

Equidistribution of dynamically small subvarieties over the function field of a curve

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1. INTRODUCTION

A typical arithmetic equidistribution theorem says something like the following: if X is an algebraic variety over a global field K that has an arithmetic height function h , then for each place v of K there exists an analytic space X_v^{an} and a measure μ_v supported on it so that the Galois orbits of any suitably generic sequence of algebraic points with heights tending to zero become equidistributed with respect to μ_v . The primary goal of this article is to prove such a theorem when K is the function field of a smooth curve and h and μ_v are intimately connected with the dynamics of a polarized endomorphism of X . Arithmetic equidistribution theorems of this type

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abound in the literature of the last decade, beginning with the pioneering work of Szpiro/Ullmo/Zhang [SUZ97]. For a nice survey, see the recent text by Silverman [Sil07, Section 3.10].

A (*polarized algebraic*) *dynamical system* (X, φ, L) defined over a field K consists of a projective variety X/K , an endomorphism φ of X , and an ample line bundle L on X such that $\varphi^*L \cong L^q$ for some integer $q \geq 2$. The line bundle L is often referred to as a *polarization* of the dynamical system. A preperiodic point $x \in X$ is a closed point such that the forward orbit $\{\varphi^n(x) : n = 1, 2, \dots\}$ is finite. In studying the arithmetic of algebraic dynamical systems, one might ask if, for example, the set of K -rational preperiodic points is Zariski dense in X . We present a criterion in §5 for non-Zariski density as a corollary to the main equidistribution theorem of this paper.

Perhaps the best-known example of an algebraic dynamical system is $(A, [2], L)$, where A is an abelian variety over K , $[2]$ is the multiplication by 2 morphism, and L is a symmetric ample line bundle on A (i.e., $[-1]^*L \cong L$). In this case, torsion points of A are the same as preperiodic points for the morphism $[2]$. Another important example of a dynamical system is $(\mathbb{P}_K^d, \varphi, \mathcal{O}(1))$ for some finite endomorphism φ of \mathbb{P}_K^d . If $(z_0 : \dots : z_d)$ are homogeneous coordinates on projective space, then φ can be described more concretely by giving $d + 1$ homogeneous polynomials f_i of degree q without a common zero over \bar{K} and setting $\varphi = (f_0 : \dots : f_d)$. This is in a sense a “universal dynamical system” because any other dynamical system can be embedded in one of this type. (See the paper of Fakhruddin [Fak03] for an explanation.)

To state a version of our theorem, we set the following notation. Let \mathcal{B} be a proper smooth geometrically connected curve over a field k . Let $K = k(\mathcal{B})$ be the function field of \mathcal{B} . The places of K —i.e., equivalence classes of non-trivial valuations on K that are trivial on k —are in bijective correspondence with the closed points of \mathcal{B} . For a place v , denote by K_v the completion of K with respect to v . The Berkovich analytification of the scheme X_{K_v} will be denoted by X_v^{an} . It is a compact Hausdorff topological space with the same number of connected components as X_{K_v} . For any dynamical system (X, φ, L) , there exists a measure $\mu_{\varphi, v}$ on X_v^{an} that is invariant under φ ; it reflects the distribution of preperiodic points for the morphism φ . Also, each closed point x of X breaks up into a finite set of points $O_v(x)$ in X_v^{an} , and we can define local degrees $\deg_v(y)$ for each $y \in O_v(x)$ such that $\sum_{y \in O_v(x)} \deg_v(y) = \deg(x)$. To the set $O_v(x)$, we can associate a probability measure on X_v^{an} by

$$\frac{1}{\deg(x)} \sum_{y \in O_v(x)} \deg_v(y) \delta_y.$$

There also exists a dynamical height function h_φ on closed points of X defined via intersection theory, so that $h_\varphi(x)$ measures the “arithmetic and dynamical complexity” of the point x . For example, $h_\varphi(x) = 0$ if x is a preperiodic point for φ .

THEOREM 1.1. *Let (X, φ, L) be an algebraic dynamical system over the function field K . Suppose $\{x_n\}$ is a sequence of closed points of X such that*

- *no infinite subsequence of $\{x_n\}$ is contained in a proper closed subset of X , and*
- *$h_\varphi(x_n) \rightarrow 0$ as $n \rightarrow \infty$.*

Then for any place v of K , and for any continuous function $f : X_v^{\text{an}} \rightarrow \mathbb{R}$, we have

$$\lim_{n \rightarrow \infty} \frac{1}{\deg(x_n)} \sum_{y \in O_v(x_n)} \deg_v(y) f(y) = \int_{X_v^{\text{an}}} f d\mu_{\varphi, v}.$$

That is, the sequence of measures $\left\{ \frac{1}{\deg(x_n)} \sum_{y \in O_v(x_n)} \deg_v(y) \delta_y \right\}_n$ converges weakly to $\mu_{\varphi, v}$.

Theorem 1.1 is a simplified form of Theorem 4.1, which is the main result of this paper. In the latter result, we prove that generic nets of small subvarieties are equidistributed. See Section 4 for the statement. The proof follows the technique espoused in the work of Yuan [Yua08] using the “classical” version of Siu’s Theorem from algebraic geometry.

Several equidistribution theorems in the function field case already exist in the literature. The work of Baker/Hsia [BH05] and that of Baker/Rumely [BR06] deal with polynomial maps and rational maps on the projective line, respectively. These papers approach equidistribution using nonarchimedean capacity and potential theory. Favre/Rivera-Letelier give a quantitative form of equidistribution for rational maps on the projective line, which should apply to the function field case despite the fact that they state their results for number fields [FRL06, FRL07]. Petsche has proved a quantitative equidistribution result for points of small Néron–Tate height on elliptic curves using Fourier analysis [Pet07]. Gubler has approached equidistribution using arithmetic intersection techniques (following Szpiro/Ullmo/Zhang). He gives an equidistribution result for abelian varieties that admit a place of totally degenerate reduction [Gub07]. He has recently and independently proved equidistribution results essentially equivalent to our Theorems 1.1 and 4.1 [Gub08]. The work of Petsche and Gubler holds also over the function field of a smooth projective variety equipped with an ample divisor class.

In Section 2 we state and prove all of the tools of arithmetic intersection theory over function fields necessary for the task at hand. This will only require the use of intersection theory of first Chern classes in classical algebraic geometry and a small amount of formal and analytic geometry. Our approach

is novel in that it is developed from global rather than local intersection theory. The associated measures and height functions will be introduced in that section as well. Section 3 is devoted to recalling the construction of invariant metrics for dynamical systems, the dynamical height h_φ , and the associated invariant measure $\mu_{\varphi,v}$. We state and prove the main equidistribution theorem in Section 4. Finally, in Section 5 we discuss applications of the equidistribution theorem to the distribution of preperiodic points and to the Zariski density of dynamically small points. Section 6 contains several auxiliary results whose proofs could not be located in the literature.

2. ARITHMETIC INTERSECTION THEORY OVER FUNCTION FIELDS

In this section we collect some definitions and facts from intersection theory needed for the calculation of heights and for the proof of the main theorem. The basic principle is that heights can be computed via (limits of) classical intersection numbers on models. Much has been written in the literature on the subject of local intersection theory. Bloch/Gillet/Soulé have studied nonarchimedean local intersection theory by formally defining arithmetic Chow groups using cycles on special fibers of models [BGS95]. Gubler carries out a very careful study of local intersection theory on special fibers of formal schemes in a more geometric fashion, and then he patches it together into a global theory of heights using his theory of M -fields [Gub97, Gub98, Gub03]. One can avoid most of the technical difficulties that arise in these approaches by working almost exclusively with classical intersection theory on models of an algebraic variety; this requires a lemma of Yuan to lift data from a local model to a global model (Lemma 2.7). The upshot is that this treatment uses only ideas from algebraic geometry and a small input from formal geometry, and it works for any field k of constants.

Most of these ideas appear in the literature on adelic metrics and height theory. We have endeavored to give complete proofs when the literature does not provide one or when the known proof for the number field case requires significant modification.

2.1. Notation and terminology

2.1.1. When we speak of a *variety*, we will always mean an integral scheme, separated and of finite type over a field. We do not require a variety to be geometrically integral. Throughout, we fix a field k of constants, and a proper smooth geometrically connected k -curve \mathcal{B} . At no point do we require that the constant field k be algebraically closed. Denote by K the field of rational functions on \mathcal{B} .

2.1.2. The field K admits a set of nontrivial normalized valuations that correspond bijectively to the closed points of \mathcal{B} . The correspondence associates to a point $v \in \mathcal{B}$ the valuation ord_v on the local ring $\mathcal{O}_{\mathcal{B},v}$ at v , which is a discrete valuation ring by smoothness. The normalization of v is such that a uniformizer in $\mathcal{O}_{\mathcal{B},v}$ has valuation 1. Extend ord_v to K by additivity. Such a valuation will be called a *place* of K . We will often identify closed points of \mathcal{B} with places of K without comment unless further clarification is necessary.

The places of K satisfy a product formula. For $a \in K$, set $|a|_v = \exp\{-\text{ord}_v(a)\}$. Then for any $a \in K^\times$, we have

$$\prod_{v \in \mathcal{B}} |a|_v^{[k(v):k]} = 1,$$

where $k(v)$ is the residue degree of the point v . This formula appears more frequently in its logarithmic form, in which it asserts that a rational function has the same number of zeros as poles when counted with the appropriate weights:

$$\sum_{v \in \mathcal{B}} [k(v) : k] \text{ord}_v(a) = 0.$$

2.1.3. Now let X be a projective variety over K . Given an open subvariety $\mathcal{U} \subset \mathcal{B}$, a \mathcal{U} -*model* of X consists of the data of a k -variety \mathcal{X} , a projective flat k -morphism $\mathcal{X} \rightarrow \mathcal{U}$, and a preferred K -isomorphism $\iota : X \xrightarrow{\sim} \mathcal{X}_K$. In most cases the morphism $\mathcal{X} \rightarrow \mathcal{U}$ and the isomorphism ι will be implicit, and we will use them to identify X with the generic fiber of \mathcal{X} . If L is a line bundle on X , a \mathcal{U} -*model of the pair* (X, L) is a pair $(\mathcal{X}, \mathcal{L})$ such that \mathcal{X} is a \mathcal{U} -model of X and \mathcal{L} is a line bundle on \mathcal{X} equipped with a preferred isomorphism $\iota^* \mathcal{L}|_{\mathcal{X}_K} \xrightarrow{\sim} L$ ⁽¹⁾. Again, this isomorphism will often be implicit. We may also say that \mathcal{L} is a *model of L* . To avoid trivialities, if we speak of a \mathcal{U} -model $(\mathcal{X}, \mathcal{L})$ of a pair (X, L^e) without extra qualifier, we will implicitly assume that $e \geq 1$.

For a line bundle \mathcal{L} on a variety \mathcal{X}/k with function field $k(\mathcal{X})$, a *rational section* of \mathcal{L} is a global section of the sheaf $\mathcal{L} \otimes k(\mathcal{X})$. Equivalently, a rational section is a choice of a nonempty open set $U \subset \mathcal{X}$ and a section $s \in \mathcal{L}(U)$.

Requiring X to be projective guarantees the existence of \mathcal{B} -models. For example, let L be a very ample line bundle on X . Let

$$X \hookrightarrow \mathbb{P}_K^n \hookrightarrow \mathbb{P}_{\mathcal{B}}^n = \mathbb{P}^n \times \mathcal{B}$$

be an embedding induced by L followed by identifying \mathbb{P}_K^n with the generic fiber of $\mathbb{P}_{\mathcal{B}}^n$, and define \mathcal{X} to be the Zariski closure of X in $\mathbb{P}_{\mathcal{B}}^n$ with the

(1) As is customary, we will use the terms “line bundle” and “invertible sheaf” interchangeably, but we will always work with them as sheaves.

reduced subscheme structure. Let $\mathcal{L} = \mathcal{O}_{\mathcal{X}}(1)$. Then $(\mathcal{X}, \mathcal{L})$ is a \mathcal{B} -model of (X, L) . More generally, if M is any line bundle on X , then we can write $M = M_1 \otimes M_2^\vee$ for some choice of very ample line bundles M_1 and M_2 . The procedure just described gives a means of constructing \mathcal{B} -models $(\mathcal{X}_i, \mathcal{M}_i)$ of (X, M_i) . By the Simultaneous Model Lemma (Lemma 2.2 below), there exists a single \mathcal{B} -model \mathcal{X} of X as well as line bundles \mathcal{M}'_i on \mathcal{X} such that $(\mathcal{X}, \mathcal{M}'_i)$ is a \mathcal{B} -model of (X, M_i) . Evidently, $(\mathcal{X}, \mathcal{M}'_1 \otimes (\mathcal{M}'_2)^\vee)$ is a \mathcal{B} -model of (X, M) .

2.1.4. For each place v of K , we let K_v be the completion of K with respect to the valuation v . Let K_v° be the valuation ring of K_v .

We now briefly recall the relevant definitions from formal and analytic geometry. For full references on these topics, see [Ber90, §2, 3.4], [Ber94, §1] and [BL93]. For each place v of K , let X_v^{an} be the Berkovich analytic space associated to the K_v -scheme X_{K_v} . It is a compact Hausdorff topological space equipped with the structure of a locally ringed space. The space X_v^{an} is covered by compact subsets of the form $\mathcal{M}(\mathcal{A})$, where \mathcal{A} is a strictly affinoid K_v -algebra and $\mathcal{M}(\mathcal{A})$ is its Berkovich spectrum. As a set, $\mathcal{M}(\mathcal{A})$ consists of all bounded multiplicative seminorms on \mathcal{A} . The definition of the analytic space X_v^{an} includes a natural surjective morphism of locally ringed spaces $\psi_v : X_v^{\text{an}} \rightarrow X_{K_v}$. If L is a line bundle on X , there is a functorially associated line bundle L_v on X_v^{an} defined by $\psi_v^*(L \otimes_K K_v)$.

A continuous metric on L_v , denoted $\|\cdot\|$, is a choice of a K_v -norm on each fiber of L_v that varies continuously on X_v^{an} . More precisely, suppose $\{(U_i, s_i)\}$ is a trivialization of L_v where $\{U_i\}$ is an open cover of X_v^{an} and s_i is a generator of the $\mathcal{O}_{X_v^{\text{an}}}(U_i)$ -module $L_v(U_i)$. Then the metric $\|\cdot\|$ is defined by a collection of continuous functions $\varrho_i : U_i \rightarrow \mathbb{R}_{>0}$ satisfying an appropriate cocycle condition by the formula $\|s(x)\| = |\sigma(x)|\varrho(x)$, where s is any section of L_v over U_i and σ is the regular function on U_i such that $s = \sigma s_i$. Here $|\sigma(x)|$ denotes the value of the seminorm corresponding to the point x at the function σ ⁽²⁾. A *formal metric* $\|\cdot\|$ on the line bundle L_v is one for which there exists a trivialization $\{(U_i, s_i)\}$ such that the metric is defined by $\varrho_i \equiv 1$. Equivalently, for this trivialization we have $\|s(x)\| = |\sigma(x)|$ with s and σ as above.

To an admissible formal K_v° -scheme \mathfrak{X}_v , one can associate in a functorial way its *generic fiber* $\mathfrak{X}_{v,\eta}$, which is an analytic space in the sense of Berkovich. For us, the most important example will be when \mathfrak{X}_v is the formal completion of a proper flat K_v° -scheme with generic fiber X/K_v . In this

⁽²⁾ Several authors (e.g., [BG06]) speak of *bounded* continuous metrics on $L \otimes \mathbb{C}_v$ over $X_{\mathbb{C}_v}$, where \mathbb{C}_v is the completion of an algebraic closure of K_v . Bounded and continuous metrics in their context are equivalent to our notion of continuous metric by compactness of the Berkovich analytic space X_v^{an} .

setting, properness implies there is a canonical isomorphism $\mathfrak{X}_{v,\eta} \xrightarrow{\sim} X_v^{\text{an}}$ [Con99, Thm. A.3.1]. A formal line bundle \mathfrak{L}_v on \mathfrak{X}_v determines a formal metric on $L_v = \mathfrak{L}_v \otimes K_v$ as follows. Let $\{(\mathfrak{U}_i, s_i)\}$ be a formal trivialization of \mathfrak{L}_v . Then the generic fiber functor gives a trivialization of L_v , namely $\{(\mathfrak{U}_{i,\eta}, s_i)\}$. We set $\varrho_i \equiv 1$. This definition works (i.e., the functions ϱ_i transform correctly) because the transition functions for the cover $\{(\mathfrak{U}_{i,\eta}, s_i)\}$ all have supremum norm 1 [Gub98, §7].

For an open subscheme $\mathcal{U} \subset \mathcal{B}$, a \mathcal{U} -model $(\mathcal{X}, \mathcal{L})$ of the pair (X, L^e) determines a family of continuous metrics, one on L_v for each place v of \mathcal{U} . Indeed, let $\widehat{\mathcal{X}}_v$ be the formal completion of the scheme $\mathcal{X}_{K_v^\circ}$ along its closed fiber, and let $\widehat{\mathcal{L}}_v$ be the formal completion of \mathcal{L} . As $\mathcal{X}_{K_v^\circ}$ is K_v° -flat, we know that $\widehat{\mathcal{X}}_v$ is flat over K_v° , hence admissible. We denote by $\|\cdot\|_{\mathcal{L},v}$ the formal metric on $\widehat{\mathcal{L}}_v \otimes K_v \cong L_v^e$. A metric $\|\cdot\|_{\mathcal{L},v}^{1/e}$ on L_v can then be given by defining $\|\ell\|_{\mathcal{L},v}^{1/e} = \|\ell^{\otimes e}\|_{\mathcal{L},v}^{1/e}$ for any local section ℓ of L_v .

There is an important subtlety here that is worth mentioning. If $(\mathcal{X}, \mathcal{L})$ is a \mathcal{B} -model of (X, O_X) and v is a place of \mathcal{B} , then we get a metric $\|\cdot\| = \|\cdot\|_{\mathcal{L},v}$ on $O_{X_v^{\text{an}}}$. However, $O_X \xrightarrow{\sim} O_X^n$ for any positive integer n via the canonical isomorphism given by $1 \mapsto 1^{\otimes n}$, and so we just as easily view $(\mathcal{X}, \mathcal{L})$ as \mathcal{B} -model of (X, O_X^n) . The metric induced on $O_{X_v^{\text{an}}}$ by this \mathcal{B} -model is given by $\|\cdot\|^{1/n}$.

