q}-\frac{r}{2}\biggr).$$

By (22), for $1\le k\le m$, $1\le l\le l_k$, it easily follows that $A_{m+k,l}=\pm(-1)^lA_{k,l}$.

To prove the theorem, we first assume that $P(qr/2)=0$. Then taking into account that $f(t)=\pm f(-t)$ and $f$ is a $q$-periodic function we have

$$\begin{aligned}U&=\sum_{t=1}^{q}f(t)\sum_{k=1}^{m}\sum_{l=1}^{l_k}\frac{(-1)^l}{(l-1)!}\,\frac{A_{k,l}}{q^l}\biggl(\psi^{(l-1)}\biggl(\frac{t-\alpha_k}{q}\biggr)\\&\qquad\pm(-1)^l\psi^{(l-1)}\biggl(\frac{t-\alpha_{m+k}}{q}\biggr)\biggr)\\&=\sum_{t=1}^{q}\sum_{k=1}^{m}\sum_{l=1}^{l_k}\frac{(-1)^lf(t)}{(l-1)!}\,\frac{A_{k,l}}{q^l}\,\psi^{(l-1)}\biggl(\frac{t-\alpha_k}{q}\biggr)\\&\qquad+\sum_{t=1}^{q}\sum_{k=1}^{m}\sum_{l=1}^{l_k}\frac{A_{k,l}f(q-t)}{(l-1)!q^l}\,\psi^{(l-1)}\biggl(\frac{t-\alpha_{m+k}}{q}\biggr)\\&=A+\sum_{t=1}^{q}f(t)\sum_{k=1}^{m}\sum_{l=1}^{l_k}\frac{(-1)^l}{(l-1)!}\\&\qquad\times\frac{A_{k,l}}{q^l}\biggl(\psi^{(l-1)}\biggl(\frac{t-\alpha_k}{q}\biggr)+(-1)^l\psi^{(l-1)}\biggl(1-r-\frac{t-\alpha_k}{q}\biggr)\biggr),\end{aligned}$$

where $A=-f(q)\sum_{k=1}^{m}\sum_{l=1}^{l_k}A_{k,l}/\alpha_{m+k}^l\in\overline{\mathbb{Q}}$. Now by (3), (7) and Lemma 3 we get

$$\begin{aligned}U=C_0&+\sum_{\substack{t=1\\t-\alpha_k\in q\mathbb{Z}}}^{q}\sum_{k=1}^{m}\sum_{l=2}^{l_k}C_{t,k,l}\pi^l\\&-\sum_{\substack{t=1\\t-\alpha_k\notin q\mathbb{Z}}}^{q}\sum_{k=1}^{m}\sum_{l=1}^{l_k}\frac{(-\pi)^lf(t)A_{k,l}}{q^l}\,p_{l-1}\biggl(\cot\biggl(\frac{\pi(t-\alpha_k)}{q}\biggr)\biggr)\end{aligned}$$

with $C_0, C_{t,k,l}\in\overline{\mathbb{Q}}$, from which it follows that $U$ is either equal to $C_0\in\overline{\mathbb{Q}}$

or transcendental. If condition (i) or (ii) holds, then arguing as in the proof of Theorem 1(i), (ii) we find that $U$ is transcendental.

If $P(qr/2) \neq 0$, then $P(-x) = P(x+qr)$ and thus $f$ is an odd function by the hypothesis. Arguing as above we deduce that $A_{k+m,l} = (-1)^{l-1}A_{k,l}$, $1 \leq k \leq m$, $1 \leq l \leq l_k$, and

$$\begin{aligned} U = \sum_{t=1}^{q-1} f(t) \sum_{k=1}^{m} \sum_{l=1}^{l_k} \frac{(-1)^l}{(l-1)!} \\ \times \frac{A_{k,l}}{q^l}\biggl(\psi^{(l-1)}\biggl(\frac{t-\alpha_k}{q}\biggr) + (-1)^l \psi^{(l-1)}\biggl(1 - r - \frac{t-\alpha_k}{q}\biggr)\biggr) \\ - \frac{A_{0,1}}{2q}\sum_{t=1}^{q-1} f(t)\biggl(\psi\biggl(\frac{t}{q} - \frac{r}{2}\biggr) - \psi\biggl(1 - \frac{t}{q} - \frac{r}{2}\biggr)\biggr). \end{aligned}$$

As is easily seen, if $q$ is even, then $f(q/2) = 0$ and we may assume that $t \neq q/2$ in the last sum. Now by (2), for a positive integer $t \leq q-1$, $t \neq q/2$, we have

$$\begin{aligned} \psi\biggl(\frac{t}{q} - \frac{r}{2}\biggr) &= C + \psi\biggl(\frac{t}{q} - \frac{r}{2} + \biggl[\frac{r+1}{2}\biggr]\biggr), \\ \psi\biggl(1 - \frac{t}{q} - \frac{r}{2}\biggr) &= \widetilde{C} + \psi\biggl(1 - \frac{t}{q} - \frac{r}{2} + \biggl[\frac{r}{2}\biggr]\biggr), \end{aligned} \tag{23}$$

where $C, \widetilde{C} \in \overline{\mathbb{Q}}$ and $[x]$ denotes the integer part of $x$. Now by (3), (23) and Lemma 3 we get

$$\begin{aligned} U = C_1 + \sum_{\substack{t=1 \\ t-\alpha_k \in q\mathbb{Z}}}^{q-1} \sum_{k=1}^{m} \sum_{l=2}^{l_k} C_{t,k,l}\pi^l \\ - \sum_{\substack{t=1 \\ t-\alpha_k \notin q\mathbb{Z}}}^{q-1} \sum_{k=1}^{m} \sum_{l=1}^{l_k} \frac{A_{k,l}\pi^l f(t)}{(-q)^l}\, p_{l-1}\biggl(\cot\biggl(\frac{\pi(t-\alpha_k)}{q}\biggr)\biggr) \\ + \frac{A_{0,1}\pi}{2q} \sum_{t=1}^{q-1} f(t) \cot\biggl(\frac{\pi t}{q} + \pi\biggl\{\frac{r}{2}\biggr\}\biggr) \end{aligned} \tag{24}$$

with $C_1, C_{t,k,l} \in \overline{\mathbb{Q}}$, and therefore $U$ is either equal to $C_1$ or transcendental. If $r = 0$, i.e., if $P(x)$ and $Q(x)$ are even and odd polynomials respectively, then $C_1 = 0$ and hence $U$ is either zero or transcendental. If condition (iii) is valid, then the coefficient of $\pi$ does not vanish in (24) and we conclude that $U$ is transcendental. This completes the proof of the theorem. ■

REMARK 3.1. If under the assumptions of Theorem 3 we have $r = 0$, then either $U = -f(q)\sum_{k=1}^{m}\sum_{l=1}^{l_k} A_{k,l}/\alpha_{k+m}^l$ or $U$ is transcendental.

THEOREM 4. *Let $k \in \mathbb{N}$, $r \in \mathbb{Z}$, $qr/2 \notin \mathbb{N}$, $P(x) \in \overline{\mathbb{Q}}[x]$ and $P(-x) = \pm P(x+qr)$. Let $f : \mathbb{Z} \to \overline{\mathbb{Q}}$ be an even or odd periodic function with period $q \in \mathbb{N}$ depending on whether $k$ and* $\deg P(x)$ *have the same parity or not. Suppose further that the series*

$$U = \sum_{n=1}^{\infty} \frac{f(n)P(n)}{(n-qr/2)^k}$$

*converges. Then the sum $U$ is either a computable algebraic number or transcendental. In particular, if $r = 0$, then $U$ is either zero or transcendental.*

*Proof.* For the rational function $P(x)/(x-qr/2)^k$ we have the following partial fraction expansion:

$$\frac{P(x)}{(x-qr/2)^k} = \sum_{l=0}^{[(\deg P)/2]} \frac{A_l}{(x-qr/2)^{k-\delta-2l}} \quad \text{with} \quad A_l = \frac{1}{(2l+\delta)!} P^{(2l+\delta)}\left(\frac{qr}{2}\right)$$

and $\delta$ equal to 0 or 1 according to whether $P(-x) = P(x+qr)$ or $P(-x) = -P(x+qr)$. Then by Lemma 1, we get

$$U = \sum_{t=1}^{q} f(t) \sum_{l=0}^{[(\deg P)/2]} \frac{(-1)^{k-\delta-1}}{(k-\delta-2l-1)!} \frac{A_l}{q^{k-\delta-2l}} \psi^{(k-\delta-2l-1)}\left(\frac{t}{q} - \frac{r}{2}\right).$$