2.1.5. An *adelic metrized line bundle* \bar{L} on X consists of the data of a line bundle L on X and, for each place $v \in \mathcal{B}$, a continuous metric $\|\cdot\|_v$ on the analytic line bundle L_v subject to the following adelic coherence condition: there exists an open subscheme $\mathcal{U} \subset \mathcal{B}$, and a \mathcal{U} -model $(\mathcal{X}, \mathcal{L})$ of the pair (X, L^e) for some positive integer e , such that at all places v in \mathcal{U} we have equality of the metrics $\|\cdot\|_v = \|\cdot\|_{\mathcal{L},v}^{1/e}$. The adelic metrized line bundle \bar{L} will be called *semipositive* if there exists an open subscheme $\mathcal{U} \subset \mathcal{B}$, a sequence of positive integers e_n and a sequence of \mathcal{B} -models $(\mathcal{X}_n, \mathcal{L}_n)$ of the pairs (X, L^{e_n}) such that

- \mathcal{L}_n is *relatively semipositive* for all n : it has nonnegative degree on any curve in a closed fiber of \mathcal{X} ⁽³⁾;
- for each place v of \mathcal{U} , we have equality of the metrics $\|\cdot\|_v = \|\cdot\|_{\mathcal{L}_n,v}^{1/e_n}$ for all n ; and
- for each place $v \notin \mathcal{U}$, the sequence of metrics $\|\cdot\|_{\mathcal{L}_n,v}^{1/e_n}$ converges uniformly to $\|\cdot\|_v$ on X_v^{an} .

⁽³⁾ Modern algebraic geometers would probably call this line bundle *relatively nef*, but we preserve the term “relatively semipositive” for historical reasons.

The distance between two metrics $\|\cdot\|_{1,v}$ and $\|\cdot\|_{2,v}$ on L_v is given by

$$\text{dist}_v(\|\cdot\|_{1,v}, \|\cdot\|_{2,v}) = \max_{x \in X_v^{\text{an}}} \left| \log \frac{\|s(x)\|_{1,v}}{\|s(x)\|_{2,v}} \right|,$$

where s is any local section of L_v that does not vanish at x . The quotient inside the logarithm is independent of the choice of s and defines a non-vanishing continuous function on the compact space X_v^{an} . This implies the existence of the maximum. A sequence $(\|\cdot\|_{n,v})_n$ of metrics on L_v is said to *converge uniformly* to the metric $\|\cdot\|_v$ if $\text{dist}_v(\|\cdot\|_{n,v}, \|\cdot\|_v) \rightarrow 0$ as $n \rightarrow \infty$. By abuse of notation, in the definition of semipositive metrized line bundle we may say that the \mathcal{B} -models \mathcal{L}_n converge uniformly to \bar{L} .

An adelic metrized line bundle is called *integrable* if it is of the form $\bar{L}_1 \otimes \bar{L}_2^\vee$ for two semipositive metrized line bundles \bar{L}_1 and \bar{L}_2 . We denote by $\text{Pic}(X)$ the group of integrable adelic metrized line bundles on X (under tensor product). Semipositive metrized line bundles form a semigroup inside $\text{Pic}(X)$. (This follows from Lemma 2.2 and the fact that the tensor product of nef line bundles is nef.)

2.2. Arithmetic intersection numbers. Given $d+1$ integrable metrized line bundles $\bar{L}_0, \dots, \bar{L}_d$, we wish to define the arithmetic intersection number $\widehat{c}_1(\bar{L}_0) \cdots \widehat{c}_1(\bar{L}_d)$. In the present situation, this is accomplished by approximating the metrics on these line bundles by using \mathcal{B} -models, performing an intersection calculation on the \mathcal{B} -models, and then passing to a limit. This procedure appears in the work of Zhang for the number field setting [Zha95b], and we devote this section to proving it works very generally for function fields of transcendence degree one.

Those well acquainted with intersection products in Arakelov theory will find little surprising in Theorem 2.1 below and probably nothing new in its proof. However, the literature on the subject appears only to contain these facts and their proofs in the case where K is a number field or when K is a function field over an algebraically closed field k [Gub07, §3]. Here we have removed the hypothesis that k is algebraically closed, with the only consequence being that a residue degree appears in some of the formulas. The benefit is that the proofs are global and algebraic in nature rather than local and formal.

For a projective variety Y over a field k , we will almost always identify a zero cycle $\sum n_P [P]$ with its degree $\sum n_P [k(P) : k]$. For a line bundle L on Y , we write $c_1(L)$ to denote the first Chern class valued in $A^1(Y)$, the group of codimension-1 cycles on Y . When Y has dimension d , we will write $\text{deg}_{L_1, \dots, L_d}(Y)$ or $c_1(L_1) \cdots c_1(L_d)$ instead of the more

correct form $\deg(c_1(L_1) \cdots c_1(L_d) \cdot [Y])$. Similarly, we may write $\deg_L(Y)$ for $c_1(L)^d$.

THEOREM 2.1. *Let X be a projective variety of dimension d over the function field K . There exists a pairing $\widehat{c}_1(\bar{L}_0) \cdots \widehat{c}_1(\bar{L}_d)$ on $\overline{\text{Pic}}(X)^{d+1}$ with the following properties:*

- (i) *Let \mathcal{X} be a \mathcal{B} -model of X and let $\mathcal{L}_0, \dots, \mathcal{L}_d$ be models on \mathcal{X} of the line bundles $L_0^{e_0}, \dots, L_d^{e_d}$ for some positive integers e_0, \dots, e_d . If $\bar{L}_0, \dots, \bar{L}_d$ are the associated adelic metrized line bundles with underlying algebraic bundles L_0, \dots, L_d , then*

$$\widehat{c}_1(\bar{L}_0) \cdots \widehat{c}_1(\bar{L}_d) = \frac{c_1(\mathcal{L}_0) \cdots c_1(\mathcal{L}_d)}{e_0 \cdots e_d}.$$

- (ii) *Let $\bar{L}_1, \dots, \bar{L}_d$ be semipositive metrized line bundles and take \bar{L}_0 and \bar{L}'_0 to be two integrable metrized line bundles with the same underlying algebraic bundle L_0 . The metrics of \bar{L}_0 and \bar{L}'_0 agree at almost all places, and if $\|\cdot\|_{0,v}$ and $\|\cdot\|'_{0,v}$ are the corresponding metrics at the place v , then*

$$\begin{aligned} &|\widehat{c}_1(\bar{L}_0 \otimes (\bar{L}'_0)^\vee) \widehat{c}_1(\bar{L}_1) \cdots \widehat{c}_1(\bar{L}_d)| \\ &\leq \deg_{L_1, \dots, L_d}(X) \sum_{v \in \mathcal{B}} [k(v) : k] \text{dist}_v(\|\cdot\|_{0,v}, \|\cdot\|'_{0,v}). \end{aligned}$$

- (iii) $\widehat{c}_1(\bar{L}_0) \cdots \widehat{c}_1(\bar{L}_d)$ is symmetric and multilinear in $\bar{L}_0, \dots, \bar{L}_d$.
- (iv) Let Y be a projective variety over K , and suppose $\varphi : Y \rightarrow X$ is a generically finite surjective morphism. Then

$$\widehat{c}_1(\varphi^* \bar{L}_0) \cdots \widehat{c}_1(\varphi^* \bar{L}_d) = \deg(\varphi) \widehat{c}_1(\bar{L}_0) \cdots \widehat{c}_1(\bar{L}_d).$$

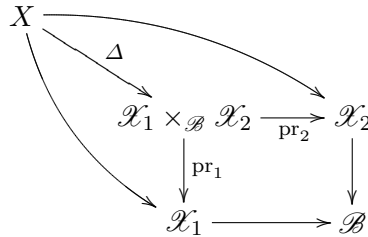
Moreover, the pairing on $\overline{\text{Pic}}(X)^{d+1}$ is uniquely defined by (i) and (ii).

The proof of this theorem will occupy the remainder of Section 2.2.

2.2.1. Preliminary lemmas. In order to define the pairing and show it is well-defined, we need a number of preliminary facts. The following result allows one to take metrics induced from line bundles on several different models of X and consolidate the data on a single \mathcal{B} -model of X .

LEMMA 2.2 (Simultaneous Model Lemma). *Let $(\mathcal{X}_1, \mathcal{L}_1), \dots, (\mathcal{X}_n, \mathcal{L}_n)$ be \mathcal{B} -models of the pairs $(X, L_1), \dots, (X, L_n)$, respectively. Then there exists a single \mathcal{B} -model \mathcal{X} along with \mathcal{B} -morphisms $\text{pr}_i : \mathcal{X} \rightarrow \mathcal{X}_i$ that restrict to isomorphisms on the generic fiber. For each i , the line bundle $\mathcal{L}'_i = \text{pr}_i^* \mathcal{L}_i$ is a model of L_i such that for each place v of K , the corresponding metrics on $L_{i,v}$ satisfy $\|\cdot\|_{\mathcal{L}_i, v} = \|\cdot\|_{\mathcal{L}'_i, v}$. If \mathcal{L}_i is nef on \mathcal{X}_i , then \mathcal{L}'_i is nef on \mathcal{X} .*

Proof. Suppose that $n = 2$; the general case is only notationally more difficult. Consider the following commutative diagram:

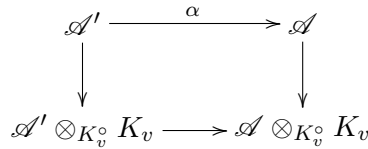


The maps $\mathcal{X}_i \rightarrow \mathcal{B}$ are the structure morphisms. The map Δ is the diagonal embedding of X in the generic fiber of $\mathcal{X}_1 \times_{\mathcal{B}} \mathcal{X}_2$, which is just $X \times_K X$, and the square is the fiber product. Let \mathcal{X} be the Zariski closure of $\Delta(X)$ in $\mathcal{X}_1 \times_{\mathcal{B}} \mathcal{X}_2$ with the reduced subscheme structure; it is a \mathcal{B} -model of X via the diagonal map as preferred isomorphism on the generic fiber. When restricted to $\Delta(X)$, the projections pr_i are isomorphisms, and so they induce isomorphisms between the generic fibers of \mathcal{X} and \mathcal{X}_i . Let $\mathcal{L}'_i = \text{pr}_i^* \mathcal{L}_i|_{\mathcal{X}}$.

The metrics on a line bundle \mathcal{L}' are unchanged by pullback through a \mathcal{B} -morphism $\mathcal{X} \rightarrow \mathcal{X}'$ that restricts to an isomorphism on the generic fiber. Indeed, completing at the closed fiber over v gives a morphism of admissible formal K_v° -schemes $f : \mathfrak{X} \rightarrow \mathfrak{X}'$. Over open sets

$$\text{Spf } \mathcal{A}' \subset \mathfrak{X}' \quad \text{and} \quad \text{Spf } \mathcal{A} \subset f^{-1}(\text{Spf } \mathcal{A}') \subset \mathfrak{X}$$

where the line bundles \mathcal{L}' and $f^* \mathcal{L}'$ are trivial, flatness of these algebras over K_v° implies that we have a commutative diagram of inclusions:



Over these open sets, the formal metrics on \mathcal{L}' and on $f^* \mathcal{L}'$ are given at x by evaluation [Gub98, Lemma 7.4]; i.e., for any local section s of \mathcal{L}' defined near x corresponding to an element $\sigma \in \mathcal{A}'$, we have

$$\|f^*(s)(x)\|_{f^* \mathcal{L}'} = |\alpha(\sigma)(x)| = |\sigma(x)| = \|s(x)\|_{\mathcal{L}'};$$

the middle equality follows because f is an isomorphism on the generic fiber.

The final claim of the lemma is simply the fact that the pullback of a nef line bundle is nef (which follows from the projection formula for classical intersection products). ■

Next we show that intersection numbers vary nicely in fibers over the base curve \mathcal{B} . This is well-known for fibers over the closed points of \mathcal{B} (cf. [Ful98, §10.2, “Conservation of Number”]), but we are also interested in comparing with the generic fiber.

LEMMA 2.3. *Let X be a projective variety over K of dimension d , let L_1, \dots, L_d be line bundles on X , $\pi : \mathcal{X} \rightarrow \mathcal{B}$ a \mathcal{B} -model of X , and $\mathcal{L}_1, \dots, \mathcal{L}_d$ models of L_1, \dots, L_d on \mathcal{X} . Then for any closed point $v \in \mathcal{B}$, we have the equality*

$$[k(v) : k] \deg_{L_1, \dots, L_d}(X) = c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_d) \cdot [\mathcal{X}_v],$$

where \mathcal{X}_v is the (scheme-theoretic) fiber over v .

Proof. As \mathcal{B} is regular, we may speak of its Cartier and Weil divisors interchangeably. Note that $[\mathcal{X}_v]$ is the cycle associated to the Cartier divisor $\pi^*[v]$. Indeed, each is cut out on \mathcal{X} by the image of a local equation for the point v under the map $\mathcal{O}_{\mathcal{B}} \rightarrow \mathcal{O}_{\mathcal{X}}$.

Now we proceed by induction on $d = \dim X$. If $d = 0$, then $X = \text{Spec } F$ corresponds to a finite extension of fields F/K . Also, $\pi : \mathcal{X} \rightarrow \mathcal{B}$ is a proper surjection of curves. Hence

$$\begin{aligned} \deg([\mathcal{X}_v]) &= \deg(\pi^*[v]) = \deg(\pi_*\pi^*[v]) \\ &= [k(\mathcal{X}) : k(\mathcal{B})] \deg([v]) \quad (\text{projection formula}) \\ &= [F : K][k(v) : k], \end{aligned}$$

which is exactly what we want.

Next assume the result holds for all K -varieties of dimension at most $d - 1$, and let $\dim X = d$. Let s_d be a rational section of \mathcal{L}_d . Write $[\text{div}(s_d)] = D_h + D_f$, where D_h is horizontal and D_f is vertical on \mathcal{X} . Then $D_f \cdot [\mathcal{X}_v] = D_f \cdot [\pi^*[v]] = 0$ in the Chow group because we may use linear equivalence on \mathcal{B} to push $\pi^*[v]$ away from the support of D_f . In the next computation, we use the letter \mathcal{Y} to denote an arbitrary horizontal prime divisor on \mathcal{X} , and we set $Y = \mathcal{Y}_K$, the generic fiber of \mathcal{Y} . This gives a bijective correspondence between horizontal prime divisors on \mathcal{X} and prime divisors on X . Moreover, $\mathcal{O}_{X,Y} = \mathcal{O}_{\mathcal{X},\mathcal{Y}}$ for any such prime divisor, and $\text{codim}(Y, X) = \text{codim}(\mathcal{Y}, \mathcal{X})$. Now we compute:

$$\begin{aligned} c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_d) \cdot [\mathcal{X}_v] &= c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_{d-1}) \cdot [\pi^*[v]] \cdot [\text{div}(s_d)] \\ &= c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_{d-1}) \cdot [\pi^*[v]] \cdot D_h \\ &= \sum_{\substack{\mathcal{Y} \subseteq \text{supp}(D_h) \\ \text{codim}(\mathcal{Y}, \mathcal{X})=1}} \text{ord}_{\mathcal{Y}}(s_d) c_1(\mathcal{L}_1|_{\mathcal{Y}}) \cdots c_1(\mathcal{L}_{d-1}|_{\mathcal{Y}}) \cdot [\pi^*[v]|_{\mathcal{Y}}] \\ &= [k(v) : k] \sum_{\substack{Y \subseteq \text{supp}(D_h \cap X) \\ \text{codim}(Y, X)=1}} \text{ord}_Y(s_d|_X) \deg_{L_1|_Y, \dots, L_{d-1}|_Y}(Y) \\ &= [k(v) : k] c_1(L_1) \cdots c_1(L_{d-1}) \cdot [\text{div}(s_d|_X)] \\ &= [k(v) : k] \deg_{L_1, \dots, L_d}(X). \end{aligned}$$

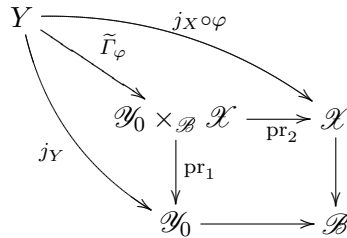
In the third to last equality we applied the induction hypothesis. ■

The next lemma allows us to see what happens to metrics after pullback through a morphism of \mathcal{B} -models.

LEMMA 2.4. *Let $\varphi : Y \rightarrow X$ be a morphism of projective K -varieties. Let L be a line bundle on X and $(\mathcal{X}, \mathcal{L})$ a \mathcal{B} -model of the pair (X, L) .*

- (i) *There exists a \mathcal{B} -model \mathcal{Y} of Y and a \mathcal{B} -morphism $\tilde{\varphi} : \mathcal{Y} \rightarrow \mathcal{X}$ such that $\tilde{\varphi}_K = \varphi$.*
- (ii) *For any \mathcal{B} -morphism $\tilde{\varphi}$ as in (i), the pair $(\mathcal{Y}, \tilde{\varphi}^* \mathcal{L})$ is a \mathcal{B} -model of the pair $(Y, \varphi^* L)$, and for each v we have the equality of metrics $\varphi^* \|\cdot\|_{\mathcal{L},v} = \|\cdot\|_{\tilde{\varphi}^* \mathcal{L},v}$.*

Proof. Let \mathcal{Y}_0 be any \mathcal{B} -model of Y . Define $j_X : X \hookrightarrow \mathcal{X}$ to be the inclusion of the generic fiber (always implicitly precomposed with the preferred isomorphism $\iota : X \xrightarrow{\sim} \mathcal{X}_K$), and similarly for $j_Y : Y \hookrightarrow \mathcal{Y}_0$. Let $\Gamma_\varphi : Y \rightarrow Y \times_K X$ be the graph morphism. Consider the commutative diagram



where $\tilde{\Gamma}_\varphi = (j_Y \times j_X) \circ \Gamma_\varphi$. Define \mathcal{Y} to be the Zariski closure of $\tilde{\Gamma}_\varphi(Y)$ in $\mathcal{Y}_0 \times_{\mathcal{B}} \mathcal{X}$ with the reduced subscheme structure, and give it the obvious structure as a flat proper \mathcal{B} -scheme. Let $\tilde{\varphi} = \text{pr}_2|_{\mathcal{Y}}$. The graph morphism Γ_φ gives the preferred isomorphism between Y and the generic fiber of \mathcal{Y} . With this identification, it is evident that $\tilde{\varphi}_K = \varphi$.

For any $\tilde{\varphi} : \mathcal{Y} \rightarrow \mathcal{X}$ such that $\tilde{\varphi}_K = \varphi$, we find that $(\mathcal{Y}, \tilde{\varphi}^* \mathcal{L})$ is a \mathcal{B} -model of $(Y, \varphi^* L)$ because passage to the generic fiber commutes with pullback of line bundles.

The final statement of the lemma is a consequence of the compatibility of the generic fiber functor for admissible formal schemes and the pullback morphism. Let $\hat{\varphi}_v : \hat{\mathcal{X}}_v \rightarrow \hat{\mathcal{Y}}_v$ be the morphism induced between the formal completions of \mathcal{X} and \mathcal{Y} , respectively, along their closed fibers over v , and let $\hat{\mathcal{L}}_v$ be the formal completion of \mathcal{L} . The metric $\|\cdot\|_{\mathcal{L},v}$ is determined by a formal trivialization $\{(\mathfrak{U}_i, s_i)\}$ of $\hat{\mathcal{L}}_v$ and a collection of functions $\varrho_i : \mathfrak{U}_{i,\eta} \rightarrow \mathbb{R}_{>0}$. (Recall that $\mathfrak{U}_{i,\eta}$ is the generic fiber of \mathfrak{U}_i , and that $\{\mathfrak{U}_{i,\eta}\}$ is a cover of X_v^{an} .) If $\varphi_v^{\text{an}} : Y_v^{\text{an}} \rightarrow X_v^{\text{an}}$ is the induced morphism between analytic spaces, we find that $(\varphi_v^{\text{an}})^{-1}(\mathfrak{U}_{i,\eta}) = \hat{\varphi}_v^{-1}(\mathfrak{U}_i)_\eta$; i.e., pullback commutes with formation of the generic fiber. Therefore both metrics $\varphi^* \|\cdot\|_{\mathcal{L},v}$

and $\|\cdot\|_{\tilde{\varphi}^* \mathcal{L}, v}$ are given by the cover $\{((\varphi_v^{\text{an}})^{-1}(\mathcal{U}_{i,\eta}), (\varphi_v^{\text{an}})^*(s_i))\}$ and the functions $\varrho_i \circ \varphi_v^{\text{an}} : (\varphi_v^{\text{an}})^{-1}(\mathcal{U}_{i,\eta}) \rightarrow \mathbb{R}_{>0}$. ■

Next we prove an estimate that indicates the dependence of intersection numbers for line bundles on the metrics induced by them. In the course of the proof we shall need to relate special values of the metric on a section of the model to the orders of vanishing of the section. We recall now how this works. Let $(\mathcal{X}, \mathcal{L})$ be a \mathcal{B} -model of (X, O_X) with \mathcal{X} normal, and let s be a rational section of \mathcal{L} that restricts to the section 1 on the generic fiber. Write $[\text{div}(s)] = \sum_v \sum_j \text{ord}_{W_{v,j}}(s)[W_{v,j}]$, where $\{W_{v,j}\}_j$ are the distinct irreducible components of the fiber \mathcal{X}_v over the point v . Let us also write $[\mathcal{X}_v] = \sum_j m(v, j)[W_{v,j}]$ for some positive integers $m(v, j)$.

There exists a surjective reduction map $r : X_v^{\text{an}} \rightarrow \mathcal{X}_v$, and for each component $W_{v,j}$, there is a unique point $\xi_{v,j} \in X_v^{\text{an}}$ mapping to the generic point of $W_{v,j}$. See [Ber94, §1] for the construction of the reduction map and an argument that shows its image is closed. We will now give an argument to conclude that its image contains the generic points of \mathcal{X}_v .

The fiber \mathcal{X}_v is unchanged if we replace \mathcal{X} by $\mathcal{X} \times_{\mathcal{B}} \text{Spec } \mathcal{O}_{\mathcal{B},v}$. Let $\eta_{v,j}$ be the generic point of $W_{v,j}$, and let $\text{Spec } A$ be an affine open subscheme containing $\eta_{v,j}$. Set \hat{A} to be the \mathfrak{m}_v -adic completion of A , where \mathfrak{m}_v is the maximal ideal of $\mathcal{O}_{\mathcal{B},v}$, and set $\mathcal{A} = \hat{A} \otimes_{K_v^\circ} K_v$ to be the corresponding strict K_v -affinoid algebra. Define a multiplicative seminorm on A by

$$|a| = \begin{cases} \exp(-\text{ord}_{W_{v,j}}(a)/m(v, j)), & a \neq 0, \\ 0, & a = 0. \end{cases}$$

Extending $|\cdot|$ to \hat{A} by continuity and then to \mathcal{A} by taking fractions gives a bounded multiplicative K_v -seminorm on \mathcal{A} ; it corresponds to a point $\xi_{v,j} \in \mathcal{M}(\mathcal{A}) \subset X_v^{\text{an}}$ such that $r(\xi_{v,j}) = \eta_{v,j}$. Uniqueness follows from the fact that any $x \in r^{-1}(\eta_{v,j})$ induces a valuation on $\text{Frac}(A)$ whose valuation ring dominates $\mathcal{O}_{\mathcal{X}, \eta_{v,j}}$. But by normality, the two valuation rings must coincide, and hence $x = \xi_{v,j}$.

Finally, note that if the rational section s of \mathcal{L} corresponds to a rational function $\sigma \in \text{Frac}(A)$, then the definitions immediately imply that

$$(1) \quad -\log \|1(\xi_{v,j})\|_{\mathcal{L}, v} = -\log |\sigma(\xi_{v,j})| = \frac{\text{ord}_{W_{v,j}}(s)}{m(v, j)}.$$

Compare with [CL06, 2.3].

LEMMA 2.5. *Let \mathcal{X} be a \mathcal{B} -model of X , and suppose $\mathcal{L}_1, \dots, \mathcal{L}_d$ on \mathcal{X} are relatively semipositive models of line bundles L_1, \dots, L_d on X . Let L_0 be another line bundle on X and \mathcal{L}_0 and \mathcal{L}'_0 two models of L_0 . Then the metrics on L_0 induced by \mathcal{L}_0 and \mathcal{L}'_0 differ at only finitely many places v ,*

and we have

$$|c_1(\mathcal{L}_0 \otimes (\mathcal{L}'_0)^\vee)c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_d)| \leq \deg_{L_1, \dots, L_d}(X) \sum_{v \in \mathcal{B}} [k(v) : k] \operatorname{dist}_v(\|\cdot\|_{\mathcal{L}_0, v}, \|\cdot\|_{\mathcal{L}'_0, v}).$$

Proof. First note that since \mathcal{L}_0 and \mathcal{L}'_0 restrict to the same line bundle on the generic fiber, they must be isomorphic over some nonempty open subset $\mathcal{U} \subset \mathcal{B}$. The places corresponding to points of $\mathcal{B} \setminus \mathcal{U}$ are finite in number, and these are the only places at which the metrics of \mathcal{L}_0 and \mathcal{L}'_0 can differ.

We may reduce to the case that \mathcal{X} is normal. Indeed, let $\tilde{\varphi} : \tilde{\mathcal{X}} \rightarrow \mathcal{X}$ be the normalization morphism. Endow $\tilde{\mathcal{X}}$ with the structure of \mathcal{B} -scheme via composition of $\tilde{\varphi}$ with the structure morphism for \mathcal{X} . Define $\varphi = \tilde{\varphi}_K$. Then φ and $\tilde{\varphi}$ have degree 1, so the projection formula for classical intersection products shows that intersection numbers in the above inequality are unaffected by pullback to $\tilde{\mathcal{X}}$ and $(\tilde{\mathcal{X}})_K$. To see that the distance between the metrics is unaffected, we note that by Lemma 2.4, $\|\cdot\|_{\tilde{\varphi}^* \mathcal{L}, v} = \varphi^* \|\cdot\|_{\mathcal{L}, v}$ for any line bundle \mathcal{L} on \mathcal{X} . Since the morphism $\varphi^{\text{an}} : \tilde{\mathcal{X}}^{\text{an}}_K \rightarrow \mathcal{X}^{\text{an}}_K$ is surjective, we find

$$\operatorname{dist}_v(\varphi^* \|\cdot\|_{\mathcal{L}_0, v}, \varphi^* \|\cdot\|_{\mathcal{L}'_0, v}) = \operatorname{dist}_v(\|\cdot\|_{\mathcal{L}_0, v}, \|\cdot\|_{\mathcal{L}'_0, v}).$$

Thus we may replace \mathcal{X} , X , and \mathcal{L}_i by $\tilde{\mathcal{X}}$, $(\tilde{\mathcal{X}})_K$, and $\tilde{\varphi}^* \mathcal{L}_i$, respectively. Note that $\tilde{\varphi}^* \mathcal{L}_i$ is relatively semipositive by the projection formula.

Finally, it suffices to prove that if \mathcal{L}_0 is any model of the trivial bundle on X , then

$$|c_1(\mathcal{L}_0) \cdots c_1(\mathcal{L}_d)| \leq \deg_{L_1, \dots, L_d}(X) \sum_v [k(v) : k] \left\{ \max_{x \in X_v^{\text{an}}} |-\log \|1(x)\|_{\mathcal{L}_0, v}| \right\}.$$

Let s be a rational section of \mathcal{L}_0 that restricts to the section 1 on the generic fiber. Write $[\operatorname{div}(s)] = \sum_v \sum_j \operatorname{ord}_{W_{v,j}}(s) W_{v,j}$ and $[\mathcal{X}_v] = \sum_j m(v, j) W_{v,j}$ as in the remarks preceding this lemma. The function $x \mapsto -\log \|1(x)\|_{\mathcal{L}_0, v}$ must assume its maximum value at one of the $\xi_{v,j}$ [Ber90, 2.4.4]. Using (1), we have

$$\begin{aligned} |c_1(\mathcal{L}_0) \cdots c_1(\mathcal{L}_d)| &\leq \sum_v \sum_j |\operatorname{ord}_{W_{v,j}}(s)| |c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_d) \cdot [W_{v,j}]| \\ &= \sum_v \sum_j |-\log \|1(\xi_{v,j})\|_{\mathcal{L}_0, v}| m(v, j) |c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_d) \cdot [W_{v,j}]| \\ &\leq \sum_v \left\{ \max_{x \in X_v^{\text{an}}} |-\log \|1(x)\|_{\mathcal{L}_0, v}| \right\} |c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_d) \cdot [\mathcal{X}_v]|. \end{aligned}$$

In the second inequality we dropped the absolute values on the intersection with $W_{v,j}$ by using the relative semipositivity of the line bundles $\mathcal{L}_1, \dots, \mathcal{L}_d$. The result now follows immediately from Lemma 2.3. ■

2.2.2. Existence of the intersection pairing. Now we define the intersection pairing. We will proceed in several steps. Unless stated otherwise, we let X be a projective variety over K , and $\bar{L}_0, \dots, \bar{L}_d$ will be adelic metrized line bundles on X . The idea is to take property (i) of Theorem 2.1 as the definition when the metrics are induced by models, and then to use Lemma 2.5 to control the intersection number when passing to limits of model metrics.

STEP 1 (Metrics induced by relatively semipositive models). Suppose that $(\mathcal{X}_i, \mathcal{L}_i)$ is a \mathcal{B} -model of $(X, L_i^{e_i})$ that induces the given metric on L_i , and suppose further that each \mathcal{L}_i is relatively semipositive. Using the Simultaneous Model Lemma (Lemma 2.2), we obtain a single \mathcal{B} -model \mathcal{X} and models of the $L_i^{e_i}$ that induce the given metric on L_i . We abuse notation and denote these models on \mathcal{X} by \mathcal{L}_i . Then we define

$$\widehat{c}_1(\bar{L}_0) \cdots \widehat{c}_1(\bar{L}_d) = \frac{c_1(\mathcal{L}_0) \cdots c_1(\mathcal{L}_d)}{e_0 \cdots e_d}.$$

To see that this is well-defined, it suffices to take \mathcal{X}' to be another \mathcal{B} -model of X and \mathcal{L}'_i to be models of $L_i^{e'_i}$ that also induce the given metrics. (Note that the exponents e'_i need not equal the e_i .) In order to prove that this data gives the same intersection number, it is enough to prove that

$$c_1((\mathcal{L}'_0)^{e_0}) \cdots c_1((\mathcal{L}'_d)^{e_d}) = c_1(\mathcal{L}_0^{e_0}) \cdots c_1(\mathcal{L}_d^{e_d}).$$

By another application of the Simultaneous Model Lemma, we can find a single \mathcal{B} -model \mathcal{Y} of X , birational morphisms $\text{pr} : \mathcal{Y} \rightarrow \mathcal{X}$ and $\text{pr}' : \mathcal{Y} \rightarrow \mathcal{X}'$ that are isomorphisms on generic fibers over \mathcal{B} , and models $\mathcal{M}_i = \text{pr}^* \mathcal{L}_i^{e_i}$ and $\mathcal{M}'_i = \text{pr}'^*(\mathcal{L}'_i)^{e_i}$ of $L_i^{e_i e'_i}$ that induce the given metrics on L_i . By the projection formula, we are reduced to showing

$$(2) \quad c_1(\mathcal{M}_0) \cdots c_1(\mathcal{M}_d) = c_1(\mathcal{M}'_0) \cdots c_1(\mathcal{M}'_d).$$

Observe that \mathcal{M}_i and \mathcal{M}'_i may be different line bundles on \mathcal{Y} , but they are models of the same line bundle on X and they induce the same metrics on it.

Proving (2) uses a telescoping sum argument. To set it up, note that since the metrics on L_i induced by \mathcal{M}_i and \mathcal{M}'_i agree, Lemma 2.5 shows that

$$c_1(\mathcal{M}_0) \cdots c_1(\mathcal{M}_{i-1}) c_1(\mathcal{M}_i \otimes (\mathcal{M}'_i)^\vee) c_1(\mathcal{M}'_{i+1}) \cdots c_1(\mathcal{M}'_d) = 0.$$

Therefore

$$\begin{aligned} & |c_1(\mathcal{M}_0) \cdots c_1(\mathcal{M}_d) - c_1(\mathcal{M}'_0) \cdots c_1(\mathcal{M}'_d)| \\ &= \left| \sum_{i=0}^d c_1(\mathcal{M}_0) \cdots c_1(\mathcal{M}_{i-1}) c_1(\mathcal{M}_i \otimes (\mathcal{M}'_i)^\vee) c_1(\mathcal{M}'_{i+1}) \cdots c_1(\mathcal{M}'_d) \right| \\ &\leq \sum_{i=0}^d |c_1(\mathcal{M}_0) \cdots c_1(\mathcal{M}_{i-1}) c_1(\mathcal{M}_i \otimes (\mathcal{M}'_i)^\vee) c_1(\mathcal{M}'_{i+1}) \cdots c_1(\mathcal{M}'_d)|. \end{aligned}$$

Every term of the final sum is zero by the previous displayed formula. This completes the first step.

STEP 2 (Arbitrary semipositive metrized line bundles). Let $\bar{L}_0, \dots, \bar{L}_d$ be semipositive metrized line bundles on X with underlying algebraic bundles L_0, \dots, L_d , respectively. For each place v , denote the metric on $L_{i,v}$ by $\|\cdot\|_{i,v}$. By definition, there exists a sequence of \mathcal{B} -models $(\mathcal{X}_{i,m}, \mathcal{L}_{i,m})$ of the pairs $(X, L_i^{e_i, m})$ such that

- $\mathcal{L}_{i,m}$ is relatively semipositive on $\mathcal{X}_{i,m}$ for every i, m ;
- there exists an open set $\mathcal{U} \subset \mathcal{B}$ such that for each place $v \in \mathcal{U}$, each index i and all m , we have $\|\cdot\|_{\mathcal{L}_{i,m},v}^{1/e_i,m} = \|\cdot\|_{i,v}$;
- for each place $v \notin \mathcal{U}$ and each i , the sequence of metrics $\|\cdot\|_{\mathcal{L}_{i,m},v}^{1/e_i,m}$ converges uniformly to $\|\cdot\|_{i,v}$.

Define $\bar{L}_{i,m}$ to be the semipositive metrized line bundle having algebraic bundle L_i and the metrics induced by $\mathcal{L}_{i,m}$. Now we define the arithmetic intersection number to be

$$(3) \quad \widehat{c}_1(\bar{L}_0) \cdots \widehat{c}_1(\bar{L}_d) = \lim_{m_0, \dots, m_d \rightarrow \infty} \widehat{c}_1(\bar{L}_{0,m_0}) \cdots \widehat{c}_1(\bar{L}_{d,m_d}).$$

Each of the intersection numbers on the right is well-defined by Step 1, so we have to show that the limit exists and that it is independent of the sequence of models.

To prove that the limit in (3) exists, we take two $(d+1)$ -tuples of positive integers (m_0, \dots, m_d) and (m'_0, \dots, m'_d) and set $\mathcal{L}_i = \mathcal{L}_{i,m_i}^{e_i, m'_i}$, $\mathcal{L}'_i = \mathcal{L}_{i,m'_i}^{e_i, m_i}$, and $e_i = e_{i,m_i} e_{i,m'_i}$. Then on the generic fibers, we have

$$\mathcal{L}_i|_X = \mathcal{L}_{i,m_i}^{e_i, m'_i}|_X = L_i^{e_i}, \quad \mathcal{L}'_i|_X = \mathcal{L}_{i,m'_i}^{e_i, m_i}|_X = L_i^{e_i}.$$

Further, we may assume that for each i , \mathcal{L}_i and \mathcal{L}'_i are line bundles on a single \mathcal{B} -model \mathcal{X} (Simultaneous Model Lemma). Then our definition of the intersection pairing in Step 1 gives

$$(4) \quad \begin{aligned} & \widehat{c}_1(\bar{L}_{0,m_0}) \cdots \widehat{c}_1(\bar{L}_{d,m_d}) - \widehat{c}_1(\bar{L}_{0,m'_0}) \cdots \widehat{c}_1(\bar{L}_{d,m'_d}) \\ &= \frac{c_1(\mathcal{L}_{0,m_0}) \cdots c_1(\mathcal{L}_{d,m_d})}{e_{0,m_0} \cdots e_{d,m_d}} - \frac{c_1(\mathcal{L}_{0,m'_0}) \cdots c_1(\mathcal{L}_{d,m'_d})}{e_{0,m'_0} \cdots e_{d,m'_d}} \\ &= \frac{c_1(\mathcal{L}_0) \cdots c_1(\mathcal{L}_d) - c_1(\mathcal{L}'_0) \cdots c_1(\mathcal{L}'_d)}{e_0 \cdots e_d}. \end{aligned}$$

Now fix $\varepsilon > 0$. For all places $v \in \mathcal{U}$, we have

$$\|\cdot\|_{\mathcal{L}_i,v}^{1/e_i} = \|\cdot\|_{\mathcal{L}'_i,v}^{1/e_i} = \|\cdot\|_{i,v}.$$

For $v \notin \mathcal{U}$ and m_i, m'_i sufficiently large, uniform convergence gives

$$\text{dist}_v(\|\cdot\|_{\mathcal{L}_i, v}^{1/e_i}, \|\cdot\|_{\mathcal{L}'_i, v}^{1/e_i}) < \varepsilon.$$

Just as in Step 1, we use a telescoping argument and apply Lemma 2.5:

$$\begin{aligned} (5) \quad & |c_1(\mathcal{L}_0) \cdots c_1(\mathcal{L}_d) - c_1(\mathcal{L}'_0) \cdots c_1(\mathcal{L}'_d)| \\ &= \left| \sum_{i=0}^d c_1(\mathcal{L}_0) \cdots c_1(\mathcal{L}_{i-1}) c_1(\mathcal{L}_i \otimes (\mathcal{L}'_i)^\vee) c_1(\mathcal{L}'_{i+1}) \cdots c_1(\mathcal{L}'_d) \right| \\ &\leq \sum_{i=0}^d |c_1(\mathcal{L}_0) \cdots c_1(\mathcal{L}_{i-1}) c_1(\mathcal{L}_i \otimes (\mathcal{L}'_i)^\vee) c_1(\mathcal{L}'_{i+1}) \cdots c_1(\mathcal{L}'_d)| \\ &\leq \sum_{i=0}^d \text{deg}_{L_0, \dots, \widehat{L}_i, \dots, L_d}^{e_0, \dots, e_i, \dots, e_d}(X) \sum_v [k(v) : k] \text{dist}_v(\|\cdot\|_{\mathcal{L}_i, v}, \|\cdot\|_{\mathcal{L}'_i, v}) \\ &< \varepsilon \prod_{i=0}^d e_i \left(\sum_i \text{deg}_{L_0, \dots, \widehat{L}_i, \dots, L_d}(X) \right) \left(\sum_{v \notin \mathcal{U}} [k(v) : k] \right). \end{aligned}$$

Combining (4) and (5) shows that the sequence $\{\widehat{c}_1(\bar{L}_{0, m_0}) \cdots \widehat{c}_1(\bar{L}_{d, m_d})\}$ is Cauchy, which is tantamount to showing that the limit in (3) exists.