Note that if $k$ and $\deg P$ have the same (distinct) parity, then $k-\delta$ is even (odd) and $f$ is an even (odd) function by the hypothesis. Thus we have $f(t) = (-1)^{k-\delta} f(q-t)$ and

$$2U = \sum_{t=1}^{q} \sum_{l=0}^{[(\deg P)/2]} \frac{(-1)^{k-\delta-1} f(t) - f(q-t)}{(k-\delta-2l-1)!} \frac{A_l}{q^{k-\delta-2l}} \psi^{(k-\delta-2l-1)}\left(\frac{t}{q} - \frac{r}{2}\right)$$

or

$$(25) \quad 2U = \sum_{t=1}^{q-1} f(t) \sum_{l=0}^{[(\deg P)/2]} \frac{(-1)^{k-\delta-1}}{(k-\delta-2l-1)!} \frac{A_l}{q^{k-\delta-2l}} \left(\psi^{(k-\delta-2l-1)}\left(\frac{t}{q} - \frac{r}{2}\right) + (-1)^{k-\delta} \psi^{(k-\delta-2l-1)}\left(1 - \frac{t}{q} - \frac{r}{2}\right)\right) + \widetilde{U},$$

where

$$(26) \quad \widetilde{U} = (f(q) + (-1)^{k-\delta} f(0)) \times \sum_{l=0}^{[(\deg P)/2]} \frac{(-1)^{k-\delta-1}}{(k-\delta-2l-1)!} \frac{A_l}{q^{k-\delta-2l}} \psi^{(k-\delta-2l-1)}\left(1 - \frac{r}{2}\right).$$

It can be easily seen that $\widetilde{U} = 0$ if $f$ is an odd function; if $f$ is even, then

$k-\delta$ is even and by (7) we have

$$\widetilde{U} = C + \sum_{l=0}^{[(\deg P)/2]} C_l \pi^{k-\delta-2l} \tag{27}$$

with algebraic coefficients $C, C_l$. From (23), (3), (7) and Lemma 3 it follows that

$$\psi^{(k-\delta-2l-1)}\left(\frac{t}{q}-\frac{r}{2}\right) + (-1)^{k-\delta}\psi^{(k-\delta-2l-1)}\left(1-\frac{t}{q}-\frac{r}{2}\right) \in \mathbb{Q}\pi^{k-\delta-2l} + \mathbb{Q}. \tag{28}$$

Finally, by (25)–(28), we find

$$U = \widetilde{C} + \sum_{l=0}^{[(\deg P)/2]} \widetilde{C}_l \pi^{k-\delta-2l}$$

with $\widetilde{C}, \widetilde{C}_l \in \overline{\mathbb{Q}}$, and therefore either $U$ is equal to $\widetilde{C}$ or $U \notin \overline{\mathbb{Q}}$. If $r = 0$, then from (25), (26) it easily follows that $U$ is either zero or transcendental. This completes the proof of the theorem. ■

The special case of Theorem 4 for the number $U = L(k,\chi) = \sum_{n=1}^{\infty} \chi(n)/n^k$, where $\chi$ is an even (or odd) Dirichlet character, was proved in [10, §6].

Now consider several applications of Theorem 3 which gives us means to construct new examples of transcendental numbers. If in Theorem 3 we put $f(n) = \chi(n)$, where $\chi(n)$ is a Dirichlet character mod $q$, then the Gauss sum

$$\tau(\chi) = \sum_{k=1}^{q} \chi(k) e^{-2\pi i k/q}$$

is never zero when $\chi$ is a primitive character (see [4, Ch. 8]). Namely, we have $|\tau(\chi)| = \sqrt{q}$. This gives us the following.