To see that the limit in (3) is independent of the sequence of models chosen, for each i we let $\{(\mathcal{X}'_{i, m}, \mathcal{L}'_{i, m})\}$ be another sequence of models with associated metrics converging to the given ones on L_i . Then we obtain a third sequence

$$\{(\mathcal{X}''_{i, m}, \mathcal{L}''_{i, m})\} = \{(\mathcal{X}_{i, 1}, \mathcal{L}_{i, 1}), (\mathcal{X}'_{i, 1}, \mathcal{L}'_{i, 1}), (\mathcal{X}_{i, 2}, \mathcal{L}_{i, 2}), (\mathcal{X}'_{i, 2}, \mathcal{L}'_{i, 2}), \dots\}.$$

This sequence also induces metrics converging uniformly to \bar{L}_i , so by our above work we know that the limit in (3) exists for this sequence. Therefore the limits over odd and even terms must agree, which is precisely what we wanted to prove.

Note that the symmetry and multilinearity of the pairing are guaranteed immediately by virtue of the same properties for the classical intersection pairing.

STEP 3 (Integrable metrized line bundles). Now we extend the pairing by linearity since any integrable metrized line bundle \bar{L} can be written $\bar{L} = \bar{L}' \otimes (\bar{L}'')^\vee$ with \bar{L}' and \bar{L}'' semipositive. As there may be multiple ways of decomposing \bar{L} as a difference of semipositive metrized line bundles, a question of uniqueness arises. This, however, is easily settled using Lemma 2.5 and we illustrate it only in the simplest case to avoid unnecessary notation.

Let $\bar{L}_1, \dots, \bar{L}_d$ be semipositive metrized line bundles and let \bar{L} be an integrable metrized line bundle with a decomposition $\bar{L} = \bar{L}' \otimes (\bar{L}'')^\vee$ as

above. Then

$$(6) \quad \widehat{c}_1(\bar{L})\widehat{c}_1(\bar{L}_1)\cdots\widehat{c}_1(\bar{L}_d) \\ = \widehat{c}_1(\bar{L}')\widehat{c}_1(\bar{L}_1)\cdots\widehat{c}_1(\bar{L}_d) - \widehat{c}_1(\bar{L}'')\widehat{c}_1(\bar{L}_1)\cdots\widehat{c}_1(\bar{L}_d).$$

If \bar{L} has a second decomposition $\bar{M}' \otimes (\bar{M}'')^\vee$ as a difference of semipositive metrized line bundles, it follows that $\bar{L}' \otimes \bar{M}'' = \bar{L}'' \otimes \bar{M}'$. Each side of this last equality is semipositive with the same underlying algebraic bundle and the same metrics, and so it follows from Lemma 2.5 that

$$\widehat{c}_1(\bar{L}' \otimes \bar{M}'')\widehat{c}_1(\bar{L}_1)\cdots\widehat{c}_1(\bar{L}_d) = \widehat{c}_1(\bar{L}'' \otimes \bar{M}')\widehat{c}_1(\bar{L}_1)\cdots\widehat{c}_1(\bar{L}_d).$$

Splitting each side into two terms using linearity and rearranging shows that the expression in (6) is indeed well-defined.

2.2.3. Proof of Theorem 2.1. In the previous section we constructed an intersection pairing on $\overline{\text{Pic}}(X)^{d+1}$, and it is straightforward to check that the construction gives properties (i)–(iii) by using the symmetry and multilinearity of the classical intersection product, Lemma 2.5, and a limiting argument. Conversely, since we used (i) as our definition, and since the behavior under limits is governed by (ii), our construction gives the only intersection product having these two properties.

Therefore, it only remains to prove property (iv). Intuitively, the idea is that if \mathcal{Y} and \mathcal{X} are \mathcal{B} -models of Y and X , respectively, then the morphism $\varphi : Y \rightarrow X$ extends to a rational map $\tilde{\varphi} : \mathcal{Y} \dashrightarrow \mathcal{X}$ that is generically finite and has degree equal to the degree of φ .

By linearity, it suffices to prove (iv) when $\bar{L}_0, \dots, \bar{L}_d$ are semipositive metrized line bundles. Let us also assume for the moment that the metrics are induced by relatively semipositive models \mathcal{L}_i of $L_i^{e_i}$. As per usual, we may use the Simultaneous Model Lemma to suppose all of these model line bundles live on a single \mathcal{B} -model \mathcal{X} of X . By Lemma 2.4 there exists a \mathcal{B} -model \mathcal{Y} of Y and a morphism $\tilde{\varphi} : \mathcal{Y} \rightarrow \mathcal{X}$ such that $\tilde{\varphi}_K = \varphi$. In this case, we find that $(\mathcal{Y}, \tilde{\varphi}^* \mathcal{L}_i)$ is a \mathcal{B} -model of the pair $(Y, \varphi^* L_i^{e_i})$. By the projection formula for classical intersection products, we get

$$\widehat{c}_1(\varphi^* \bar{L}_0)\cdots\widehat{c}_1(\varphi^* \bar{L}_d) = \frac{c_1(\tilde{\varphi}^* \mathcal{L}_0)\cdots c_1(\tilde{\varphi}^* \mathcal{L}_d)}{e_0 \cdots e_d} = \text{deg}(\tilde{\varphi}) \frac{c_1(\mathcal{L}_0)\cdots c_1(\mathcal{L}_d)}{e_0 \cdots e_d} \\ = \text{deg}(\tilde{\varphi}) \widehat{c}_1(\bar{L}_0)\cdots\widehat{c}_1(\bar{L}_d).$$

Passing to the generic fibers of \mathcal{Y} and \mathcal{X} over \mathcal{B} does not change their function fields; hence, $\text{deg}(\tilde{\varphi}) = \text{deg}(\varphi)$. This completes the proof when the metrics are all induced by models.

Since $\varphi : Y \rightarrow X$ is surjective, so is the induced morphism of analytic spaces $\varphi^{\text{an}} : Y_v^{\text{an}} \rightarrow X_v^{\text{an}}$ [Ber90, Prop. 3.4.6]. Therefore, if \bar{L}_i is an arbitrary semipositive metrized line bundle and $(\mathcal{X}_i, \mathcal{L}_i)$ is a model of $(X, L_i^{e_i})$, we

have

$$\text{dist}_v(\varphi^* \|\cdot\|_{\mathcal{L}_{i,v}}^{1/e_i}, \varphi^* \|\cdot\|_{\bar{L}_{i,v}}) = \text{dist}_v(\|\cdot\|_{\mathcal{L}_{i,v}}^{1/e_i}, \|\cdot\|_{\bar{L}_{i,v}}),$$

where $\|\cdot\|_{\bar{L}_{i,v}}$ denotes the given metric on L_i at the place v . This shows that if $\{\bar{L}_{i,m}\}_m$ is a sequence of semipositive metrized line bundles induced from models for which the metrics converge uniformly to those of \bar{L}_i , then $\{\varphi^* \bar{L}_{i,m}\}_m$ is a sequence of semipositive metrized line bundles for which the metrics converge uniformly to those of $\varphi^* \bar{L}_i$. This observation coupled with the above work in the model case gives the desired conclusion.

2.3. Model functions. In this section we fix a projective variety X over K and a place v of K . As always, we will identify v with a closed point of the curve \mathcal{B} .

A function $f : X_v^{\text{an}} \rightarrow \mathbb{R}$ will be called a *model function* if it is of the form

$$f(x) = -\log \|1(x)\|^{1/e}$$

for some formal metric $\|\cdot\|$ on $O_{X_v^{\text{an}}}$ and some positive integer e . Every such function is continuous on X_v^{an} . The set of model functions forms a \mathbb{Q} -vector space. (The tensor product and inverse of formal metrics is again a formal metric.) The importance of model functions stems from the following result:

LEMMA 2.6 (Gubler). *The space of model functions is uniformly dense in the space of real-valued continuous functions on X_v^{an} .*

Proof. This is the content of Theorem 7.12 of [Gub98]. One should note that, while the author assumes at the outset of Section 7 that the field K is algebraically closed, he makes no use of this (nor is it needed for his references to the papers of Bosch and Lütkebohmert). ■

As our intersection theory for adelic metrized line bundles is defined via intersection theory on \mathcal{B} -models, we need to be able to relate model functions to global models in order to perform computations with them. A result of Yuan provides this relation.

LEMMA 2.7 (Yuan). *Suppose $\|\cdot\|_v$ is a formal metric on the trivial bundle over X_v^{an} , $e \geq 1$ is an integer, and $f(x) = -\log \|1(x)\|_v^{1/e}$ is a model function. Then there exists a \mathcal{B} -model $(\mathcal{X}, \mathcal{O}(f))$ of (X, O_X^e) such that the metrics on O_X determined by $\mathcal{O}(f)$ are trivial at all places $w \neq v$, and at the place v the metric is $\|\cdot\|_v^{1/e}$. The line bundle $\mathcal{O}(f)$ admits a rational section s such that $s|_X = 1$ and such that the support of the divisor of s lies entirely in the fiber over the point $v \in \mathcal{B}$.*

Proof. When $e = 1$, this is precisely the content of the statement and proof of [Yua08, Lemma 3.5]. The proof executed there in the case where K is a number field applies equally well to the present situation. To extend

to the case $e > 1$, we start with a model function $f(x) = -\log \|1(x)\|_v^{1/e}$. Let $g = ef$. We may apply the case $e = 1$ to the model function g to get a \mathcal{B} -model $(\mathcal{X}, \mathcal{O}(g))$ of (X, O_X) with trivial metrics at all places $w \neq v$ and the metric $\|\cdot\|_v$ at v . Now view $\mathcal{O}(g)$ as a model of the bundle O_X^e via the canonical isomorphism $O_X \xrightarrow{\sim} O_X^e$ sending 1 to $1^{\otimes e}$. This is precisely the \mathcal{B} -model we seek, as the metric at v is now $\|\cdot\|_v^{1/e}$. ■

Momentarily we will prove a fundamental formula for intersecting the line bundle $\mathcal{O}(f)$ with the Zariski closure of a point of x in some model \mathcal{X} . First we need some notation. If x is any closed point of X , note that x breaks up into finitely many closed points over K_v —one for each extension of the valuation v to the residue field $K(x)$ at x (cf. Proposition 6.1 below). Let $O_v(x)$ be the image of this set of points under the canonical inclusion of $|X_{K_v}| \hookrightarrow X_v^{\text{an}}$, where $|X_{K_v}|$ is the set of closed points of X_{K_v} . Another way to view $O_v(x)$ is as the image of $\{x\}_v^{\text{an}}$ in X_v^{an} . Let $\deg_v(y) = [K_v(y) : K_v]$ be the degree of the residue field of the closed point $y \in X_{K_v}$ as an extension over K_v . These degrees satisfy the relation $\deg(x) = \sum_{y \in O_v(x)} \deg_v(y)$. (This is a classical fact, proved in the appendix. It can also be deduced from the discussion at the end of Section 2.4 by integrating a nonzero constant function.)

LEMMA 2.8. *Suppose f is a model function on X_v^{an} induced by a formal metric on $O_{X_v^{\text{an}}}$. Let $(\mathcal{X}, \mathcal{O}(f))$ be a \mathcal{B} -model of (X, O_X) as in Yuan’s lemma. If x is a closed point of X , then*

$$(7) \quad c_1(\mathcal{O}(f)) \cdot [\bar{x}] = [k(v) : k] \sum_{y \in O_v(x)} \deg_v(y) f(y),$$

where \bar{x} is the Zariski closure of x in \mathcal{X} and $k(v)$ is the residue field of the point $v \in \mathcal{B}$.

Proof. We begin the proof by interpreting the contribution of a point $y \in O_v(x)$ to the sum in (7) in terms of lengths of modules over a neighborhood in the formal completion of \mathcal{X} along the closed fiber over the point v . Then we interpret the intersection number $c_1(\mathcal{O}(f)) \cdot \bar{x}$ in terms of the same quantities by working on the formal completion of \bar{x} along its closed fiber over v .

A point $y \in O_v(x)$ corresponds to a finite extension of fields $K_v(y)/K_v$ and a K_v -morphism $\text{Spec } K_v(y) \rightarrow X_{K_v}$. Let R be the valuation ring of $K_v(y)$, i.e., the integral closure of K_v° in $K_v(y)$. By properness, we obtain a lift to a K_v° -morphism $\tilde{y} : \text{Spec } R \rightarrow \mathcal{X}_{K_v^\circ}$. Now take an open affine $W = \text{Spec } A$ around the image of the closed point of $\text{Spec } R$ via \tilde{y} over which $\mathcal{O}(f) \otimes K_v^\circ$ is trivial. For topological reasons, \tilde{y} factors through W . Set $M = (\mathcal{O}(f) \otimes K_v^\circ)|_W$; it is a free A -module of rank 1. Then there exists a

section $u \in M$ that generates M as an A -module and, by Yuan’s lemma, a rational section s such that $s|_{W_{K_v}} = 1$. Write $s = (\gamma_1/\gamma_2)u$ for some $\gamma_1, \gamma_2 \in A$.

Let \mathfrak{m}_v and \mathfrak{m}_R be the maximal ideals of K_v° and R , respectively. Both of the latter are discrete valuation rings; we denote the corresponding normalized valuations by ord_v and ord_R , respectively. (Recall that this means a uniformizer has valuation 1.) Extend these valuations to the fraction fields K_v and $K_v(y)$ in the usual way. The normalized absolute value on K_v is given by $|c| = \exp(-\text{ord}_v(c))$ for any $c \in K_v$. If e_y is the ramification index of $K_v(y)$ over K_v , then the absolute value extends uniquely to $K_v(y)$, and is given by the formula $|c| = \exp(-\text{ord}_R(c)/e_y)$.

Passing to the \mathfrak{m}_v -adic completion of everything in sight gives a morphism $\widehat{y} : \text{Spf}(R) \rightarrow \text{Spf}(\widehat{A})$, and we denote by $\widehat{\gamma}_i$ the image of γ_i in \widehat{A} . Let $\alpha_{\widehat{y}} : \widehat{A} \rightarrow R$ be the induced morphism of complete K_v° -algebras. By definition, we now have

$$\begin{aligned} f(y) &= -\log \|1(y)\| = -\log \left| \frac{\widehat{\gamma}_1}{\widehat{\gamma}_2}(y) \right| = -\log \left| \frac{\alpha_{\widehat{y}}(\widehat{\gamma}_1)}{\alpha_{\widehat{y}}(\widehat{\gamma}_2)} \right| \\ &= e_y^{-1} \text{ord}_R(\alpha_{\widehat{y}}(\widehat{\gamma}_1)/\alpha_{\widehat{y}}(\widehat{\gamma}_2)). \end{aligned}$$

Letting $k_R = R/\mathfrak{m}_R$ and $k_v = K_v^\circ/\mathfrak{m}_v$ be the relevant residue fields, we have $[K_v(y) : K_v] = e_y[k_R : k_v]$. This follows, for example, from the degree formula for extensions of Dedekind rings. Therefore

$$(8) \quad \deg_v(y)f(y) = [k_R : k_v] \text{ord}_R(\alpha_{\widehat{y}}(\widehat{\gamma}_1)/\alpha_{\widehat{y}}(\widehat{\gamma}_2)).$$

Note that k_v is canonically isomorphic to the residue field of $\mathcal{O}_{\mathcal{B},v}$ because the completion of this local ring is precisely K_v° , i.e., $k_v \cong k(v)$.

Now we turn to the intersection number $c_1(\mathcal{O}(f)) \cdot [\bar{x}]$. Let $\pi : \mathcal{X} \rightarrow \mathcal{B}$ be the structure morphism, and let $j : \bar{x} \hookrightarrow \mathcal{X}$ be the closed immersion of \bar{x} with its reduced subscheme structure. There exists a rational section s such that $\text{supp}[\text{div}(s)]$ is contained entirely in $\pi^{-1}(v)$, and $s|_X = 1$. This is the same section s that was used above (prior to restriction and base change). As \bar{x} is proper and quasi-finite over \mathcal{B} , it is finite over \mathcal{B} . To compute the intersection number $c_1(\mathcal{O}(f)) \cdot [\bar{x}]$ we may restrict to an affine neighborhood $U = \text{Spec } C$ of v . Let $\bar{x}_U = \text{Spec } T$, where T is a finite domain over C . We also denote by j the morphism $\text{Spec } T \hookrightarrow \mathcal{X}_U$. Let $N = j^*(\mathcal{O}(f)|_{\mathcal{X}_U})$ be the corresponding T -module, and we will also write s for the image of our rational section in N . By definition, we have

$$(9) \quad c_1(\mathcal{O}(f)) \cdot [\bar{x}] = \sum_{\substack{t \in \text{Spec } T \\ \text{closed points}}} [k(t) : k] \text{ord}_t(s).$$

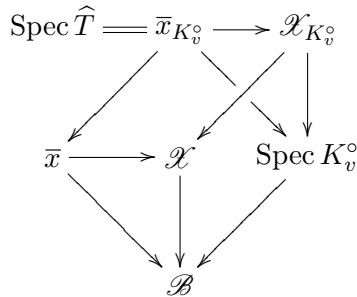
Here $\text{ord}_t(s) = l_{T_{\mathfrak{m}_t}}(T_{\mathfrak{m}_t}/\sigma_1 T_{\mathfrak{m}_t}) - l_{T_{\mathfrak{m}_t}}(T_{\mathfrak{m}_t}/\sigma_2 T_{\mathfrak{m}_t})$, where \mathfrak{m}_t is the maximal ideal of T corresponding to the point t , and s corresponds to σ_1/σ_2 for some $\sigma_1, \sigma_2 \in T_{\mathfrak{m}_t}$ under an isomorphism $N \otimes T_{\mathfrak{m}_t} \xrightarrow{\sim} T_{\mathfrak{m}_t}$. It is independent of the choice of isomorphism and of the choice of σ_i (cf. [Ful98, Appendix A.3]).