Corollary 6. *Let $q > 1$ be an integer and $\chi$ be a primitive character mod $q$. Suppose that $P(x) \in \overline{\mathbb{Q}}[x]$, $P(-x) = \pm P(x+qr)$, $Q(x) = \prod_{k=1}^{2m}(x-\alpha_k)^{l_k}$ for some $m, l_1, \ldots, l_{2m} \in \mathbb{N}, r \in \mathbb{Z}$, where $\alpha_k = a_k + i b_k\sqrt{d} \in \mathbb{Q}(i\sqrt{d}) \setminus \mathbb{N}$, $k = 1, \ldots, 2m$, are distinct numbers such that $\alpha_{m+k} = qr - \alpha_k$, $b_k \geq 0$, $l_{m+k} = l_k$, $k = 1, \ldots, m$, and $\chi$ is an even (resp. odd) character if $\deg P$ is even (resp. odd). If $b_{k_0} := \min\{b_k > 0\}$ is a unique minimum of the positive numbers $b_k$ and $\operatorname{res}_{z=\alpha_{k_0}} P(z)/Q(z) \neq 0$, then the sum*

$$\sum_{n=1}^{\infty} \frac{P(n)}{Q(n)} \chi(n)$$

*is transcendental.*

Corollary 7. *Let $q>1$ be a square-free integer with $q\equiv 1 \pmod 4$, and let $\left(\frac{n}{q}\right)$ denote Jacobi's symbol. Then*

$$\sum_{n=1}^{\infty}\frac{P(n)}{Q(n)}\left(\frac{n}{q}\right)\notin\overline{\mathbb{Q}},$$

*where $P(x)\in\overline{\mathbb{Q}}[x]$, $P(-x)=P(x+qr)$ and $Q(x)$ is as in Corollary* 6. *In particular*, *the sum*

$$\sum_{n=1}^{\infty}\frac{\left(\frac{n}{q}\right)}{(n^2+qrn+b)^m}$$

*is transcendental for any $m\in\mathbb{N}$, $b,r\in\mathbb{Z}$ such that $q^2r^2<4b$.*

Corollary 8. *Let $q>1$ be a square-free integer with $q\equiv 3 \pmod 4$. Then*

$$\sum_{n=1}^{\infty}\frac{P(n)}{Q(n)}\left(\frac{n}{q}\right)\notin\overline{\mathbb{Q}},$$

*where $P(x)\in\overline{\mathbb{Q}}[x]$, $P(-x)=-P(x+qr)$ and $Q(x)$ is as in Corollary* 6. *In particular*, *the sum*

$$\sum_{n=1}^{\infty}\left(\frac{n}{q}\right)\frac{(n+qr/2)^{2m-1}}{(n^2+qrn+b)^m}$$

*is transcendental for any $m\in\mathbb{N}$, $b,r\in\mathbb{Z}$ such that $q^2r^2<4b$.*

If $\chi_0$ is the principal character mod $q$, then

$$\sum_{n=1}^{q}\chi_0(n)=\varphi(q),\qquad \tau(\chi_0)=\sum_{\substack{k=1\\(k,q)=1}}^{q}e^{-2\pi ik/q}=\mu(q),$$

where $\varphi$ and $\mu$ are the Euler and Möbius functions, respectively (see [11, Ch. 3]) and we have

Corollary 9. *If $q>1$ is a square-free integer and $\chi_0$ is the principal character mod $q$, then the sum*

$$\sum_{n=1}^{\infty}\frac{P(n)}{Q(n)}\chi_0(n)$$

*is transcendental*, *where $P(x)\in\overline{\mathbb{Q}}[x]$, $P(-x)=P(x+qr)$ and the polynomial $Q(x)$ is as in Corollary* 6. *In particular*, *the sum of the series*

$$\sum_{n=1}^{\infty}\frac{\chi_0(n)}{(n^2+qrn+b)^m}$$

*is transcendental for any $m\in\mathbb{N}$, $b,r\in\mathbb{Z}$ such that $q^2r^2<4b$.*

COROLLARY 10. *Let $q > 1$ be an integer and $\chi_0$ the principal character mod $q$. Suppose that $P(x), Q(x) \in \overline{\mathbb{Q}}[x]$, $P(-x) = P(x + qr)$ and $Q(x) = \prod_{k=1}^{2m}(x-\alpha_k)^{l_k}$ for some $m, l_1, \ldots, l_{2m} \in \mathbb{N}, r \in \mathbb{Z}$, where $\alpha_k = a_k + ib_k\sqrt{d} \in \mathbb{Q}(i\sqrt{d}) \setminus \mathbb{Q}$, $k = 1, \ldots, 2m$, are distinct and such that $\alpha_{k+m} = \overline{\alpha_k}$, $b_k \geq 0$, $l_{k+m} = l_k$, $k = 1, \ldots, m$. If $\sum_{k=1}^{m} \mathrm{res}_{z=\alpha_k} P(z)/Q(z) \neq 0$, then the sum*

$$\sum_{n=1}^{\infty} \frac{P(n)}{Q(n)} \chi_0(n)$$

*is transcendental.*

COROLLARY 11. *Let $f : \mathbb{Z} \to \overline{\mathbb{Q}}$ be odd, periodic with period $q \in \mathbb{N}$. Then the sum*

$$\sum_{n=1}^{\infty} \frac{P(n)f(n)}{n(n^2+b)^m}$$

*is either zero or transcendental for any $m, b \in \mathbb{N}$ and any even polynomial $P(x)$ with $\deg P \leq 2m$.*

## 4. Transcendence of trigonometric series

THEOREM 5. *Suppose that $\beta_1, \ldots, \beta_s \in [0, 2)$ are distinct rational numbers, $Q(x), P_1(x), \ldots, P_s(x) \in \overline{\mathbb{Q}}[x]$, $Q(x) = (x-\alpha_1)^{l_1} \ldots (x-\alpha_m)^{l_m}$, where $\alpha_1, \ldots, \alpha_m \in \mathbb{Q}(i\sqrt{d}) \setminus \mathbb{Z}$ are distinct, $l_1, \ldots, l_m \in \mathbb{N}$, $h(n) = \sum_{j=1}^{s} P_j(n) e^{i\pi\beta_j n}$, and for $1 \leq j \leq s$,*

$$\deg P_j(x) \leq \begin{cases} \deg Q(x) - 1 & \text{if } \beta_j > 0, \\ \deg Q(x) - 2 & \text{if } \beta_j = 0. \end{cases}$$

*Then the sum*

$$V = \sum_{n=-\infty}^{\infty} \frac{h(n)}{Q(n)}$$

*is either zero or transcendental.*

*Proof.* We consider the complex integral

$$I_N = \frac{1}{2\pi i} \int_{L_N} \frac{h^-(z)}{Q(z)} \frac{\pi}{\sin \pi z} \, dz,$$

where $h^-(z) = \sum_{j=1}^{s} P_j(z) e^{i\pi(\beta_j - 1)z}$, $L_N$ is a square contour with vertices $(N + 1/2)(\pm 1 \pm i)$, and $N$ is a large positive integer such that $\alpha_1, \ldots, \alpha_m$ are inside $L_N$. For $z = \pm(N + 1/2) + iy$, $y \in [-N - 1/2, N + 1/2]$, we have

$$\left| \frac{1}{\sin \pi z} \right| = \frac{2}{e^{\pi y} + e^{-\pi y}},$$

and therefore,

$$\left|\frac{P_j(z)e^{i\pi(\beta_j-1)z}}{Q(z)\sin\pi z}\right| = \frac{2|P_j(z)|}{|Q_j(z)|(e^{\pi\beta_j y}+e^{\pi(\beta_j-2)y})} \le 2\,\frac{|P_j(z)|}{|Q_j(z)|}\,e^{-\pi|y|\min\{\beta_j,2-\beta_j\}}. \tag{29}$$

If $\beta_j = 0$, then from (29) it follows that

$$\left|\frac{1}{2\pi i}\int\limits_{\substack{z=\pm(N+1/2)+iy\\ -N-1/2\le y\le N+1/2}} \frac{P_j(z)e^{i\pi(\beta_j-1)z}}{Q(z)}\,\frac{\pi}{\sin\pi z}\,dz\right| \le \int\limits_{-N-1/2}^{N+1/2}\frac{|P_j(z)|}{|Q_j(z)|}\,dy = O\Big(\frac{1}{N}\Big). \tag{30}$$

If $0<\beta_j<2$, then (29) implies

$$\left|\frac{1}{2\pi i}\int\limits_{\substack{z=\pm(N+1/2)+iy\\ -N-1/2\le y\le N+1/2}} \frac{P_j(z)e^{i\pi(\beta_j-1)z}}{Q(z)}\,\frac{\pi}{\sin\pi z}\,dz\right| \le O\Big(\frac{1}{N}\Big)\int\limits_{-N-1/2}^{N+1/2} e^{-\pi|y|\min\{\beta_j,2-\beta_j\}}\,dy = O\Big(\frac{1}{N}\Big). \tag{31}$$