To say that $\text{supp}[\text{div}(s)]$ lies in the fiber over v means that $\text{ord}_t(s) = 0$ whenever the closed point t does not lie over v . Thus localizing on the base U in (9) preserves all of the quantities in the sum, and so we may replace T by $T \otimes_C \mathcal{O}_{\mathcal{B},v}$. We continue to call this semilocal ring T . But length is preserved by flat residually trivial base extension, so we may even pass to the \mathfrak{m}_v -adic completion without affecting the quantities in (9). Now $\widehat{\mathcal{O}}_{\mathcal{B},v} = K_v^\circ$, and $\widehat{T} = T \otimes_{\mathcal{O}_{\mathcal{B},v}} K_v^\circ = \prod_{i=1}^r \widehat{T}_i$, where the maximal ideals of T are $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ and \widehat{T}_i is the \mathfrak{m}_v -adic completion of $T_{\mathfrak{m}_i}$ (cf. [Mat89, Thm. 8.15]). Note also that the residue fields of $\mathcal{O}_{\mathcal{B},v}$ and K_v° are canonically isomorphic. Equation (9) now becomes

$$(10) \quad c_1(\mathcal{O}(f)) \cdot [\bar{x}] = [k(v) : k] \sum_{i=1}^r [k(\widehat{t}_i) : k(v)] \text{ord}_{\widehat{t}_i}(\widehat{s}),$$

where $\widehat{t}_1, \dots, \widehat{t}_r$ are the closed points of $\text{Spec } \widehat{T}$, and \widehat{s} is the image of s in the \mathfrak{m}_v -adic completion of N .

By construction, base changing $\bar{x} \hookrightarrow \mathcal{X}$ to K_v° gives a commutative diagram



The horizontal maps are closed immersions. By virtue of this diagram, we see that the points of $O_v(x)$ are in bijective correspondence with the generic points of $\text{Spec } \widehat{T}$, which in turn are in bijective correspondence with the generic points of the disjoint closed subschemes $\text{Spec } \widehat{T}_i$. Moreover, given $y \in O_v(x)$ corresponding to the generic point of $\text{Spec } \widehat{T}_i$, the K_v° -morphism $\tilde{y} : \text{Spec } R \rightarrow \mathcal{X}_{K_v^\circ}$ constructed above factors through the closed immersion $\text{Spec } \widehat{T}_i \hookrightarrow \mathcal{X}_{K_v^\circ}$. This gives the equality of residue fields $[k_R : k_v] = [k_R : k(\widehat{t}_i)][k(\widehat{t}_i) : k(v)]$.

Comparing (8) and (10), we see the proof will be complete once we show that

$$\text{ord}_{\widehat{t}_i}(\widehat{s}) = [k_R : k(\widehat{t}_i)] \text{ord}_R(\alpha_{\widehat{\gamma}}(\widehat{\gamma}_1)/\alpha_{\widehat{\gamma}}(\widehat{\gamma}_2)).$$

Again for topological reasons, $\text{Spec } R \rightarrow \text{Spec } \widehat{A}$ factors through $\text{Spec } \widehat{T}_i$, so we find that the composition $\widehat{A} \rightarrow \widehat{T}_i \rightarrow R$ equals the above homomorphism $\alpha_{\widehat{\gamma}} : \widehat{A} \rightarrow R$. Furthermore, the definitions are such that the elements $\widehat{\sigma}_j$ used to compute $\text{ord}_{\widehat{t}_i}(\widehat{s})$ may be chosen to correspond to $\alpha_{\widehat{\gamma}}(\widehat{\gamma}_j)$ under the homomorphism $\widehat{T}_i \rightarrow R$. As R is the integral closure of \widehat{T}_i in $K_v(y)$, the desired equality is easily deduced from the well-known formula given below upon setting $S = \widehat{T}_i$ and $a = \widehat{\sigma}_i$. ■

LEMMA 2.9 ([Ful98, Example A.3.1]). *Let S be a one-dimensional local noetherian domain. For any $a \in S$, we have*

$$l_S(S/aS) = \sum_R l_R(R/aR)[R/\mathfrak{m}_R : S/\mathfrak{m}_S],$$

where the sum is over all discrete valuation rings of the fraction field of S that dominate S .

2.4. Associated measures. As before, let X be a projective variety of dimension d over the function field K , and fix a place v for the entirety of this section. For semipositive metrized line bundles $\overline{L}_1, \dots, \overline{L}_d$ on X , we will define a bounded Borel measure $c_1(\overline{L}_1) \cdots c_1(\overline{L}_d)$ on X_v^{an} . In order to avoid extra notation, we do not indicate the dependence of the measure on the place v as it will be apparent from the context. Any model function $f : X_v^{\text{an}} \rightarrow \mathbb{R}$ induces an integrable metrized line bundle $\overline{O_X}(f)$ on X , and the measure is defined by

$$\int_{X_v^{\text{an}}} f c_1(\overline{L}_1) \cdots c_1(\overline{L}_d) = \widehat{c}_1(\overline{O_X}(f)) \widehat{c}_1(\overline{L}_1) \cdots \widehat{c}_1(\overline{L}_d).$$

This approach through global intersection theory has the advantage of being technically easy to define. However, it obscures the fact (which we shall prove) that the measure depends only on the metrics of the \overline{L}_i at the place v . One could also develop local intersection theory on formal schemes over K_v° and define the associated measures purely in terms of local intersection products. This is the viewpoint taken by Gubler; for a nice synopsis of the properties of local intersection theory, see [Gub07, §2]. These measures were originally defined by Chambert-Loir [CL06] in the number field case using the formula of Theorem 2.12(i) below, and then by passing to the limit using the local intersection theory of Gubler [Gub98].

For f a continuous function on X_v^{an} , define $\overline{O_X}(f)$ to be the adelic metrized line bundle with underlying bundle O_X , the trivial metric at all places $w \neq v$ and the metric $\|1(x)\|_v = e^{-f(x)}$ at v .

LEMMA 2.10. *If f is a model function on X_v^{an} , then the adelic metrized line bundle $\overline{O_X(f)}$ is integrable.*

Proof. If $f = -\log \|\cdot\|_v^{1/e}$, then Yuan’s lemma (Lemma 2.7) allows one to construct a \mathcal{B} -model $(\mathcal{X}, \mathcal{O}(f))$ of (X, O_X^e) such that the associated adelic metrized line bundle is $\overline{O_X(f)}$. As \mathcal{X} is projective, we can write $\mathcal{O}(f) = \mathcal{L}_1 \otimes \mathcal{L}_2^\vee$ for some ample line bundles $\mathcal{L}_1, \mathcal{L}_2$ on \mathcal{X} . They are *a fortiori* relatively semipositive, and so they induce semipositive metrized line bundles \bar{L}_1, \bar{L}_2 such that $\overline{O_X(f)} = \bar{L}_1 \otimes \bar{L}_2^\vee$. Thus $\overline{O_X(f)}$ is integrable. ■

LEMMA 2.11. *Let X be a projective variety of dimension d over the function field K and v a place of K . For any choice of semipositive metrized line bundles $\bar{L}_1, \dots, \bar{L}_d$, the association*

$$f \mapsto \widehat{c}_1(\overline{O_X(f)})\widehat{c}_1(\bar{L}_1) \cdots \widehat{c}_1(\bar{L}_d)$$

defines a bounded linear functional on the \mathbb{Q} -vector space of model functions on X_v^{an} with the uniform norm.

Proof. It is apparent from the definition that if f and g are two model functions, then $\overline{O_X(f)} \otimes \overline{O_X(g)} = \overline{O_X(f+g)}$. Thus the map in question is additive. Next note that $\frac{1}{n}f$ is a model function for any $n \geq 1$ whenever f is a model function (view O_X as O_X^n via the isomorphism $1 \mapsto 1^{\otimes n}$). It is therefore an easy consequence of additivity that the map in question is \mathbb{Q} -linear as desired.

The map is bounded by Theorem 2.1(ii). ■

The space of model functions on X_v^{an} is dense in the linear space $\mathcal{C}(X_v^{\text{an}}, \mathbb{R})$ of real-valued continuous functions endowed with the uniform norm by Lemma 2.6, so the association in the previous lemma extends to a bounded linear functional on $\mathcal{C}(X_v^{\text{an}}, \mathbb{R})$. By the Riesz representation theorem, we may identify it with a Borel measure on X_v^{an} . Denote this measure by $c_1(\bar{L}_1) \cdots c_1(\bar{L}_d)$. Evidently, we require $d = \dim X \geq 1$ for this notation to be sensible, an annoyance we will remedy at the end of this section.

THEOREM 2.12. *Let X be a projective variety of dimension d over the function field K and v a place of K . If $\bar{L}_1, \dots, \bar{L}_d$ denote semipositive metrized line bundles on X , then the following properties hold for the measures $c_1(\bar{L}_1) \cdots c_1(\bar{L}_d)$:*

- (i) *Suppose that X is normal, that \mathcal{X} is a normal \mathcal{B} -model of X , and that $\mathcal{L}_1, \dots, \mathcal{L}_d$ are models on \mathcal{X} of $L_1^{e_1}, \dots, L_d^{e_d}$, respectively, that induce the metrized line bundles $\bar{L}_1, \dots, \bar{L}_d$. Let $[\mathcal{X}_v] = \sum m(j)[W_j]$ with each W_j irreducible, and let δ_{ξ_j} denote the Dirac measure at the unique point $\xi_j \in X_v^{\text{an}}$ that reduces to the generic point of W_j .*

(See the remarks preceding Lemma 2.5.) Then

$$c_1(\bar{L}_1) \cdots c_1(\bar{L}_d) = \sum_j m(j) \frac{c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_d) \cdot [W_j]}{e_1 \cdots e_d} \delta_{\xi_j}.$$

- (ii) $c_1(\bar{L}_1) \cdots c_1(\bar{L}_d)$ is symmetric and multilinear in $\bar{L}_1, \dots, \bar{L}_d$.
- (iii) $c_1(\bar{L}_1) \cdots c_1(\bar{L}_d)$ is a nonnegative measure ⁽⁴⁾.
- (iv) If \bar{L}_1 and \bar{L}'_1 have the same underlying algebraic bundle and identical metrics at the place v , then

$$c_1(\bar{L}_1) \cdots c_1(\bar{L}_d) = c_1(\bar{L}'_1) \cdots c_1(\bar{L}_d)$$

as measures on X_v^{an} .

- (v) The measure $c_1(\bar{L}_1) \cdots c_1(\bar{L}_d)$ has total mass

$$[k(v) : k] \deg_{L_1, \dots, L_d}(X).$$

- (vi) If Y is another projective K -variety and $\varphi : Y \rightarrow X$ is a generically finite surjective morphism, then

$$\varphi_* \{c_1(\varphi^* \bar{L}_1) \cdots c_1(\varphi^* \bar{L}_d)\} = \deg(\varphi) c_1(\bar{L}_1) \cdots c_1(\bar{L}_d).$$

Proof. (i) It suffices to prove that both measures integrate the same way against a model function of the form $f = -\log \|1\|_v$. Use Yuan’s lemma to choose a model $(\mathcal{X}, \mathcal{O}(f))$ of (X, \mathcal{O}_X) and a rational section s of $\mathcal{O}(f)$ such that the support of $[\text{div}(s)]$ is contained in the fiber \mathcal{X}_v . Then we have

$$\begin{aligned} \int_{X_v^{\text{an}}} f c_1(\bar{L}_1) \cdots c_1(\bar{L}_d) &= \frac{c_1(\mathcal{O}(f)) c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_d)}{e_1 \cdots e_d} \\ &= \sum_j \text{ord}_{W_j}(s) \frac{c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_d) \cdot [W_j]}{e_1 \cdots e_d}. \end{aligned}$$

Applying (1) of Section 2.2 shows us that $\text{ord}_{W_j}(s) = m(j)f(\xi_j)$, which implies the result.

- (ii) This follows immediately from Theorem 2.1(iii).

(iii) It suffices to show that if $f = -\log \|1\|_v$ is a nonnegative model function, then $\int_{X_v^{\text{an}}} f c_1(\bar{L}_1) \cdots c_1(\bar{L}_d) \geq 0$. We may also assume that all of our metrized line bundles are induced by models $\mathcal{L}_1, \dots, \mathcal{L}_d$ by using a limit argument. Apply Yuan’s lemma to get a line bundle $\mathcal{O}(f)$ that induces the metrized line bundle $\overline{\mathcal{O}_X(f)}$, and let s be a rational section of $\mathcal{O}(f)$ whose associated divisor is supported in \mathcal{X}_v . As $f \geq 0$, we deduce that $[\text{div}(s)]$ is effective (cf. (1)). Then

$$\int_{X_v^{\text{an}}} f c_1(\bar{L}_1) \cdots c_1(\bar{L}_d) = c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_d) \cdot [\text{div}(s)] \geq 0$$

⁽⁴⁾ Measure theory texts would call this a positive measure.

because the intersection of relatively semipositive line bundles on components of the fiber \mathcal{X}_v is nonnegative.

(iv) It suffices to show that

$$\int_{X_v^{\text{an}}} gc_1(\bar{L}_1) \cdots c_1(\bar{L}_d) = \int_{X_v^{\text{an}}} gc_1(\bar{L}'_1) \cdots c_1(\bar{L}_d)$$

for any model function $g : X_v^{\text{an}} \rightarrow \mathbb{R}$. In terms of intersection numbers, we must show

$$\widehat{c}_1(\overline{O_X(g)})\widehat{c}_1(\bar{L}_1) \cdots \widehat{c}_1(\bar{L}_d) = \widehat{c}_1(\overline{O_X(g)})\widehat{c}_1(\bar{L}'_1) \cdots \widehat{c}_1(\bar{L}_d).$$

By linearity, this reduces to proving that if \bar{L}_1 is an integrable metrized line bundle with underlying bundle O_X and the trivial metric at v , and if $\bar{L}_2, \dots, \bar{L}_d$ are arbitrary semipositive metrized line bundles, then

$$\widehat{c}_1(\overline{O_X(g)})\widehat{c}_1(\bar{L}_1) \cdots \widehat{c}_1(\bar{L}_d) = 0.$$

We know \bar{L}_1 must have the trivial metric at almost all places, so there exist finitely many places w_1, \dots, w_n of K and corresponding continuous functions $f_{w_i} : X_{w_i}^{\text{an}} \rightarrow \mathbb{R}$ such that

$$\bar{L}_1 = \overline{O_X(f_{w_1})} \otimes \cdots \otimes \overline{O_X(f_{w_n})}.$$

We may assume that no w_i equals v . Again by linearity, we may reduce to the case $\bar{L}_1 = \overline{O_X(f_w)}$ for some continuous function f_w with $w \neq v$. By a limit argument, we may further suppose that f_w is a model function and that $\bar{L}_2, \dots, \bar{L}_d$ are induced by models $\mathcal{L}_2, \dots, \mathcal{L}_d$ on some \mathcal{B} -model \mathcal{X} . Using Yuan’s lemma (and the Simultaneous Model Lemma), we can find a line bundle $\mathcal{O}(f_w)$ on \mathcal{X} that induces $\overline{O_X(f_w)}$ and a rational section s of $\mathcal{O}(f_w)$ with associated divisor supported entirely in the fiber \mathcal{X}_w . Finally, use Yuan’s lemma again to get a line bundle $\mathcal{O}(g)$ on \mathcal{X} (perhaps after replacing \mathcal{X} with a dominating model) equipped with a rational section t whose divisor is supported in \mathcal{X}_v . Then

$$\widehat{c}_1(\overline{O_X(g)})\widehat{c}_1(\bar{L}_1) \cdots \widehat{c}_1(\bar{L}_d) = c_1(\mathcal{O}(g))c_1(\mathcal{O}(f_w))c_1(\mathcal{L}_2) \cdots c_1(\mathcal{L}_d) = 0,$$

since the section t is regular and invertible when restricted to \mathcal{X}_w . Thus (iv) is proved.

(v) Take any \mathcal{B} -model \mathcal{X} of X . The cycle $[\mathcal{X}_v]$ is a Cartier divisor, and we can use it to define the constant model function 1. Indeed, if π is a uniformizer of $\mathcal{O}_{\mathcal{B},v}$, then π is a local equation for \mathcal{X}_v on \mathcal{X} . Consider the line bundle $\mathcal{O}_{\mathcal{X}}([\mathcal{X}_v])$; it induces the metrized line bundle $\overline{O_X(f)}$, where $f(x) = -\log \|1(x)\|_v = -\log |\pi|_v = 1$.

By a limiting argument we may assume that the metrized line bundles $\bar{L}_1, \dots, \bar{L}_d$ are induced by models $\mathcal{L}_1, \dots, \mathcal{L}_d$ of $L_1^{e_1}, \dots, L_d^{e_d}$, respectively.

Now Lemma 2.3 shows

$$\begin{aligned} \int_{X_v^{\text{an}}} 1 c_1(\bar{L}_1) \cdots c_1(\bar{L}_d) &= \frac{c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_d) \cdot [\mathcal{X}_v]}{e_1 \cdots e_d} \\ &= \frac{[k(v) : k] \deg_{L_1^{e_1}, \dots, L_d^{e_d}}(X)}{e_1 \cdots e_d}. \end{aligned}$$

(vi) This is an easy consequence of Theorem 2.1(iv) and the fact that $\overline{O_Y(f \circ \varphi)} = \varphi^* \overline{O_X(f)}$ for any continuous function f . ■

Parts (iii) and (v) of the above theorem indicate a natural normalization for these measures. For a semipositive metrized line bundle \bar{L} with ample underlying bundle \bar{L} and a place v of K , define a probability measure by

$$\mu_{\bar{L},v} = \frac{c_1(\bar{L})^d}{[k(v) : k] \deg_L(X)}.$$

Given any subvariety $Y \subset X$, we can similarly define a probability measure supported on Y_v^{an} since $\bar{L}|_Y$ is also semipositive. If $j : Y_v^{\text{an}} \hookrightarrow X_v^{\text{an}}$ is the canonical inclusion, then we set

$$\mu_{Y,\bar{L},v} = \frac{j_* \{c_1(\bar{L}|_Y)^{\dim Y}\}}{[k(v) : k] \deg_L(Y)}.$$

When $Y = X$, we see immediately that $\mu_{Y,\bar{L},v} = \mu_{\bar{L},v}$.