If $z = x\pm i(N+1/2)$, $x\in[-N-1/2, N+1/2]$, then

$$\left|\frac{1}{\sin\pi z}\right| = \frac{2}{e^{\pi(N+1/2)}-e^{-\pi(N+1/2)}}$$

and

$$\left|\frac{P_j(z)e^{i\pi(\beta_j-1)z}}{Q(z)\sin\pi z}\right| \le \frac{2|P_j(z)|}{|Q(z)|}\,\frac{e^{\pi|\beta_j-1|(N+1/2)}}{e^{\pi(N+1/2)}-e^{-\pi(N+1/2)}} = \begin{cases} O\Big(\dfrac{1}{N^2}\Big) & \text{if } \beta_j=0,\\ O\Big(\dfrac{1}{Ne^{\pi(1-|\beta_j-1|)N}}\Big) & \text{if } 0<\beta_j<2.\end{cases} \tag{32}$$

Therefore, by (30)–(32), we conclude that $I_N = O(N^{-1})$ as $N\to\infty$. On the other hand, by the residue theorem we have

$$I_N - \sum_{k=1}^{m}\operatorname*{res}_{z=\alpha_k}\Big(\frac{h^-(z)}{Q(z)}\,\frac{\pi}{\sin\pi z}\Big) = \sum_{k=-N}^{N}\operatorname*{res}_{z=k}\Big(\frac{h^-(z)}{Q(z)}\,\frac{\pi}{\sin\pi z}\Big) = \sum_{k=-N}^{N}\frac{h(k)}{Q(k)}.$$

Now letting $N$ tend to infinity we get

$$V = -\sum_{k=1}^{m} \operatorname*{res}_{z=\alpha_k} \left( \frac{\pi h^-(z)}{Q(z)\sin \pi z} \right) = \sum_{k=1}^{m} \frac{-\pi}{(l_k-1)!} \left( \frac{h^-(z)(z-\alpha_k)^{l_k}}{Q(z)\sin \pi z} \right)^{(l_k-1)} \Bigg|_{z=\alpha_k},$$

which implies that $V \in \overline{\mathbb{Q}}(\pi, e^{\pi\sqrt{d}/B})$ for some $B \in \mathbb{N}$, and hence either $V = 0$ or $V \notin \overline{\mathbb{Q}}$. ■

COROLLARY 12. *If in addition to the assumptions of Theorem* 5, $Q(x)$ *is an even polynomial*, *then the sum*

$$W = \sum_{n=0}^{\infty} \frac{h(n) + h(-n)}{Q(n)}$$

*is either* $h(0)/Q(0)$ *or transcendental.*

COROLLARY 13. *Suppose that* $\beta_1, \beta_2 \in (0,1) \cup (1,2)$ *are rational numbers*, $Q(x)$, $P_1(x)$, $P_2(x) \in \overline{\mathbb{Q}}[x]$ *such that* $P_1(x), Q(x)$ *are even polynomials*, $P_2(x)$ *is an odd polynomial*, $\deg P_j(x) \le \deg Q(x) - 1$, $j = 1, 2$, *and all roots of* $Q(x)$ *belong to* $\mathbb{Q}(i\sqrt{d}) \setminus \mathbb{Z}$. *Then the trigonometric series*

$$W = \frac{P_1(0)}{2Q(0)} + \sum_{n=1}^{\infty} \frac{P_1(n)\cos(\pi\beta_1 n) + P_2(n)\sin(\pi\beta_2 n)}{Q(n)}$$

*is either zero or transcendental.*

*Proof.* We define

$$h(n) = \begin{cases} \frac{1}{2}P_1(n)e^{i\pi\beta_1 n} - \frac{1}{2}iP_2(n)e^{i\pi\beta_2 n} & \text{if } \beta_1 \ne \beta_2, \\ \frac{1}{2}P_1(n)e^{i\pi\beta_1 n} + \frac{1}{2}iP_2(n)e^{i\pi(2-\beta_1)n} & \text{if } \beta_1 = \beta_2, \end{cases}$$

and consider the sum

$$\sum_{n=0}^{\infty} \frac{h(n) + h(-n)}{Q(n)} - \frac{h(0)}{Q(0)} = \frac{P_1(0)}{2Q(0)} + \sum_{n=1}^{\infty} \frac{P_1(n)\cos(\pi\beta_1 n) + P_2(n)\sin(\pi\beta_2 n)}{Q(n)},$$

which, by Corollary 12, is either zero or transcendental. ■

**5. Schanuel's conjecture and infinite sums.** For more general set of roots of the polynomials $Q_j(x)$, when not all $\alpha_{j,k}$ are in $\mathbb{Q}(i\sqrt{d})$, we give some statements on the transcendence of the sums $S, T, U, V$ provided that the Schanuel conjecture holds (see [12, §3.1], [8, §10.7.G]).

SCHANUEL CONJECTURE (S). *If* $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ *are linearly independent over* $\mathbb{Q}$, *then the transcendence degree over* $\mathbb{Q}$ *of the field* $\mathbb{Q}(\alpha_1, \ldots, \alpha_n, e^{\alpha_1}, \ldots, e^{\alpha_n})$ *is at least* $n$.

We formulate the following propositions, which are consequences of (S):

CONJECTURE ($S_1$). *Let* $P_1, \dots, P_s$, $Q_1, \dots, Q_s \in \overline{\mathbb{Q}}[x]$, $r_1, \dots, r_s \in \mathbb{Z}$, *where for any* $1 \le j \le s$ *the polynomials* $P_j, Q_j$ *satisfy the following conditions*: $\deg P_j \le \deg Q_j - 2$, $Q_j(r_j/2) \ne 0$, $Q_j(n) \ne 0$, $n = 0, 1, \dots,$ *and*

$$\frac{P_j(-x)}{Q_j(-x)} = \frac{P_j(r_j + x)}{Q_j(r_j + x)}.$$

*Then the sum*

$$S = \sum_{n=0}^{\infty} \left( \frac{P_1(n)}{Q_1(n)} + \dots + \frac{P_s(n)}{Q_s(n)} \right)$$

*is either a computable algebraic number or transcendental.*

*Proof.* Under the conditions stated above, we see that for $1 \le j \le s$, $Q_j(x) = \prod_{k=1}^{2m_j} (x - \alpha_{j,k})^{l_{j,k}}$, where $\alpha_{j,k}$ are distinct algebraic numbers distinct from non-negative integers and such that $\alpha_{j,m_j+k} = r_j - \alpha_{j,k}$, $l_{j,m_j+k} = l_{j,k} \in \mathbb{N}$, $k = 1, \dots, m_j$. Therefore, from (16) we have

$$S = C_0 + \sum_{j=1}^{s} \sum_{\substack{k=1 \\ \alpha_{j,k} \in \mathbb{Z}}}^{m_j} \sum_{\substack{l=1 \\ l \text{ even}}}^{l_{j,k}} C_{j,k,l} \pi^l + \sum_{j=1}^{s} \sum_{\substack{k=1 \\ \alpha_{j,k} \notin \mathbb{Z}}}^{m_j} \sum_{l=1}^{l_{j,k}} \frac{(-1)^{l-1} A_{j,k,l}}{(l-1)!} \pi^l p_{l-1}(-\cot \pi \alpha_{j,k}), \tag{33}$$

where $C_0$ and all the coefficients $C_{j,k,l}, A_{j,k,l}$ are algebraic numbers. From (33) it follows that $S$ is equal to $C_0$ or transcendental by (S). Indeed, suppose that $S \ne C_0$ and $S$ is algebraic. Assume that the numbers

$$\frac{1}{\lambda}, \ \frac{\alpha_{j_1,k_1}}{\lambda_1}, \ \dots, \ \frac{\alpha_{j_l,k_l}}{\lambda_l}, \tag{34}$$

where $\lambda_1, \dots, \lambda_l \in \mathbb{N}$, are linearly independent over $\mathbb{Q}$ and all the other roots $\alpha_{j,k}$ are $\mathbb{Z}$-linear combinations of (34). Then the numbers

$$\frac{\pi i}{\lambda}, \ \frac{\pi i \alpha_{j_1,k_1}}{\lambda_1}, \ \dots, \ \frac{\pi i \alpha_{j_l,k_l}}{\lambda_l}$$