Finally, we want to define $\mu_{\{x\},\bar{L},v}$ for a closed point $x \in X$. For any model function f on X_v^{an} , define

$$\int_{X_v^{\text{an}}} f d\mu_{\{x\},\bar{L},v} = \frac{\widehat{c}_1(\overline{O_X(f)})|_{\{x\}}}{[k(v) : k] \deg(x)}.$$

The proofs of Lemma 2.11 and Theorem 2.12 apply here to show that $\mu_{\{x\},\bar{L},v}$ extends to a Borel probability measure on X_v^{an} . Evidently, it is independent of the semipositive line bundle \bar{L} , but we have chosen to retain it in the notation to preserve symmetry with $\mu_{Y,\bar{L},v}$ when Y is a higher-dimensional subvariety. By Lemma 2.8, we have the appealing formula

$$\mu_{\{x\},\bar{L},v} = \frac{1}{\deg(x)} \sum_{y \in O_v(x)} \deg_v(y) \delta_y,$$

where δ_y is the point measure supported at y .

2.5. Global height functions. In this section we will define normalized height functions associated to semipositive metrized line bundles. One of the most useful properties of height functions with regard to arithmetic intersection theory is the transformation law that they satisfy when one changes some of the metrics by a constant. For example, this property will

allow us to define canonical height functions and invariant measures associated to a dynamical system.

Suppose L is an ample line bundle on X and \bar{L} is a semipositive metrized line bundle with underlying bundle L . Then $\bar{L}|_Y$ is semipositive for any subvariety $Y \subset X$. We define the height of such a subvariety by

$$h_{\bar{L}}(Y) = \frac{\widehat{c}_1(\bar{L}|_Y)^{\dim Y + 1}}{(\dim Y + 1) \deg_L(Y)}.$$

Recall that $\overline{O_X(b)}$ is defined to be the adelic metrized line bundle with underlying bundle O_X , the trivial metric at all places $w \neq v$ and the metric $\|1(x)\|_v = e^{-b}$ at v .

THEOREM 2.13. *Suppose X is a projective variety over K , L is an ample line bundle on X , and \bar{L} is any semipositive metrized line bundle with underlying bundle L .*

- (i) *If \bar{L}' is another semipositive metrized line bundle with the same underlying algebraic bundle L , then there exists a positive constant C such that for any subvariety Y of X ,*

$$|h_{\bar{L}}(Y) - h_{\bar{L}'}(Y)| \leq C.$$

In fact, we may take

$$(11) \quad C = \sum_v [k(v) : k] \operatorname{dist}_v(\|\cdot\|_{\bar{L},v}, \|\cdot\|_{\bar{L}',v}).$$

- (ii) *Fix a real number b and a place v of K . Then the adelic metrized line bundle $\bar{L} \otimes \overline{O_X(b)}$ is semipositive, and for any subvariety Y , we have*

$$h_{\bar{L} \otimes \overline{O_X(b)}}(Y) = h_{\bar{L}}(Y) + b[k(v) : k].$$

- (iii) *Given any closed point x and rational section s of L such that $x \notin \operatorname{supp}(\operatorname{div}(s))$, we have the following local decomposition:*

$$h_{\bar{L}}(x) = \frac{-1}{\deg(x)} \sum_{v \in \mathcal{B}} [k(v) : k] \sum_{y \in O_v(x)} \deg_v(y) \log \|s(y)\|_{\bar{L},v}.$$

Proof. To prove (i), we use the telescoping sum trick from (5), and Theorem 2.1(ii). Set $r = \dim Y$. Then

$$\begin{aligned} |\widehat{c}_1(\bar{L}|_Y)^{r+1} - \widehat{c}_1(\bar{L}'|_Y)^{r+1}| &\leq \sum_{j=0}^r |\widehat{c}_1(\bar{L}|_Y)^j \widehat{c}_1((\bar{L} \otimes (\bar{L}')^\vee)|_Y) \widehat{c}_1(\bar{L}'|_Y)^{r-j}| \\ &\leq \sum_{j=0}^r \deg_L(Y) \sum_v [k(v) : k] \operatorname{dist}_v(\|\cdot\|_{\bar{L},v}, \|\cdot\|_{\bar{L}',v}) \\ &= C(r + 1) \deg_L(Y), \end{aligned}$$

where C is the constant in (11). Dividing both sides by $(r + 1) \deg_L(Y)$ and using the definition of height gives the result.

For (ii), we note that arithmetic intersection numbers are continuous with respect to change of metric (Theorem 2.1(ii)). Therefore it suffices to assume that \bar{L} is induced by a relatively semipositive \mathcal{B} -model $(\mathcal{X}, \mathcal{L})$ of (X, L^e) . Let us also assume that $b = m/n \in \mathbb{Q}$. As in the proof of Theorem 2.12, we can construct a \mathcal{B} -model of O_X that induces the metrized line bundle $\overline{O_X(b)}$ by taking the line bundle $\mathcal{M} = \mathcal{O}_{\mathcal{X}}(m[\mathcal{X}_v])$ associated to the Cartier divisor $m[\mathcal{X}_v]$ on \mathcal{X} . It is a \mathcal{B} -model of the trivial bundle O_X , and we may view it as a \mathcal{B} -model of O_X^n via the isomorphism $O_X \xrightarrow{\sim} O_X^n$ carrying 1 to $1^{\otimes n}$. If π is a local equation for \mathcal{X}_v on \mathcal{X} , then we see that the metric on O_X at v induced by \mathcal{M} is given by $\|1(x)\|_{\mathcal{M},v} = |\pi^m|_v^{1/n} = e^{-m/n}$. The metrics at all of the other places are evidently trivial.

The line bundle \mathcal{M} is relatively semipositive on \mathcal{X} . Indeed, take any curve C supported in the fiber over a point $w \in \mathcal{B}$. If $w \neq v$, then C and $[\mathcal{X}_v]$ are disjoint and $[\mathcal{X}_v] \cdot C = 0$. If $w = v$, then we note that $[\mathcal{X}_v] = \pi^*[v]$, where $\pi : \mathcal{X} \rightarrow \mathcal{B}$ is the structure morphism. Let D be a divisor on \mathcal{B} linearly equivalent to $[v]$ such that $v \notin \text{supp}(D)$. Then π^*D is a divisor with support disjoint from C , and so $[\mathcal{X}_v] \cdot C = \pi^*D \cdot C = 0$.

Notice that, as an operator on codimension-two cycles, $c_1(\mathcal{M})^2 = 0$ by a linear equivalence argument similar to the one at the end of the last paragraph. Hence, for any subvariety $Y \subset X$ of dimension r , we have

$$\begin{aligned} \widehat{c}_1((\bar{L} \otimes \overline{O_X(b)})|_Y)^{r+1} &= \sum_{j=0}^{r+1} \binom{r+1}{j} \widehat{c}_1(\bar{L}|_Y)^j \widehat{c}_1(\overline{O_X(b)}|_Y)^{r+1-j} \\ &= \sum_{j=0}^{r+1} \binom{r+1}{j} \frac{c_1(\mathcal{L})^j c_1(\mathcal{M})^{r+1-j} \cdot [\bar{Y}]}{e^{jn^{r+1-j}}} \\ &= \frac{c_1(\mathcal{L})^{r+1} \cdot [\bar{Y}]}{e^{r+1}} + (r+1) \frac{c_1(\mathcal{L})^r c_1(\mathcal{M}) \cdot [\bar{Y}]}{e^{rn}} \\ &= \widehat{c}_1(\bar{L}|_Y)^{r+1} + \frac{m}{n} (r+1) \frac{c_1(\mathcal{L})^r \cdot [\bar{Y}] \cdot [\mathcal{X}_v]}{e^r} \\ &= \widehat{c}_1(\bar{L}|_Y)^{r+1} + \frac{m}{n} (r+1) \frac{c_1(\mathcal{L}|_{\bar{Y}})^r \cdot [(\bar{Y})_v]}{e^r} \\ &= \widehat{c}_1(\bar{L}|_Y)^{r+1} + \frac{m}{n} (r+1) [k(v) : k] \deg_L(Y). \end{aligned}$$

The last equality follows from Lemma 2.3. Applying the definition of height immediately gives the result in the case $b = m/n$. The general case follows by continuity of arithmetic intersection numbers when we take a limit over rational approximations of b .

The proof of (iii) is similar to the proof of Lemma 2.8, so we omit it. ■

3. ALGEBRAIC DYNAMICAL SYSTEMS

In this section we review the facts necessary to work with algebraic dynamical systems defined over a function field, including the construction of the invariant metrics on the polarization of a dynamical system, the theory of (canonical) dynamical heights, and the invariant measures for the dynamical system.

3.1. Invariant metrics. Here we are concerned with the existence and uniqueness properties of invariant metrics on the polarization of an algebraic dynamical system. This will give us a natural semipositive metrized line bundle with which to define heights related to a dynamical system.

Let (X, φ, L) be an algebraic dynamical system over K as in the introduction. Suppose $\theta : \varphi^*L \xrightarrow{\sim} L^q$ is an isomorphism with $q > 1$. For a place v of K , choose any initial metric $\|\cdot\|_{1,v}$ on L_v . For example, it could be the metric induced by a \mathcal{B} -model of L . We can construct an *invariant metric* on L_v by Tate’s limit process: by induction, define

$$\|\cdot\|_{n+1,v} = (\varphi^* \|\cdot\|_{n,v} \circ \theta^{-1})^{1/q}.$$

Here $\varphi^* \|\cdot\|_{n,v}$ denotes the metric on φ^*L_v induced by pullback. It is well-known (cf. [BG06, §9.5] or [Zha95b, §2]) that this sequence of metrics converges uniformly to a continuous metric $\|\cdot\|_{0,v}$ on L_v with the following properties:

- (i) The pullback by φ agrees with the q th tensor power (up to the isomorphism θ):

$$\|\cdot\|_{0,v}^{\otimes q} \circ \theta = \varphi^* \|\cdot\|_{0,v}.$$

- (ii) If θ is replaced by $\theta' = a\theta$ for some $a \in K^\times$, then the corresponding metric constructed by Tate’s limit process satisfies

$$\|\cdot\|'_{0,v} = |a|_v^{1/(q-1)} \|\cdot\|_{0,v}.$$

Property (i) uniquely determines the metric. In the literature this metric is sometimes called the “canonical metric” or an “admissible metric”. We adhere to the term *invariant metric* because it is only canonical up to a choice of isomorphism θ by property (ii), and we feel the term “admissible” is already overused in nonarchimedean geometry. We can interpret property (i) by saying that the family $\{\|\cdot\|_{0,v}\}_v$ of invariant metrics provides the unique adelic metric structure on L such that the isomorphism $\theta : \varphi^*L \xrightarrow{\sim} L^q$ becomes an isometry.

The above discussion settles the existence of invariant metrics at each place $v \in \mathcal{B}$, but we still need to show that they fit together to give a semipositive adelic metrized line bundle:

- (iii) There exists a sequence of \mathcal{B} -models $(\mathcal{X}_n, \mathcal{L}_n)$ of (X, L^{e_n}) such that each \mathcal{L}_n is nef, $\|\cdot\|_{\mathcal{L}_n, v}^{1/e_n} = \|\cdot\|_{0, v}$ for almost all v , and $\|\cdot\|_{\mathcal{L}_n, v}^{1/e_n} \rightarrow \|\cdot\|_{0, v}$ uniformly for every other place v . In particular, the metrized line bundle \bar{L} with underlying bundle L and the family of metrics $\{\|\cdot\|_{0, v}\}_v$ is semipositive.

The first step in this direction is to construct a sequence of \mathcal{B} -models that determine metrics on L according to Tate’s limit process.

LEMMA 3.1. *Let X be a projective variety over K and L an ample line bundle on X . Then there exists a positive integer e and a \mathcal{B} -model $(\mathcal{X}, \mathcal{L})$ of (X, L^e) such that \mathcal{L} is nef.*

Proof. This proof was adapted from a remark in the Notation and Conventions section of [Yua08, §2.1]. Choose e so that L^e is very ample. Let

$$X \hookrightarrow \mathbb{P}_K^N \hookrightarrow \mathbb{P}_{\mathcal{B}}^N = \mathbb{P}^N \times \mathcal{B}$$

be an embedding induced by L^e followed by identifying \mathbb{P}_K^N with the generic fiber of $\mathbb{P}_{\mathcal{B}}^N$, and set \mathcal{X} to be the Zariski closure of X in $\mathbb{P}_{\mathcal{B}}^N$ with the reduced structure. Let $\pi : \mathcal{X} \rightarrow \mathcal{B}$ be the restriction of the second projection. Choose a collection of basepoint free global sections s_0, \dots, s_N of L^e , and let \tilde{s}_i be the section s_i viewed as a rational section of $\mathcal{O}_{\mathcal{X}}(1)$. Let D be an ample Cartier divisor \mathcal{B} such that $[\pi^*D] + [\text{div}(\tilde{s}_i)]$ is effective for all i . Finally, define $\mathcal{L} = \mathcal{O}_{\mathcal{X}}(1) \otimes \pi^*\mathcal{O}_{\mathcal{B}}(D)$.

We claim that \mathcal{L} is nef. Indeed, suppose \mathcal{Y} is an irreducible curve on \mathcal{X} . If \mathcal{Y} is vertical—i.e., $\pi(\mathcal{Y}) = \{v\}$ for some closed point $v \in \mathcal{B}$ —then

$$c_1(\mathcal{L}) \cdot [\mathcal{Y}] = \text{deg}(\mathcal{O}_{\mathcal{X}}(1)|_{\mathcal{Y}}) > 0,$$

since $\mathcal{O}_{\mathcal{X}}(1)$ is relatively ample. If \mathcal{Y} is horizontal, then choose one of the sections s_i of L^e such that $\mathcal{Y}_K \not\subset \text{supp}[\text{div}(s_i)]$. Then \mathcal{Y} is not contained in the support of $[\text{div}(\tilde{s}_i)]$, and it intersects properly with any subvariety of a vertical fiber. Hence,

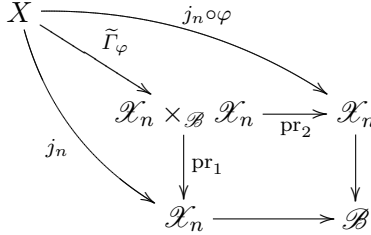
$$c_1(\mathcal{L}) \cdot [\mathcal{Y}] = [\text{div}(\tilde{s}_i)]_h \cdot [\mathcal{Y}] + ([\text{div}(\tilde{s}_i)]_f + c_1(\pi^*D)) \cdot [\mathcal{Y}] \geq 0,$$

where $[\text{div}(\tilde{s}_i)] = [\text{div}(\tilde{s}_i)]_h + [\text{div}(\tilde{s}_i)]_f$ is the decomposition of this cycle into its horizontal and vertical parts. ■

Returning to our construction, choose an initial \mathcal{B} -model $(\mathcal{X}_1, \mathcal{L}_1)$ of (X, L^{e_1}) such that \mathcal{L}_1 is nef on \mathcal{X}_1 . The above lemma guarantees the existence of such a \mathcal{B} -model. Define the metric on L_v to be $\|\cdot\|_{1, v} = \|\cdot\|_{\mathcal{L}_1, v}^{1/e_1}$.

We proceed by induction. Suppose $(\mathcal{X}_n, \mathcal{L}_n)$ is a \mathcal{B} -model of (X, L^{e_n}) , where $e_n = q^{n-1}e_1$ and \mathcal{L}_n is nef. Let $j_n : X \hookrightarrow \mathcal{X}_n$ be the inclusion of the generic fiber (always implicitly precomposed with the preferred isomorphism $\iota_n : X \xrightarrow{\sim} (\mathcal{X}_n)_K$). Let $\Gamma_\varphi : X \rightarrow X \times_K X$ be the graph morphism. Consider

the commutative diagram



where $\tilde{\Gamma}_\varphi = (j_n \times j_n) \circ \Gamma_\varphi$. Define \mathcal{X}_{n+1} to be the Zariski closure of $\tilde{\Gamma}_\varphi(X)$ in $\mathcal{X}_n \times_{\mathcal{B}} \mathcal{X}_n$ with the reduced subscheme structure. Take $\mathcal{L}_{n+1} = \text{pr}_2^* \mathcal{L}_n|_{\mathcal{X}_{n+1}}$. The graph morphism Γ_φ and the tensor power $\theta^{\otimes e_n}$ give the preferred isomorphisms between X and the generic fiber of \mathcal{X}_{n+1} and between $\varphi^* L^n$ and L^{q^n} , respectively. As always, we will make these identifications without comment in what follows. Set $e_{n+1} = qe_n$, and define a metric on L_v by $\|\cdot\|_{n+1,v} = \|\cdot\|_{\mathcal{L}_{n+1},v}^{1/e_{n+1}}$. Observe that \mathcal{L}_{n+1} is nef since it is the pullback of a nef line bundle.

The metrics on L_v are, by construction, exactly as given by Tate’s limit process. This follows from the fact that formation of formal metrics commutes with formal pullback (Lemma 2.4), and a small computation:

$$\begin{aligned}
 \|\cdot\|_{n+1,v} &= \|\cdot\|_{\mathcal{L}_{n+1},v}^{1/e_{n+1}} = (\|\cdot\|_{\text{pr}_2^* \mathcal{L}_n|_{\mathcal{X}_{n+1},v}}^{1/e_n})^{1/q} = (\varphi^* \|\cdot\|_{\mathcal{L}_n,v}^{1/e_n} \circ \theta^{-1})^{1/q} \\
 &= (\varphi^* \|\cdot\|_{n,v} \circ \theta^{-1})^{1/q}.
 \end{aligned}$$

Moreover, we now show that almost all of the metrics constructed are stable under this pullback procedure. As X is of finite type over K , there exists an open subset $\mathcal{U} \subset \mathcal{B}$ such that the endomorphism φ extends to a \mathcal{U} -morphism $\varphi_{\mathcal{U}} : (\mathcal{X}_1)_{\mathcal{U}} \rightarrow (\mathcal{X}_1)_{\mathcal{U}}$, and the isomorphism $\theta : \varphi^* L \xrightarrow{\sim} L^q$ extends to an isomorphism $\theta_{\mathcal{U}} : \varphi_{\mathcal{U}}^* \mathcal{L}_1|_{\pi^{-1}(\mathcal{U})} \xrightarrow{\sim} \mathcal{L}_1^q|_{\pi^{-1}(\mathcal{U})}$. The graph morphism $\tilde{\Gamma}_\varphi$ extends over \mathcal{U} to give a closed immersion

$$(\mathcal{X}_1)_{\mathcal{U}} \hookrightarrow (\mathcal{X}_1)_{\mathcal{U}} \times_{\mathcal{U}} (\mathcal{X}_1)_{\mathcal{U}}.$$

Consequently, its (scheme-theoretic) image is exactly $(\mathcal{X}_2)_{\mathcal{U}}$, so that \mathcal{X}_1 and \mathcal{X}_2 are isomorphic when restricted over \mathcal{U} . Pulling back \mathcal{L}_2 via this isomorphism and applying $\theta_{\mathcal{U}}$ shows $\Gamma_{\varphi_{\mathcal{U}}}^* \mathcal{L}_2 = \varphi_{\mathcal{U}}^* \mathcal{L}_1 \cong \mathcal{L}_1^q$ over \mathcal{U} . As \mathcal{L}_2 is a model of L^{e_2} via the graph morphism and the isomorphism θ , we conclude that for each place v corresponding to a closed point of \mathcal{U} , we have

$$\|\cdot\|_{2,v} = \|\cdot\|_{\mathcal{L}_2}^{1/e_2} = \|\cdot\|_{\mathcal{L}_1^q}^{1/qe_1} = \|\cdot\|_{\mathcal{L}_1}^{1/e_1} = \|\cdot\|_{1,v}.$$

The isomorphism between $(\mathcal{X}_1)_{\mathcal{U}}$ and $(\mathcal{X}_2)_{\mathcal{U}}$ allows us to extend the work in the previous paragraph by induction to conclude that for each place v of \mathcal{U} , the metrics $\|\cdot\|_{n,v}$ on L_v are equal for all n .