are also linearly independent over $\mathbb{Q}$. Put

$$\begin{aligned} K &= \overline{\mathbb{Q}}\left( \frac{\pi i}{\lambda}, \frac{\pi i \alpha_{j_1,k_1}}{\lambda_1}, \dots, \frac{\pi i \alpha_{j_l,k_l}}{\lambda_l}, e^{\pi i \alpha_{j_1,k_1}/\lambda_1}, \dots, e^{\pi i \alpha_{j_l,k_l}/\lambda_l} \right) \\ &= \overline{\mathbb{Q}}\left( \frac{\pi i}{\lambda}, e^{\pi i \alpha_{j_1,k_1}/\lambda_1}, \dots, e^{\pi i \alpha_{j_l,k_l}/\lambda_l} \right). \end{aligned}$$

Then by (S), it follows that $\operatorname{tr\,deg}(K : \overline{\mathbb{Q}}) = l + 1$. From (33) we see that $S - C_0 \in K$. If $S - C_0 \in \overline{\mathbb{Q}} \setminus \{0\}$, then there exists a non-zero polynomial

$A(x) \in \mathbb{Z}[x]$ such that $A(S - C_0) = 0$. Hence $\operatorname{tr}\deg(K : \overline{\mathbb{Q}}) \le l$ and the contradiction obtained proves $(S_1)$. ∎

REMARK 5.1. If all $\alpha_{j,k} \in \mathbb{Q}(i\sqrt{d})$, then $(S_1)$ is true by Theorem 1.

By a similar argument we have

CONJECTURE $(S_2)$. *Let* $P_1, \dots, P_s, Q_1, \dots, Q_s \in \overline{\mathbb{Q}}[x]$, $r_1, \dots, r_s \in \mathbb{Z}$, *where for any* $1 \le j \le s$ *the polynomials* $P_j, Q_j$ *satisfy the following conditions*: $\deg P_j \le \deg Q_j - 1$, $Q_j(r_j/2) \ne 0$, $Q_j(n) \ne 0$, $n = 0, 1, \dots,$ *and*

$$\frac{P_j(-x)}{Q_j(-x)} = (-1)^{r_j} \frac{P_j(r_j + x)}{Q_j(r_j + x)}.$$

*Then the sum*

$$T = \sum_{n=0}^{\infty} \left( \frac{P_1(n)}{Q_1(n)} + \dots + \frac{P_s(n)}{Q_s(n)} \right) (-1)^n$$

*is either a computable algebraic number or transcendental.*

CONJECTURE $(S_3)$. *Let* $f : \mathbb{Z} \to \overline{\mathbb{Q}}$ *be periodic with period* $q \in \mathbb{N}$. *Suppose that* $r \in \mathbb{Z}$, $P(x), Q(x) \in \overline{\mathbb{Q}}[x]$, $(Q'(qr/2))^2 + (Q(qr/2))^2 \ne 0$, $Q(n) \ne 0$, $n = 1, 2, \dots,$

$$\frac{P(-x)}{Q(-x)} = \pm \frac{P(x + qr)}{Q(x + qr)} \tag{35}$$

*and* $f$ *is an even or odd function according to whether we have the "plus" or "minus" sign in* (35). *Suppose further that the series*

$$U = \sum_{n=1}^{\infty} \frac{P(n)}{Q(n)} f(n)$$

*converges. Then* $U$ *is either a computable algebraic number or transcendental.*

CONJECTURE $(S_4)$. *Suppose that* $\beta_1, \dots, \beta_s \in [0, 2)$ *are distinct rational numbers*, $Q(x), P_1(x), \dots, P_s(x) \in \overline{\mathbb{Q}}[x]$, $Q(n) \ne 0$, $n \in \mathbb{Z}$, $h(n) = \sum_{j=1}^{s} P_j(n) e^{i\pi\beta_j n}$, *and for* $1 \le j \le s$, $\deg P_j(x) \le \deg Q(x) - 1$ *if* $0 < \beta_j < 2$ *and* $\deg P_j(x) \le \deg Q(x) - 2$ *if* $\beta_j = 0$. *Then the sum*

$$V = \sum_{n=-\infty}^{\infty} \frac{h(n)}{Q(n)}$$

*is either zero or transcendental.*

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