3.2. Dynamical heights. Let the data (X, φ, L) , $\theta : \varphi^*L \xrightarrow{\sim} L^q$, and \bar{L} be as in the previous section. For a subvariety $Y \subset X$, we can define its dynamical height with respect to the dynamical system (X, φ, L) by the formula

$$h_\varphi(Y) = h_{\bar{L}}(Y) = \frac{\widehat{c}_1(\bar{L}|_Y)^{\dim Y + 1}}{(\dim Y + 1) \deg_L(Y)}.$$

THEOREM 3.2. *Let (X, φ, L) be a dynamical system over K , $\theta : \varphi^*L \xrightarrow{\sim} L^q$ an isomorphism, and \bar{L} the line bundle L equipped with the corresponding invariant metrics $\{\|\cdot\|_{0,v}\}_v$.*

- (i) *The height h_φ is independent of the choice of isomorphism θ .*
- (ii) *For any subvariety $Y \subset X$, $h_\varphi(Y) \geq 0$.*
- (iii) *For any subvariety $Y \subset X$, $h_\varphi(\varphi(Y)) = qh_\varphi(Y)$.*
- (iv) *If Y is preperiodic for the map φ , then $h_\varphi(Y) = 0$. (Recall that preperiodic means the forward orbit $\{\varphi^n(Y) : n = 1, 2, \dots\}$ is finite.)*

Before turning to the proof, we will need the following

LEMMA 3.3. *Let (X, φ, L) be a dynamical system defined over K , and let $\varphi^*L \cong L^q$ for some integer $q > 1$. Then for any subvariety $Y \subset X$, the induced morphism $Y \rightarrow \varphi(Y)$ is finite of degree $q^{\dim Y}$.*

Proof. Let $\psi : Y \rightarrow \varphi(Y)$ be the morphism induced by φ . First note that $\psi^*(L|_{\varphi(Y)})$ is ample on Y since the restriction of an ample bundle to a subvariety is still ample, and

$$\psi^*(L|_{\varphi(Y)}) = (\varphi^*L)|_Y \cong L^q|_Y = (L|_Y)^q.$$

If $\psi(Z) = \{p\}$ for some subvariety $Z \subset Y$ and some point p , then we find $\psi^*(L|_{\varphi(Z)}) \cong \mathcal{O}_Z$, which can only be ample if Z is reduced to a point. Hence $\psi : Y \rightarrow \varphi(Y)$ has finite fibers. As X is projective, we see ψ is a projective quasi-finite morphism, and so it must be finite.

If $r = \dim Y$, the projection formula gives

$$c_1(L)^r \cdot [Y] = q^{-r} c_1(\varphi^*L)^r \cdot [Y] = q^{-r} c_1(L)^r \cdot \varphi_*([Y]) = \frac{\deg(\psi)}{q^r} c_1(L)^r \cdot [Y].$$

As L is ample, we may divide by $c_1(L)^r \cdot [Y]$ to conclude $\deg(\psi) = q^r$. ■

Proof of Theorem 3.2. (i) Let $\theta' = a\theta$ for some $a \in K^\times$. If \bar{M} is the metrized line bundle with underlying bundle \mathcal{O}_X and metric at the place v given by $\|1(x)\|_v = |a|_v^{1/(q-1)}$, then the invariant metrized line bundle corresponding to θ' is $\bar{L}' = \bar{L} \otimes \bar{M}$ (property (ii) in Section 3.1). Take any model \mathcal{X} of X and consider the line bundle $\mathcal{O}(\text{div}(a))$ associated to the principal divisor $\text{div}(a)$ on \mathcal{X} . We can view it as a model of \mathcal{O}_X^{q-1} on the generic fiber via isomorphisms $\mathcal{O}_X(\text{div}(a)) \cong \mathcal{O}_X \cong \mathcal{O}_X^{q-1}$. It is easy to check

that the metric on O_X at v given by $\mathcal{O}(\operatorname{div}(a))$ coincides with that of \bar{M} . Letting $r = \dim Y$, we find that

$$\begin{aligned} \widehat{c}_1(\bar{L}'|_Y)^{r+1} - \widehat{c}_1(\bar{L}|_Y)^{r+1} &= \widehat{c}_1(\bar{L}|_Y \otimes \bar{M}|_Y)^{r+1} - \widehat{c}_1(\bar{L}|_Y)^{r+1} \\ &= \sum_{i=0}^r \binom{r+1}{i} \widehat{c}_1(\bar{L}|_Y)^i \widehat{c}_1(\bar{M}|_Y)^{r+1-i}. \end{aligned}$$

All of the terms in this sum involve an intersection with $\widehat{c}_1(\bar{M})$, and if we compute this intersection on a model, we are forced to intersect with the principal divisor $\operatorname{div}(a)$. Thus each term in the sum vanishes.

(ii) Arithmetic intersection numbers are continuous with respect to change of metric, so it suffices to prove $c_1(\mathcal{L})^{\dim Y+1} \cdot [\bar{Y}] \geq 0$, whenever $(\mathcal{X}, \mathcal{L})$ is a \mathcal{B} -model of (X, L^e) , \mathcal{L} is nef and \bar{Y} is the Zariski closure of Y in \mathcal{X} . Kleiman’s theorem on intersections with nef divisors implies the desired inequality [Laz04, Thm. 1.4.9].

(iii) Let $r = \dim Y$. By Lemma 3.3, the morphism φ restricts to a finite morphism $Y \rightarrow \varphi(Y)$ of degree q^r . Theorem 2.1(iv) implies

$$\begin{aligned} h_\varphi(\varphi(Y)) &= \frac{\widehat{c}_1(\bar{L}|_{\varphi(Y)})^{r+1}}{(r+1) \operatorname{deg}_L(\varphi(Y))} = \frac{\widehat{c}_1(\varphi^* \bar{L}|_Y)^{r+1}}{(r+1) \operatorname{deg}_{\varphi^* L}(Y)} \\ &= q \frac{\widehat{c}_1(\bar{L}|_Y)^{r+1}}{(r+1) \operatorname{deg}_L(Y)} = q h_\varphi(Y). \end{aligned}$$

(iv) If the set $\{\varphi^n(Y) : n = 1, 2, \dots\}$ is finite, then $\varphi^n(Y) = \varphi^m(Y)$ for some $m > n \geq 1$. By the previous part, we have

$$q^n h_\varphi(Y) = h_\varphi(\varphi^n(Y)) = h_\varphi(\varphi^m(Y)) = q^m h_\varphi(Y).$$

As $q > 1$, we are forced to conclude that $h_\varphi(Y) = 0$. ■

As a special case of part (iv) of the previous theorem, we note that

$$h_\varphi(X) = \frac{\widehat{c}_1(\bar{L})^{d+1}}{(d+1) \operatorname{deg}_L(X)} = 0.$$

3.3. The invariant measure $\mu_{\varphi,v}$. Let us choose an isomorphism $\theta : \varphi^* L \xrightarrow{\sim} L^q$, and let \bar{L} be the line bundle L equipped with the invariant metrics constructed as above. Fix a place v of K . Define a Borel probability measure on X_v^{an} by the formula

$$\mu_{\varphi,v} = \mu_{\bar{L},v} = \frac{c_1(\bar{L})^d}{[k(v) : k] \operatorname{deg}_L(X)}.$$

Here $\mu_{\bar{L},v}$ and $c_1(\bar{L})^d$ are the measures constructed in Section 2.4. An argument similar to the one that proved Theorem 3.2(i) shows that $\mu_{\varphi,v}$ is independent of the choice of isomorphism θ . Although it is not logically

necessary for what follows, we give some further commentary on these measures.

Since φ is finite of degree q^d (Lemma 3.3), we see that the measure $\mu_{\varphi,v}$ has the following invariance property:

$$\varphi_*\mu_{\varphi,v} = \mu_{\varphi,v}.$$

Indeed, this follows immediately from Theorem 2.12 and the fact that $\varphi^*\bar{L}$ is isometric to \bar{L}^q .

Given any subvariety $Y \subset X$, we can also define the measure $\mu_{Y,\varphi,v} = \mu_{Y,\bar{L},v}$ as in Section 2.4. Theorem 2.12(vi) can be used to show

$$\varphi_*\mu_{Y,\varphi,v} = \mu_{\varphi(Y),\varphi,v}.$$

An important example is the case when X is a smooth geometrically connected projective variety over K and v is a place of *good reduction* for (X, φ, L) , i.e., there exists an open subvariety $\mathcal{U} \subset \mathcal{B}$ containing the point v , a smooth \mathcal{U} -model $(\mathcal{X}, \mathcal{L})$ of (X, L) , a \mathcal{U} -morphism $\varphi_{\mathcal{U}} : \mathcal{X} \rightarrow \mathcal{X}$ whose restriction to the generic fiber is precisely φ , and an isomorphism $\varphi_{\mathcal{U}}^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}^q$. Roughly, the dynamical system can be reduced (mod v). One can see from Theorem 2.12(i) and our description of Tate's limit process that there exists a point $\zeta \in X_v^{\text{an}}$ such that $\mu_{\varphi,v} = \delta_{\zeta}$. The point ζ is the unique point mapping to the generic point of the special fiber \mathcal{X}_v under the reduction map $X_v^{\text{an}} \rightarrow \mathcal{X}_v$. Moreover, the forward invariance of the measure $\mu_{\varphi,v}$ implies that ζ is a fixed point of the analytification of φ : $\varphi_v^{\text{an}}(\zeta) = \zeta$.

As a final remark, we mention a backward invariance property the measure $\mu_{\varphi,v}$ presumably has based on the work of Chambert-Loir [CL06, §2.8] and others, although we do not provide any proof in the present article. There is a way to define a trace map φ_* on the space of continuous functions on X_v^{an} , and by duality a pullback measure $\varphi^*\mu_{\varphi,v}$. It should then be true that $\varphi^*\mu_{\varphi,v} = q^d\mu_{\varphi,v}$. As a consequence of this backward invariance property, if (X, φ, L) has good reduction at a place v , then $(\varphi_v^{\text{an}})^{-1}(\zeta) = \zeta$, where $\mu_{\varphi,v} = \delta_{\zeta}$ as in the previous paragraph. That is, ζ is a totally invariant point for the morphism φ_v^{an} . By analogy with the case of complex dynamical systems, we expect that the invariant measure $\mu_{\varphi,v}$ can be completely characterized as the unique Borel probability measure on X_v^{an} such that

- $\varphi^*\mu_{\varphi,v} = q^d\mu_{\varphi,v}$, and
- $\mu_{\varphi,v}$ does not charge any proper subvariety of X : $\mu_{\varphi,v}(Y_v^{\text{an}}) = 0$ for any proper subvariety $Y \subset X$.

See the articles of Chambert-Loir [CL06] and Chambert-Loir/Thuillier [CLT08] for proofs that the above properties hold for the measure $\mu_{\varphi,v}$. It is not yet known if they determine the measure. See the article of Briend and Duval [BD01] for a discussion of such a characterization in the setting of complex dynamics.

4. PROOF OF THE EQUIDISTRIBUTION THEOREM

Our goal for this section is to prove Theorem 1.1. We will deduce it from a stronger result that is more flexible for applications and also gives equidistribution of small subvarieties. As always, we let X be a variety over the function field K . Let \bar{L} be a semipositive metrized line bundle on X with ample underlying bundle L satisfying the following two conditions:

- (S1) There exists a sequence of \mathcal{B} -models $(\mathcal{X}_n, \mathcal{L}_n)$ of (X, L^{e_n}) such that each \mathcal{L}_n is nef, $\|\cdot\|_{\mathcal{L}_n, v}^{1/e_n} = \|\cdot\|_{0, v}$ for almost all v , and $\|\cdot\|_{\mathcal{L}_n, v}^{1/e_n} \rightarrow \|\cdot\|_{0, v}$ uniformly for every other place v .
- (S2) The height of X is zero: $h_{\bar{L}}(X) = 0$.

A net of subvarieties of X consists of an infinite directed set A and a subvariety $Y_\alpha \subset X$ for each $\alpha \in A$. A net of subvarieties $(Y_\alpha)_{\alpha \in A}$ is called *generic* if for any proper closed subset $V \subset X$, there exists $\alpha_0 \in A$ so that $Y_\alpha \not\subset V$ whenever $\alpha \geq \alpha_0$. Equivalently, there does not exist a cofinal subset $A' \subset A$ such that $Y_\alpha \subset V$ for all $\alpha \in A'$. The net is called *small* if $\lim_{\alpha \in A} h_{\bar{L}}(Y_\alpha) = 0$.

THEOREM 4.1. *Let X be a projective variety over the function field K equipped with a semipositive metrized line bundle \bar{L} with ample underlying bundle L satisfying conditions (S1) and (S2). Let $(Y_\alpha)_{\alpha \in A}$ be a generic small net of subvarieties of X . Then for any place v of K , and for any continuous function $f : X_v^{\text{an}} \rightarrow \mathbb{R}$, we have*

$$\lim_{\alpha \in A} \int_{X_v^{\text{an}}} f d\mu_{Y_\alpha, \bar{L}, v} = \int_{X_v^{\text{an}}} f d\mu_{\bar{L}, v}.$$

That is, the net of measures $(\mu_{Y_\alpha, \bar{L}, v})_{\alpha \in A}$ converges weakly to $\mu_{\bar{L}, v}$.

Before turning to the proof, let us indicate why Theorem 1.1 follows from Theorem 4.1. Let (X, φ, L) be a dynamical system defined over the function field K . Choose an isomorphism $\theta : \varphi^*L \xrightarrow{\sim} L^q$. Let \bar{L} be the semipositive metrized line bundle with underlying bundle L and the associated invariant metrics at all places as defined in Section 3.1. Then property (iii) of the same section is precisely condition (S1). As $h_\varphi = h_{\bar{L}}$ (by definition), the discussion at the end of Section 3.2 shows condition (S2). Thus the hypotheses of Theorem 4.1 on \bar{L} are satisfied. Upon unraveling all of the definitions, the conclusion of Theorem 1.1 follows immediately from that of the above theorem.

In order to see why the above theorem is more useful in practice, consider a dynamical system (X, φ, L) , and let Y be any subvariety of X such that $h_\varphi(Y) = 0$. If $\varphi(Y) \neq Y$, then Y cannot be considered as a dynamical system on its own. Nevertheless, we find that $\bar{L}|_Y$ is a semipositive metrized line bundle satisfying conditions (S1) and (S2), and so we can use the above

theorem to deduce equidistribution statements for generic small nets of subvarieties of Y .

Proof of Theorem 4.1. Fix a place v of K . By Lemma 2.6 and a limiting argument, it suffices to prove the theorem when $f = -\log \|1\|_v^{1/n}$ is a model function. By linearity of the integral, we may take $n = 1$. Lemma 2.7 allows us to assume that f is induced by a \mathcal{B} -model $(\mathcal{X}, \mathcal{O}(f))$ of (X, O_X) . We also choose ample line bundles \mathcal{M}_1 and \mathcal{M}_2 on \mathcal{X} so that $\mathcal{O}(f) = \mathcal{M}_1 \otimes \mathcal{M}_2^\vee$. Let \bar{M}_i be the metrized line bundle on X determined by \mathcal{M}_i . Finally, we assume that $\int_{X_v^{\text{an}}} f d\mu_{\bar{L},v} > 0$ for the moment and remove this hypothesis at the end of the proof.

For any $N \geq 1$, we define $\bar{L}^N(f) := \bar{L}^N \otimes \overline{O_X(f)}$. We wish to compute the degree of this metrized line bundle in two ways. For the first, we have

$$\begin{aligned}
 (12) \quad \widehat{c}_1(\bar{L}^N(f))^{d+1} &= (N\widehat{c}_1(\bar{L}) + \widehat{c}_1(\overline{O_X(f)}))^{d+1} \\
 &= N^{d+1}\widehat{c}_1(\bar{L})^{d+1} + N^d(d+1) \deg_L(X)[k(v) : k] \int_{X_v^{\text{an}}} f d\mu_{\bar{L},v} + O(N^{d-1}) \\
 &= N^d(d+1) \deg_L(X)[k(v) : k] \int_{X_v^{\text{an}}} f d\mu_{\bar{L},v} + O(N^{d-1}).
 \end{aligned}$$

The integral appears by the definition of the measure $\mu_{\bar{L},v}$. The term $\widehat{c}_1(\bar{L})^{d+1}$ vanishes because it is the numerator of $h_{\bar{L}}(X)$ (condition (S2)). The constant in the error term depends on \bar{L} and f .

On the other hand, we see that

$$\begin{aligned}
 (13) \quad \widehat{c}_1(\bar{L}^N(f))^{d+1} &= \widehat{c}_1(\bar{L}^N \otimes \bar{M}_1 \otimes \bar{M}_2^\vee)^{d+1} \\
 &= \sum_{i=0}^{d+1} \binom{d+1}{i} (-1)^{d+1-i} \widehat{c}_1(\bar{L}^N \otimes \bar{M}_1)^i \widehat{c}_1(\bar{M}_2)^{d+1-i} \\
 &= \widehat{c}_1(\bar{L}^N \otimes \bar{M}_1)^{d+1} - (d+1)\widehat{c}_1(\bar{L}^N \otimes \bar{M}_1)^d \widehat{c}_1(\bar{M}_2) + O(N^{d-1}).
 \end{aligned}$$

Recall that we assumed $\int_{X_v^{\text{an}}} f d\mu_{\bar{L},v} > 0$. Comparing (12) and (13) shows that for N sufficiently large,

$$\widehat{c}_1(\bar{L}^N \otimes \bar{M}_1)^{d+1} - (d+1)\widehat{c}_1(\bar{L}^N \otimes \bar{M}_1)^d \widehat{c}_1(\bar{M}_2) > 0.$$

We may fix such an N for the remainder of the argument, and as it will have no effect on the proof, we will replace \bar{L}^N by \bar{L} .

Choose $\varepsilon > 0$. By condition (S1) we may select a \mathcal{B} -model $(\mathcal{X}', \mathcal{L})$ of (X, L^ε) such that \mathcal{L} is nef, the metrics on the associated adelic metrized line bundle \bar{L}' with underlying bundle L are equal to those of \bar{L} at almost all places, and the sum of the weighted distances $[k(v) : k] \text{dist}_v(\|\cdot\|_{\bar{L},v}, \|\cdot\|_{\bar{L}',v})$ at the other places is bounded by ε . By the Simultaneous Model Lemma, we may assume that $\mathcal{X}' = \mathcal{X}$ so that $\mathcal{O}(f)$ and \mathcal{L} are line bundles on \mathcal{X} . Furthermore, continuity of intersection numbers with respect to changes in

the metric allows us to assume that

$$c_1(\mathcal{L} \otimes \mathcal{M}_1^e)^{d+1} - (d+1)c_1(\mathcal{L} \otimes \mathcal{M}_1^e)^d c_1(\mathcal{M}_2^e) > 0.$$

The necessary tool from algebraic geometry needed to move forward at this point is

SIU'S THEOREM ([Laz04, Theorem 2.2.15]). *Let \mathcal{Y} be a projective variety of dimension n over the field k and suppose \mathcal{N}_1 and \mathcal{N}_2 are nef line bundles on \mathcal{Y} . If*

$$c_1(\mathcal{N}_1)^n - n c_1(\mathcal{N}_1)^{n-1} c_1(\mathcal{N}_2) > 0,$$

then $(\mathcal{N}_1 \otimes \mathcal{N}_2^\vee)^r$ has nonzero global sections for $r \gg 0$.

We are in a position to apply Siu's theorem with $\mathcal{Y} = \mathcal{X}$, $n = d + 1$, $\mathcal{N}_1 = \mathcal{L} \otimes \mathcal{M}_1^e$ and $\mathcal{N}_2 = \mathcal{M}_2^e$. It follows that the line bundle

$$(\mathcal{L} \otimes \mathcal{M}_1^e \otimes \mathcal{M}_2^{(-e)})^r = (\mathcal{L} \otimes \mathcal{O}(f)^e)^r$$

admits global sections for all r sufficiently large. Fix such an r and a nonzero global section s . As $(Y_\alpha)_{\alpha \in A}$ is a generic net in X , there exists α_0 such that \bar{Y}_α does not lie in the support of $\text{div}(s)$ for any $\alpha \geq \alpha_0$. This means $c_1((\mathcal{L} \otimes \mathcal{O}(f)^e)^r) \cdot [\bar{Y}_\alpha]$ is an effective cycle. As \mathcal{L} is nef, Kleiman's theorem [Laz04, Thm. 1.4.9] shows

$$c_1(\mathcal{L})^{\dim Y_\alpha} c_1((\mathcal{L} \otimes \mathcal{O}(f)^e)^r) \cdot [\bar{Y}_\alpha] \geq 0,$$

or equivalently,

$$(14) \quad \widehat{c}_1(\bar{L}'|_{Y_\alpha})^{\dim Y_\alpha} \widehat{c}_1((\bar{L}' \otimes \overline{O_X(f)})|_{Y_\alpha}) \geq 0.$$

Our precision in picking the metrics on \bar{L}' and Theorem 2.1(ii) show that

$$(15) \quad \begin{aligned} &|\widehat{c}_1(\bar{L}'|_{Y_\alpha})^{\dim Y_\alpha} \widehat{c}_1((\bar{L}' \otimes \overline{O_X(f)})|_{Y_\alpha}) \\ &\quad - \widehat{c}_1(\bar{L}|_{Y_\alpha})^{\dim Y_\alpha} \widehat{c}_1((\bar{L} \otimes \overline{O_X(f)})|_{Y_\alpha})| \\ &\leq \varepsilon(\dim Y_\alpha + 1) \text{deg}_L(Y_\alpha). \end{aligned}$$

From (14) and (15) we now get

$$\begin{aligned} h_{\bar{L}}(Y_\alpha) + [k(v) : k](\dim Y_\alpha + 1)^{-1} \int_{X_\alpha^{\text{an}}} f d\mu_{Y_\alpha, \bar{L}, v} \\ = \frac{\widehat{c}_1(\bar{L}|_{Y_\alpha})^{\dim Y_\alpha} \widehat{c}_1((\bar{L} \otimes \overline{O_X(f)})|_{Y_\alpha})}{(\dim Y_\alpha + 1) \text{deg}_L(Y_\alpha)} \geq -\varepsilon \end{aligned}$$

for all $\alpha \geq \alpha_0$. Taking the limit over $\alpha \in A$ in this last expression and recalling $h_{\bar{L}}(Y_\alpha) \rightarrow 0$ proves that

$$\liminf_{\alpha \in A} \int_{X_\alpha^{\text{an}}} f d\mu_{Y_\alpha, \bar{L}, v} \geq -\frac{\varepsilon(d+1)}{[k(v) : k]}.$$

Finally, ε is independent of f , so we conclude that

$$(16) \quad \liminf_{\alpha \in A} \int_{X_v^{\text{an}}} f \, d\mu_{Y_\alpha, \bar{L}, v} \geq 0.$$

This last inequality holds for any model function f with the added hypothesis $\int_{X_v^{\text{an}}} f \, d\mu_{\bar{L}, v} > 0$. In order to lift this restriction, we take an arbitrary model function f and consider the function $f_1 = f - \varrho$, where $\varrho \in \log \sqrt{|K_v^\times|} = \mathbb{Q}$ is such that $\int_{X_v^{\text{an}}} f_1 \, d\mu_{\bar{L}, v} > 0$. Constant functions of this form are model functions, and so nf_1 satisfies all of the necessary hypotheses to make the above argument go through for some positive integer n . (We need nf_1 to be the model function associated to a formal metric—not just the root of a formal metric.) Applying (16) to nf_1 shows that

$$\liminf_{\alpha \in A} \int_{X_v^{\text{an}}} f \, d\mu_{Y_\alpha, \bar{L}, v} \geq \varrho.$$

Letting $\varrho \rightarrow \int_{X_v^{\text{an}}} f \, d\mu_{\bar{L}, v}$ from below preserves the positivity of the integral of f_1 and shows

$$\liminf_{\alpha \in A} \int_{X_v^{\text{an}}} f \, d\mu_{Y_\alpha, \bar{L}, v} \geq \int_{X_v^{\text{an}}} f \, d\mu_{\bar{L}, v}.$$

Finally, we may replace f with $-f$ in this argument to obtain the opposite inequality. The proof is now complete. ■

5. COROLLARIES OF THE EQUIDISTRIBUTION THEOREM

Our first corollary of the equidistribution theorem shows that for a dynamical system (X, φ, L) , the invariant measures $\mu_{\varphi, v}$ reflect the v -adic distribution of the preperiodic points of the morphism φ . Recall that a closed point $x \in X$ is called *preperiodic* if its (topological) forward orbit $\{\varphi^n(x) : n = 1, 2, \dots\}$ is a finite set.

COROLLARY 5.1. *Let (X, φ, L) be an algebraic dynamical system over the function field K . For any generic net of preperiodic closed points $(x_\alpha)_{\alpha \in A}$ in X and any place v , we have the following weak convergence of measures on X_v^{an} :*

$$\lim_{\alpha \in A} \frac{1}{\deg(x_\alpha)} \sum_{y \in O_v(x_\alpha)} \deg_v(y) \delta_y = \mu_{\varphi, v}.$$

Proof. This is immediate from Theorem 1.1 upon noting that preperiodic points have dynamical height zero (Theorem 3.2(iv)). ■

The preceding corollary is meaningless unless we can find generic nets of preperiodic points. However, it is not difficult to show that preperiodic points in $X(\bar{K})$ are Zariski dense in X . Once Zariski density is established, it is not hard to construct a generic net of preperiodic points by a diagonalization argument; for example, see the beginning of the proof of Corollary 5.2.

If E is a finite extension of K , we let $[E : K]_s$ be the separable degree of E over K . Write $|X|$ for the set of closed points of a variety X .

COROLLARY 5.2. *Let (X, φ, L) be an algebraic dynamical system defined over the function field K , let Y be any subvariety of X , and let n be a positive integer. Suppose there exists a place v of K such that the support of the probability measure $\mu_{Y, \varphi, v}$ on X_v^{an} contains at least $n + 1$ points. Then there exists a positive number ε such that the set*

$$Y_n(\varepsilon) := \{y \in |Y| : h_\varphi(y) \leq \varepsilon \text{ and } [K(y) : K]_s \leq n\}$$

is not Zariski dense in Y .

Proof. If the theorem fails, then $Y_n(\varepsilon)$ is Zariski dense for each $\varepsilon > 0$. We begin by constructing a generic small net. Let A be the collection of all ordered pairs (F, ε) consisting of a proper Zariski closed subset F of Y and a positive real number ε . Then A becomes a directed set when we endow it with the partial ordering

$$(F, \varepsilon) \leq (F', \varepsilon') \Leftrightarrow F \subseteq F' \text{ and } \varepsilon \geq \varepsilon'.$$

For each pair $(F, \varepsilon) \in A$, select a point $y_{F, \varepsilon} \in Y_n(\varepsilon) \cap (Y \setminus F)$, a feat that is possible because $Y_n(\varepsilon)$ is Zariski dense. One checks easily that the net of points $(y_{F, \varepsilon})$ is generic and $h_\varphi(y_{F, \varepsilon}) \rightarrow 0$. For ease of notation, we now relabel this net as $(y_\alpha)_{\alpha \in A}$.

Let p_0, \dots, p_n be distinct points of Y_v^{an} in the support of $\mu_{Y, \varphi, v}$. By topological normality of analytic spaces associated to proper varieties [Ber90, Thm. 3.5.3], we can choose an open neighborhood U_i of p_i for each i with pairwise disjoint closures. Fix an index i_0 . Inside U_{i_0} , choose a compact neighborhood W of p_{i_0} . By Urysohn’s lemma we may find a continuous function $f : X_v^{\text{an}} \rightarrow [0, 1]$ such that $f|_W \equiv 1$ and $f|_{X_v^{\text{an}} \setminus U_{i_0}} \equiv 0$. Then Theorem 4.1 shows

$$\lim_{\alpha \in A} \frac{1}{\deg(y_\alpha)} \sum_{z \in O_v(y_\alpha)} \deg_v(z) f(z) = \int_{X_v^{\text{an}}} f d\mu_{Y, \varphi, v} \geq \mu_{Y, \varphi, v}(W) > 0.$$

Hence there exists $\alpha_0 \in A$ such that $O_v(y_\alpha) \cap U_{i_0} \neq \emptyset$ for all $\alpha \geq \alpha_0$. Repeating this argument for each index i , we can find $\alpha_1 \in A$ so that for any $i = 0, \dots, n$ and $\alpha \geq \alpha_1$, we have $O_v(y_\alpha) \cap U_i \neq \emptyset$.

For each point y_α , the set $O_v(y_\alpha)$ consists of at most n points by Corollary 6.2 in the appendix. But the $n + 1$ sets U_i are disjoint by construction, so we have a contradiction. ■

When $h_\varphi(Y) > 0$, the last corollary can be proved using the Theorem of Successive Minima with $\varepsilon = h_\varphi(Y)/2$. We recall the statement of the theorem and indicate how this works. Let Y be a variety defined over the function field K , and let \bar{L} be a semipositive metrized line bundle on Y with

ample underlying bundle L . Define the quantity

$$e_1(Y, \bar{L}) = \sup_{\substack{V \subset Y \\ \text{codim}(V, Y) = 1}} \left\{ \inf_{y \in |Y \setminus V|} h_{\bar{L}}(y) \right\},$$

where the supremum is over all closed subsets V of Y of pure codimension 1, and the infimum is over closed points of $Y \setminus V$. The Theorem of Successive Minima tells us that

$$e_1(Y, \bar{L}) \geq h_{\bar{L}}(Y).$$

This inequality was originally discovered by Zhang when K is replaced by a number field [Zha95a, Thm. 5.2], and it was proved by Gubler when K is a function field [Gub07, Lem. 4.1].

Now let \bar{L} be the semipositive metrized line bundle associated to a dynamical system (X, φ, L) and an isomorphism $\theta : \varphi^* L \xrightarrow{\sim} L^q$. By the Theorem of Successive Minima, given any $\delta > 0$ there exists a closed codimension-1 subset $V \subset Y$ so that

$$\inf\{h_{\varphi}(y) : y \in |Y \setminus V|\} > h_{\varphi}(Y) - \delta.$$

If $h_{\varphi}(Y) > 0$, then we may take $\delta = h_{\varphi}(Y)/2$. Corollary 5.2 follows immediately with $\varepsilon = h_{\varphi}(Y)/2$ since $Y_n(\varepsilon) \subset V$. In fact, this shows that $\bigcup_{n \geq 1} Y_n(\varepsilon)$ is not Zariski dense in Y when $h_{\varphi}(Y) > 0$.

COROLLARY 5.3. *Let (X, φ, L) be an algebraic dynamical system over the function field K , let Y be any subvariety, and let n be a positive integer. Suppose there exists a place v of K such that the support of the probability measure $\mu_{Y, \varphi, v}$ on X_v^{an} contains at least $n + 1$ points. Then the set of preperiodic closed points contained in Y of separable degree at most n over K is not Zariski dense in Y .*

The problem with these last two results is that one must have some knowledge of the support of the measure $\mu_{Y, \varphi, v}$ in order to utilize them. As we indicated at the end of Section 3.3, the support of the measure $\mu_{\varphi, v}$ is precisely one point if X is smooth and the dynamical system (X, φ, L) has good reduction at the place v . So we cannot apply the corollaries in the case of good reduction.

We expect a converse to be true. Suppose that X is geometrically connected and smooth over K (e.g., the projective space \mathbb{P}_K^d). If E is a finite extension of K and v is a place of K , we say that the dynamical system (X, φ, L) has *potential good reduction at v* if there exists a place w of E lying over v so that the base-changed dynamical system $(X_E, \varphi_E, L \otimes E)$ has good reduction at w . If (X, φ, L) does not have potential good reduction at v , then it has *genuinely bad reduction at v* . With these definitions in mind, we present the following folk conjecture which, when combined with Corollaries 5.2 and 5.3, would yield very pleasing arithmetic results:

CONJECTURE 5.4. *Let (X, φ, L) be a dynamical system defined over the function field K , and suppose X is smooth and geometrically connected. Then the support of the measure $\mu_{\varphi, v}$ is either a single point or a Zariski dense set according to whether (X, φ, L) has potential good reduction or genuinely bad reduction.*

The conjecture is true when X is a curve. See for example the manuscript of Baker and Rumely [BR08, §10.4] for the case $X = \mathbb{P}_K^1$. (Compare the article [Bak09] of Baker for a similar statement and arithmetic consequence.) In [Gub07] Gubler's work shows that if X is an abelian variety with totally degenerate reduction at a place v , then $\mu_{\varphi, v}$ has Zariski dense support. For an elliptic curve, totally degenerate reduction is the same as genuinely bad reduction. In fact, in this case there is a topological subspace of X_v^{an} homeomorphic to a circle in such a way that $\mu_{\varphi, v}$ is a Haar measure on this circle.

6. APPENDIX

PROPOSITION 6.1. *Let K be a field that is finitely generated over its prime field. Let E be a finite extension of K , v a discrete valuation of K , and K_v the completion of K with respect to v . Then there are at most $[E : K]_s$ valuations w extending v to E , and if E_w is the completion of E with respect to the valuation w , then there exists an isomorphism of K_v -algebras*

$$K_v \otimes_K E \cong \prod_{w|v} E_w.$$

Proof. If E is a separable extension of K , this is proved in [CF67, II.9-10]. Any algebraic extension can be decomposed as $K \subset E^s \subset E$, where E^s is the separable closure of K in E , and E/E^s is a purely inseparable extension. By tensoring first up to the separable closure, we may apply the result in the separable case and reduce to the situation where E/K is a purely inseparable extension. Thus we may suppose K has positive characteristic p . It now suffices to show that the valuation v extends in exactly one way to E , and that $K_v \otimes_K E \cong E_w$ holds. To that end, we may even reduce to the case where E is a simple nontrivial extension of K , i.e., there is $\gamma \in E \setminus K$ such that $E = K(\gamma)$.

We first argue $K_v \otimes_K E$ is a field. The valuation ring $O_v \subset K$, being the localization of an algebra of finite type over \mathbb{F}_p , is a G-ring [Mat89, §32]. Hence $O_v \rightarrow O_v^\wedge = K_v^\circ$ is a regular homomorphism, which implies $K_v = \text{Frac}(K_v^\circ)$ is geometrically regular over $K = \text{Frac}(O_v)$. In particular, $K_v \otimes_K E$ is a reduced ring.

On the other hand, as E is a simple purely inseparable extension of K , we may write $E = K[x]/(f(x))$ for some irreducible polynomial

$$f(x) = x^{p^n} - a = (x - \gamma)^{p^n},$$

some positive integer n , $a \in K$ and $\gamma \in E \setminus K$. Evidently, $K_v \otimes_K E = K_v[x]/(f(x))$ is reduced if and only if $\gamma \notin K_v$. Thus $f(x)$ is irreducible over K_v and $K_v \otimes_K E$ is a field. Note that it is the (unique) minimal extension of K_v containing E .

If F is any finite extension of K_v , then F inherits the unique extension of the valuation v and is complete with respect to the extended valuation. Therefore $K_v \otimes_K E$ is a complete field under the unique extension of v . Let w be the restriction of the extended valuation to $E \subset K_v \otimes_K E$. By continuity the completion E_w injects canonically into $K_v \otimes_K E$, and since $K_v \otimes_K E$ is the minimal extension of K_v containing E , we must have $E_w = K_v \otimes_K E$. We have already mentioned that v extends uniquely to $K_v \otimes_K E$, so the proof is complete. ■

COROLLARY 6.2. *Let X be a variety over the function field K as in previous sections. If $x \in |X|$ is a closed point, v is a place of K and $\psi : X_{K_v} \rightarrow X$ is the base change morphism, then there are at most $[K(x) : K]_s$ points in $\psi^{-1}(x)$, and*

$$[K(x) : K] = \sum_{y \in \psi^{-1}(x)} [K_v(y) : K_v].$$

Proof. The ring of functions on the scheme-theoretic fiber $\psi^{-1}(x)$ is $K_v \otimes_K K(x)$. Use Proposition 6.1 and compute dimensions over K_v . ■

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